Filomat 39:33 (2025), 11669–11676
https://doi.org/10.2298/FIL2533669M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A note on closed ∗-paranormal operators and Weyl's theorem

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Abstract. In this note we generalize the definition of ∗-paranormal operator to the case of unbounded operators. We show that every closed symmetric operator as well as every hyponormal operator is ∗-paranormal. Later we discuss a few spectral properties of this class and show that every ∗-paranormal operator satisfy the Weyl's theorem. We also prove that the Riesz idempotent corresponding to an isolated an eigenvalue of a ∗-paranormal operator is self-adjoint.

Introduction

Let $B(H)$ denote the algebra of all bounded linear operators acting on infinite dimensional separable complex Hilbert space H. We say $T \in \mathcal{B}(H)$ satisfy the *Weyl's theorem* if

$$\sigma(T) \setminus \omega(T) = \pi_{00}(T),$$

where $\sigma(T), \omega(T)$ and $\pi_{00}(T)$ denote the spectrum, the Weyl's spectrum and the set consisting of all isolated eigenvalues with finite multiplicity, respectively.

It is clear that normal operators satisfy the Weyl's theorem. We say $T \in \mathcal{B}(H)$ to be hyponormal if $\|Tx\| \geq \|T^*x\|$ for all $x \in H$. Here T^* is the adjoint of A. This class is bigger than the class of normal operators. Coburn [5] proved that hyponormal and Toeplitz operators satisfy the Weyl's theorem. Later this result is extended to paranormal operators by Uchiyama [24]. Recall that $T \in \mathcal{B}(H)$ is said to be paranormal if

$$\|Tx\|^2 \leq \|T^2x\|\|x\|, \text{ for all } x \in H.$$

We refer to [6, 7, 12] for more information about hyponormal and paranormal operators and some more generalizations of these classes. Another important class of operators which contains the hyponormal operators is the class of ∗-paranormal operators, which was introduced by S. M. Patel [21]. An operator $T \in \mathcal{B}(H)$ is said to be ∗-paranormal if

$$\|T^*x\|^2 \leq \|T^2x\|\|x\|, \text{ for all } x \in H.$$

2020 *Mathematics Subject Classification*. Primary 47A10, 47A53; Secondary 47B20.
Keywords. paranormal operator, ∗-paranormal operator, Closed operator.
Received: 15 October 2023; Accepted: 30 September 2025
Communicated by Dragan S. Djordjević
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The class of $*$-paranormal operators is independent of the class of paranormal operators. For more information about this class we refer to [2, 22]. Recently, Tanahashi and Uchiyama has shown that $*$-paranormal operators satisfy the Weyl's theorem [22]. The Weyl's theorem for quasi $*$-paranormal and k-quasi $*$-paranormal operators is discussed in [15] and [16], respectively. In [4], the authors proved that closed paranormal operators satisfy the Weyl's theorem and the Riesz idempotent corresponding to the isolated eigenvalue is self-adjoint. In the present article we prove these results for $*$-paranormal operators which are not necessarily bounded.

Hyponormal operators are paranormal and $*$-paranormal. An operator $T \in B(H)$ is said to be normaloid if $\|T\| = r(T)$ (the spectral radius of A) . Paranormal operators are normaloid and $*$-paranormal operators are normaloid ([2, 6, 10, 21]).

1. Preliminaries

Here we recall a few basic definitions and results related to the class of closed operators. In what follows, H denotes a separable Hilbert space, and $B(H)$ is the algebra of all bounded linear operators on H. For an operator $A \in B(H)$, $\ker(A)$ and $ran(A)$ denote respectively, the null space and the range of A. An operator A is said to be densely defined linear operator on H, if the domain $\mathcal{D}(A)$ of A is dense in H, i.e., $\overline{\mathcal{D}(A)} = H$. This condition is necessary for the existence of the adjoint A^* of A defined by

$$\langle Ax, y\rangle = \langle x, A^*y\rangle,\ x \in \mathcal{D}(A), y \in \mathcal{D}(A^*),$$

where

$$\mathcal{D}(A^*) = \{y \in H : \text{ the map } x \mapsto \langle Ax, y\rangle \text{ is continuous on } \mathcal{D}(A)\}.$$

