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Property (ω) and (W_E) under topological uniform descent

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Abstract. In this paper, we introduce a novel characterization for operators and their functional calculus which simultaneously satisfy property (ω) and property (W_E) by employing the concept of topological uniform descent. Additionally, we discuss the compact perturbation of operators that concurrently fulfill two properties.

1. Introduction

As a basic branch of functional analysis, the spectral theory of linear operators not only has direct theoretical applications in modern mathematics, computational mathematics, and nonlinear science, but also occupies a fundamental position in mathematical physics, quantum mechanics, and engineering science. For example, the topological structure and distribution characteristics of spectra are fundamental to the frequency analysis of structural vibration modes in engineering and the establishment of stability criteria for dynamical systems in physics. Therefore, the study of spectral theory has long been closely followed by researchers, especially Weyl-type theorems, which are closely related to the eigenvalue distribution problem in quantum mechanics, has evolved into a cutting-edge research topic.

The historical development of Weyl-type theorems traces its origins to 1909, when Weyl, in his foundational paper ([16]), examined the spectra of compact perturbations applied to a Hermitian operator acting on a Hilbert space. He established that a point lies within the spectra of all compact perturbations of an operator if and only if it fails to be an isolated eigenvalues of finite multiplicity. This foundational result is now recognized as Weyl's theorem. Rakočević further contributed to this field in 1989 ([13]) by introducing a stronger property, known as a-Weyl's theorem. In recent years, numerous variations of Weyl's theorem, now recognized as Weyl-type theorems, have emerged, such as property (ω), property (R), property (W_E), and more ([1, 2, 12]). The research on Weyl-type theorems centers on three pivotal domains: operator judgement, functional calculus, and perturbation theory. For instance, in addition to traditional definition, scholars have employed consistent invertibility and consistent in Fredholm and index to characterize Weyl-type theorems ([17, 18]). Moreover, researchers have explored whether any functional calculus of an operator satisfies Weyl-type theorems, provided the operator itself adheres to such theorems ([14, 18]). The perturbation problem has also garnered significant attention ([4, 15, 17]).

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Weyl-type theorems exhibit a profound connection with topological uniform descent (TUD) in operator theory. This relationship has been extensively explored in recent research: Rashid [14] demonstrated that the certain operator which has TUD satisfied generalized Weyl's theorem; Xin and Jiang [17] established necessary and sufficient conditions for property (ω) in terms of TUD; Ren, Jiang and Kong [15] provided a characterization of operators satisfying property (W_E) using the TUD resolvent set; Zhang and Cao [19] characterized property (UW_{Π}) and a-Weyl's theorem by TUD. These studies collectively highlight TUD as a unifying framework for analyzing spectral properties in Weyl-type theorems.

Building upon the foundational work in [15, 17, 19], we establish an equivalent characterization for operators T and their functional calculus to simultaneously satisfy property (ω) and property (W_E), utilizing spectral sets arising from topological uniform descent. Before presenting our main results, we recall some essential notation and fundamental concepts from spectral theory.

Let $\mathcal{H}$ be an infinite-dimensional complex separable Hilbert space, and let $\mathcal{B}(\mathcal{H})$ denote the Banach algebra of all bounded linear operators on $\mathcal{H}$. For any $T \in \mathcal{B}(\mathcal{H})$, let $N(T)$, $R(T)$, $\sigma(T)$ and $\sigma_p(T)$ denote the null space, the range, the spectrum and the point spectrum of T, respectively. Set $n(T) = \dim N(T)$ and $d(T) = \dim H/R(T) = \operatorname{codim} R(T)$.

An operator T is called:

- bounded from below if $n(T) = 0$ and $R(T)$ is closed;
- upper semi-Fredholm if $n(T) < \infty$ and $R(T)$ is closed;
- lower semi-Fredholm if $d(T) < \infty$;
- Fredholm if both $n(T)$ and $d(T)$ are finite.

For a semi-Fredholm operator T, its index is given by $\operatorname{ind}(T) = n(T) - d(T)$. A Fredholm operator with $\operatorname{ind}(T) = 0$ is called Weyl. The ascent and the descent of T are defined as:

$$\begin{aligned} asc(T) &= \inf\{n \in \mathbb{N} : N(T^n) = N(T^{n+1})\}, \\ des(T) &= \inf\{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\}, \end{aligned}$$

where $\inf\emptyset = \infty$. Moreover, $\lambda \in \mathbb{C}$, the condition $0 < asc(T - \lambda I) = des(T - \lambda I) < \infty$ characterizes poles of the resolvent (see [9]). An operator T is Drazin invertible if both $asc(T)$ and $des(T)$ are finite; is Browder if it is Fredholm and has finite ascent and descent. Finally, an operator $R \in \mathcal{B}(\mathcal{H})$ is called Riesz if $R - \lambda I$ is Fredholm for all $\lambda \in \mathbb{C}\backslash\{0\}$.

The semi-Fredholm spectrum $\sigma_{SF}(T)$, the upper semi-Fredholm spectrum $\sigma_{SF_+}(T)$, the essential approximate point spectrum $\sigma_{ea}(T)$, the approximate point spectrum $\sigma_a(T)$, the Weyl spectrum $\sigma_w(T)$, the Drazin spectrum $\sigma_D(T)$ and the Browder spectrum $\sigma_b(T)$ are defined by:

$$\begin{aligned} \sigma_{SF}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not semi-Fredholm}\}, \\ \sigma_{SF_+}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not upper semi-Fredholm}\}, \\ \sigma_{ea}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not upper semi-Fredholm with } \operatorname{ind}(T) \leq 0\}, \\ \sigma_a(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not bounded from below}\}, \\ \sigma_w(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\}, \\ \sigma_D(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Drazin invertible}\}, \\ \sigma_b(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder}\}. \end{aligned}$$

Their corresponding resolvent sets are denoted by $\rho_{SF}(T), \rho_{SF_+}(T), \rho_{ea}(T), \rho_a(T), \rho_w(T), \rho_b(T)$. Additionally, we define the following subsets of the semi-Fredholm resolvent set:

$$\begin{aligned} \rho_{SF}^-(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is semi-Fredholm with } \operatorname{ind}(T) < 0\}, \\ \rho_{SF}^+(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is semi-Fredholm with } \operatorname{ind}(T) > 0\}. \end{aligned}$$

Definition 1.1. *An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be consistent in Fredholm and index (abbreviated as CFI) if for every $S \in \mathcal{B}(\mathcal{H})$, the following holds:*
(1) Both ST and TS are Fredholm and ind(ST)=ind(TS)=ind(S);
(2) Neither ST nor TS are Fredholm.

The CFI spectrum of T is defined as

$$\sigma_{CFI}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not CFI}\}.$$

Unlike standard spectra, the CFI spectrum may be non-closed or even empty (see [3]).

