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On Hermite-Hadamard-Fejér type inequalities without symmetry condition and applications

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Abstract. In this paper, with a new approach, a new Hermite-Hadamard-Fejér inequalities without symmetry condition for convex functions is obtained by using not only Riemann integrals but also the Riemann-Liouville fractional integral. Also, to have new fractional trapezoid and midpoint type inequalities for the differentiable convex functions, some new equalities are proved. Additionally, we will provide some examples of special cases that arise from these inequalities.

1. Introduction

The theory of convex functions plays a pivotal role in mathematics, with widespread applications in areas such as optimization theory, control theory, operations research, geometry, functional analysis, and information theory. Beyond mathematics, convexity is also fundamental in various applied fields, including economics, finance, engineering, and management sciences.

Among the most prominent results in the literature is the Hermite–Hadamard integral inequality (see [9]), which serves as a powerful tool in the study of convex functions. This inequality has profound implications and has been extensively examined, leading to the development of numerous techniques in mathematical analysis.

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{1}$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$.

The inequalities in (1) are reversed if f is concave. The double inequality (1) was first introduced by Ch. Hermite in 1881, published in the journal *Mathesis*. However, it remained largely unrecognized in the mathematical community until it was attributed to J. Hadamard, who proved the same inequality in 1893 (see [2]). Later, in 1974, D. S. Mitrinovic rediscovered Hermite's original note (see [9], [19]). As a result, the inequality is now known as the Hermite–Hadamard inequality.

Since its discovery, the Hermite–Hadamard inequality ([23], p. 139) has been considered one of the most fundamental results in mathematical analysis. It serves as a cornerstone for deriving specific inequalities

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involving various types of means by choosing appropriate forms of the function f. Moreover, the inequality in (1) also provides a necessary and sufficient condition for convexity on an interval (a, b). In [1], Azpeitia generalized the Hadamard inequality to Stieltjes integrals by assuming convexity on closed intervals or Jensen convexity under integrability assumptions. Additionally, new formulations and operators inspired by (1) have been presented in [5]–[8].

Fejér, in his 1906 study on trigonometric polynomials, proposed a weighted version of inequality (1). His generalization reads as follows (see [23], p. 138):

Theorem 1.1. *If $f : [a, b] \to \mathbb{R}$ is a convex function, then the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \leq \frac{1}{b-a}\int_a^b f(x)g(x)\,dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx, \tag{2}$$

where g is nonnegative, integrable, and symmetric about $x = \frac{a+b}{2}$.

Many mathematicians have addressed the important problem of estimating bounds for the following quantities appearing in inequalities (1) and (2):

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right|, \quad \left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right|,$$

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx - f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx\right|,$$

$$\left|\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)g(x)\,dx\right|.$$

In [10], Farid established Hadamard and Fejér–Hadamard inequalities for generalized fractional integrals involving the Mittag–Leffler function and derived related inequalities for special cases. Sarikaya, in [27], proved several weighted inequalities for differentiable mappings connected to the Hermite–Hadamard–Fejer type integral inequality, thus extending earlier results. Further extending these ideas, in [29], the authors obtained estimates on the right-hand side of Hermite–Hadamard–Fejér type inequalities for functions whose first derivatives' absolute values are s-convex, providing new generalizations.

In [4], Dragomir and Agarwal proved results related to the right part of inequality (1), often referred to as the trapezoid inequality.

Lemma 1.2. *Let $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ be differentiable function on I°, the interior of the interval I, where $a, b \in I^\circ$ with $a < b$. If $f' \in L[a, b]$, then the following identity holds;*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\int_a^b \left(x - \frac{a+b}{2}\right) f'(x)\,dx. \tag{3}$$

Theorem 1.3. *Let $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ be differentiable function on I°, the interior of the interval I, where $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds;*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{b-a}{4}\left(\frac{|f'(a)| + |f'(b)|}{2}\right). \tag{4}$$

As similar result to that of Theorem 1.3, Kirmaci gave the following results connected with the left part of (1) which is named Midpoint inequality in [13].