Also, a densely defined linear operator A with $\mathcal{D}(A) \subset H$ is said to be closed if its graph $G(A) = \{(x, Ax),\ x \in \mathcal{D}(A)\}$ is closed, i.e., for each sequence $(x_n, Ax_n) \in G(A)$ that converges to (x, y), then $x \in \mathcal{D}(A)$ and $y = Ax$. We denote the set of all closed operators defined in H by $C(H)$. If $A, B \in C(H)$, then we define

$$\mathcal{D}(A + B) = \mathcal{D}(A) \cap \mathcal{D}(B)$$
$$\mathcal{D}(AB) = \{x \in \mathcal{D}(B) : Bx \in \mathcal{D}(A)\}$$

and $(A + B)x = Ax + Bx$ for all $x \in \mathcal{D}(A + B)$ and $(AB)(x) = A(Bx)$ for all $x \in \mathcal{D}(AB)$.

We propose the definition of $*$-paranormal operator in the unbounded case.

Definition 1.1. *A densely defined linear operator $A : \mathcal{D}(A) \subseteq H \to H$ is said to be $*$-paranormal if $\mathcal{D}(A^2) \subseteq \mathcal{D}(A^*)$ and for every $x \in D(A^2)$, we have*

$$\|A^*x\|^2 \leq \|A^2x\|\|x\|.$$

In the present paper, we are interested to the study of defined densely closed $*$-paranormal operators. We show certain fundamental and spectral properties. The Weyl's theorem, as well as the self-adjointness of the Riesz projection with respect to an isolated point of the spectrum of a such class of operators are also established.

Definition 1.2. *A densely defined linear operator A on $\mathcal{D}(A)$ is said to be symmetric if $A \subseteq A^*$, i.e.,*

$$\mathcal{D}(A) \subseteq \mathcal{D}(A^*) \text{ and } Ax = A^*x,\ x \in \mathcal{D}(A).$$

For a densely defined closed operator A the resolvent set is defined by

$$\rho(A) := \{\lambda \in \mathbb{C} : (A - \lambda I)^{-1} \text{ exists and } (A - \lambda I)^{-1} \in \mathcal{B}(H)\}$$

and the spectrum of A is defined by $\sigma(A) = \mathbb{C} \setminus \rho(T)$. The point spectrum of A is defined by

$$\sigma_p(A) := \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not one-to-one}\}.$$

Note that $\sigma(T)$ is a non-empty compact subset of $\mathbb{C}$. The number $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$ is called the spectral radius of T. We say $T \in \mathcal{B}(H)$ to be normaloid if $r(T) = \|T\|$.

The set of isolated spectral points of A is denoted by $iso(A)$ and $\pi_{00}(A) = \{\mu \in iso(\sigma(A)) : 0 < \dim(\ker(A - \mu I)) < \infty\}$. A densely defined closed operator $A \in C(H)$ is said to be Fredholm if $\ker(A)$ and $\ker(A^*)$ are finite dimensional and $ran(A)$ is closed. In this case, the index of A is defined by

$$ind(A) = \dim\ker(A) - \dim\ker(A^*).$$

The Weyl spectrum of A is defined by

$$\omega(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not a Fredholm operator with } ind(A - \lambda I) = 0\}.$$

2. Main results

In this section we prove our main results. First, we define the $*$-paranormal operator in the unbounded case and show that every closed symmetric operator as well as every hyponormal operator is $*$-paranormal. Later we show that that every $*$-paranormal operator satisfy the Weyl's theorem.