Let $T \in \mathcal{B}(\mathcal{H})$. For each positive integer n, the operator T induces a linear mapping $\Gamma_n : R(T^n)/R(T^{n+1}) \to R(T^{n+1})/R(T^{n+2})$ defined by $\Gamma_n(x + R(T^{n+1})) = Tx + R(T^{n+2})$. We say that T has uniform descent for $n \geq d$ if there exists a nonnegative integer d such that $\dim N(\Gamma_n) = 0$ for all $n \geq d$. And T has topological uniform descent if $R(T^n)$ is closed in the operator range topology of $R(T^d)$ for $n \geq d$ ([5]).

It is well-known that every semi-Fredholm operator has topological uniform descent. We define the topological uniform descent resolvent set as

$$\rho_\tau(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ has topological uniform descent}\}.$$

For operators with topological uniform descent, the following fundamental property holds (see [5]):

Lemma 1.2. *Suppose that $\lambda \in \partial\sigma(T)$, and $T-\lambda I$ has topological uniform descent. Then $asc(T-\lambda I) = des(T-\lambda I) < \infty$, so λ is a pole of the resolvent, where $\partial\sigma(T)$ denotes the boundary of $\sigma(T)$.*

For $T \in \mathcal{B}(\mathcal{H})$, if σ is a clopen subset of $\sigma(T)$, then there exists an analytic Cauchy domain Ω such that $\sigma \subseteq \Omega$ and $[\sigma(T)\backslash\sigma] \cap \overline{\Omega} = \emptyset$, where $\overline{\Omega}$ is the closure of Ω. The Riesz idempotent of T associated with σ, denoted $E(\sigma; T)$, is given by:

$$E(\sigma; T) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma (\lambda I - T)^{-1} d\lambda,$$

where $\Gamma = \partial\Omega$ is positively oriented with respect to Ω (in the sense of complex analysis). The corresponding spectral subspace is $\mathcal{H}(\sigma; T) = R(E(\sigma; T))$. Note that if $\lambda \in \operatorname{iso}\sigma(T)$, then the singleton $\{\lambda\}$ is a clopen in $\sigma(T)$. We write $\mathcal{H}(\lambda; T)$=$\mathcal{H}(\{\lambda\}; T)$. Moreover, if $\dim \mathcal{H}(\lambda; T) < \infty$, then λ is a normal eigenvalue of T, i.e., $\lambda \in \sigma_0(T)$, where $\sigma_0(T)$ denotes the set of all normal eigenvalues of T. It is well-known that $\sigma_0(T) = \sigma(T)\backslash\sigma_b(T)$.

Following [12], an operator $T \in \mathcal{B}(\mathcal{H})$ is said to satisfy property (ω) if

$$\sigma_a(T)\backslash\sigma_{ea}(T) = \pi_{00}(T),$$

where $\pi_{00}(T) = \{\lambda \in \operatorname{iso}\sigma(T) : 0 < n(T - \lambda I) < \infty\}$.

A modified version of Weyl's theorem, introduced by M.Berkani and M.Kachad [2], defines property (W_E) for $T \in \mathcal{B}(\mathcal{H})$ as

$$\sigma(T)\backslash\sigma_w(T) = E(T),$$

where $E(T) = \{\lambda \in \operatorname{iso}\sigma(T) : n(T - \lambda I) > 0\}$.

The paper is organized as follows: Section 2 utilizes the spectrum arising from topological uniform descent to characterize operators and their functional calculus that simultaneously satisfy both property (ω) and property (W_E). Section 3 investigates compact perturbations of operators preserving these two properties concurrently.

2. Property (ω) and property (W_E)

In this section, we first establish a characterization of operators simultaneously satisfying properties (ω) and (W_E) by means of topological uniform descent, and subsequently investigate the stability of these properties for the functional calculus of such operators. For convenience, set

$$(\omega \cap W_E) = \{T \in \mathcal{B}(\mathcal{H}) : T \text{ satisfies both properties } (\omega) \text{ and } (W_E)\}.$$

Theorem 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $T \in (\omega \cap W_E)$ if and only if $\sigma_b(T) = [\sigma_\tau(T) \cap acc\sigma(T)] \cup acc\sigma_{ea}(T) \cup [\partial\sigma_{CFI}(T) \cap acc\sigma(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$.*

Proof. ($\Rightarrow$) The inclusion "$\supseteq$" is clear. Assume that $\lambda_0 \notin [\sigma_\tau(T) \cap \text{acc}\sigma(T)] \cup \text{acc}\sigma_{ea}(T) \cup [\partial\sigma_{CFI}(T) \cap \text{acc}\sigma(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$. Without loss of generality, let $\lambda_0 \in \sigma(T)$, then $n(T - \lambda_0 I) > 0$. Again, $\lambda_0 \notin \text{acc}\sigma_{ea}(T)$, there exists $\epsilon > 0$ such that $\lambda \notin \sigma_{ea}(T)$ if $0 < |\lambda - \lambda_0| < \epsilon$.

Case 1. Let $\lambda_0 \notin \sigma_\tau(T) \cup \partial\sigma_{CFI}(T)$. If there is $\lambda \in \sigma_a(T)$ for $0 < |\lambda - \lambda_0| < \epsilon$, then $\lambda \notin \sigma_b(T)$ since T satisfies property (ω). According to the definition of boundary points, we have $\lambda_0 \in \partial\sigma(T)$. It follow from Theorem 4.9 in [12] that $\lambda_0 \notin \sigma_D(T)$. Then $\lambda_0 \in \text{iso}\sigma(T)$. If $\lambda \notin \sigma_a(T)$ for $0 < |\lambda - \lambda_0| < \epsilon$, then $\lambda_0 \in \text{iso}\sigma_a(T)$. Claim that $\lambda_0 \notin \sigma_{CFI}(T)$, otherwise $\lambda_0 \in \sigma_{CFI}(T)$, then $\lambda_0 \in \sigma_a(T) \backslash \sigma_{ea}(T)$. Observing that property (ω) holds for T yields that $\lambda_0 \notin \sigma_b(T)$, which means $\lambda_0 \notin \sigma_{CFI}(T)$. It is a contradiction. Again $\lambda_0 \notin \partial\sigma_{CFI}(T)$, we have $\lambda_0 \notin \overline{\sigma_{CFI}(T)}$, and then $T - \lambda I$ is invertible. Hence $\lambda_0 \in \text{iso}\sigma(T)$.

Case 2. Let $\lambda_0 \notin \text{acc}\sigma(T)$, then $\lambda_0 \in \text{iso}\sigma(T)$.

In either case mentioned above, it follows that $\lambda_0 \in \text{iso}\sigma(T)$, which together with $n(T - \lambda_0 I) > 0$ implies that $\lambda_0 \in E(T)$. Since T satisfies property (W_E), then $\lambda_0 \notin \sigma_b(T)$.

