Filomat 39:33 (2025), 11717–11731
https://doi.org/10.2298/FIL2533717L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Pseudo S-asymptotically antiperiodic solutions to Volterra difference equations with infinite delay

Penghui Lü[a], Yong-Kui Chang[a,*]

[a]*School of Mathematics and Statistics, Xidian University, Xi'an 710071, Shaanxi, P. R. China*

Abstract. In this paper, we firstly present notions and basic properties of m-(weighted) pseudo S-asymptotically ω-antiperiodic sequence and ∞-(weighted) pseudo S-asymptotically ω-antiperiodic sequence, which are applicable to the investigation of delayed difference equations. We secondly show some existence and uniqueness of pseudo S-asymptotically ω-antiperiodic solutions to Volterra difference equation with infinite delay as an application of aforementioned sequences.

1. Introduction

When modelling discrete time phenomena or discretized stimulations differential equations with historical effects, delayed difference equation can be regarded as a powerful and applicable tool, see monographs [13, 15, 16] and references cited therein. Asymptotical behavior on solutions has been one of the most attractive topics in the qualitative theory of (delayed) difference equations in the past ten years, see for instance [1, 4, 5, 8, 17]. Particularly, much attention has been paid to studies on the existence of almost periodic, asymptotically almost periodic, almost automorphic and pseudo S-asymptotically ω-periodic solutions to various delayed difference equations in [4, 6, 8, 10, 22]. It is noted that the existence of antiperiodic type solution for differential or integro-differential equations has also been considerably investigated, for example [2, 3, 7, 21]. As a discretized analogue, Chang and Lü [9] introduced the notion of weighted pseudo S-asymptotically ω-antiperiodic sequences and considered the existence of weighted pseudo S-asymptotically ω-antiperiodic sequential solutions to a semilinear difference equation. However, the sequence defined in [9] cannot be directly applied to delayed difference equations.

In this paper, we present notions and fundamental properties of m-(weighted) pseudo S-asymptotically ω-antiperiodic sequence and ∞-(weighted) pseudo S-asymptotically ω-antiperiodic sequence, which are applicable to the study of delayed difference equations. As an application, we investigate the existence of pseudo S-asymptotically ω-antiperiodic solutions to the following Volterra difference equation with infinite

2020 *Mathematics Subject Classification*. Primary 39A23; Secondary 39A24, 34G20.
Keywords. $m(\infty)$-pseudo S-asymptotically ω-antiperiodic sequence; Volterra difference equation; infinite delay; existence and uniqueness.
Received: 17 January 2023; Accepted: 12 October 2025
Communicated by Dragan S. Djordjević
Research partially supported by NSF of Shaanxi Province (2023-JC-YB-011).
* Corresponding author: Yong-Kui Chang
Email addresses: lphlvpenghui@163.com (Penghui Lü), lzchangyk@163.com (Yong-Kui Chang)
ORCID iD: https://orcid.org/0000-0001-5500-826X (Yong-Kui Chang)

delay

$$u(n+1) = \lambda \sum_{j=-\infty}^{n} a(n-j)u(j) + p(n, u_n), \ n \in \mathbb{Z}, \tag{1}$$

where λ is a complex number, $a(n)$ is a $\mathbb{C}$-value summable function, $u_n : \mathbb{Z}_- \to X$ is defined by $u_n(\theta) = u(n+\theta)$ for all $\theta \in \mathbb{Z}_-$, $p : \mathbb{Z}\times\mathscr{B} \to \mathbb{X}$, $\mathscr{B}$ is a phase space, all of which will be specified later. Some interesting results for Eq. (1) has been established in [4, 8], however the existence and uniqueness of pseudo S-asymptotically ω-antiperiodic sequential solutions to Eq. (1) is an untreated topic yet. Our main results can be seen as a supplement to works in [4, 8, 9].

The paper is outlined as follows. Section 2 is to recall some basic notations and notions which will be used later. Section 3 is mainly concerned with notions and properties of m-(weighted) pseudo S-asymptotically ω-antiperiodic sequence and ∞-(weighted) pseudo S-asymptotically ω-antiperiodic sequence. Section 4 is mainly to deal with the existence and uniqueness of pseudo S-asymptotically ω-antiperiodic sequential solutions to Eq. (1).