2.1. Closed $*$-paranormal operators

Theorem 2.1. *A symmetric closed operator A on $\mathcal{D}(A) \subseteq H$ is $*$-paranormal.*

Proof. Let $x \in \mathcal{D}(A^2)$. Since A is symmetric we have $\mathcal{D}(A) \subseteq \mathcal{D}(A^*)$. It is clear that $\mathcal{D}(A^2) \subseteq \mathcal{D}(A)$ by definition. So for $x \in \mathcal{D}(A^2)$, we have

$$\begin{aligned} \|A^*x\|^2 = \|Ax\|^2 = \langle Ax, Ax\rangle &= \langle x, A^*Ax\rangle \\ &= \langle x, A^2x\rangle \\ &\le \|A^2x\|\|x\|, \end{aligned}$$

by the Cauchy-Schwarz inequality. This shows that A is $*$-paranormal. □

Definition 2.2. *[11] Let A be a densely defined operator in H. Then A is said to be hyponormal if*

1. $\mathcal{D}(A) \subseteq \mathcal{D}(A^*)$
2. $\|A^*x\| \le \|Ax\|$ *for all* $x \in \mathcal{D}(A)$.

Lemma 2.3. *Let A be a closed hyponormal operator. Then A is $*$-paranormal.*

Proof. Since A is hyponormal, we have $\mathcal{D}(A) \subseteq \mathcal{D}(A^*)$. So if $x \in \mathcal{D}(A^2)$, we have

$$\begin{aligned} \|A^*x\|^2 \le \|Ax\|^2 = \langle Ax, Ax\rangle &= \langle A^*Ax, x\rangle \quad (\text{since } Ax \in \mathcal{D}(A) \subseteq \mathcal{D}(A^*)) \\ &\le \|A^*Ax\|\|x\| \\ &\le \|A(Ax)\|\|x\|, \quad (\text{since } A \text{ hyponormal}) \\ &= \|A^2x\|\|x\|. \end{aligned}$$

□

The following result generalizes that of [2, Lemma 2.1].

Lemma 2.4. *Let $A \in C(H)$ be $*$-paranormal and $\lambda \in \sigma_p(A)$. Then* $\ker(A - \lambda I) \subseteq \ker(A^* - \bar{\lambda} I)$.

Proof. Let $x \in \ker(A - \lambda I)$ with $\|x\| = 1$. Then $Ax = \lambda x$. It is clear that $x \in \mathcal{D}(A^2) \subseteq \mathcal{D}(A^*)$. Hence we have

$$\|A^*x\|^2 \leq \|A^2x\| = |\lambda|^2.$$

Thus

$$\begin{aligned} 0 \leq \|A^*x - \bar{\lambda}x\|^2 &= \|A^*x\|^2 - 2\text{Re}\langle A^*x, \bar{\lambda}x\rangle + |\lambda|^2 \\ &\leq |\lambda|^2 - 2\text{Re}\langle x, \bar{\lambda}x\rangle + |\lambda|^2 \\ &= 0. \end{aligned}$$

This imply that $x \in \ker(A^* - \bar{\lambda}I)$. This completes the proof. □

Recall that a closed subspace M of H is said to be invariant under $A \in C(H)$ if $A(\mathcal{D}(A) \cap M) \subseteq M$, that is, $Ax \in M$ whenever $x \in \mathcal{D}(A) \cap M$.

Theorem 2.5. *Let $A \in C(H)$ be a $*$-paranormal operator. Then, the restriction of A on a closed invariant subspace $M \subset H$ is also $*$-paranormal.*

Proof. Let M be an invariant subspace for A and let $A_M = A|_M : \mathcal{D}(A) \cap M \to M$. First we want to show that $\mathcal{D}(A_M^2) \subseteq \mathcal{D}(A_M^*)$. Note that $\mathcal{D}(A_M^2) = \mathcal{D}(A^2) \cap M$. So, if $x \in \mathcal{D}(A_M^2)$, then

$$x \in \mathcal{D}(A^2) \cap M \subseteq \mathcal{D}(A^*) \cap M. \tag{1}$$

By definition we have that $y \in \mathcal{D}(A_M^*)$ if and only if the map $x \mapsto \langle A_M x, y\rangle$ is continuous for all $x \in \mathcal{D}(A_M)$. First, note that A_M^* is a map from M into $\mathcal{D}(A) \cap M$ with $\mathcal{D}(A_M^*) \subseteq M$. Hence for $x \in \mathcal{D}(A_M)$ and $y \in \mathcal{D}(A_M^*)$, we have

$$\langle Ax, y\rangle = \langle A_M x, y\rangle,$$

that is, $\langle x, A^*y\rangle = \langle x, A_M^* y\rangle$. That is, the map $x \mapsto \langle Ax, y\rangle$ is continuous for all $y \in \mathcal{D}(A_M^*)$. Hence A^*y exists for all $y \in \mathcal{D}(A_M^*)$. That is, $y \in \mathcal{D}(A^*) \cap M$. Hence $\mathcal{D}(A_M^*) \subseteq \mathcal{D}(A^*) \cap M$.