($\Leftarrow$) Since $\{[\sigma_a(T) \backslash \sigma_{ea}(T)] \cup E(T)\} \cap [\sigma_\tau(T) \cap \text{acc}\sigma(T)] = \emptyset = \{[\sigma_a(T) \backslash \sigma_{ea}(T)] \cup E(T)\} \cap \{\text{acc}\sigma_{ea}(T) \cup [\partial\sigma_{CFI}(T) \cap \text{acc}\sigma(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}\}$, it follows from the condition (2) in Theorem 2.1 that $\{[\sigma_a(T) \backslash \sigma_{ea}(T)] \cup E(T)\} \cap \sigma_b(T) = \emptyset$, and then $[\sigma_a(T) \backslash \sigma_{ea}(T)] \cup E(T) \subseteq \rho_b(T)$. Again, $\sigma(T) \backslash \sigma_w(T) \subseteq \sigma_a(T) \backslash \sigma_{ea}(T)$, and $\pi_{00}(T) \subseteq E(T)$, hence $\sigma_a(T) \backslash \sigma_{ea}(T) = \pi_{00}(T)$, and $\sigma(T) \backslash \sigma_w(T) = E(T)$. □

Remark 2.2. *Each of the four terms contributing to $\sigma_b(T)$ in Theorem 2.1 is essential, as demonstrated by the following examples.*

- *(i) Let $A : \ell^2 \to \ell^2$ be the right shift operator defined by $A(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, \cdots)$ and let $B : \ell^2 \to \ell^2$ be the weighted right shift operator given by $B(x_1, x_2, x_3, \cdots) = (0, 0, \frac{x_2}{2}, \frac{x_3}{3}, \cdots)$. Then $T = A \otimes B \in (\omega \cap W_E)$. Nevertheless, $\sigma_b(T) \neq acc\sigma_{ea}(T) \cup [\partial\sigma_{CFI}(T) \cap acc\sigma(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$.*
- *(ii) Consider the left shift operator $T : \ell^2 \to \ell^2$ defined by $T(x_1, x_2, x_3, \cdots) = (x_2, x_3, x_4, \cdots)$. Obviously $T \in (\omega \cap W_E)$. But $\sigma_b(T) \neq [\sigma_\tau(T) \cap acc\sigma(T)] \cup [\partial\sigma_{CFI}(T) \cap acc\sigma(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$.*
- *(iii) For the operator T defined in (i), the Browder spectrum further fails to satisfy $\sigma_b(T) \neq [\sigma_\tau(T) \cap acc\sigma(T)] \cup acc\sigma_{ea}(T) \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$.*
- *(iv) Let $T : \ell^2 \to \ell^2$ be the weighted right shift operator: $T(x_1, x_2, x_3, \cdots) = (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots)$. Through direct computation, we verify that $T \in (\omega \cap W_E)$. However, $\sigma_b(T) \neq [\sigma_\tau(T) \cap acc\sigma(T)] \cup acc\sigma_{ea}(T) \cup [\partial\sigma_{CFI}(T) \cap acc\sigma(T)]$.*

Corollary 2.3. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $T \in (\omega \cap W_E)$ if and only if $E(T) \subseteq \rho_\tau(T) \subseteq acc\sigma_{ea}(T) \cup [\partial\sigma_{CFI}(T) \cap acc\sigma(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup \rho_b(T)$.*

Proof. ($\Rightarrow$) Since T satisfies property (W_E), then $E(T) \subseteq \sigma_a(T) \backslash \sigma_{ea}(T) \subseteq \rho_\tau(T)$. Let $\lambda_0 \notin \text{acc}\sigma_{ea}(T) \cup [\partial\sigma_{CFI}(T) \cap \text{acc}\sigma(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup \rho_b(T)$, then $\lambda_0 \in \sigma_b(T) \backslash [\text{acc}\sigma_{ea}(T) \cup [\partial\sigma_{CFI}(T) \cap \text{acc}\sigma(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}]$. It follows from Theorem 2.1 that $\lambda_0 \in \sigma_\tau(T) \cap \text{acc}\sigma(T)$. Hence $\lambda_0 \notin \rho_\tau(T)$.

($\Leftarrow$) By the given condition, we know that $\sigma_b(T) \subseteq \sigma_\tau(T) \cup \text{acc}\sigma_{ea}(T) \cup [\partial\sigma_{CFI}(T) \cap \text{acc}\sigma(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$ and $\sigma_b(T) \subseteq \text{acc}\sigma(T) \cup \text{acc}\sigma_{ea}(T) \cup [\partial\sigma_{CFI}(T) \cap \text{acc}\sigma(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$. Hence $\sigma_b(T) = [\sigma_\tau(T) \cap \text{acc}\sigma(T)] \cup \text{acc}\sigma_{ea}(T) \cup [\partial\sigma_{CFI}(T) \cap \text{acc}\sigma(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$. □

Applying $\{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} = \{\lambda \in \sigma_a(T) : n(T - \lambda I) = 0\} \cup [\rho_a(T) \cap \sigma(T)]$, we obtain the following result.

Corollary 2.4. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $T \in (\omega \cap W_E)$ if and only if $\sigma_b(T) = [\sigma_\tau(T) \cap acc\sigma(T)] \cup acc\sigma_{ea}(T) \cup [\partial\sigma_{CFI}(T) \cap acc\sigma(T)] \cup \{\lambda \in \sigma_a(T) : n(T - \lambda I) = 0\} \cup [\rho_a(T) \cap \sigma(T)]$.*

We recall that T is isoloid if $\text{iso}\sigma(T) \subseteq \sigma_p(T)$.

Corollary 2.5. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $T \in (\omega \cap W_E)$ and T is isoloid if and only if $\sigma_b(T) = [\sigma_\tau(T) \cap acc\sigma(T)] \cup acc\sigma_{ea}(T) \cup [\partial\sigma_{CFI}(T) \cap acc\sigma(T)] \cup [\rho_a(T) \cap \sigma(T)]$.*

Proof. (⇒) Corollary 2.4 tells us that $\sigma_b(T) = [\sigma_\tau(T) \cap \text{acc}\sigma(T)] \cup \text{acc}\sigma_{ea}(T) \cup [\partial\sigma_{CFI}(T) \cap \text{acc}\sigma(T)] \cup \{\lambda \in \sigma_a(T) : n(T - \lambda I) = 0\} \cup [\rho_a(T) \cap \sigma(T)]$. It suffices to show $\{\lambda \in \sigma_a(T) : n(T - \lambda I) = 0\} \subseteq [\sigma_\tau(T) \cap \text{acc}\sigma(T)] \cup \text{acc}\sigma_{ea}(T) \cup [\partial\sigma_{CFI}(T) \cap \text{acc}\sigma(T)] \cup [\rho_a(T) \cap \sigma(T)]$. Assume that $\lambda_0 \notin \sigma_\tau(T) \cup \text{acc}\sigma_{ea}(T) \cup [\partial\sigma_{CFI}(T) \cap \text{acc}\sigma(T)] \cup [\rho_a(T) \cap \sigma(T)]$, it follows from the proof of Theorem 2.1 that $\lambda_0 \in \text{iso}\sigma(T) \cup \rho(T)$. Combining that T is isoloid, we know $n(T - \lambda_0 I) > 0$, which means $\lambda_0 \notin \{\lambda \in \sigma_a(T) : n(T - \lambda I) = 0\}$.