Lemma 1.4. *Let $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ be differentiable function on I°, the interior of the interval I, where $a, b \in I^\circ$ with $a < b$. If $f' \in L[a,b]$, then the following identity holds;*

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{(b-a)^2}\left[\int_a^{\frac{a+b}{2}} (x-a) f'(x)\,dx - \int_{\frac{a+b}{2}}^b (b-x) f'(x)\,dx\right]. \tag{5}$$

Theorem 1.5. *Let $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ be differentiable function on I°, the interior of the interval I, where $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds;*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{4}\left(\frac{|f'(a)| + |f'(b)|}{2}\right). \tag{6}$$

Subsequently, numerous researchers have made significant contributions to the development of new results concerning the Trapezoid and Midpoint inequalities by utilizing various classes of convex functions. These advancements have greatly broadened the theoretical understanding and practical applications of these inequalities in mathematical analysis; see, for example, [3], [18], [27], [36] and the references therein. Moreover, analogous techniques have been employed to derive Fejér-type inequalities, which serve as weighted generalizations of the classical Trapezoid and Midpoint inequalities. Such weighted inequalities have been particularly effective in scenarios involving non-uniform distributions or where terms carry different levels of importance; see, e.g., [22], [24], [27], [31], [37] and related references.

Regarding other generalizations and extensions of (1), a natural question arises whether the symmetry condition imposed on the weight function p in (2) can be relaxed or removed. In [22], this question is answered affirmatively. Specifically, it is shown that if $f : [a,b] \to \mathbb{R}$ is a convex function and $g : [a,b] \to \mathbb{R}$ is an integrable, positive, and normalized function (i.e., $\int_a^b g(t)\,dt = 1$), then the following double inequality holds:

$$f(\lambda a + \mu b) \le \int_a^b g(x) f(x)\,dx \le \lambda f(b) + \mu f(a) \tag{7}$$

where

$$\lambda = \frac{1}{b-a}\int_a^b (b-t)\, g(t)\,dt \quad \text{and} \quad \mu = \frac{1}{b-a}\int_a^b (t-a)\, g(t)\,dt. \tag{8}$$

It is straightforward to verify that if the function g is symmetric with respect to the midpoint $\frac{a+b}{2}$, then inequality (7) reduces to the classical form (2).

We now recall the definitions of the Riemann-Liouville fractional integral operators:

Definition 1.6. *Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \ge 0$ are defined by*

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively where $\Gamma(\alpha) = \int_0^\infty e^{-t} u^{\alpha-1} du$. Here is $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$.

The following Hermite-Hadamard inequality for fractional integrals was established by Sarikaya et al. in [26]:

Theorem 1.7. *Let $f : [a, b] \to \mathbb{R}$ be a function with $a < b$ and $f \in L_1([a, b])$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{9}$$

with $\alpha > 0$.

Fractional calculus, with its rich historical background, has witnessed a resurgence of interest in recent years, particularly within applied sciences. This revival is largely driven by the introduction of novel fractional derivative and integral operators, which have substantially enriched the theoretical landscape. These developments have yielded numerous new fractional operators in the literature, motivated by the exploration of properties such as singularity, locality, and variations in kernel structures of fractional derivatives and integrals.

Applications of fractional derivatives and integrals have demonstrated their effectiveness in modeling various real-world phenomena (see [25]). Consequently, the study of fractional differential equations necessitates further advances in inequalities of fractional type. For instance, in [12], new Hermite-Hadamard type inequalities are derived for generalized fractional integrals introduced by Mubeen et al. [21], along with results for specific fractional integrals from this generalization. In [14], a new fractional Hermite-Hadamard inequality is established based on the right Riemann-Liouville fractional integral; furthermore, equalities for fractional trapezoid and midpoint inequalities are proven, suggesting directions for future research on fractional Hermite-Hadamard inequalities. Additional works such as [15]-[17] extend Hermite-Hadamard-Fejér type inequalities to various convexities, derive related integral identities and inequalities, and generalize previous results for convex, harmonically convex, and p-convex functions. In [20], the concept of the λ-incomplete gamma function is introduced, Hermite-Hadamard inequalities involving tempered fractional integrals for convex functions are established, and examples involving modified Bessel functions and the q-digamma function are provided. For more examples of such inequalities, see also [27], [30], [32]-[35].

In the present paper, we propose a novel approach to derive new Hermite–Hadamard–Fejér inequalities for convex functions by relaxing the classical symmetry condition. These results are developed using both traditional Riemann integrals and Riemann–Liouville fractional integrals, showcasing the flexibility and generality of the method. Furthermore, we rigorously establish several new fractional trapezoid and midpoint-type inequalities for differentiable convex functions. The theoretical contributions are supported by illustrative examples to demonstrate the broad applicability and effectiveness of the proposed results.