2. Preliminaries

We denote by $\mathbb{Z}(\mathbb{R}), \mathbb{Z}^+(\mathbb{R}^+), \mathbb{Z}_+(\mathbb{R}_+), \mathbb{Z}_-$ and $\mathbb{C}$ the set of all integers (real numbers), positive integers (real numbers), nonnegative integers (real numbers), nonpositive integers and complex numbers respectively. The notation card A stands for the number of elements in any finite set $A \subset \mathbb{R}$. Let X be a Banach space. The set $\mathcal{B}(X)$ denotes the space of all bounded linear operators from X to X and $l^\infty(\mathbb{Z}, X)$ is the Banach space consisting of all bounded sequences $u : \mathbb{Z} \to X$ with sup-norm $\|u\|_\infty = \sup_{n\in\mathbb{Z}} \|u(n)\|$. The space $l^\infty(\mathbb{Z} \times Y, X)$ represents the set formed by all sequences $u : \mathbb{Z} \times Y \to X$ such that $u(\cdot, x) \in l^\infty(\mathbb{Z}, X)$ uniformly for each x in any bounded subset of the normed space Y. A sequence $v : \mathbb{Z} \to \mathbb{C}$ is called q-th summable if $\sum_{k\in\mathbb{Z}}^{\infty} |v(k)|^q < \infty (1 \le q < \infty)$, and v is simply called summable if $q = 1$.

In order to deal with the existence of antiperiodic solution for difference equation with infinite delay, we recall the axiomatically-defined phase space $\mathscr{B}$, see details in [4, 8, 19, 20]. The space $\mathscr{B}$ stands for a vector space consists of all sequences $v : \mathbb{Z}_- \to X$. In the sense of norm $\|\cdot\|_{\mathscr{B}}$, it can be a Banach space and verifies the following axioms:

Axiom 2.1. If $x : \mathbb{Z} \to X$ is a sequence such that $x_m \in \mathscr{B}, m \in \mathbb{Z}$, then for all $n \ge m, n \in \mathbb{Z}$, the following conditions are fulfilled:

(1) $x_n \in \mathscr{B}$;

(2) $\|x(n)\|_X \le C\|x_n\|_{\mathscr{B}}$;

(3) $\|x_n\|_{\mathscr{B}} \le N(n-m)\max_{m\le i\le n}\|x(i)\|_X + M(n-m)\|x_m\|_{\mathscr{B}}$.

where C is a positive constant, $N(\cdot)$ and $M(\cdot)$ are nonnegative sequences defined on $\mathbb{Z}_+$.

Axiom 2.2. If $(\varphi^n)_{n\in\mathbb{N}}$ is a uniformly bounded sequence in $\mathscr{B}$, which converges pointwise to φ, then $\varphi \in \mathscr{B}$ and $\|\varphi^n - \varphi\|_{\mathscr{B}} \to 0$ as $n \to \infty$.

Remark 2.3. [8] From Axiom 2.2, we can get $l^\infty(\mathbb{Z}_-, X)$ is continuously included in $\mathscr{B}$. In the rest of paper, we assume that $\|\varphi\|_{\mathscr{B}} \le H\|\varphi\|_\infty$ for every $\varphi \in l^\infty(\mathbb{Z}_-, X)$, where $H > 0$ is a constant.

Example 2.4. Let $\alpha > 0$. $\mathscr{B}^\alpha(X)$ denote the space consisting of all sequences $\phi : \mathbb{Z}_- \to X$ with the norm $\|\phi\|_{\mathscr{B}^\alpha(X)} = \sup_{i\in\mathbb{Z}_-} |\phi(i)|e^{\alpha i} < \infty$. It is well-known that this space satisfies Axioms 2.1 and 2.2, see [8, 19] for details.

