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# Approximation by Szász-type operators based on Bernoulli polynomials of negative order

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**Abstract.** In this study, we define a Szász-type operator sequence derived from Bernoulli polynomials of negative order. By computing the moments and central moments of the operator, we show via the Korovkin-type method the uniform convergence of the operator. We then determine the operator's rate of convergence by means of the modulus of continuity, Peetre's $\mathcal{K}$-function, and the Lipschitz class. We also look at their local as well as global convergence. For the smooth functions, we derive estimates of the rate of convergence and asymptotic behavior of the operator. Furthermore, we establish a generalization of the operator we are currently dealing with by using the Taylor series. We conclude by providing examples to validate and contrast the research's conclusions.

## 1. Introduction

Algebraic polynomials represent a large family of functions, as they are flexible structures expressible through finite combinations of powers and coefficients. The Weierstrass theorem illustrates both the prevalence of polynomials in this context and their capacity to approximate a wide range of functions. In 1912, Bernstein constructed a family of concrete and practical polynomials bearing his name. Researchers have constructed various modifications of Bernstein polynomials to investigate approximations in different areas. Researchers such as Wright, Cholodovsky, Szász and Meyer-König and Zeller [12, 24, 35, 38] have contributed to these modifications. Bohman and Korovkin extended Bernstein's theorem to monotone operators, demonstrating a practical method for approximating functions in the space of continuous functions using polynomials. Today, the study of approximation using these theorems is known as the Korovkin-type approximation (see [5]). Other significant works on monotone operators have been carried out [9–11, 16]. Otherside Bernoulli numbers and polynomials essentially play a central role in number theory and combinatorics. They appear naturally in formulas such as the Euler–Maclaurin summation formula and are essential in evaluating the Riemann zeta function at negative integers. In number theory, they are linked to sums of powers of integers and various congruence relations. For further studies in these areas, we refer to

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[1, 13, 28]. A key concern in approximation theory is the rate of convergence, which quantifies how quickly an operator approaches a given function. The rapidity is a crucial aspect in evaluating the efficiency and accuracy of approximation methods. Popoviciu and Voronovskaya [36, 37] studied the rate of convergence of Bernstein polynomials. Appell [6] introduced a sequence of polynomials, $\{A_k(x)\}_{k\geq 0}$, now known as Appell polynomials, which satisfy the recurrence relation:

$$\frac{d}{dx}A_k(x) = kA_{k-1}(x), a_0 \neq 0, k \in \mathbb{N},$$

and the generating function of these polynomials is as follows:

$$A(t)e^{xt} = \sum_{k=0}^{\infty} A_k(x)\frac{t^k}{k!},$$

where $A(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!}, a_0 \neq 0$. For detail information for Appell polynomial see [23]. Jakimovski and Leviatan [18] constructed a generalization of Szász operators using Appell polynomials. Later, Ismail [17] introduced a generalization of Szász and Jakimovski–Leviatan operators. These works have pioneered the study of approximation of linear operators constructed using Appell polynomials and their various modifications. In [25], the adjunction form of Bernoulli polynomials was given, and it was easily shown that they belong to the Appell class via their generating function. The use of Bernoulli polynomials of negative order as a basis for constructing an operator was first introduced in [39]. For further studies on various aspects of negative order Bernoulli polynomials, we recommend the works in [2–4, 14, 15, 19, 26, 29–31]. Bernoulli polynomials of negative order, $\{B_k^{(-1)}(x)\}_{k\geq 0}$, with the exponential generating function as follows:

$$\frac{e^t-1}{t}e^{xt} = \sum_{k=0}^{\infty} B_k^{(-1)}(x)\frac{t^k}{k!} \tag{1}$$

Equation (1) is derived within the framework of array polynomials. A more general version can be observed in [33]. Using Taylor series of $\frac{e^t-1}{t}e^{xt}$, the following is obtained:

$$\begin{aligned} &1 + \left(x + \frac{1}{2}\right)t + \left(x^2 + x + \frac{1}{3}\right)\frac{t^2}{2!} + \left(x^3 + \frac{3}{2}x^2 + x + \frac{1}{4}\right)\frac{t^3}{3!} \\ &+ \left(x^4 + 2x^3 + 2x^2 + x + \frac{1}{5}\right)\frac{t^4}{4!} + O(t^5), \end{aligned} \tag{2}$$

where fancy $O$ denotes big-O notation.
Utilizing equation (2), the first five Bernoulli polynomials of negative order are presented as follows

$$\begin{aligned} B_0^{(-1)}(x) &= 1 \\ B_1^{(-1)}(x) &= x + \frac{1}{2} \\ B_2^{(-1)}(x) &= x^2 + x + \frac{1}{3} \\ B_3^{(-1)}(x) &= x^3 + \frac{3}{2}x^2 + x + \frac{1}{4} \\ B_4^{(-1)}(x) &= x^4 + 2x^3 + 2x^2 + x + \frac{1}{5} \end{aligned}$$

For this family of polynomials, the evaluation at $x = 0$ produces the Bernoulli numbers of negative order. As shown in [20], for each $n$ these values are given explicitly by $\frac{1}{n+1}$, and it is evident that they are strictly positive. For more, see [21].

Figure 1 illustrates the graph of the Bernoulli polynomials of negative order listed above.

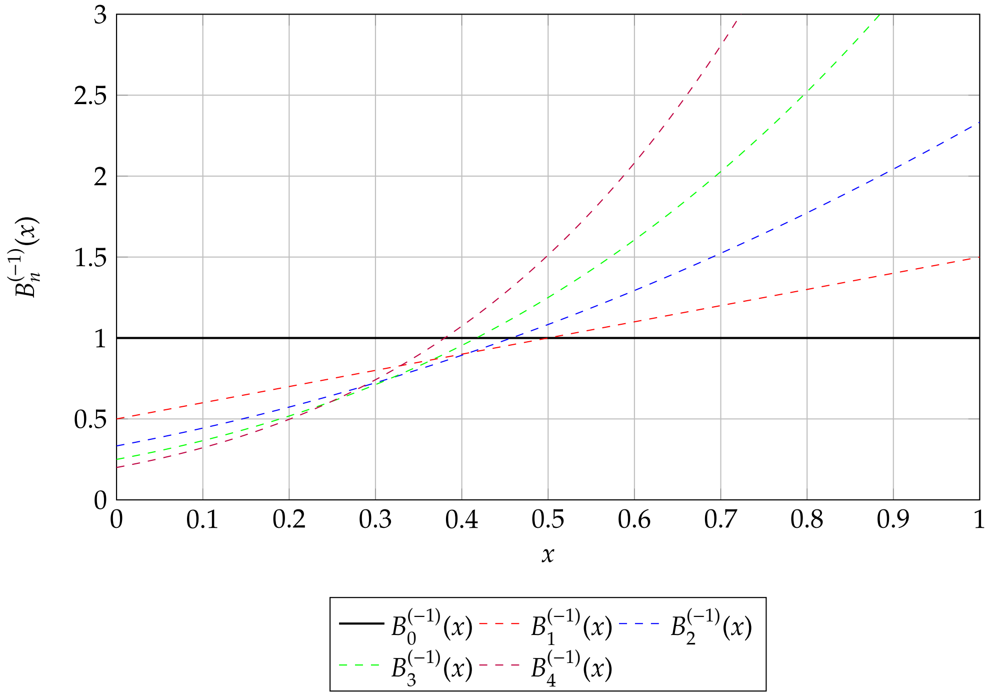


Figure 1: Bernoulli polynomials of negative order, $B_n^{(-1)}(x)$ for $n = 0$ to 4, plotted on the interval $[0, 1]$.