2. Main results

To prove our main results, we require the following lemma:

Lemma 2.1. *Let $f : I \subset \mathbb{R}^{+} \to \mathbb{R}$ be differentiable function on I°, the interior of the interval I, where $a, b \in I^{\circ}$ with $a < b$, and $g : [a, b] \to (0, \infty)$ is integrable and normalized function. If $f' \in L[a, b]$, then the following identities hold,*

$$f(\lambda a + \mu b) - \int_{a}^{b} g(x) f(x)\,dx = \int_{a}^{\lambda a+\mu b}\left(\int_{a}^{x} g(t)\,dt\right) f'(x)\,dx - \int_{\lambda a+\mu b}^{b}\left(\int_{x}^{b} g(t)\,dt\right) f'(x)\,dx \tag{10}$$

and

$$\lambda f(b) + \mu f(a) - \int_{a}^{b} g(x) f(x)\,dx \tag{11}$$

$$= \int_a^b \left(\frac{1}{b-a} \int_a^x (b-t)\, g(t)\, dt \right) f'(x)\, dx - \int_a^b \left(\frac{1}{b-a} \int_x^b (t-a)\, g(t)\, dt \right) f'(x)\, dx$$

where λ and μ are defined as in (8).

Proof. By integration by parts, we have

$$\int_a^{\lambda a+\mu b} \left(\int_a^x g(t)\, dt \right) f'(x)\, dx - \int_{\lambda a+\mu b}^b \left(\int_x^b g(t)\, dt \right) f'(x)\, dx$$

$$= \left(\int_a^{\lambda a+\mu b} g(t)\, dt \right) f(\lambda a+\mu b) - \int_a^{\lambda a+\mu b} g(x) f(x)\, dx + \left(\int_{\lambda a+\mu b}^b g(t)\, dt \right) f(\lambda a+\mu b) - \int_{\lambda a+\mu b}^b g(x) f(x)\, dx$$

$$= f(\lambda a+\mu b) - \int_a^b g(x) f(x)\, dx,$$

which is completed the proof of (10). By similar way, we get

$$\int_a^b \left(\frac{1}{b-a} \int_a^x (b-t)\, g(t)\, dt \right) f'(x)\, dx \tag{12}$$

$$= \left(\frac{1}{b-a} \int_a^b (b-t)\, g(t)\, dt \right) f(b) - \frac{1}{b-a} \int_a^b (b-x)\, g(x) f(x)\, dx$$

and

$$\int_a^b \left(\frac{1}{b-a} \int_a^x (b-t)\, g(t)\, dt \right) f'(x)\, dx \tag{13}$$

$$= -\left(\frac{1}{b-a} \int_a^b (t-a)\, g(t)\, dt \right) f(a) + \frac{1}{b-a} \int_a^b (x-a)\, g(x) f(x)\, dx.$$

By subtracting from (12) to (13), we get the desired equality (11). □

Remark 2.2. *In Lemma 2.1, if we choose $g(x) = \frac{1}{b-a}$, then the equalities (10) and (11) reduce to (5) and (3), respectively.*

In Lemma 2.1, if we choose $g(x) = \frac{\alpha}{(b-a)^\alpha}(x-a)^{\alpha-1}$, then we obtain

$$\lambda = \frac{1}{\alpha+1} \quad \text{and} \quad \mu = \frac{\alpha}{\alpha+1}.$$

Thus, equality (10) reduces to the following form:

$$f(b) - \frac{\alpha}{(b-a)^\alpha} \int_a^b (x-a)^{\alpha-1} f(x)\, dx = \frac{1}{(b-a)^\alpha} \int_a^b (x-a)^\alpha f'(x)\, dx.$$

By multiplying both sides of this equation by $\frac{1}{\Gamma(\alpha+1)}$, we obtain

$$\frac{f(b)}{\Gamma(\alpha+1)} - \frac{1}{(b-a)^\alpha} J_{b-}^\alpha f(a) = \frac{1}{(b-a)^\alpha} J_{b-}^{\alpha+1} f'(a). \tag{14}$$