On the other hand, if $y \in \mathcal{D}(A^*) \cap M$, then the map

$$x \mapsto \langle Ax, y\rangle$$

is continuous for all $x \in \mathcal{D}(A)$. That is, $y \in \mathcal{D}(A_M^*)$. Therefore, $\mathcal{D}(A_M^*) \subseteq \mathcal{D}(A^*) \cap M$. Hence we can conclude that $\mathcal{D}(A_M^2) \subseteq \mathcal{D}(A_M) \subseteq \mathcal{D}(A_M^*)$.

Next, for $y \in \mathcal{D}(A_M^2)$, we have

$$\begin{aligned} \|A_M^* y\|^2 &= \|A^*y\|^2 \text{ (as } \mathcal{D}(A_M^*) \subseteq \mathcal{D}(A^*)) \\ &\leq \|A^2y\|\|y\| \text{ (as } A \text{ is } *-\text{paranormal)} \\ &\leq \|A_M^2 y\|\|y\|, \text{ since } \mathcal{D}(A_M^2) \subseteq \mathcal{D}(A^2) \cap M. \end{aligned}$$

Therefore, A_M is $*$-paranormal. □

Let $iso(\sigma(A))$ denote the isolated spectrum of an operator A. If $\mu \in iso(\sigma(A))$, then the Riesz idempotent E_μ with respect to μ is defined by

$$E_\mu = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} (z - A)^{-1} dz,$$

where $\mathbb{D}$ is the closed disk with center μ and having a small enough radius such that $\mathbb{D} \cap \sigma(A) = \{\mu\}$. It's known that $E_\mu^2 = E_\mu$, E_μ commutes with A, $\sigma(A\big|_{E_\mu(H)}) = \{\mu\}$ and $\sigma(A\big|_{(I-E_\mu)(H)}) = \sigma(A) \setminus \{\mu\}$. Also, E_μ is an orthogonal projection if and only if E_μ coincides with its adjoint. Reader can see [8] for further information. We've then the following important result.

Theorem 2.6. *Let $A \in C(H)$ be a $*$-paranormal operator and let $\mu \in iso(\sigma(A))$. Then,* $\ker(A - \mu I) = ran(E_\mu)$.

Proof. It suffices to show that $\mathrm{ran}(E_\mu) \subset \ker(A - \mu I)$ according to [4, Lemma 2.2]. The restriction $A\big|_{ran(E_\mu)}$ is a bounded $*$-paranormal operator by Theorem 2.5 and [8, Theorem 2.2]. Then, $A\big|_{ran(E_\mu)}$ is normaloid by [22, Proposition 1]. If $\mu = 0$, then $A\big|_{ran(E_\mu)}$ is quasinilpotent. Thus, $\|A\big|_{ran(E_\mu)}\| = 0$, and so $ran(E_0) \subset \ker(A)$. Assume that $\mu \neq 0$. Then, $\sigma(\mu^{-1}A\big|_{ran(E_\mu)}) = \{1\}$. Hence, $A\big|_{ran(E_\mu)} = \mu I\big|_{ran(E_\mu)}$ by [22, Corollary 1], that is, $ran(E_\mu) \subset \ker(A - \mu I)$. □

Corollary 2.7. *If $A \in C(H)$ is a $*$-paranormal operator, then $ran(A - \mu I) = \ker(E_\mu)$ for each isolated point $\mu \in iso(\sigma(A))$.*

Proof. We have $\mu \notin \sigma(A\big|_{\ker(E_\mu)}$ by [8, Theorem 2.2]. Then, $ran(A-\mu I)\big|_{\ker(E_\mu)} = \ker(E_\mu)$. Since $ran(A-\mu I)\big|_{\ker(E_\mu)} \subset ran(A - \mu I)$,, it follows that $\ker(E_\mu) \subset ran(A - \mu I)$.