Let U denote the set of weights $\rho : \mathbb{Z} \to \mathbb{R}^+$. Set $U_\infty := \{\rho \in U : \lim_{n\to\infty} s(n,\rho) = \infty\}$ and $U_B := \{\rho \in U_\infty : 0 < \inf_{k\in\mathbb{Z}} \rho(k) \le \sup_{k\in\mathbb{Z}} \rho(k) < \infty\}$, where $s(n,\rho) = \sum_{k=-n}^{n} \rho(k)$ for $n \in \mathbb{Z}^+$. For $\rho_1, \rho_2 \in U_\infty$, we denote $\rho_1 \sim \rho_2$ if $\rho_1/\rho_2 \in U_B$. It is known from [23] that the relation "$\sim$" forms a binary equivalence relation on U_∞ and $U_\infty = \bigcup_{\rho\in U_\infty} \mathrm{cl}(\rho) := \{\varrho \in U_\infty : \rho \sim \varrho\}$. For given $\rho \in U_\infty$, $j \in \mathbb{Z}$, let $\rho_j(k) = \rho(k+j)$ for $k \in \mathbb{Z}$. We set $U_T = \{\rho \in U_\infty : \rho \sim \rho_j$ for each $j \in \mathbb{Z}\}$. We also define sets $\Sigma(n,m)$ and $\Sigma(n,\rho,m)$ for $\rho \in U_\infty$ and $m \in \mathbb{Z}^+$ as following:

$$\Sigma(n,m) := \left\{\nu : \mathbb{Z} \to \mathbb{R}_+ \,\middle|\, \lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} \max_{s\in[k-m,k]\cap\mathbb{Z}} \nu(s) < \infty\right\}, \tag{2}$$

$$\Sigma(n,\rho,m) := \left\{\nu : \mathbb{Z} \to \mathbb{R}_+ \,\middle|\, \lim_{n\to\infty} \frac{1}{s(n,\rho)} \sum_{k=-n}^{n} \rho(k) \max_{s\in[k-m,k]\cap\mathbb{Z}} \nu(s) < \infty\right\}, \tag{3}$$

which are useful to establish superposition theorems of parameterized ∞-(weighted) pseudo S-asymptotically ω-antiperiodic sequence.

3. Asymptotically antiperiodic sequences

In this section, we introduce notions of $m(\infty)$-pseudo S-asymptotically ω-antiperiodic sequence and $m(\infty)$-weighted pseudo S-asymptotically ω-antiperiodic sequence and establish some properties of such sequences.

We first recall the following definitions.

Definition 3.1. [9] A sequence $u \in l^\infty(\mathbb{Z}, X)$ is called pseudo S-asymptotically ω-antiperiodic if there exists $\omega \in \mathbb{Z}^+$ such that

$$\lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} \|u(k+\omega) + u(k)\| = 0.$$

The set of such sequences is denoted by $PSAP_{-\omega}(\mathbb{Z}, X)$.

Definition 3.2. [9] Let $\rho \in U_\infty$. A sequence $u \in l^\infty(\mathbb{Z}, X)$ is said weighted pseudo S-asymptotically ω-antiperiodic if there exists $\omega \in \mathbb{Z}^+$ such that

$$\lim_{n\to\infty} \frac{1}{s(n,\rho)} \sum_{k=-n}^{n} \rho(k)\|u(k+\omega) + u(k)\| = 0.$$

The set of such sequences is denoted by $WPSAP_{-\omega}(\mathbb{Z}, X, \rho)$.

In order to deal with pseudo S-asymptotically ω-antiperiodicity with infinite delay, we introduce the following notions.

Definition 3.3. A sequence $u \in l^\infty(\mathbb{Z}, X)$ is called m-pseudo S-asymptotically ω-antiperiodic sequence if there exists $\omega \in \mathbb{Z}^+$ such that

$$\lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} \max_{s\in[k-m,k]\cap\mathbb{Z}} \|u(s+\omega) + u(s)\| = 0$$

for $m \in \mathbb{Z}^+$. The set of such sequences is denoted by $PSAP_{-\omega}(\mathbb{Z}, X, m)$.