(⇐) Evidently, $T \in (\omega \cap W_E)$. It remains to show that T is isoloid. Note that $\text{iso}\sigma(T) \cap \{[\sigma_\tau(T) \cap \text{acc}\sigma(T)] \cup \text{acc}\sigma_{ea}(T) \cup [\partial\sigma_{CFI}(T) \cap \text{acc}\sigma(T)] \cup [\rho_a(T) \cap \sigma(T)]\} = \emptyset$, which shows that $\text{iso}\sigma(T) \cap \sigma_b(T) = \emptyset$, then $\text{iso}\sigma(T) \subseteq \rho_b(T)$, which means that $n(T - \lambda_0 I) > 0$. □

Corollary 2.6. *Let $T \in \mathcal{B}(\mathcal{H})$. Then the following are equivalent:*

(1) $T \in (\omega \cap W_E)$ and $\sigma(T) = \sigma_a(T)$;

(2) $\sigma_b(T) = [\sigma_\tau(T) \cap acc\sigma(T)] \cup acc\sigma_{ea}(T) \cup \{\lambda \in \sigma_a(T) : n(T - \lambda I) = 0\}$;

(3) $E(T) \subseteq \rho_\tau(T) \subseteq acc\sigma_{ea}(T) \cup \{\lambda \in \sigma_a(T) : n(T - \lambda I) = 0\} \cup \rho_b(T)$;

(4) $\sigma_b(T) = [\sigma_\tau(T) \cap acc\sigma(T)] \cup int\sigma_{ea}(T) \cup \{\lambda \in \sigma_a(T) : n(T - \lambda I) = 0\}$;

(5) $E(T) \subseteq \rho_\tau(T) \subseteq int\sigma_{ea}(T) \cup \{\lambda \in \sigma_a(T) : n(T - \lambda I) = 0\} \cup \rho_b(T)$.

Corollary 2.7. *Let $T \in \mathcal{B}(\mathcal{H})$. Then the following are equivalent:*

(1) $T \in (\omega \cap W_E)$ is isoloid and $\sigma(T) = \sigma_a(T)$;

(2) $\sigma_b(T) = [\sigma_\tau(T) \cap acc\sigma(T)] \cup acc\sigma_{ea}(T)$;

(3) $iso\sigma(T) \subseteq \rho_\tau(T) \subseteq acc\sigma_{ea}(T) \cup \rho_b(T)$;

(4) $\sigma_b(T) = [\sigma_\tau(T) \cap acc\sigma(T)] \cup int\sigma_{ea}(T)$;

(5) $iso\sigma(T) \subseteq \rho_\tau(T) \subseteq int\sigma_{ea}(T) \cup \rho_b(T)$.

Let $H(T) = \{f : \Omega \to \mathbb{C} | \ \sigma(T) \subseteq \Omega$ and f is holomorphic on $\Omega\}$, and $f(T)$ be the holomorphic functional calculus of T with respect to f.

Theorem 2.8. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $f(T) \in (\omega \cap W_E)$ for all $f \in H(T)$ if and only if*

(1) $T \in (\omega \cap W_E)$;

(2) If $\rho_{SF}^-(T) \neq \emptyset$, then there exists no $\lambda \in \rho_{SF}(T)$ such that $0 < ind(T - \lambda I) < \infty$;

(3) If $\sigma_0(T) \neq \emptyset$, then $\sigma_b(T) = [\sigma_\tau(T) \cap acc\sigma(T)] \cup acc\sigma_{ea}(T)$.

Proof. (⇒) Suppose that $f(T) \in (\omega \cap W_E)$.

(1) Take $f(\lambda) = \lambda$, then $T = f(T) \in (\omega \cap W_E)$.

(2) We suppose on the contrary that $\lambda_1, \lambda_2 \in \rho_{SF}(T)$ with $0 < \text{ind}\,(T - \lambda_1 I) = n < \infty$ and $-\infty \leq \text{ind}\,(T - \lambda_2 I) = -m < 0$. Define $f(z) = (z - \lambda_1)^m (z - \lambda_2)^n$ if m is a positive integer, otherwise let $f(z) = (z - \lambda_1)(z - \lambda_2)$ if $m = +\infty$. Then $f(T)$ is a Weyl operator or $0 \in \sigma_a(f(T)) \backslash \sigma_{ea}(f(T))$. Since $f(T)$ satisfies property (ω), it follows that $0 \notin \sigma_b(f(T))$, and $\lambda_1 \notin \sigma_b(T)$, which contradicts with "$\text{ind}\,(T - \lambda_1 I) > 0$ ".

(3) If $\sigma_0(T) \neq \emptyset$, then $\sigma_a(T) = \sigma(T)$. Otherwise, we can choose $\lambda_1 \in \sigma_0(T)$ and $\mu \in \sigma(T) \backslash \sigma_a(T)$. Let $f(z) = (z - \lambda_1)(z - \mu)$, it follows from the proof of (2) that $0 \notin \sigma_b(f(T))$, which is a contradiction. In this case, let $\lambda_2 \in \sigma_b(T)$ with $\lambda_2 \notin [\sigma_\tau(T) \cap \text{acc}\sigma(T)] \cup \text{acc}\sigma_{ea}(T)$. It follows from the proof of Theoreom 2.1 that $\lambda_2 \in \text{iso}\sigma(T) \cup \rho(T)$. Define $f(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)$. Then, $n(f(T)) \geq n(T - \lambda_1 I) > 0$. By Corollary 2.10 [20], it follows from $\lambda_1, \lambda_2 \in \text{iso}\sigma(T)$ that $0 \in \text{iso}\sigma\,(f(T))$. Thus, it follows that $0 \in E\,(f(T))$. Given that $f(T)$ possesses property (W_E), it is evident that $f(T)$ is a Browder operator. Consequently, $\lambda_2 \notin \sigma_b(T)$. It is a contradiction.

(⇐) Choose an $f \in H(T)$ arbitrarily. It suffices to prove that $f(T) \in (\omega \cap W_E)$.

Step 1. Let $\mu_0 \in \sigma_a(f(T)) \backslash \sigma_{ea}(f(T))$. Then $0 < n(f(T) - \mu_0 I) < \infty$ and $\text{ind}(f(T) - \mu_0 I) \leq 0$. we may assume that

$$f(z) - \mu_0 = (z - \lambda_1)^{n_1} \cdots (z - \lambda_n)^{n_k}\, g(z)$$

where $\lambda_i \neq \lambda_j$ and $g(z) \neq 0$ for all $z \in \sigma(T)$. Hence

$$f(T) - \mu_0 I = (T - \lambda_1 I)^{n_1} (T - \lambda_2 I)^{n_2} \cdots (T - \lambda_k I)^{n_k} g(T),$$

where $g(T)$ is invertible. Then $\lambda_i \in \rho_{SF}(T)$ and $n(T - \lambda_i I) < \infty$. By condition (2), we get $\text{ind}(T - \lambda_i I) \leq 0$, and hence $\lambda_i \notin \sigma_a(T)$ or $\lambda_i \in \sigma_a(T) \backslash \sigma_{ea}(T)$.