Conversely, let $z = (A - \mu I)(t) \in ran(A - \mu I)$, for some $t \in \mathcal{D}(A)$. According to the decomposition $H = ran(E_\mu) \oplus \ker(E_\mu)$, we can write $t = x + y$, where $x \in ran(E_\mu)$ and $y \in \ker(E_\mu)$. By the previous Theorem, $x \in \ker(A - \mu I)$. Then, $y = (t - x) \in \mathcal{D}(A)$. On the other hand, $\ker(E_\mu)$ is invariant for A by [8, Theorem 2.2]. It follows that $z = (A - \mu I)y \in \ker(E_\mu)$. □

Corollary 2.8. *Let $A \in C(H)$ be $*$-paranormal. If $0 \in iso(\sigma(A))$, then $ran(A)$ is closed.*

Proof. In view of [8, Theorem 2.2], $0 \notin \sigma(A\big|_{\ker(E_0)})$. Furthermore, $ran(A) = \ker(E_0)$ by Corollary 2.7. Hence the subspace $ran(A)$ is then closed. □

Theorem 2.9. *Let $A \in C(H)$ be a $*$-paranormal operator such that $\ker(A^*) \subset \ker(A)$ and $0 \in \sigma(A)$. If $ran(A)$ is closed, then $0 \in iso(\sigma(A))$.*

Proof. The operator $A\big|_{\ker(A)^\perp} : \ker(A)^\perp \cap \mathcal{D}(A) \to \ker(A)^\perp$ is one-to-one with closed range since $ran(A\big|_{\ker(A)^\perp}) = ran(A)$ which is closed. By the definition of a $*$-paranormal operator and the hypothesis we have $\ker(A^*) \subset \ker(A)$, so we get that $\ker(A^*) = \ker(A)$. Then, $ran(A\big|_{\ker(A)^\perp}) = \ker(A^*)^\perp = \ker(A)^\perp$. Hence, the operator $A\big|_{\ker(A)^\perp}$ is invertible and its inverse $(A\big|_{\ker(A)^\perp})^{-1}$ is bounded on $\ker(A)^\perp$. Thus, $0 \notin \sigma(A\big|_{\ker(A)^\perp})$, which entails that $\sigma(A) \subset \sigma(A\big|_{\ker(A)^\perp}) \cup \{0\}$ by [23, Theorem 5.4]. Since $0 \in \sigma(A)$, $\sigma(A) = \sigma(A\big|_{\ker(A)^\perp}) \cup \{0\}$. This achieves the proof. □

Definition 2.10. *Let $A \in C(H)$ be densely defined. Then*

1. *the minimum modulus of A is the nonnegative real number*

$$m(A) := \inf\{\|Ax\| : x \in \mathcal{D}(A), \|x\| = 1\}$$

2. *the reduced minimum modulus of A is the nonnegative real number*

$$\gamma(A) := \inf\{\|Ax\| : x \in C(A), \|x\| = 1\},$$

where $C(A) = \mathcal{D}(A) \cap \ker(A)^\perp$ is said to be the carrier of A. We refer to [3, 9] for more details.

Obviously, $m(A) \leq \gamma(A)$.

Theorem 2.11. *Let A be a densely closed defined $*$-paranormal operator such that $\ker(A^*) \subset \ker(A)$. Then we have the following:*

1. *if A not one-to-one, then $m(A) = dist(0, \sigma(A))$.*
2. *if A is injective, then $0 \notin \sigma(A)$. If in addition, A^{-1} is normaloid, then $m(A) = dist(0, \sigma(A))$.*

Proof. First note that by the definition of paranormality of A, we get $ker(A) \subseteq ker(A^*)$. By the hypothesis, we get that $ker(A) = ker(A^*)$.

Proof of (1): In this case $m(A) = 0$. Since A is not-injective, we have $0 \in \sigma(A)$. Hence we have $m(A) = d(0, \sigma(A)) = 0$.