The notations employed throughout this paper are defined below.
A:=[a, b], $0 \leq a \leq b$, is any arbitrary closed subset in $\mathbb{R}^+ \cup \{0\}$. The real space $C(A)$ involves of real valued continuous functions with the norm $\|h\| = \max_{x\in A} \|h(x)\|$.
$C^r(A) = \left\{ h \in C(A) : h^{(r)} \in C(A) \right\}$.
Substituting $x \to nx$ and taking $t = 1$ at (1), we construct the Szász-type operators based on Bernoulli polynomials of negative order at the following:

**Definition 1.1.** *Let $L_n : C(A) \to C(A)$. For $n \in \mathbb{N}$ and $h \in C(A)$, the operators $\{L_n\}_{n\geq 0}$ defined by*

$$L_n(h; x) := \frac{e^{-nx}}{e-1} \sum_{k=0}^{\infty} \frac{B_k^{(-1)}(nx)}{k!} h\left(\frac{k}{n}\right), \tag{3}$$

*where $\{B_k^{(-1)}(x)\}_{k\geq 0}$ are the Szász-type sequence of Bernoulli polynomials of negative order.*

In this paper, we construct a Szász-type sequence of operators using the generating function of the Bernoulli polynomials of negative order. We analyze some approximation properties of this operator sequence within the framework of Korovkin-type approximation. The paper is organized as follows:

In Section 2, we present some auxiliary lemmas that are essential for proving the main theorems.

In Section 3, we examine the rate of convergence of the sequence $\{L_n\}_{n\geq 0}$, which we have defined and shown to be uniformly convergent. We analyze this convergence from multiple perspectives. First, we investigate the rate of convergence both locally and globally using the modulus of continuity. Next, we estimate the rate of convergence for smooth functions by applying the Shisha-Mond theorem [34]. Finally, we provide further estimates for the rate of convergence using Peetre's $\mathcal{K}$-functional and Lipschitz class concepts.

In Section 4, we derive an asymptotic formula for the sequence $\{L_n\}_{n\geq 0}$ in the sense of Voronovskaya.

In Section 5, we introduce a generalization of $\{L_n\}_{n\geq 0}$ that incorporates derivatives, extending the applicability of our results.

In Section 6, we provide two examples related to the deviation of the operator. In these examples, we compare the theoretical results obtained in the previous sections, demonstrating the practical implications of our findings.

## 2. Auxiliary Results

In this section, we introduce the key properties of the operators $\{L_n\}_{n\geq 0}$ which will be utilized in the proofs of the main theorems.

**Lemma 2.1.** *Let $x \in \mathbb{R}^+$. The Bernoulli polynomials of negative order $\{B_k^{(-1)}(x)\}_{k\geq 0}$ are positive, i.e.,*

$$B_k^{(-1)}(x) \geq 0, \quad \forall k \geq 0.$$

*Proof.* Recall that the Bernoulli polynomials of negative order are defined via the generating function

$$\sum_{k=0}^{\infty} \frac{B_k^{(-1)}(x)}{k!} t^k = \frac{e^t - 1}{t} e^{xt}.$$

Using the identity

$$\frac{e^t - 1}{t} = \int_0^1 e^{ut}\, du,$$

we obtain an integral representation for $B_k^{(-1)}(x)$:

$$\sum_{k=0}^{\infty} \frac{B_k^{(-1)}(x)}{k!} t^k = \int_0^1 e^{(x+u)t}\, du = \int_0^1 \sum_{k=0}^{\infty} \frac{(x+u)^k}{k!} t^k\, du.$$

Comparing coefficients of $t^k$ on both sides yields

$$\frac{B_k^{(-1)}(x)}{k!} = \int_0^1 \frac{(x+u)^k}{k!}\, du \geq 0,$$

since $x > 0$ and $u \in [0, 1]$. Multiplying both sides by $k!$ gives

$$B_k^{(-1)}(x) \geq 0, \quad \forall k \geq 0.$$

This completes the proof. □

**Theorem 2.2.** *The operators $L_n(h; x)$ defined in (3) are linear and positive.*

*Proof.* Let $h \in C(A)$ and define the weights

$$w_{n,k}(x) := \frac{e^{-nx}}{e-1} \frac{B_k^{(-1)}(nx)}{k!}, \quad k \geq 0.$$

By Lemma 2.1, $B_k^{(-1)}(nx) \geq 0$ for all $x \in A$, hence $w_{n,k}(x) \geq 0$. The weights are normalized:

$$\sum_{k=0}^{\infty} w_{n,k}(x) = \frac{e^{-nx}}{e-1} \int_0^1 \sum_{k=0}^{\infty} \frac{(nx+u)^k}{k!}\, du = 1.$$

Consequently, for any nonnegative $h$, we have

$$L_n(h;x) = \sum_{k=0}^{\infty} w_{n,k}(x)\, h\left(\frac{k}{n}\right) \geq 0,$$

showing that $L_n$ is positive.
Linearity follows immediately from the linearity of the sum:

$$L_n(\alpha h + \beta g; x) = \alpha L_n(h;x) + \beta L_n(g;x), \quad \alpha, \beta \in \mathbb{R},\ h, g \in C(A).$$

□

**Theorem 2.3.** *For every nonnegative integer m and for all real x, one has*

$$\sum_{k=0}^{\infty} k^m \frac{B_k^{(-1)}(nx)}{k!} = \int_{nx}^{nx+1} e^u \mathcal{B}_m(u)\, du,$$

*where $\mathcal{B}_m(u)$ denotes the exponential Bell polynomial*

$$\mathcal{B}_m(u) = \sum_{j=0}^{m} S(m,j)\, u^j,$$

*and $S(m,j)$ are the Stirling numbers of the second kind.*

*Proof.* From the exponential generating function and the identity

$$\frac{e^t - 1}{t} = \int_0^1 e^{st}\, ds,$$

we obtain the integral representation

$$B_k^{(-1)}(x) = \int_0^1 (x+s)^k\, ds.$$

Hence,

$$\sum_{k=0}^{\infty} k^m \frac{B_k^{(-1)}(nx)}{k!} = \int_0^1 \sum_{k=0}^{\infty} \frac{k^m}{k!} (nx+s)^k\, ds.$$