In a similar way, (11) becomes

$$\begin{aligned}
&\frac{f(b)+\alpha f(a)}{\alpha+1}-\frac{\alpha}{(b-a)^{\alpha}}\int_a^b (x-a)^{\alpha-1}f(x)\,dx\\
=\;& \int_a^b\left(\frac{\alpha}{(b-a)^{\alpha+1}}\int_a^x (b-t)(t-a)^{\alpha-1}\,dt\right)f'(x)\,dx-\int_a^b\left(\frac{\alpha}{(b-a)^{\alpha+1}}\int_x^b (t-a)^{\alpha}\,dt\right)f'(x)\,dx\\
=\;& \frac{1}{(b-a)^{\alpha}}\int_a^b (x-a)^{\alpha}f'(x)\,dx-\frac{\alpha}{\alpha+1}\left[f(b)-f(a)\right].
\end{aligned}$$

By multiplying both sides by $\frac{1}{\Gamma(\alpha+1)}$, we obtain the same result as in (14).

Similarly, in Lemma 2.1, if we choose $g(x)=\frac{\alpha}{(b-a)^{\alpha}}(b-x)^{\alpha-1}$, then

$$\lambda=\frac{\alpha}{\alpha+1}\quad\text{and}\quad \mu=\frac{1}{\alpha+1}.$$

Thus, equality (10) becomes

$$f(a)-\frac{\alpha}{(b-a)^{\alpha}}\int_a^b (b-x)^{\alpha-1}f(x)\,dx=-\frac{1}{(b-a)^{\alpha}}\int_a^b (b-x)^{\alpha}f'(x)\,dx.$$

By multiplying both sides of this equation by $\frac{1}{\Gamma(\alpha+1)}$, we obtain

$$\frac{f(a)}{\Gamma(\alpha+1)}-\frac{1}{(b-a)^{\alpha}}J_{a+}^{\alpha}f(b)=-\frac{1}{(b-a)^{\alpha}}J_{a+}^{\alpha+1}f'(b). \tag{15}$$

Similarly, (11) becomes

$$\begin{aligned}
&\frac{\alpha f(b)+f(a)}{\alpha+1}-\frac{\alpha}{(b-a)^{\alpha}}\int_a^b (b-x)^{\alpha-1}f(x)\,dx\\
=\;& \int_a^b\left(\frac{\alpha}{(b-a)^{\alpha+1}}\int_a^x (b-t)^{\alpha}\,dt\right)f'(x)\,dx-\int_a^b\left(\frac{\alpha}{(b-a)^{\alpha+1}}\int_x^b (t-a)(b-t)^{\alpha-1}\,dt\right)f'(x)\,dx\\
=\;& \frac{\alpha}{\alpha+1}\left[f(b)-f(a)\right]-\frac{1}{(b-a)^{\alpha}}\int_a^b (b-x)^{\alpha}f'(x)\,dx.
\end{aligned}$$

By multiplying both sides by $\frac{1}{\Gamma(\alpha+1)}$, we obtain the same result as in (15).

Now, by adding (14) and (15), we derive the following equality for fractional integrals:

$$\frac{f(a)+f(b)}{\Gamma(\alpha+1)}-\frac{1}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]=\frac{1}{(b-a)^{\alpha}}\left[J_{b-}^{\alpha+1}f'(a)-J_{a+}^{\alpha+1}f'(b)\right].$$

Finally, by multiplying both sides by $\frac{\Gamma(\alpha+1)}{2}$, we obtain

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]=\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{b-}^{\alpha+1}f'(a)-J_{a+}^{\alpha+1}f'(b)\right],$$

which was proved by Sarikaya et al. in [26].

Example 2.3. *In Lemma 2.1, we choose the normalized function* $g(x)=\sin x$ *on the interval* $\left[0,\frac{\pi}{2}\right]$*. Then, we have*

$$\lambda=\frac{2}{\pi}\left(\frac{\pi}{2}-1\right)\quad\textit{and}\quad \mu=\frac{2}{\pi}.$$