If $\lambda_i \notin \sigma_a(T)$, then $\mu_0 \notin \sigma_a(f(T))$. And if $\lambda_i \in \sigma_a(T) \backslash \sigma_{ea}(T)$, which together with the fact property (ω) holds for T implies that $T - \lambda_i I$ is Browder, then $\lambda_i \in \text{iso}\sigma(T)$. Therefore $\mu_0 \in \text{iso}\sigma(f(T))$. Since $0 < n(f(T) - \mu_0) < \infty$, we can conclude that $\mu_0 \in \pi_{00}(f(T))$.

Step 2. Arbitrarily choose a $\mu_0 \in E(f(T))$. Then $\mu_0 \in$ iso$\sigma(f(T))$ and $n(f(T) - \mu_0 I) > 0$. Suppose that

$$f(T) - \mu_0 I = (T - \lambda_1 I)^{n_1} (T - \lambda_2 I)^{n_2} \cdots (T - \lambda_k I)^{n_k} g(T),$$

where $\lambda_i \neq \lambda_j$ and $g(T)$ is invertible. Then $\lambda_i \in \text{iso}\sigma(T) \cup \rho(T)$. In this case, $\lambda_i \notin [\sigma_\tau(T) \cap \text{acc}\sigma(T)] \cup \text{acc}\sigma_{ea}(T)$. Now, without loss of generality, we assume that $\lambda_i \in \text{iso}\sigma(T)$. Since $n(f(T) - \mu_0 I) > 0$, there exists some i_0 $(1 \leq i_0 \leq n)$ such that $n(T - \lambda_{i_0} I) > 0$. Hence, $\lambda_{i_0} \in E(T)$. Since T satisfies property (W_E), we have $\lambda_{i_0} \notin \sigma_b(T)$ and hence $\lambda_{i_0} \in \sigma_0(T)$. By condition (3), we get $\lambda_i \notin \sigma_b(T)$. Hence $\mu_0 \notin \sigma_b(f(T))$.

Since $\sigma(f(T)) \backslash \sigma_w(f(T)) \subseteq \sigma_a(f(T)) \backslash \sigma_{ea}(f(T)) \subseteq \pi_{00}(f(T)) \subseteq E(f(T))$, and $E(f(T)) \subseteq \sigma(f(T)) \backslash \sigma_b(f(T))$, we know $f(T) \in (\omega \cap W_E)$. □

Corollary 2.9. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $f(T) \in (\omega \cap W_E)$ is isoloid for all $f \in H(T)$ if and only if*
(1) If $\rho_{SF}^-(T) \neq \emptyset$, then there exists no $\lambda \in \rho_{SF}(T)$ such that $0 < ind(T - \lambda I) < \infty$;
(2) $\sigma_b(T) = [\sigma_\tau(T) \cap acc\sigma(T)] \cup acc\sigma_{ea}(T)$.

Proof. (⇒) (1) It is evident from Theorem 2.8 (2) that this conclusion holds.

(2) We have two cases:

Case 1. $\sigma_0(T) \neq \emptyset$. By Theorem 2.8, we get $\sigma_b(T) = [\sigma_\tau(T) \cap \text{acc}\sigma(T)] \cup \text{acc}\sigma_{ea}(T)$.

Case 2. $\sigma_0(T) = \emptyset$. Since $T \in (\omega \cap W_E)$, we get $\sigma_{ea}(T) = \sigma_a(T) = \sigma_w(T) = \sigma(T)$. It follows from $T \in (\omega \cap W_E)$ is isoloid that $\sigma_b(T) = [\sigma_\tau(T) \cap \text{acc}\sigma(T)] \cup \text{acc}\sigma_{ea}(T)$.

(⇐) By condition (2) and Corollary 2.5, we know that $T \in (\omega \cap W_E)$ is isoloid, and then $f(T)$ is isoloid for all $f \in H(T)$. We have two cases:

Case 1. $\sigma_0(T) = \emptyset$. By Theorem 2.8, $f(T) \in (\omega \cap W_E)$.

Case 2. $\sigma_0(T) \neq \emptyset$. Combined with condition (2), it follows from Theorem 2.8 that $f(T) \in (\omega \cap W_E)$. □

Example 2.10. *Let $T : \ell^2 \to \ell^2$ be defined by $T(x_1, x_2, x_3, \cdots) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots)$. Then $\sigma(T) = \left\{\frac{1}{n} : n = 1, 2, \cdots\right\} \cup \{0\}$. For $\lambda = 0$, the range of T is not closed. For $\lambda = \frac{1}{n}$, the operator $T - \lambda I$ is Weyl. If $\lambda \notin \sigma(T)$, then $T - \lambda I$ is invertible. Therefore, condition (1) in Corollary 2.9 is satisfied. Moreover, we have $\sigma_b(T) = \sigma_\tau(T) = acc\sigma(T) = acc\sigma_{ea}(T) = \{0\}$, which implies that condition (2) in Corollary 2.9 is also satisfied. By Corollary 2.9, it follows that for every $f \in H(T)$, the operator $f(T) \in (\omega \cap W_E)$ is isoloid.*

3. Properties (ω) and (W_E) under perturbations

Feng and Li (see [4]) presented an example to demonstrate that when an operator T has both properties (ω) and (W_E), the operator $T + K$, where K is a finite rank operator that commutes with T, does not necessarily possess properties (ω) and (W_E). In the following section, we will characterize those operators for which properties (ω) and (W_E) remain stable under power finite rank perturbations. To begin with, let us review the concept of power finite rank operators, which can be regarded as an extended version of finite rank operators, along with some of their notable properties.

An operator $K \in \mathcal{B}(\mathcal{H})$ is said to be a power finite rank operator if there exists a positive integer m such that K^m is of finite rank. It is evident that finite rank operators are a special case of power finite rank operators.

Lemma 3.1. *[17] Let $T \in \mathcal{B}(\mathcal{H})$. If $K \in \mathcal{B}(\mathcal{H})$ is a power finite rank operator commuting with T, then*
(1) K is a Riesz operator;
(2) $n(T + K) < \infty \Leftrightarrow n(T) < \infty$;
(3) $iso\sigma(T + K) \subseteq iso\sigma(T) \cup \rho(T)$.

Theorem 3.2. *Let $T \in \mathcal{B}(\mathcal{H})$. If $K \in \mathcal{B}(\mathcal{H})$ is a power finite rank operator which commutes with T, then $T + K \in (\omega \cap W_E)$ if and only if*
$\sigma_b(T) = [\sigma_\tau(T) \cap acc\sigma(T)] \cup acc\sigma_{ea}(T) \cup [\partial\sigma_{CFI}(T) \cap acc\sigma(T)] \cup \{\lambda \in \sigma(T + K) : n(T + K - \lambda I) = 0\}$.