Next, by the Stirling decomposition

$$k^m = \sum_{j=0}^{m} S(m,j)\, (k)_j, \qquad (k)_j = k(k-1)\cdots(k-j+1),$$

and the generating identity

$$\sum_{k=0}^{\infty} \frac{(k)_j}{k!} z^k = z^j e^z,$$

it follows that

$$\sum_{k=0}^{\infty} \frac{k^m}{k!} z^k = e^z \sum_{j=0}^{m} S(m,j)\, z^j = e^z \mathcal{B}_m(z).$$

Substituting $z = nx + s$ and integrating over $s \in [0,1]$ gives

$$\sum_{k=0}^{\infty} k^m \frac{B_k^{(-1)}(nx)}{k!} = \int_0^1 e^{nx+s}\, \mathcal{B}_m(nx+s)\, ds.$$

Finally, with the change of variables $u = nx + s$ we obtain

$$\sum_{k=0}^{\infty} k^m \frac{B_k^{(-1)}(nx)}{k!} = \int_{nx}^{nx+1} e^{u}\, \mathcal{B}_m(u)\, du,$$

which completes the proof. □

Next Corollary gives explicit evaluations for small $m$.

**Corollary 2.4.** *Let $y := nx$. For $m = 0,1,2,3,4$ one has*

$$\sum_{k=0}^{\infty} k^m \frac{B_k^{(-1)}(nx)}{k!} = e^{y}\, p_m(y),$$

*where the polynomials $p_m$ are given explicitly as*

*i.* $p_0(y) = e - 1.$

*ii.* $p_1(y) = (e-1)\, y + 1.$

*iii.* $p_2(y) = (e-1)\, y^2 + (e+1)\, y + (e-1).$

*vi.* $p_3(y) = (e-1)\, y^3 + 3e\, y^2 + (4e-1)\, y + (e+1).$

*v.* $p_4(y) = (e-1)\, y^4 + (6e-2)\, y^3 + (13e-1)\, y^2 + (11e+1)\, y + (4e-1).$

**Lemma 2.5.** *Let $e_i(t) = t^i \in C(A)$, $i = \overline{0,4}$. For all $x \in A$ and $n \in \mathbb{N}$, the operators $\{L_n\}_{n\geq 0}$ satisfy the following*

*i.* $L_n(e_0; x) = 1,$

*ii.* $L_n(e_1; x) = x + \frac{1}{n(e-1)},$

*iii.* $L_n(e_2; x) = x^2 + \frac{e+1}{n(e-1)}x + \frac{1}{n^2},$

*iv.* $L_n(e_3; x) = x^3 + \frac{1}{n(e-1)}\left(3ex^2 + \frac{4e-1}{n}x + \frac{e+1}{4n^2}\right),$

*v.* $L_n(e_4; x) = x^4 + \frac{1}{n(e-1)}\left((6e-2)\, x^3 + \frac{13e-1}{n}x^2 + \frac{11e+1}{n^2}x + \frac{4e-1}{n^3}\right)$

*Proof.* We give the proof for only (iii). Using Corollary 2.4-(iii) and (1), we have,

$$\begin{aligned} L_n(e_2; x) &= \frac{e^{-nx}}{e-1} \sum_{k=0}^{\infty} \frac{B_k^{(-1)}(nx)}{k!} \left(\frac{k}{n}\right)^2 \\ &= \frac{e^{-nx}}{(e-1)\, n^2} e^{nx} \left(n^2 x^2 (e-1) + nx(e+1) + (e-1)\right) \\ &= x^2 + \frac{e+1}{n(e-1)}x + \frac{1}{n^2}. \end{aligned}$$

The proofs for (i), (ii), (iv) and (v) are similar to proof of (iii). □

**Lemma 2.6.** *Let $\psi_i = (e_1 - x)^i \in C(A), i = \overline{1,4}$, be central moments of the $\{L_n\}_{n\geq 0}$. Then, for all $x \in A$ and $n \in \mathbb{N}$, we have*

*i.* $L_n(\psi_1; x) = \frac{1}{(e-1)n}$,
*ii.* $L_n(\psi_2; x) = \frac{x}{n} + \frac{1}{n^2}$,
*iii.* $L_n(\psi_3; x) = \frac{e+2}{(e-1)n^2}x + \frac{e+1}{(e-1)n^3}$,
*iv.* $L_n(\psi_4; x) = \frac{3x^2}{n^2} + \frac{7e-3}{n^3(e-1)}x + \frac{4e-1}{(e-1)n^4}$.

*Proof.* We present the proof only for $L_n(\psi_2; x)$, as it serves as a representative case and provides insight for other scenarios. Taking into account the linearity of $\{L_n\}_{n\geq 0}$, we obtain the following result:

$$L_n(\psi_2; x) = L_n(e_2; x) - 2xL_n(e_1; x) + x^2 L_n(e_0; x).$$

If we rewrite the above expressions by using Lemma 2.5, we obtain the desired result. □

**Theorem 2.7.** *For every $h \in C(A)$,*

$$\lim_{n\to\infty} L_n(h; x) = h(x)$$

*converge uniformly on A.*

*Proof.* Our aim is to prove the following uniform convergence condition:

$$\lim_{n\to\infty} L_n(e_i; x) = x^i, \qquad i = 0, 1, 2,$$

where $e_i(t) = t^i$, $t \in \mathbb{R}$. From Lemma 2.5, we obtain

$$\lim_{n\to\infty} L_n(e_0; x) = 1, \qquad \lim_{n\to\infty} L_n(e_1; x) = x.$$

Moreover, as $n \to \infty$, we have

$$L_n(e_2; x) = x^2 + \frac{e+1}{n(e-1)}x + \frac{1}{n^2} \to x^2.$$

Hence, all three test functions satisfy the required convergence. Therefore, by the Bohman–Korovkin theorem [5], we conclude that

$$\lim_{n\to\infty} L_n(h; x) = h(x)$$

uniformly on $A$. □

## 3. Estimating the rate of convergence

In order to complement the qualitative convergence guaranteed by Korovkin-type theorems, it is essential to investigate quantitative error estimates. Such estimates describe the rate of convergence of positive linear operators in terms of the smoothness of the target function. Tools such as the modulus of continuity, the Peetre $K$-functional, and Lipschitz classes play a central role in this context, as they establish precise relations between approximation error and functional regularity. Moreover, both global and local approximation properties are considered: global estimates provide uniform error bounds over the entire domain, while local estimates capture the behavior of the error near a point, often yielding sharper bounds for smooth functions. In this section, we derive error estimates for the operators $L_n$ by employing these classical tools.