Definition 3.4. A sequence $u \in l^\infty(\mathbb{Z}, X)$ is called ∞-pseudo S-asymptotically ω-antiperiodic sequence if $u \in PSAP_{-\omega}(\mathbb{Z}, X, m)$ for all $m \in \mathbb{Z}^+$. The set of such sequences is denoted by $PSAP_{-\omega}(\mathbb{Z}, X, \infty)$.

Definition 3.5. Let $\rho \in U_\infty$. A sequence $u \in l^\infty(\mathbb{Z}, X)$ is called m-weighted pseudo S-asymptotically ω-antiperiodic if there exists $\omega \in \mathbb{Z}^+$ such that

$$\lim_{n\to\infty} \frac{1}{s(n,\rho)} \sum_{k=-n}^{n} \rho(k) \max_{s\in[k-m,k]\cap\mathbb{Z}} \|u(s+\omega) + u(s)\| = 0$$

for $m \in \mathbb{Z}^+$. The set of such sequences is denoted by $WPSAP_{-\omega}(\mathbb{Z}, X, \rho, m)$.

Definition 3.6. Let $\rho \in U_\infty$. A sequence $u \in l^\infty(\mathbb{Z}, X)$ is called ∞-weighted pseudo S-asymptotically ω-antiperiodic if $u \in WPSAP_{-\omega}(\mathbb{Z}, X, \rho, m)$ for all $m \in \mathbb{Z}^+$. The set of such sequences is denoted by $WPSAP_{-\omega}(\mathbb{Z}, X, \rho, \infty)$.

We have the following basic properties for the m-pseudo S-asymptotically ω-antiperiodic sequence and m-weighted pseudo S-asymptotically ω-antiperiodic sequence:

Theorem 3.7. Let $u = (u(n)),\ v = (v(n)) \in PSAP_{-\omega}(\mathbb{Z}, X, m)$. Then we have:

(1) $u + v \in PSAP_{-\omega}(\mathbb{Z}, X, m)$, $\lambda u \in PSAP_{-\omega}(\mathbb{Z}, X, m)$ for any $\lambda \in \mathbb{C}$;

(2) For given $j \in \mathbb{Z}$, let $u_j(n) = u(n+j)$. Then $u_j \in PSAP_{-\omega}(\mathbb{Z}, X, m)$;

(3) Let $\mathcal{A} \in \mathcal{B}(X)$, then $\mathcal{A}u \in PSAP_{-\omega}(\mathbb{Z}, X, m)$.

Proof. If $u = (u(n)),\ v = (v(n)) \in PSAP_{-\omega}(\mathbb{Z}, X, m)$, then

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} \max_{s\in[k-m,k]\cap\mathbb{Z}} \|(u(s+\omega) + v(s+\omega)) + (u(s) + v(s))\| \\
\leq\ & \lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} \max_{s\in[k-m,k]\cap\mathbb{Z}} \|u(s+\omega) + u(s)\| \\
&+ \lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} \max_{s\in[k-m,k]\cap\mathbb{Z}} \|v(s+\omega) + v(s)\| = 0; \\
&\lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} \max_{s\in[k-m,k]\cap\mathbb{Z}} \|\lambda u(s+\omega) + \lambda u(s)\| \\
=\ & |\lambda| \lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} \max_{s\in[k-m,k]\cap\mathbb{Z}} \|u(s+\omega) + u(s)\| = 0.
\end{aligned}$$

Thus the conclusion (1) holds and $PSAP_{-\omega}(\mathbb{Z}, X, m)$ is a vector space.

For $j \in \mathbb{Z}^+$ we have

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} \max_{s\in[k-m,k]\cap\mathbb{Z}} \|u(s+j+\omega) + u(s+j)\| \\
=\ & \lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} \max_{s\in\left[k-m+j,k+j\right]\cap\mathbb{Z}} \|u(s+\omega) + u(s)\| \\
=\ & \lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=-n+j}^{n+j} \max_{s\in[k-m,k]\cap\mathbb{Z}} \|u(s+\omega) + u(s)\| \\
=\ & \lim_{n\to\infty} \Bigg(\frac{1}{2n+1} \sum_{k=-n-j}^{n+j} \max_{s\in[k-m,k]\cap\mathbb{Z}} \|u(s+\omega) + u(s)\|
\end{aligned}$$