Proof. By Theorem 2.1, we need to prove that the following statements are equivalent:

(1) $\sigma_b(T) = [\sigma_\tau(T) \cap \mathrm{acc}\sigma(T)] \cup \mathrm{acc}\sigma_{ea}(T) \cup [\partial\sigma_{CFI}(T) \cap \mathrm{acc}\sigma(T)] \cup \{\lambda \in \sigma(T + K) : n(T + K - \lambda I) = 0\}$.

(2) $\sigma_b(T + K) = [\sigma_\tau(T + K) \cap \mathrm{acc}\sigma(T + K)] \cup \mathrm{acc}\sigma_{ea}(T + K) \cup [\partial\sigma_{CFI}(T + K) \cap \mathrm{acc}\sigma(T + K)] \cup \{\lambda \in \sigma(T + K) : n(T + K - \lambda I) = 0\}$.

By symmetry, it suffices to verify that (1) $\Rightarrow$ (2).

Assume that $\lambda_0 \notin [\sigma_\tau(T + K) \cap \mathrm{acc}\sigma(T + K)] \cup \mathrm{acc}\sigma_{ea}(T + K) \cup [\partial\sigma_{CFI}(T + K) \cap \mathrm{acc}\sigma(T + K)] \cup \{\lambda \in \sigma(T + K) : n(T + K - \lambda I) = 0\}$. Without loss of generality, let $\lambda_0 \in \sigma(T + K)$, then $n(T + K - \lambda_0 I) > 0$, and there is $\epsilon > 0$ such that $T + K - \lambda I \in SF_+^-(H)$ if $0 < |\lambda - \lambda_0| < \epsilon$. Now we know $\lambda \notin \sigma_{ea}(T)$, hence $\lambda \notin \sigma_\tau(T) \cup \mathrm{acc}\sigma_{ea}(T) \cup \partial\sigma_{CFI}(T)$.

Case 1. Assume that $\lambda_0 \notin \sigma_\tau(T + K) \cup \partial\sigma_{CFI}(T + K)$. There is $\lambda \in \sigma_a(T + K)$, then $\lambda \notin \{\lambda \in \sigma(T + K) : n(T + K - \lambda I) = 0\}$. By the condition (1), we know that $\lambda \notin \sigma_b(T) = \sigma_b(T + K)$, which implies $\lambda_0 \in \partial\sigma(T + K)$. It follows from $\lambda_0 \notin \sigma_\tau(T + K)$ that $\lambda_0 \in \mathrm{iso}\sigma(T + K)$.

If $\lambda \notin \sigma_a(T + K)$ for any λ with $0 < |\lambda - \lambda_0| < \epsilon$, which shows $\lambda_0 \in \mathrm{iso}\sigma_a(T + K)$. Claim that $\lambda_0 \notin \sigma_{CFI}(T + K)$. Otherwise, $\lambda_0 \in \sigma_{CFI}(T + K)$, then $T + K - \lambda_0 I$ is semi-Fredholm and $\mathrm{ind}(T + K - \lambda_0 I) \neq 0$. By the perturbation theory of semi-Freholm operators and $\lambda \notin \sigma_{ea}(T + K)$, we obtain $\lambda_0 \notin \sigma_{ea}(T + K)$, which induces $\lambda_0 \notin \sigma_{ea}(T)$. Thus $\lambda_0 \notin \sigma_\tau(T) \cup \mathrm{acc}\sigma_{ea}(T) \cup \partial\sigma_{CFI}(T)$. By the condition (1), we know that $\lambda_0 \notin \sigma_b(T) = \sigma_b(T + K)$. This is a contradiction. Hence $\lambda_0 \notin \overline{\sigma_{CFI}(T + K)}$. Combining with the condition $\lambda_0 \in \mathrm{iso}\sigma_a(T + K)$, we have $\lambda_0 \in \mathrm{iso}\sigma(T + K)$.

Case 2. Assume that $\lambda_0 \notin \mathrm{acc}\sigma(T + K)$, then $\lambda_0 \in \mathrm{iso}\sigma(T + K) \cup \rho(T + K)$.

By Lemma 3.1 (3), $\lambda_0 \in \mathrm{iso}\sigma(T) \cup \rho(T)$, and then $\lambda_0 \notin \mathrm{acc}\sigma(T) \cup \mathrm{acc}\sigma_{ea}(T) \cup [\partial\sigma_{CFI}(T) \cap \mathrm{acc}\sigma(T)] \cup \{\lambda \in \sigma(T + K) : n(T + K - \lambda I) = 0\}$. Hence $\lambda_0 \notin \sigma_b(T) = \sigma_b(T + K)$. □

Theorem 3.3. *Let $T \in \mathcal{B}(\mathcal{H})$. If $K \in \mathcal{B}(\mathcal{H})$ is a power finite rank operator which commutes with T, then $T + K \in (\omega \cap W_E)$ is isoloid if and only if $\sigma_b(T) = [\sigma_\tau(T) \cap acc\sigma(T)] \cup acc\sigma_{ea}(T) \cup [\partial\sigma_{CFI}(T) \cap acc\sigma(T)] \cup [\rho_a(T + K) \cap \sigma(T + K)]$.*

Proof. Theorem 3.2 tells us that $T + K \in (\omega \cap W_E)$, so we only need to prove that $T + K$ is isoloid in the following. Let $\lambda_0 \in \mathrm{iso}\sigma(T + K)$, it follows from Lemma 3.1 (3) that $\lambda_0 \in \mathrm{iso}\sigma(T) \cup \rho(T)$. From the given condition, we know that $\lambda_0 \notin \sigma_b(T) = \sigma_b(T + K)$, which yields that $n(T + K - \lambda_0 I) > 0$.

Conversely, assume that $\lambda_0 \notin [\sigma_\tau(T) \cap \mathrm{acc}\sigma(T)] \cup \mathrm{acc}\sigma_{ea}(T) \cup [\partial\sigma_{CFI}(T) \cap \mathrm{acc}\sigma(T)]$, then there exists $\epsilon > 0$ such that $\lambda_0 \notin \sigma_{ea}(T) = \sigma_{ea}(T + K)$ if $0 < |\lambda - \lambda_0| < \epsilon$.