**Definition 3.1.** *The first-order modulus of continuity is defined by*

$$\omega_1(h; \delta) := \sup\{|h(t) - h(x)|, t, x \in A, |t - x| \leq \delta\},$$

*where $\delta \geq 0, h \in C(A)$.*

**Lemma 3.2.** *Let $h$ be bounded function on $A$ and let $\delta \geq 0$.*

$$\omega_1(h; m\delta) \leq (1+m)\,\omega_1(h;\delta),$$

for any $m > 0$. The following theorem states the rate of local approximation of $h$ by $\{L_n\}_{n\geq 0}$, since it contains the rate of approximation for each $x \in A$.

**Theorem 3.3.** *Let $h \in C(A)$. For each $x \in A$ we have*

$$|L_n(h;x) - h(x)| \leq 2\omega_1\left(h; \sqrt{\sigma_n(x)}\right),$$

*where*

$$\sigma_n(x) := L_n(\psi_2; x). \tag{4}$$

*Proof.* The linearity property of $\{L_n\}_{n\geq 0}$ and (1) give us the possibility to do the following

$$|L_n(h;x) - h(x)| \leq \frac{e^{-nx}}{e-1}\sum_{k=0}^{\infty}\frac{B_k^{(-1)}(nx)}{k!}\left|h\left(\frac{k}{n}\right) - h(x)\right|$$

And then, using the definiton of $\omega_1$ and Lemma 3.2, we get

$$\left|h\left(\frac{k}{n}\right) - h(x)\right| \leq \omega_1\left(h; \left|\frac{k}{n} - x\right|\frac{\delta}{\delta}\right) \leq \left(1 + \delta^{-2}\left(\frac{k}{n} - x\right)^2\right)\omega_1(h;\delta)$$

For $\left|\frac{k}{n} - x\right| \geq \delta$,

$$\omega_1\left(h; \left|\frac{k}{n} - x\right|\right) \leq \left(1 + \delta^{-1}\left(\frac{k}{n} - x\right)\right)\omega_1(h;\delta) \leq \left(1 + \delta^{-2}\left(\frac{k}{n} - x\right)^2\right)\omega_1(h;\delta).$$

For $\left|\frac{k}{n} - x\right| < \delta$, the inequality is trivial.

$$\begin{aligned}|L_n(h;x) - h(x)| &\leq \frac{e^{-nx}}{e-1}\sum_{k=0}^{\infty}\frac{B_k^{(-1)}(nx)}{k!}\left(1 + \delta^{-2}\left(\frac{k}{n} - x\right)^2\right)\omega_1(h;\delta)\\ &\leq \left(L_n(e_0;x) + \delta^{-2}L_n\left(\left(\frac{k}{n} - x\right)^2; x\right)\right)\omega_1(h;\delta)\end{aligned}$$

for any $\delta > 0$ and each $x \in A$. It is clear from Lemma 2.6 that $\sigma_n(x) \geq 0$, for each $x$, so we can choose positive $\delta$ as $\sigma_n(x)$, hence

$$|L_n(h;x) - h(x)| \leq \left(1 + \delta^{-2}\sigma_n(x)\right)\omega_1(h;\delta) = 2\omega_1(h;\delta)$$

for any $\delta > 0$ and each $x \in A$. □

**Theorem 3.4.** *For any $h \in C(A)$ and any $x \in A$, we have*

$$|L_n(h;x) - h(x)| \leq 2\omega_1\left(h; \sqrt{\sigma_{n,b}}\right),$$

*where $\sigma_{n,b} = \frac{b}{n} + \frac{1}{n^2}$.*

*Proof.* For any $x \in A$, $L_n(\psi_2; x) \leq \frac{b}{n} + \frac{1}{n^2}$ holds. Then using Theorem 3.3, we get

$$|L_n(h;x) - h(x)| \leq 2\omega_1\left(h; \sqrt{\sigma_{n,b}}\right).$$

□

Differently from the Theorem 3.3, to find the rate of global approximations, i.e. for any $x$ in the interval $A$, we choose $\delta := \max_{x \in A} \sqrt{\sigma_n(x)}$ (cf.[7, 8]). The following theorem gives an estimate the rate of convergence for smooth functions. (cf.[7, 8, 34]).

**Theorem 3.5.** *Let $h \in C^1(A)$ and each $x \in A$, then the operator sequence $\{L_n\}_{n \geq 0}$ holds*

$$|L_n(h;x) - h(x)| \leq |h'(x)| \frac{1}{n(e-1)} + 2\sqrt{\sigma_n(x)}\omega_1\left(h'; \sqrt{\sigma_n(x)}\right),$$

*where $\sigma_n(x)$ is given in* (4).

*Proof.* By applying a well-known result due to [34], it is possible to write the following inequality:

$$\begin{aligned} |L_n(h;x) - h(x)| &\leq |h(x)| \, |L_n(e_0;x) - 1| + |h'(x)| \, |L_n(e_1;x) - xL_n(e_0;x)| \\ &\quad + \sqrt{L_n(\psi_2;x)} \times \left(\sqrt{L_n(e_0;x)} + \delta^{-1}\sqrt{L_n(\psi_2;x)}\right)\omega_1(h';\delta). \end{aligned} \tag{5}$$

One uses Lemma 2.6-(ii) in (5) and chooses $\delta = \sqrt{\frac{x}{n} + \frac{1}{n^2}}$, we obtain desired result. □

The following corollary gives the rate of global approximation for smooth functions.

**Corollary 3.6.** *For any $h \in C^1(A)$ and any $x \in A$, the operator sequence $\{L_n\}_{n \geq 0}$ satisfies the following inequality:*

$$|L_n(h;x) - h(x)| \leq U,$$

*where*

$$U := \max_{x \in A} |h'(x)| \frac{1}{n(e-1)} + 2\max_{x \in A} \sqrt{\sigma_n(x)}\omega_1(h';\delta). \tag{6}$$

Let $C^2(A)$ denote space of functions $h$ such that $h, h', h'' \in C(A)$ and endowed with

$$\|h\|_{C^2(A)} = \|h\|_{C(A)} + \|h'\|_{C(A)} + \|h''\|_{C(A)}.$$

Now we estimate the rate of convergence using the Peetre's $\mathcal{K}$-functional defined by

$$K(h;\delta) = \inf_{g \in C^2(A)} \left\{ \|h - g\|_{C(A)} + \delta \|g\|_{C^2(A)} \right\}.$$

**Theorem 3.7.** *If $h \in C(A)$, then the following holds*

$$|L_n(h;x) - h(x)| \leq 2K(h;\delta),$$

*where $\delta := \delta_{n,b} = \frac{1}{(e-1)n} + \frac{1}{2!}\left(\frac{b}{n} + \frac{1}{n^2}\right)$.*