Thus, equality (10) reduces to

$$f(1)-\int_0^{\frac{\pi}{2}} f(x)\sin x\,dx=\int_0^1(1-\cos x)f'(x)\,dx-\int_1^{\frac{\pi}{2}}\cos x\,f'(x)\,dx,$$

and similarly, equality (11) reduces to

$$\begin{aligned}&\left(\frac{\pi}{2}-1\right)f\left(\frac{\pi}{2}\right)+f(0)-\frac{\pi}{2}\int_0^{\frac{\pi}{2}} f(x)\sin x\,dx\\ =\ &\int_0^{\frac{\pi}{2}}\left(\frac{\pi}{2}-\left(\frac{\pi}{2}-x\right)\cos x-\sin x\right)f'(x)\,dx-\int_0^{\frac{\pi}{2}}(x\cos x+1-\sin x)\,f'(x)\,dx.\end{aligned}$$

Theorem 2.4. *Under the assumptions of Lemma 2.1, if $|f'|$ is convex on $[a,b]$ and g is a bounded function on $[a,b]$, then the following inequalities hold:*

$$\begin{aligned}&\left|f(\lambda a+\mu b)-\int_a^b g(x)f(x)\,dx\right| \\ \leq\ &\|g\|_\infty|f'(\lambda a+\mu b)|\left[\frac{(\lambda a+\mu b-a)^2}{3}+\frac{(b-(\lambda a+\mu b))^2}{3}\right]\\ &+\|g\|_\infty\frac{(\lambda a+\mu b-a)^2}{6}|f'(a)|+\|g\|_\infty\frac{(b-(\lambda a+\mu b))^2}{6}|f'(b)|,\end{aligned} \tag{16}$$

and

$$\begin{aligned}&\left|\lambda f(b)+\mu f(a)-\int_a^b g(x)f(x)\,dx\right| \\ \leq\ &\frac{(b-a)^2}{6}\|g\|_\infty\left(\frac{|f'(a)|+|f'(b)|}{2}+\frac{1}{2}|f'(\frac{a+b}{2})|\right)\\ \leq\ &\frac{(b-a)^2}{4}\|g\|_\infty\frac{|f'(a)|+|f'(b)|}{2},\end{aligned} \tag{17}$$

where $\|g\|_\infty:=\sup_{x\in[a,b]}|g(x)|<+\infty$.

Proof. Since $|f'|$ is convex on $[a,b]$, for $x\in[a,\lambda a+\mu b]$ we have

$$\begin{aligned}|f'(x)| &= \left|f'\left(\frac{\lambda a+\mu b-x}{\lambda a+\mu b-a}a+\frac{x-a}{\lambda a+\mu b-a}(\lambda a+\mu b)\right)\right|\\ &\leq \frac{\lambda a+\mu b-x}{\lambda a+\mu b-a}|f'(a)|+\frac{x-a}{\lambda a+\mu b-a}|f'(\lambda a+\mu b)|,\end{aligned}$$

and for $x\in[\lambda a+\mu b,b]$,

$$\begin{aligned}|f'(x)| &= \left|f'\left(\frac{b-x}{b-(\lambda a+\mu b)}(\lambda a+\mu b)+\frac{x-(\lambda a+\mu b)}{b-(\lambda a+\mu b)}b\right)\right|\\ &\leq \frac{b-x}{b-(\lambda a+\mu b)}|f'(\lambda a+\mu b)|+\frac{x-(\lambda a+\mu b)}{b-(\lambda a+\mu b)}|f'(b)|.\end{aligned}$$

Using identity (10) and the convexity of $|f'|$, we get

$$\begin{aligned}&\left|f(\lambda a+\mu b)-\int_a^b g(x)f(x)\,dx\right|\\ \leq\ &\int_a^{\lambda a+\mu b}\left(\int_a^x|g(t)|\,dt\right)|f'(x)|\,dx+\int_{\lambda a+\mu b}^b\left(\int_x^b|g(t)|\,dt\right)|f'(x)|\,dx\end{aligned}$$

$$\leq \quad \|g\|_\infty \int_a^{\lambda a+\mu b} (x-a)|f'(x)|\,dx + \|g\|_\infty \int_{\lambda a+\mu b}^{b} (b-x)|f'(x)|\,dx,$$

and after computing the integrals using the convex combination, we obtain (16).