Case 1. Let $\lambda_0 \notin \sigma_\tau(T) \cup \partial\sigma_{CFI}(T)$. If there is $\lambda \in \sigma_a(T + K)$ for $0 < |\lambda - \lambda_0| < \epsilon$, then $\lambda \notin \sigma_b(T + K) = \sigma_b(T)$ since $T + K$ satisfies property (ω). According to the definition of boundary points, we have $\lambda_0 \in \partial\sigma(T)$. It follow from Theorem 4.9 in [12] that $\lambda_0 \in \sigma_D(T)$. Then $\lambda_0 \in \mathrm{iso}\sigma(T)$. If $\lambda \notin \sigma_a(T + K)$ for $0 < |\lambda - \lambda_0| < \epsilon$, then $\lambda_0 \in \mathrm{iso}\sigma_a(T + K)$. Claim that $\lambda_0 \notin \sigma_{CFI}(T)$, otherwise $\lambda_0 \in \sigma_{CFI}(T)$, then $\lambda_0 \notin \sigma_{ea}(T) = \sigma_{ea}(T + K)$. Observing that property (ω) holds for $T + K$ yields that $\lambda_0 \notin \sigma_b(T)$. It is a contradiction. Again $\lambda_0 \notin \partial\sigma_{CFI}(T)$, we have $\lambda_0 \notin \overline{\sigma_{CFI}(T)}$, and then $T - \lambda I$ is invertible. Hence $\lambda_0 \in \mathrm{iso}\sigma(T)$.

Case 2. Let $\lambda_0 \notin \mathrm{acc}\sigma(T)$, then $\lambda_0 \in \mathrm{iso}\sigma(T)$.

According to Lemma 3.1 (3), $\lambda_0 \in \mathrm{iso}\sigma(T + K) \cup \rho(T + K)$. Since $T + K$ is isoloid, $\lambda_0 \in E(T + K)$, which together with $T + K \in (\omega \cap W_E)$ yields that $\lambda_0 \notin \sigma_b(T + K) = \sigma_b(T)$. □

Remark 3.4. *The statement that "T + K is isoloid" constitutes an essential condition in Theorem 3.3. Let $T \in B(\ell^2)$ be defined by*

$$T(x_1, x_2, x_3, \cdots) = (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots),$$

and $K = 0 \in B(\ell^2)$. Clearly $T + K \in (\omega \cap W_E)$ is not isoloid since $n(T + K) = 0$. In this case, $\sigma_b(T) \neq [\sigma_\tau(T) \cap acc\sigma(T)] \cup acc\sigma_{ea}(T) \cup \partial\sigma_{CFI}(T)$.

Corollary 3.5. *Let $T \in \mathcal{B}(\mathcal{H})$. If $K \in \mathcal{B}(\mathcal{H})$ is a power finite rank operator which commutes with T, then $T + K \in (\omega \cap W_E)$ is isoloid and $\sigma_a(T) = \sigma(T)$ if and only if $\sigma_b(T) = [\sigma_\tau(T) \cap acc\sigma(T)] \cup acc\sigma_{ea}(T)$.*

In [4], Feng and Li studied the perturbation of property $(\omega \cap W_E)$ under finite rank operators, and the main result is as follows:

Corollary 3.6. *Let $T \in \mathcal{B}(\mathcal{H})$. If $T \in (\omega \cap W_E)$ is isoloid with $\sigma_a(T) = \sigma(T)$, then $T + K \in (\omega \cap W_E)$ for any finite rank operator K commuting with T.*

Proof. Given that $T \in (\omega \cap W_E)$ is isoloid with $\sigma_a(T) = \sigma(T)$, we deduce from Corollary 2.7 that $\sigma_b(T) = [\sigma_\tau(T) \cap \mathrm{acc}\sigma(T)] \cup \mathrm{acc}\sigma_{ea}(T)$. By Corollary 3.5, we know that $T + K \in (\omega \cap W_E)$. □

Theorem 3.7. *Suppose that $T \in (\omega \cap W_E)$ is isoloid and K is finite rank commuting with T, then the followings are equivalent:*

(1) For any $f \in H(T)$, $f(T) + K \in (\omega \cap W_E)$ and $\sigma_a(f(T) + K) = \sigma(f(T) + K)$;

(2) For any $\lambda \in \rho_{SF_+}(T)$, $ind(T - \lambda I) \geq 0$;

(3) $\sigma_{CFI}(T) \cap \rho_a(T) = \emptyset$.

Proof. (1) ⇒ (2). If $\lambda_0 \in \rho_{SF_+}(T)$ with $\mathrm{ind}(T - \lambda_0 I) < 0$, then $\lambda_0 \notin \sigma_a(T)$ or $\lambda_0 \in \sigma_a(T) \backslash \sigma_{ea}(T)$. Since $\sigma_a(T) = \sigma(T)$ and $T \in (\omega \cap W_E)$, then $\lambda_0 \notin \sigma_b(T)$, which contradicts with $\mathrm{ind}(T - \lambda_0 I) < 0$.

(2) ⇒ (3). If there exists $\lambda_0 \in \sigma_{CFI}(T) \cap \rho_a(T)$, it follows from the definition of the CFI spectrum that $\mathrm{ind}(T - \lambda_0 I) < 0$. However, this contradicts the fact that $\mathrm{ind}(T - \lambda I) \geq 0$ for any $\lambda \in \rho_{SF_+}(T)$.

(3) ⇒ (1). Assume that $\sigma_{CFI}(T) \cap \rho_a(T) = \emptyset$. This implies $\sigma(T) = \sigma_a(T)$. Combining this with the condition $T \in (\omega \cap W_E)$, we deduce that $\mathrm{ind}(T - \lambda I) \geq 0$ for any $\lambda \in \rho_{SF_+}(T)$. We now claim that for any $f \in H(T)$, the operator $f(T) \in (\omega \cap W_E)$ is isoloid. Indeed, let $\mu_0 \in \sigma_a(f(T)) \backslash \sigma_{ea}(f(T))$. Suppose $f(T) - \mu_0 I$ admits the factorization: $f(T) - \mu_0 I = (T - \lambda_1 I)^{n_1}(T - \lambda_2 I)^{n_2} \cdots (T - \lambda_k I)^{n_k} g(T)$, where $\lambda_i \neq \lambda_j$ for $i \neq j$, and $g(T)$ is invertible. Since $T \in (\omega \cap W_E)$, then $\lambda_i \notin \sigma_b(T)$. Consequently, $f(T) - \mu_0 I$ is Browder as well, which means that $\mu_0 \in \pi_{00}(f(T))$. Suppose $\mu_0 \in E(f(T))$, and let $f(T) - \mu_0 I$ admit the factorization: $f(T) - \mu_0 I = (T - \lambda_1 I)^{n_1}(T - \lambda_2 I)^{n_2} \cdots (T - \lambda_k I)^{n_k} g(T)$, where $\lambda_i \neq \lambda_j$ for $i \neq j$, and $g(T)$ is invertible. it follows that $\lambda_i \in \mathrm{iso}\sigma(T) \cup \rho(T)$. Since T is isoloid, each $\lambda_i \in E(T)$, and consequently, $\lambda_i \notin \sigma_b(T)$ since $T \in (\omega \cap W_E)$. This implies $\mu_0 \notin \sigma_b(f(T))$. Thus, $f(T) \in (\omega \cap W_E)$. Moreover, since $\mathrm{iso}\sigma(f(T)) \subseteq f(\mathrm{iso}\sigma(T))$ and T is isoloid, it follows that $f(T)$ is isoloid. By Corollaries 2.7 and 3.5, we deduce that $f(T) + K \in (\omega \cap W_E)$ and $\sigma_a(f(T) + K) = \sigma(f(T) + K)$. □

Finally, we give the necessary and sufficient conditions for compact perturbations of operators that simultaneously satisfy both property (ω) and property (W_E).