$$
\begin{aligned}
&\quad -\frac{1}{2n+1}\sum_{k=-n-j}^{-n+j}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|u(s+\omega)+u(s)\|\Big) \\
&\le \lim_{n\to\infty}\frac{1}{2n+1}\sum_{k=-(n+j)}^{n+j}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|u(s+\omega)+u(s)\| \\
&= \lim_{n\to\infty}\frac{2(n+j)+1}{2n+1}\cdot\frac{1}{2(n+j)+1}\sum_{k=-(n+j)}^{n+j}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|u(s+\omega)+u(s)\| \\
&= 0;
\end{aligned}
$$

Similarly, for $j \in \mathbb{Z}_-$ we have

$$
\begin{aligned}
&\lim_{n\to\infty}\frac{1}{2n+1}\sum_{k=-n}^{n}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|u(s+j+\omega)+u(s+j)\| \\
&\le \lim_{n\to\infty}\frac{2(n-j)+1}{2n+1}\cdot\frac{1}{2(n-j)+1}\sum_{k=-(n-j)}^{n-j}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|u(s+\omega)+u(s)\| \\
&= 0,
\end{aligned}
$$

which implies the conclusion (2) is true.

Since $\mathcal{A} \in \mathcal{B}(X)$, we have

$$
\begin{aligned}
&\lim_{n\to\infty}\frac{1}{2n+1}\sum_{k=-n}^{n}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|\mathcal{A}u(s+\omega)+\mathcal{A}u(s)\| \\
&\le \lim_{n\to\infty}\frac{1}{2n+1}\sum_{k=-n}^{n}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|\mathcal{A}\|\|u(s+\omega)+u(s)\| \\
&\le \|\mathcal{A}\|\lim_{n\to\infty}\frac{1}{2n+1}\sum_{k=-n}^{n}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|u(s+\omega)+u(s)\| = 0.
\end{aligned}
$$

Thus the conclusion (3) holds. □

Theorem 3.8. The vector space $PSAP_{-\omega}(\mathbb{Z}, X, m)$ is a Banach space with the sup-norm.

Proof. Let $u_{(k)} := \left(u_{(k)}(n)\right)_{n\in\mathbb{Z}} \in PSAP_{-\omega}(\mathbb{Z}, X, m)$, and $\|u_{(k)} - u\|_\infty = 0$ as $k \to \infty$. Then for any $\varepsilon > 0$, we can choose a suitable $N \in \mathbb{Z}^+$ such that for $n, k > N$,

$$
\frac{1}{2n+1}\sum_{i=-n}^{n}\max_{s\in[i-m,i]\cap\mathbb{Z}}\|u_{(k)}(s+\omega)+u_{(k)}(s)\| \le \frac{\varepsilon}{2},\ \|u_{(k)}-u\|_\infty \le \frac{\varepsilon}{4}.
$$

Thus we have

$$
\begin{aligned}
&\frac{1}{2n+1}\sum_{i=-n}^{n}\max_{s\in[i-m,i]\cap\mathbb{Z}}\|u(s+\omega)+u(s)\| \\
&= \frac{1}{2n+1}\sum_{i=-n}^{n}\max_{s\in[i-m,i]\cap\mathbb{Z}}\|u(s+\omega)-u_{(k)}(s+\omega)+u_{(k)}(s+\omega)+u_{(k)}(s)-u_{(k)}(s)+u(s)\| \\
&\le \frac{1}{2n+1}\sum_{i=-n}^{n}\max_{s\in[i-m,i]\cap\mathbb{Z}}\|u(s+\omega)-u_{(k)}(s+\omega)\|
\end{aligned}
$$

$$+\frac{1}{2n+1}\sum_{i=-n}^{n}\max_{s\in[i-m,i]\cap\mathbb{Z}}\|u_{(k)}(s+\omega)+u_{(k)}(s)\|+\frac{1}{2n+1}\sum_{i=-n}^{n}\max_{s\in[i-m,i]\cap\mathbb{Z}}\|u_{(k)}(s)-u(s)\|$$
$$\leq \frac{\varepsilon}{4}+\frac{\varepsilon}{2}+\frac{\varepsilon}{4}=\varepsilon,$$

which implies that $PSAP_{-\omega}(\mathbb{Z},X,m)$ is a closed subspace of $l^{\infty}(\mathbb{Z},X)$ and thus a Banach space. □

For the set $WPSAP_{-\omega}(\mathbb{Z},X,\rho,m)$, we have the following corollary by similar proofs of Theorems 3.7-3.8 and [9, Theorem 3.1].