*Proof.* Let $g \in C^2(A)$. Taylor series gives the following

$$\left|L_n(g;x) - g(x)\right| \leq |L_n(s-x;x)| \left|g'(x)\right| + \frac{1}{2!}\left\|\frac{x}{n} + \frac{1}{n^2}\right\| \left\|g''(x)\right\|_{C([a,b])}.$$

$$\left\|L_n(g;x)-g(x)\right\|_{C([a,b])}\leq\left\|\frac{1}{(e-1)n}\right\|_{C(A)}\left\|g'(x)\right\|_{C(A)}+\frac{1}{2!}\left\|\frac{x}{n}+\frac{1}{n^2}\right\|\left\|g''(x)\right\|_{C(A)}$$
$$\leq\left(\frac{1}{(e-1)n}+\frac{1}{2!}\left(\frac{b}{n}+\frac{1}{n^2}\right)\right)\left\|g(x)\right\|_{C^2(A)} \tag{7}$$

On the other hand, since $\{L_n\}_{n\geq0}$ is the linear operator sequence, we get

$$|L_n(h;x)-h(x)|\leq\left|L_n(h-g;x)\right|+\left|h-g\right|+\left|L_n(g;x)-g(x)\right|$$

Hence using $L_n(e_0;x)=1$ and finding maximum over $A$ in both sides of the above inequality, we get

$$\|L_n(h;x)-h(x)\|_{C(A)}\leq2\left\|h(x)-g(x)\right\|_{C(A)}+\left\|L_n(g;x)-g(x)\right\|_{C(A)} \tag{8}$$

When we use (7) in (8), we have final form of (8) :

$$\|L_n(h;x)-h(x)\|_{C(A)}\leq2\left\|h(x)-g(x)\right\|_{C(A)}+2\left[\frac{1}{(e-1)n}+\frac{1}{2!}\left(\frac{b}{n}+\frac{1}{n^2}\right)\right]\left\|g(x)\right\|_{C^2(A)}. \tag{9}$$

Assume that the coefficient of $\left\|g(x)\right\|_{C^2(A)}$ is denoted by $\delta_{n,b}:=\frac{1}{(e-1)n}+\frac{1}{2!}\left(\frac{b}{n}+\frac{1}{n^2}\right)$. Then if we take infimum over $g\in C^2(A)$ in both sides of (9), we obtain

$$\|L_n(h;x)-h(x)\|_{C(A)}\leq2K(h;\delta).$$

□

In the theorem to be given now, the rate of convergence will be given with the help of the Lipschitz class, denoted by $Lip_M(\alpha)$, i.e., for $0<\alpha\leq1$,

$$Lip_M(\alpha)=\{h\in C(A):|h(s)-h(x)|\leq M|s-x|^{\alpha},s,x\in A\}.$$

**Theorem 3.8.** *If $h$ belongs to the Lipschitz class, then we have*

$$\|L_n(h;x)-h(x)\|_{C(A)}\leq M\left(\frac{x}{n}+\frac{1}{n^2}\right)^{\frac{\alpha}{2}}.$$

*Proof.* Suppose that $h$ belongs to the Lipschitz class. By definition of $Lip_M(\alpha)$, we have

$$|L_n(h;x)-h(x)|\leq|L_n(|h(t)-h(x)|;x)|$$
$$\leq M\frac{e^{-nx}}{e-1}\sum_{k=0}^{\infty}\frac{B_k^{(-1)}(nx)}{k!}\left|h\left(\frac{k}{n}\right)-h(x)\right|^{\alpha}.$$

Using Holder inequality with $p=\frac{2}{\alpha}$ and $q=\frac{2}{2-\alpha}$, we have

$$|L_n(h;x)-h(x)|\leq M\left[L_n\left((t-x)^2;x\right)\right]^{\frac{\alpha}{2}}.$$

Next, taking into consideration Lemma 2.6, we complete the proof. □

## 4. Asymptotic Approximation of the Operators $L_n$

Asymptotic approximation studies the behavior of a sequence of operators $\{L_n\}_{n\geq0}$ as $n\to\infty$, focusing on the difference

$$L_n(h;x)-h(x),\qquad h\in C(A),$$

and providing explicit estimates of the error in terms of $n$ and the smoothness of $h$. This approach extends the qualitative convergence guaranteed by Korovkin-type theorems by revealing the rate at which the approximation improves. A classical example of asymptotic approximation is given by the Bernstein polynomials. Voronovskaya's theorem states that if $h$ is twice continuously differentiable, then

$$\lim_{n\to\infty} n\Big(B_n(h;x)-h(x)\Big)=\frac{1}{2}x(1-x)h''(x),$$

providing an explicit asymptotic expression for the error. In a similar fashion, the operators $\{L_n\}_{n\geq 0}$ considered in this work exhibit asymptotic behavior. For smooth functions, the difference $L_n(h;x)-h(x)$ can be expressed as an asymptotic series in powers of $1/n$. For the basic test functions $e_i(x)=x^i$ $(i=0,1,2)$, the approximation errors satisfy

$$L_n(e_0;x)-e_0(x)=0,\quad L_n(e_1;x)-e_1(x)=O\Big(\frac{1}{n}\Big),\quad L_n(e_2;x)-e_2(x)=\frac{e+1}{n(e-1)}x+O\Big(\frac{1}{n^2}\Big),$$

uniformly on $A$. These examples illustrate the asymptotic nature of the approximation and the rate of convergence for more general functions. This discussion motivates the precise asymptotic result formalized in the following theorem, which gives an explicit formula for $n(L_n(h;x)-h(x))$ for functions $h\in C^2(A)$.

**Theorem 4.1.** *If $h\in C^2(A)$, then for every $x\in A$, then*

$$\lim_{n\to\infty} n\left(L_n(h;x)-h(x)\right)=\frac{1}{e-1}h'(x)+xh''(x)$$

*Proof.* Let $h,h',h''\in C(A)$ and let $x\in A$ be fixed. The Taylor formula using the Peano remainder term is as follows

$$h(t)=h(x)+(t-x)h'(x)+\frac{(t-x)^2}{2}h''(x)+p(t,x)(t-x)^2, \tag{10}$$

where $p(t,x)\in C(A)$ is Peano remainder and $\lim_{t\to x} p(t,x)=0$. At this stage, we will first apply the $L_n$ operator to (10) and then use the linearity of the operator:

$$\begin{aligned} n\left(L_n(h;x)-h(x)\right) &= nh'(x)L_n((t-x);x)+\frac{n}{2}h''(x)L_n\left((t-x)^2;x\right)\\ &\quad + nL_n\left(p(t,x)(t-x)^2;x\right). \end{aligned} \tag{11}$$

We want to prove that the last term in (11) converges to zero, so we appeal the Cauchy-Schwarz inequality and obtain the following:

$$L_n\left(p(t,x)(t-x)^2;x\right)\leq\sqrt{L_n\left(p^2(t,x);x\right)}\times\sqrt{L_n\left((t-x)^4;x\right)}.$$