Theorem 3.8. *If $T \in \mathcal{B}(\mathcal{H})$, then $T + K \in (\omega \cap W_E)$ for every compact operator K if and only if*

(1) $\rho_{ea}(T)$ is connected;

(2) $iso\sigma_w(T) = \emptyset$.

Proof. (⇒) (1) Let us proceed under the assumption that the contrary holds true, namely, that $\rho_{ea}(T)$ is not a connected set. Then, we can choose a bounded connected component Ω within $\rho_{ea}(T)$. Consequently, the boundary $\partial\Omega$ is a subset of $\sigma_{SF}(T)$.

According to Lemma 2.10 [10], there exists a compact operator $K_1 \in \mathcal{B}(\mathcal{H})$ such that

$$T + K_1 = \begin{pmatrix} N & * \\ 0 & A \end{pmatrix} \begin{matrix} H_1 \\ H_2 \end{matrix},$$

where N is a diagonal normal operator and with $\sigma(N) = \sigma_{SF}(N) = \partial\Omega$. Furthermore, by Theorem 3.1 [8], there exists a compact operator K_2 on H_1 such that $\sigma(N + K_2) = \overline{\Omega}$. Now, let

$$K_0 = K_1 + \begin{pmatrix} K_2 & 0 \\ 0 & 0 \end{pmatrix}.$$

It follows that K_0 is a compact operator on $\mathcal{H}$, and

$$T + K_0 = \begin{pmatrix} N + K_2 & * \\ 0 & A \end{pmatrix} \begin{matrix} H_1 \\ H_2 \end{matrix}.$$

Given that $T+K_0 \in (\omega)$, it follows that there exists some $\lambda_0 \in \Omega$ for which $T+K_0-\lambda_0 I$ is bounded from below. As a consequence, we have $n(N+K_2-\lambda_0 I)=0$. Since $N+K_2-\lambda_0 I$ is a Weyl operator, we can infer that $N+K_2-\lambda_0 I$ is invertible. However, this directly contradicts the assertion that "$\sigma(N+K_2)=\overline{\Omega}$".

(2) Assume that $\lambda_0 \in \text{iso}\,\sigma_w(T)$. By the continuity of the index function, it follows that $\lambda_0 \in \sigma_{SF}(T)$.

By Lemma 3.2.6 [11], there exists a compact operator $K_1 \in \mathcal{B}(\mathcal{H})$ satisfying the following decomposition:

$$T+K_1=\begin{pmatrix} \lambda_0 I_0 & E \\ 0 & A \end{pmatrix}\begin{matrix} M \\ M^\perp \end{matrix}.$$

where $\dim M=\infty=\dim M^\perp, I_0$ denotes the identity operator on M, and the following spectral and index relations hold: $\sigma(T)=\sigma(A), \sigma_{SF}(T)=\sigma_{SF}(A)$ and $\text{ind}\,(T-\lambda I)=\text{ind}\,(A-\lambda I)$ for all $\lambda \in \rho_{SF}(T)$.

Let $\{e_n\}_{n=1}^{\infty}$ denote an orthonormal basis for the Hilbert subspace M. For any $x \in M$ represented as $x=\sum_{i=1}^{\infty}\alpha_i e_i$, we focus on a sequence of finite rank operators $\overline{K_{2,N}}$, defined as follows:

$$\overline{K_{2,N}}x=\sum_{i=1}^{N}\frac{\alpha_{i+1}}{i}e_i,$$

and define $\overline{K_2}$ as the limit of a sequence of $\overline{K_{2,N}}$ in the operator norm. Then $\overline{K_2}$ is compact, $\ker\overline{K_2}=\text{span}\{e_1\}$ and $\sigma\left(\overline{K_2}\right)=\{0\}$.

Now, define $B=\lambda_0 I_0+\overline{K_2}$. So there exists a compact operator K_2 on $\mathcal{H}$ such that the following decomposition holds:

$$T+K_1+K_2=\begin{pmatrix} B & E \\ 0 & A \end{pmatrix}\begin{matrix} M \\ M^\perp \end{matrix}.$$

According to Proposition 3.4 [6], there exists a compact operator $\overline{K_3}$ acting on the orthogonal complement $M^\perp$ satisfying $\sigma_p\left(A+\overline{K_3}\right)=\rho_{SF}^{+}\left(A+\overline{K_3}\right)$. Let us define $A_1=A+\overline{K_3}$. Then there is a compact operator K_3 on $\mathcal{H}$ such that the following decomposition holds:

$$T+K_1+K_2+K_3=\begin{pmatrix} B & E \\ 0 & A_1 \end{pmatrix}\begin{matrix} M \\ M^\perp \end{matrix}.$$

It is worth noting that $\sigma_w(T)=\sigma_w(A)=\sigma_w(A_1)$. Consequently, $\lambda_0 \in \text{iso}\sigma_w(A_1)$. Given that $\sigma_p(A_1)=\rho_{SF}^{+}(A_1)$, we conclude that $\sigma_w(A_1)=\sigma(A_1)$, which means that $\lambda_0 \in \text{iso}\sigma(A_1)$. Take $K=K_1+K_2+K_3$. Given $n(B-\lambda_0 I)>0$, it follows from the decomposition that $n(T+K-\lambda_0 I)>0$. Coupled with $\sigma(B)=\{\lambda_0\}$, this implies $\lambda_0 \in E(T+K)$. Since $T+K \in (W_E)$, we have $\lambda_0 \in \sigma(T+K)\backslash\sigma_w(T+K)$. However, $\lambda_0 \in \sigma_{SF}(T)=\sigma_{SF}(T+K)$, which contradicts the assertion that $\lambda_0 \notin \sigma_w(T+K)$.

($\Leftarrow$) For any compact operator $K \in \mathcal{B}(\mathcal{H})$. Since $\rho_{ea}(T+K)$ is connected and coincides with $\rho_{ea}(T)$, and given that $\rho(T+K) \subseteq \rho_{ea}(T+K)$, we can deduce, via Corollary 1.14 in [7], that $\rho_{ea}(T+K)=\rho(T+K)\cup E$, where E is a subset of $\text{iso}\sigma(T+K)$. As a result, it holds that $\rho_{ea}(T+K)=\rho(T+K)\cup\sigma_0(T+K)$, which implies that $\sigma_a(T+K)\backslash\sigma_{ea}(T+K)\subseteq\rho_b(T+K)$.

Now, consider any $\lambda_0 \in E(T+K)$, we have $\lambda_0 \in \text{iso}\sigma_w(T)\cup\rho_w(T)$. It follows from $\text{iso}\sigma_w(T)=\emptyset$ that $\lambda_0 \in \rho_w(T)=\rho_w(T+K)$. Hence $\lambda_0 \in \sigma(T+K)\backslash\sigma_w(T+K)$. □

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