Proof of (2): If A is one-to-one, then $\ker(A) = \ker(A^*) = \{0\}$ since A is $*$-paranormal, and $m(A) = \gamma(A)$. If $\gamma(A) = 0$, then by [3, Page 334], $ran(A)$ is not closed. Hence we have $0 \in \sigma(A)$ and in this case, $m(A) = 0 = d(0, \sigma(A))$. Next assume that, $\gamma(A) > 0$. Then $ran(A)$ is closed, by [3, Page 334]. We've necessarily, $0 \notin \sigma(A)$. Otherwise, 0 is an eigenvalue of A by Theorem 2.6, Theorem 2.9 and Corollary 2.8, and this yields to a contradiction since A is injective. Since, A^{-1} is normaloid and by [13, Proposition 2.12]

$$\begin{aligned}\gamma(A) = \frac{1}{\|A^{-1}\|} = \frac{1}{r(A^{-1})} &= \frac{1}{\sup\{|\lambda| : \lambda \in \sigma(A^{-1})\}} \\ &= \inf\{|\mu| : \mu \in \sigma(A)\} \\ &= dist(0, \sigma(A)).\end{aligned}$$

□

2.2. Weyl's theorem for densely closed $*$-paranormal operators

In [22], it's proved that bounded $*$-paranormal operators satisfy Weyl's theorem. Also, authors in [4] showed that Weyl's theorem holds for closed paranormal operators. In the present section, we show that a closed $*$-paranormal operator satisfy the Weyl's theorem.

Recall that

$$\begin{aligned}&w(A) = \{\mu \in \sigma(A) : (A - \mu I) \text{ is not Fredholm and } ind(A) = 0\} \\ &\pi_{00}(A) = \{\mu \in iso(\sigma(A)) :\ 0 < \dim(\ker(A - \mu I)) < \infty\}.\end{aligned}$$

An operator A is said to satisfy Weyl's theorem if $\sigma(A) \setminus w(A) = \pi_{00}(A)$. For the definition and more information on the Weyl's spectrum we refer to [1, Page 132].

Theorem 2.12. *Let $A \in C(H)$ be a $*$-paranormal operator. Then A satisfies the Weyl's theorem, i.e., $\sigma(A) \setminus w(A) = \pi_{00}(A)$.*

Proof. Let $\mu \in \sigma(A) \setminus w(A)$. Then, $\dim(\ker(A - \mu I)) = \dim(\ker(A - \mu I)^*) < \infty$ and $ran(A - \mu I)$ is closed. Hence,

$$A - \mu I = \begin{pmatrix} 0 & A_1 \\ 0 & A_2 - \mu I \end{pmatrix}$$

on $H = \ker(A - \mu I) \oplus \ker(A - \mu I)^{\perp}$, $A_2 - \mu I = P_{\ker(A - \mu I)^{\perp}}(A - \mu I)$. We've $A_2 - \mu I$ is densely defined closed operator with closed range and has the domain $\mathcal{D}(A - \mu I) \cap \ker(A - \mu I)^{\perp}$. Since $\dim(\ker(A - \mu I)) < \infty$, A_2 is finite rank with null index. Thus, $ind(A - \mu I) = ind(A_2 - \mu I) = 0$. As $\ker((A_2 - \mu I)^*) = \ker(A_2 - \mu I) = \{0\}$, $\overline{ran(A_2 - \mu I)} = \ker(A - \mu I)^{\perp}$. Therefore, $A_2 - \mu I$ is invertible and its inverse is bounded. Furthermore, $\mu \notin \sigma(A_2)$. Since $\sigma(A) \subseteq \{\mu\} \cup \sigma(A_2)$, $\mu \in iso(\sigma(A)$. Consequently, $\mu \in \pi_{00}(A)$.