Observe that $p^2(x,x)=0$ and $p^2(.,x)\in C(A)$. Using uniform convergence of the operator $\{L_n\}_{n\geq 0}$, we may write

$$\lim_{n\to\infty} L_n\left(p^2(t,x);x\right)=p^2(x,x)=0, \tag{12}$$

uniformly for $x\in A$. On the other hand, Lemma 2.6-(iv) says that for $x\in A$, $\lim_{n\to\infty} n^2L_n\left((t-x)^4;x\right)$ is nonnegative and finite. Thus, taking into account(12), we get

$$\lim_{n\to\infty} nL_n\left(p(t,x)(t-x)^2;x\right)\leq\lim_{n\to\infty}\sqrt{L_n\left(p^2(t,x);x\right)}\times\lim_{n\to\infty}\sqrt{n^2L_n\left((t-x)^4;x\right)}=0.$$

Finally the proof is completed by substituting the expressions in Lemma 2.6-(i) and (ii) into (11). □

## 5. Generalization of the r-th order of $\{L_n\}$

In this section we introduce a generalization of the linear positive operator defined in (3) based on Kirov (1993) using the Taylor polynomial of degree r of $h$. Then we will use this operator to investigate the smoothness of $h$.

If a function whose first r derivatives exist at $x = c$, then the Taylor polynomial of degree r of $h$ at $x = c$ is defined by $p_r(x) = \sum_{i=0}^{r} \frac{h^{(i)}(c)}{i!}(x-c)^i$. Starting from this notion, if $h$ is differentiable only r times and if its r-th derivative is continuous, then it is possible to construct a combined operator as below. For similar works see [27, 32].

**Definition 5.1.** *Let $L_n^{(r)} : C^r(A) \to C(A)$. For $n \in \mathbb{N}$ and $h \in C^r(A)$, the operators $L_n^{(r)}$ defined by*

$$L_n^{(r)}(h;x) := \frac{e^{-nx}}{e-1} \sum_{k=0}^{n} \sum_{i=0}^{r} \frac{h^{(i)}(k/n)}{i!}(x-k/n)^i \frac{B_k^{(-1)}(nx)}{k!}, \tag{13}$$

*where $\left\{B_k^{(-1)}(x)\right\}_{k\geq 0}$ are the sequence of Bernoulli polynomials of negative order.*

**Remark 5.2.** *Since $h^{(0)} = h, L_n^0 = L_n$, then the operator returns to* (1), *so the operator $L_n^{(r)}$ is a generalization of $L_n$. The linearity of $L_n^{(r)}$ is obvious, but it may not be possitive even if h is positive.*

**Theorem 5.3.** *Let $\left\{L_n^{(r)}\right\}_{n\geq 0}$ be sequence of operators given by* (13) *and $\{L_n\}_{n\geq 0}$ is given in* (1). *If $h \in C^r(A)$ and $h^{(r)} \in Lip_M(\alpha)$, then we have*

$$\left\|L_n^{(r)}(h;x) - h(x)\right\|_{C(A)} \leq \left\|L_n\left(\left|x-\frac{k}{n}\right|^{\alpha+r};x\right)\right\|_{C(A)} \frac{M}{(r-1)!}\frac{\alpha}{\alpha+r}B(\alpha,r),$$

*where $B(\alpha, r)$ is beta function.*

*Proof.* Let $h \in C^r(A)$. From (13) we can write

$$f(x) - L_n^{(r)}(h;x) = \frac{e^{-nx}}{e-1}\sum_{k=0}^{\infty}\left[h(x) - \sum_{i=0}^{r}\frac{h^{(i)}(k/n)}{i!}(x-k/n)^i\right]\frac{B_k^{(-1)}(nx)}{k!} \tag{14}$$

for every $x \in A$ and for every $\delta \in A$.
Using the Taylor's formula, we obtain the following

$$h(x) - \sum_{i=0}^{r}\frac{h^{(i)}(k/n)}{i!}(x-k/n)^i \tag{15}$$

$$= \frac{(x-k/n)^r}{(r-1)!}\int_0^1 (1-t)^{r-1}\left[h^{(r)}(k/n + t(x-k/n)) - h^{(r)}(k/n)\right]dt$$

Since $h^{(r)} \in Lip_M(\alpha)$, we have

$$\left|h^{(r)}\left(\frac{k}{n} + t\left(x-\frac{k}{n}\right)\right) - h^{(r)}\left(\frac{k}{n}\right)\right| \leq Mt^\alpha\left|x-\frac{k}{n}\right|^\alpha \tag{16}$$

Now putting (16) in (15) we get

$$\begin{aligned}\left|h(x)-\sum_{i=0}^{r}\frac{h^{(i)}(k/n)}{i!}(x-k/n)^{i}\right| &\leq \left|x-\frac{k}{n}\right|^{r}\frac{1}{(r-1)!}\int_{0}^{1}(1-t)^{r-1}Mt^{\alpha}\left|x-\frac{k}{n}\right|^{\alpha}dt \\ &= \left|x-\frac{k}{n}\right|^{r}\frac{M}{(r-1)!}\int_{0}^{1}(1-t)^{r-1}t^{\alpha}dt \\ &= \frac{\alpha}{\alpha+1}\frac{M}{(r-1)!}B(\alpha,r)\left|x-\frac{k}{n}\right|^{\alpha+r}.\end{aligned} \tag{17}$$

Substituting (17) in (14), we get the following desired result:

$$\begin{aligned}\left|h(x)-L_{n}^{(r)}(h;x)\right| &\leq \frac{\alpha}{\alpha+1}\frac{M}{(r-1)!}B(\alpha,r)\frac{e^{-nx}}{e-1}\sum_{k=0}^{\infty}\left|x-\frac{k}{n}\right|^{\alpha+r}\frac{B_{k}^{(-1)}(nx)}{k!} \\ &= \frac{\alpha}{\alpha+1}\frac{M}{(r-1)!}B(\alpha,r)L_{n}\left(\left|x-\frac{k}{n}\right|^{\alpha+r};x\right).\end{aligned}$$

□

Now, let $f\in C(A)$ be a function defined by

$$f(s)=|s-x|^{r+\alpha}. \tag{18}$$

Since $f(x)=0$, by Theorem 2.7, we have

$$\lim_{n\to\infty}\left\|L_{n}(f;x)\right\|_{C(A)}=0.$$

Consequently, according to Theorem 5.3, for all $h\in C^{r}(A)$ such that $h^{(r)}\in Lip_{M}(\alpha)$, it follows that

$$\lim_{n\to\infty}\left\|L_{n}^{(r)}(h;x)-h(x)\right\|_{C(A)}=0.$$

The following observations show the rate at which the sequence $L_{n}^{r}$ converges to the function $h$ using the modulus of continuity and Pierre's K-functional, respectively.