Similarly, taking the absolute value of (11) and using the convexity of $|f'|$, we get (17). □

Remark 2.5. *In Theorem 2.4, we choose $g(x) = \frac{1}{b-a}$, then the inequalities (16) and (17) become the inequalities (6) and (4) respectively.*

Example 2.6. *We apply Theorem 2.4 with the normalized function $g(x) = \sin x$ on $[0, \frac{\pi}{2}]$ and $f(x) = x^2$ on $[0, \frac{\pi}{2}]$. Then we get*

$$\lambda = \frac{2}{\pi}\left(\frac{\pi}{2}-1\right) \quad \text{and} \quad \mu = \frac{2}{\pi}.$$

Thus, the inequality (16) becomes

$$0 \leq \frac{1}{3} + \frac{(\frac{\pi}{2}-1)^2}{2}.$$

Moreover, (17) yields

$$\left|\frac{2}{\pi}\left(\frac{\pi}{2}-1\right) f\left(\frac{\pi}{2}\right) + \frac{2}{\pi} f(0) - \int_0^{\frac{\pi}{2}} \sin x\, f(x)\,dx\right| \leq \frac{(\frac{\pi}{2})^2}{4}\left(\frac{|f'(0)| + |f'(\frac{\pi}{2})|}{2}\right),$$

which gives

$$\left|\frac{\pi}{2}-2\right| \leq \frac{\pi^2}{32}.$$

Theorem 2.7. *Suppose that all the assumptions of Lemma 2.1 hold. If $\left|f'\right|^q$ is convex on $[a,b]$ for some fixed $q > 1$, and g is a bounded function on $[a,b]$, then the following inequalities hold:*

$$\left|f(\lambda a+\mu b) - \int_a^b g(x) f(x)\,dx\right| \tag{18}$$

$$\leq \quad \frac{\left\|g\right\|_\infty}{(p+1)^{\frac{1}{p}}}\left\{(\lambda a+\mu b-a)^2\left(\frac{\left|f'(a)\right|^q + \left|f'(\lambda a+\mu b)\right|^q}{2}\right)^{\frac{1}{q}} + (b-\lambda a-\mu b)^2\left(\frac{\left|f'(\lambda a+\mu b)\right|^q + \left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}\right\}$$

and

$$\left|\lambda f(b) + \mu f(a) - \int_a^b g(x) f(x)\,dx\right| \tag{19}$$

$$\leq \quad (b-a)^2\left(\frac{2p}{2p+1}\right)^{\frac{1}{p}}\left\|g\right\|_\infty\left(\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. We use the identity (10), apply Hölder's inequality, and, by using the convexity of $\left|f'\right|^q$, we have:

$$\left|f(\lambda a+\mu b) - \int_a^b g(x) f(x)\,dx\right|$$

$$\leq \left(\int_a^{\lambda a+\mu b}\left(\int_a^x |g(t)|\,dt\right)^p dx\right)^{\frac{1}{p}}\left(\int_a^{\lambda a+\mu b}|f'(x)|^q dx\right)^{\frac{1}{q}}+\left(\int_{\lambda a+\mu b}^b\left(\int_x^b |g(t)|\,dt\right)^p dx\right)^{\frac{1}{p}}\left(\int_{\lambda a+\mu b}^b |f'(x)|^q dx\right)^{\frac{1}{q}}$$

$$\leq \|g\|_\infty \frac{(\lambda a+\mu b-a)^{1+\frac{1}{p}}}{(p+1)^{\frac{1}{p}}}\left(\int_a^{\lambda a+\mu b}\left[\frac{\lambda a+\mu b-x}{\lambda a+\mu b-a}|f'(a)|^q+\frac{x-a}{\lambda a+\mu b-a}|f'(\lambda a+\mu b)|^q\right]dx\right)^{\frac{1}{q}}$$

$$+\|g\|_\infty \frac{(b-\lambda a-\mu b)^{1+\frac{1}{p}}}{(p+1)^{\frac{1}{p}}}\left(\int_{\lambda a+\mu b}^b\left[\frac{b-x}{b-(\lambda a+\mu b)}|f'(\lambda a+\mu b)|^q+\frac{x-(\lambda a+\mu b)}{b-(\lambda a+\mu b)}|f'(b)|^q\right]dx\right)^{\frac{1}{q}}$$

$$= \|g\|_\infty \frac{(\lambda a+\mu b-a)^{1+\frac{1}{p}}}{(p+1)^{\frac{1}{p}}}\left(\frac{(\lambda a+\mu b-a)}{2}\left[|f'(a)|^q+|f'(\lambda a+\mu b)|^q\right]\right)^{\frac{1}{q}}$$

$$+\|g\|_\infty \frac{(b-\lambda a-\mu b)^{1+\frac{1}{p}}}{(p+1)^{\frac{1}{p}}}\left(\frac{(b-(\lambda a+\mu b))}{2}\left[|f'(\lambda a+\mu b)|^q+|f'(b)|^q\right]\right)^{\frac{1}{q}}.$$

This proves inequality (18).