Now, if $\mu \in \pi_{00}(A)$, then by [8, Theorem 2.2] and Corollary 2.7, $\mu \notin \sigma(A\big|_{\ker(E_\mu}})$ and

$$ran(A - \mu I) = ran((A - \mu I)\big|_{\ker(E_\mu}}) = \ker(E_\mu)$$

Since $\mu \notin \sigma(A\big|_{\ker(E_\mu}})$, $ran((A - \mu I)\big|_{\ker(E_\mu}})$ is closed and so is $ran(A - \mu I)$. Thus,

$$\begin{aligned}\dim(\ker(A - \mu I)^*) = \dim(ran(A - \mu I)^{\perp}) &= \dim(\ker(E_\mu)^{\perp}) \\ &= \dim(ran(E_\mu)) \\ &= \dim(\ker(A - \mu I)).\end{aligned}$$

Hence $A - \mu I$ is Fredholm and $ind(A - \mu I) = 0$. Finally, $\mu \in \sigma(A) \setminus w(A)$. □

Theorem 2.13. *Let $A \in \mathcal{C}(H)$ be a $*$-paranormal operator and $\mu \in iso(\sigma(A))$. Then*

$$ran(E_\mu) = \ker(A-\mu I) = \ker(A-\mu I)^*.$$

In addition, E_μ is self-adjoint.

Proof. By [8, Theorem 2.2] and Corollary 2.7, $\mu \notin \ker(E_\mu)$ and $ran(A-\mu I) = \ker(E_\mu)$. Since $A\big|_{\ker(A-\mu I)^\perp}$: $\ker(A-\mu I)^\perp \cap \mathcal{D}(A) \to ran(A-\mu I)$ is bijective,

$$\ker(E_\mu) \cap \mathcal{D}(A) \subseteq \ker(A-\mu I)^\perp \cap \mathcal{D}(A).$$

Likewise, if $x \in \ker(A-\mu I)^\perp \cap \mathcal{D}(A)$. Put $E_\mu x = u+v$, $u \in \ker(A-\mu I)$ and $v \in \ker(A-\mu I)^\perp$. Then, $E_\mu x = u+v = (E_\mu)^2 x = u + E_\mu v$. Thus, $E_\mu v = v \in ran(E_\mu) \cap \ker(A-\mu I) \perp = \{0\}$ by Theorem 2.6. This implies $E_\mu x = E_\mu u = u$. That is,

$$x - u \in \ker(E_\mu) \cap \mathcal{D}(A) \subseteq \ker(A-\mu I)^\perp \cap \mathcal{D}(A).$$

Since $x \in \ker(A-\mu I)^\perp$, $u \in \ker(A-\mu I)^\perp \cap \ker(A-\mu I) = \{0\}$. This shows that $E_\mu x = 0$ and then

$$\ker(A-\mu I)^\perp \cap \mathcal{D}(A) \subseteq \ker(E_\mu) \cap \mathcal{D}(A).$$

Consequently,

$$\ker(A-\mu I)^\perp \cap \mathcal{D}(A) = \ker(E_\mu) \cap \mathcal{D}(A).$$

By [14, Lemma 3.3] and Corollary 2.7,

$$\begin{aligned}\ker(A-\mu I)^\perp = \overline{\ker(A-\mu I)^\perp \cap \mathcal{D}(A)} &= \overline{ran(A-\mu I) \cap \mathcal{D}(A)}\\ &= \overline{(\ker(A-\mu I)^*)^\perp \cap \mathcal{D}(A)}\\ &\subseteq \overline{(\ker(A-\mu I)^*)^\perp \cap \mathcal{D}(A^*)}\\ &\subseteq (\ker(A-\mu I)^*)^\perp,\end{aligned}$$

by [14, Lemma 3.3] again.

Then, $(\ker(A-\mu I)^*) \subseteq \ker(A-\mu I)$. Hence, $\ker(E_\mu)^\perp \subseteq ran(E_\mu)$ by Corollary 2.7.

Let $x \in ran(E_\mu$. Then, $x = a+b$, $a \in \ker(E_\mu$ and $b \in \ker(E_\mu)^\perp$. Since $\ker(E_\mu)^\perp \subseteq ran(E_\mu)$, $a = x-b \in \ker(E_\mu) \cap ran(E_\mu) = \{0\}$. Thus,

$$\ker(E_\mu)^\perp = ran(E_\mu) \tag{2}$$

i.e., $\ker(A-\mu I) = \ker(A-\mu I)^*$.

By Equation (2), E_μ is an orthogonal projection. Hence, $E_\mu = E_\mu^*$. This completes the proof. □

Acknowledgements We would like to thank the referee for his insightful suggestions that helped improve a manuscript or document.

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