**Corollary 5.4.** *For all $h\in C^{r}(A)$ such that $h^{(r)}\in \mathrm{Lip}_{M}(\alpha)$, we have*

$$\|L_{n}^{(r)}(h;x)-h(x)\|_{C(A)}\leq\frac{2M}{(r-1)!}\cdot\frac{\alpha}{\alpha+1}B(\alpha,r)\cdot\omega(f,\sigma_{n}^{\alpha}(x)),$$

*where $\sigma_{n}(x)$ is given in* (4) *and $f$ is given in 18.*

**Corollary 5.5.** *For all $h\in C^{r}(A)$ such that $h^{(r)}\in \mathrm{Lip}_{M}(\alpha)$, we have*

$$\|L_{n}^{(r)}(h;x)-h(x)\|_{C(A)}\leq\frac{2M}{(r-1)!}\cdot\frac{\alpha}{\alpha+1}B(\alpha,r)\cdot\mathcal{K}(f;\delta),$$

*where $\delta:=\delta_{n,b}=\frac{1}{(e-1)n}+\frac{1}{2!}\left(\frac{b}{n}+\frac{1}{n^{2}}\right)$, and $f$ is as defined in Equation 18.*

## 6. Examples

In this section, we present two examples for a given function $f$, under the same constraints for $x$ and $n$. We derive upper bounds for the error term $E_n = |L_n(f;x) - f(x)|$ using both the usual modulus of continuity and the estimates provided by Corollary 3.6, which is a consequence of the Shisha-Mond theorem.
Consider the function $f(x) = x^2e^x$. The error estimates for $x \in [0,1]$ are computed by evaluating specific values of $x$ for $n \in \{30, 50, 100, 500, 1000\}$. All computations are carried out using Maple 2021.

**Example 6.1.** *Using the first-modulus of continuity $\omega_1$, we compute the values presented in Table 1. A clear trend is observed upon comparing these values: $E_n$ decreases as $n$ increases. This behavior is further illustrated graphically in Figure 2.*

Table 1: Error of approximation operators $L_n$ to $f(x) = x^2e^x$ using $\omega_1(f;.)$.

| x | $E_{30}$ | $E_{50}$ | $E_{100}$ | $E_{500}$ | $E_{1000}$ |
|---|---|---|---|---|---|
| 0 | 0.0829374778 | 0.04706410696 | 0.022335590 | 0.004191499 | 0.00206635309 |
| 0.1 | 0.1949434910 | 0.12824146850 | 0.0761457458 | 0.026339979 | 0.01743017678 |
| 0.2 | 0.3395549714 | 0.23347423100 | 0.1461925554 | 0.055337638 | 0.03757325014 |
| 0.3 | 0.5234391710 | 0.36771323960 | 0.2358393028 | 0.092615406 | 0.06349607700 |
| 0.4 | 0.7544152994 | 0.53676760260 | 0.3490342502 | 0.139854182 | 0.09637411660 |
| 0.5 | 1.0416346750 | 0.7474391846 | 0.4904024930 | 0.199024186 | 0.13758534500 |
| 0.6 | 1.3957872420 | 1.0076768120 | 0.6653512916 | 0.272430096 | 0.18874186880 |
| 0.7 | 1.8293381430 | 1.3267529570 | 0.8801907792 | 0.362762789 | 0.25172616520 |
| 0.8 | 2.3567985280 | 1.7154660040 | 1.1422721680 | 0.473158592 | 0.32873258000 |
| 0.9 | 2.9950353220 | 2.1863716760 | 1.4601459100 | 0.607267098 | 0.42231484400 |
| 1 | 3.7636253380 | 2.7540476020 | 1.8437425270 | 0.769328733 | 0.53544039140 |

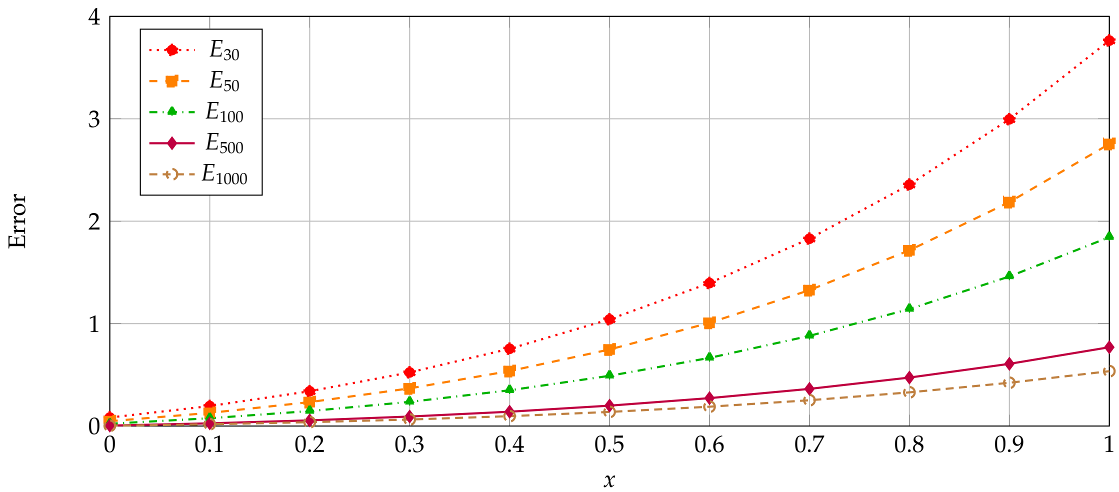


Figure 2: Error estimates by $\omega_1(f;\delta)$.

In the upper estimate obtained in Corolary 3.6, it can be seen that the derivative of the function is needed. Since the upper estimate $U$ given in (6) will be used in the next example, the values in the Table are calculated considering $f'(x) = 2xe^x + x^2e^x$.

**Example 6.2.** *Using the $U$, we compute the values presented in Table 2. For a fixed value of $x$, the deviation $E_n$ decreases as $n$ increases. This shows that the operator converges better at higher values of $n$. At high values such as*

$n = 1000$, *the operator produces very accurate results for all x values. This is further illustrated graphically in Figure 3.*

Table 2: Error of approximation operators $L_n$ to $f(x) = x^2e^x$ using $U$.

| x | $E_{30}$ | $E_{50}$ | $E_{100}$ | $E_{500}$ | $E_{1000}$ |
|---|---|---|---|---|---|
| 0 | 0.45309566690 | 0.26394721300 | 0.04691587854 | 0.00857063486 | 0.01235292768 |
| 0.1 | 0.50805973550 | 0.29513522140 | 0.14326724830 | 0.02765950609 | 0.01372324287 |
| 0.2 | 0.57442916830 | 0.33283922920 | 0.16118400270 | 0.03102859853 | 0.01538504482 |
| 0.3 | 0.65418950830 | 0.37819753080 | 0.18276042060 | 0.03509147570 | 0.01738971407 |
| 0.4 | 0.74963824510 | 0.43252762920 | 0.20862844240 | 0.03996844300 | 0.01979676410 |
| 0.5 | 0.86343060790 | 0.49735258480 | 0.23951879700 | 0.04579866104 | 0.02267504151 |
| 0.6 | 0.99863178790 | 0.57443106120 | 0.27627546150 | 0.05274290982 | 0.02610409484 |
| 0.7 | 1.15877642600 | 0.66579155440 | 0.31987213420 | 0.06098673968 | 0.03017573446 |
| 0.8 | 1.34793635600 | 0.77377137980 | 0.37143100080 | 0.07074405940 | 0.03499581010 |
| 0.9 | 1.57079770700 | 0.90106105680 | 0.43224410140 | 0.08226122366 | 0.04068623502 |
| 1 | 1.83274864400 | 1.05075479700 | 0.50379763740 | 0.09582168406 | 0.04738728886 |