Now, taking the absolute value of (11) and using the convexity of $|f'|^q$, we have:

$$\left|\lambda f(b)+\mu f(a)-\int_a^b g(x)f(x)\,dx\right|$$

$$\leq \|g\|_\infty\left\{\left(\int_a^b\left(\frac{(b-a)^2-(b-x)^2}{2(b-a)}\right)^p dx\right)^{\frac{1}{p}}+\left(\int_a^b\left(\frac{(b-a)^2-(x-a)^2}{2(b-a)}\right)^p dx\right)^{\frac{1}{p}}\right\}\left(\int_a^b |f'(x)|^q dx\right)^{\frac{1}{q}}$$

$$\leq (b-a)^2\left(\frac{2p}{2p+1}\right)^{\frac{1}{p}}\|g\|_\infty\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}.$$

Here, we use

$$(A-B)^p \leq A^p - B^p$$

for any $A > B \geq 0$ and $p \geq 1$. This proves inequality (19). □

Remark 2.8. *Under assumption of Theorem 2.7, we choose $g(x)=\frac{1}{b-a}$, then the inequalities (18) becomes the following inequality,*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$

$$\leq \frac{(b-a)}{4(p+1)^{\frac{1}{p}}}\left[\left(\frac{|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{(b-a)}{4(p+1)^{\frac{1}{p}}}\left[\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right]$$

which is proved by Kirmaci in [13].

Corollary 2.9. *Suppose that all the assumptations of Theorem 2.7. Then, the following inequalities hold*

$$\left|\frac{f(b)+f(a)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le (b-a)\left(\frac{2p}{2p+1}\right)^{\frac{1}{p}}\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}.$$

Proof. We apply Theorem 2.7 with the normalized function $g(x)=\frac{1}{b-a}$, then the proof follows directly. □

Corollary 2.10. *Suppose that all the assumptations of Theorem 2.7. Then, the following inequalities hold*

$$\begin{aligned}&\left|f\left(\frac{\alpha a+b}{\alpha+1}\right)-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}J_{b-}^\alpha f(a)\right|\\ \le\ & \frac{\alpha(b-a)}{(p+1)^{\frac{1}{p}}}\left\{\left(\frac{1}{\alpha+1}\right)^2\left(\frac{\left|f'(a)\right|^q+\left|f'\left(\frac{\alpha a+b}{\alpha+1}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\alpha}{\alpha+1}\right)^2\left(\frac{\left|f'\left(\frac{\alpha a+b}{\alpha+1}\right)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}\right\}\end{aligned}$$

and

$$\left|\frac{f(a)+\alpha f(b)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}J_{b-}^\alpha f(a)\right| \le \alpha(b-a)\left(\frac{2p}{2p+1}\right)^{\frac{1}{p}}\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}$$

with $\alpha>0$.

Proof. We apply Theorem 2.7 with the normalized function $g(x)=\frac{\alpha}{(b-a)^\alpha}(x-a)^{\alpha-1}$, then the proof follows directly. □

Corollary 2.11. *Suppose that all the assumptations of Theorem 2.7. Then, the following inequalities hold*

$$\begin{aligned}&\left|f\left(\frac{\alpha a+b}{\alpha+1}\right)-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}J_{a+}^\alpha f(b)\right|\\ \le\ & \frac{\alpha(b-a)}{(p+1)^{\frac{1}{p}}}\left\{\left(\frac{1}{\alpha+1}\right)^2\left(\frac{\left|f'(a)\right|^q+\left|f'\left(\frac{\alpha a+b}{\alpha+1}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\alpha}{\alpha+1}\right)^2\left(\frac{\left|f'\left(\frac{\alpha a+b}{\alpha+1}\right)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}\right\}\\ \le\ & \frac{\alpha(b-a)}{(\alpha+1)^{2+\frac{1}{q}}(p+1)^{\frac{1}{p}}}\left\{\left(\frac{(2\alpha+1)\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}+\alpha^2\left(\frac{\left|f'(a)\right|^q+(2\alpha+1)\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}\right\}\end{aligned}$$

and

$$\begin{aligned}&\left|\frac{\alpha f(a)+f(b)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}J_{a+}^\alpha f(b)\right|\\ \le\ & \alpha(b-a)\left(\frac{2p}{2p+1}\right)^{\frac{1}{p}}\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}\end{aligned}$$

with $\alpha>0$.

Proof. We apply Theorem 2.7 with the normalized function $g(x)=\frac{\alpha}{(b-a)^\alpha}(b-x)^{\alpha-1}$, then the proof follows directly. □

Example 2.12. *We apply Corollary 2.10 and Corollary 2.11 with the function $f(x) = x^n$ and $\alpha = n(n \in N)$. By change of variable $x = tb + (1-t)a$ and using Binomial formula, we have*

$$\begin{aligned}
J_{b-}^{n} f(a) &= \frac{1}{\Gamma(n)} \int_a^b (x-a)^{n-1} x^n dx = \frac{(b-a)^n}{\Gamma(n)} \int_a^b t^{n-1} (tb + (1-t)a)^n dt \\
&= \frac{(b-a)^n}{\Gamma(n)} \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \int_a^b t^{2n-k-1} (1-t)^k dt \\
&= \frac{(b-a)^n}{\Gamma(n)} \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} B(2n-k, k+1) \\
&= \frac{(b-a)^n}{\Gamma(n)} \sum_{k=0}^{n} \binom{n}{k} \frac{(k)!\,(2n-k-1)!}{(2n)!} a^k b^{n-k}.
\end{aligned}$$

Thus, we obtain that

$$\begin{aligned}
&\sum_{k=0}^{n} \frac{(n)!\,(2n-k-1)!}{(n-k)!\,(2n-1)!} a^k b^{n-k} \\
\leq\ & \frac{n^2(b-a)}{(p+1)^{\frac{1}{p}}} \left\{ \left(\frac{1}{n+1}\right)^2 A^{\frac{1}{q}} \left(a^{(n-1)q}, W^{(n-1)q}(b,a)\right) + \left(\frac{n}{n+1}\right)^2 A^{\frac{1}{q}} \left(b^{(n-1)q}, W^{(n-1)q}(b,a)\right) \right\}
\end{aligned}$$

and

$$\left| W(a^n, b^n) - \frac{1}{2} \sum_{k=0}^{n} \frac{(n)!\,(2n-k-1)!}{(n-k)!\,(2n-1)!} a^k b^{n-k} \right| \leq n^2 (b-a) \left(\frac{2p}{2p+1}\right)^{\frac{1}{p}} A^{\frac{1}{q}} \left(a^{(n-1)q}, b^{(n-1)q}\right).$$

where $W(a,b) = \frac{a+nb}{n+1}$ is the weighted arithmetic mean and $A(a,b) = \frac{a+b}{2}$ is the arithmetic mean for $a, b > 0$.

By similar way we get

$$J_{a+}^{\alpha} f(b) = \frac{(b-a)^n}{\Gamma(n)} \sum_{k=0}^{n} \binom{n}{k} \frac{(n-k)!\,(n+k-1)!}{(2n)!} a^k b^{n-k}.$$

Therefore, we have

$$\begin{aligned}
&\left| W^n(a,b) - \frac{1}{2} \sum_{k=0}^{n} \frac{(n)!\,(n+k-1)!}{(k)!\,(2n-1)!} a^k b^{n-k} \right| \\
\leq\ & \frac{n^2(b-a)}{(p+1)^{\frac{1}{p}}} \left\{ \left(\frac{1}{n+1}\right)^2 A^{\frac{1}{q}} \left(a^{(n-1)q}, W^{(n-1)q}(a,b)\right) + \left(\frac{n}{n+1}\right)^2 A^{\frac{1}{q}} \left(b^{(n-1)q}, W^{(n-1)q}(a,b)\right) \right\}
\end{aligned}$$

and

$$\left| W(b^n, a^n) - \frac{1}{2} \sum_{k=0}^{n} \frac{(n)!\,(n+k-1)!}{(k)!\,(2n-1)!} a^k b^{n-k} \right| \leq n^2 (b-a) \left(\frac{2p}{2p+1}\right)^{\frac{1}{p}} A^{\frac{1}{q}} \left(a^{(n-1)q}, b^{(n-1)q}\right).$$

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