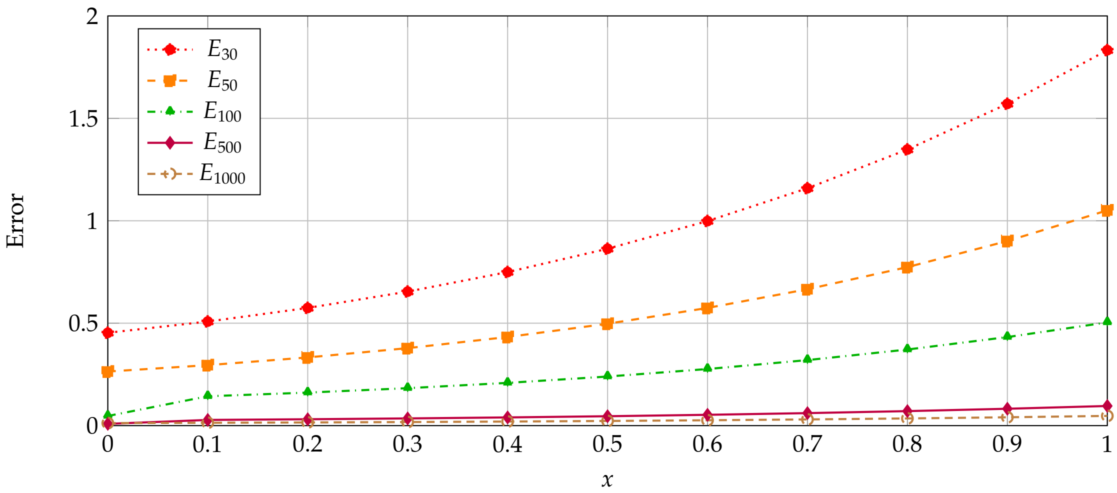


Figure 3: Error estimates by $U$.

***Analysis of Approximation Behavior:*** *The comparison of error estimates using two different upper bounds $w_1$ and U reveals significant differences in the convergence behavior of the linear positive operator to the target function. In Table 1, where the $w_1$ usual modulus of continuity is employed, the error estimates decrease as the indices increase; however, this reduction occurs more slowly, particularly for higher values of x. In contrast, Table 2 demonstrates that the upper estimate U yields a more rapid and pronounced decrease in error estimates, especially for higher index values (e.g., 500 and 1000), resulting in more precise approximations. These findings indicate that the upper bound U provides stronger convergence properties and is particularly effective for larger values of x. On the other hand, while the $w_1$ modulus is suitable for simpler cases, it exhibits limitations in terms of convergence speed and precision compared to the U. Therefore, depending on the requirements of the problem, the use of the U is recommended for achieving more accurate and efficient results. The data presented in Table 1 and Table 2 are graphically represented in Figure 2 and Figure 3, respectively. These figures provide a visual comparison of the error estimates obtained using the $w_1$ modulus of continuity and the upper bound U, highlighting their distinct convergence behaviors.*

## Conclusion

In problems related to the uniform approximation of continuous functions by a set of operators, Korovkin-type approximation provides significant simplification. However, such an approximation requires the operator to be both linear and positive. Bernoulli polynomials can be either positive or negative depending on the domain, which makes it challenging for an operator involving Bernoulli polynomials to remain positive across all domains. In contrast, Bernoulli polynomials of negative order are strictly positive on the positive axis. In this study, for the first time, we construct a Szász-type operator incorporating these polynomials and ensure that the operator remains positive in the domain where the polynomial is positive. As a result, we are able to establish the uniform approximation of the operator effortlessly using Korovkin's theorem. Subsequently, we employ the concepts of the modulus of continuity, Peetre's $\mathcal{K}$-functional, and Lipschitz classes to determine the deviation in the approximation of our operator to a continuous function, analyzing the rate of convergence both locally and globally. Since Voronovskaya-type results are a crucial tool for analyzing the rate of convergence and the limiting behavior of operator sequences, in this study, we derive such a result for the sequence $\{L_n\}_{n\geq 0}$. This result enables us to examine the effects of the operators on smooth functions and to quantitatively evaluate the rate of convergence. Within the framework of Kirov's work [22], we extend the definition of the operator by incorporating the Taylor series and the operator as a basis. Interestingly, unlike in the previous section, the new operator does not necessarily need to be linear or positive. This leads us to investigate the convergence rate of the operator without initially analyzing its uniform approximation. Through examples, we compare the upper bounds obtained in Theorem 3.3 and Corollary 3.6, demonstrating that the upper bound derived in Corollary 3.6 provides a better error estimate compared to the classical modulus of continuity.

## Further Research

As a suggestion for future studies, one may further investigate new families of operators constructed by means of Bernoulli polynomials of order $(-m)$. In this direction, we propose the following polynomial and associated operator. The following result can be regarded as a special case of the main theorem established in Theorem 6.1 in [33].

**Theorem 6.3.**

$$\left(\frac{e^t-1}{t}\right)^m e^{xt} = \sum_{n=0}^{\infty} \frac{S_k^{n+m}(x,1,e,1)}{\binom{n+m}{m}} \frac{t^n}{n!},$$

*where*

$$S_m^n(x) := S_m^n(x,1,e,1) = \frac{1}{m!}\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}(x+j)^n.$$

*These polynomials coincide with the Bernoulli polynomials of negative order $(-m)$, $m>0$.*

**Definition 6.4.** *For $f \in C[0,1]$ we define the Szász type operators based on Bernoulli polynomials of order $(-m)$:*

$$(T_{m,t}f)(x) = \frac{e^{-nx}}{(e-1)^m}\sum_{n=0}^{\infty} f\left(\frac{n}{n+m}\right)\frac{S_m^n(nx)}{\binom{n+m}{m}}\frac{t^n}{n!}, \qquad x \geq 0,\ t>0.$$

*Here,*

$$W_{n,m}(x;t) = \frac{S_m^{n+m}(nx,1,e,1)}{\binom{n+m}{m}}\frac{t^n}{n!}$$

*is nonnegative and normalized, so $T_{m,t}$ is a positive linear operator.*

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