Corollary 3.9. Let $\rho\in U_{\infty}$ and $u=(u(n)),\ v=(v(n))\in WPSAP_{-\omega}(\mathbb{Z},X,\rho,m)$. Then the following results are satisfied:

(a) $u+v\in WPSAP_{-\omega}(\mathbb{Z},X,\rho,m)$, $\lambda u\in WPSAP_{-\omega}(\mathbb{Z},X,\rho,m)$ for each $\lambda\in\mathbb{C}$;

(b) For given $j\in\mathbb{Z}$, let $u_j(n)=u(n+j)$. Then $u_j\in WPSAP_{-\omega}(\mathbb{Z},X,\rho,m)$ for $\rho\in U_T$;

(c) Let $\mathcal{A}\in\mathcal{B}(X)$, then $\mathcal{A}u\in WPSAP_{-\omega}(\mathbb{Z},X,\rho,m)$;

(d) The vector space $WPSAP_{-\omega}(\mathbb{Z},X,\rho,m)$ is a Banach space with the sup-norm.

By Definitions 3.4 and 3.6 combined with Theorem 3.8 and Corollary 3.9(d), we easily obtain the following corollary.

Corollary 3.10. The vector space $PSAP_{-\omega}(\mathbb{Z},X,\infty)$ (or $WPSAP_{-\omega}(\mathbb{Z},X,\rho,\infty)$) is a Banach space with the sup-norm.

We have the following properties for sequences defined in a phase space.

Lemma 3.11. Assume that $\mathscr{B}$ is a phase space and satisfies the following condition:

(F) There is a constant N_{∞} such that $N(\cdot)\leq N_{\infty}$ and $\lim_{n\to\infty}M(n)=0$;

Then $\mathbb{Z}\to\mathscr{B}, n\mapsto f_n\in PSAP_{-\omega}(\mathbb{Z},\mathscr{B},\infty)$ whenever $f\in PSAP_{-\omega}(\mathbb{Z},X,\infty)$.

Proof. For arbitrary given $m\in\mathbb{Z}^+$ and $\varepsilon>0$. By the $\lim_{n\to\infty}M(n)=0$, there is $n_0>m$ such that $M(l)\leq\varepsilon$ for all $l\geq n_0$. For $n\in\mathbb{Z}^+$ and $l\geq n_0$, we have

$$\frac{1}{2n+1}\sum_{k=-n}^{n}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|f_{s+\omega}+f_s\|_{\mathscr{B}}$$
$$\leq \frac{1}{2n+1}\sum_{k=-n}^{n}\max_{s\in[k-m,k]\cap\mathbb{Z}}\Big(M(l)\|f_{s-l+\omega}+f_{s-l}\|_{\mathscr{B}}+N(l)\max_{i\in[s-l,s]\cap\mathbb{Z}}\|f(i+\omega)+f(i)\|\Big)$$
$$\leq 2H\varepsilon\|f\|_{\infty}+\frac{N_{\infty}}{2n+1}\sum_{k=-n}^{n}\max_{i\in[k-(m+l),k]\cap\mathbb{Z}}\|f(i+\omega)+f(i)\|,$$

which shows $n\mapsto f_n\in PSAP_{-\omega}(\mathbb{Z},\mathscr{B},\infty)$ by $f\in PSAP_{-\omega}(\mathbb{Z},X,\infty)$. □

Remark 3.12. [14] The condition (F) is verified if $\mathscr{B}$ is a uniformly fading memory space.

Corollary 3.13. Let $\rho\in U_{\infty}$. Assume that $\mathscr{B}$ is a phase space and satisfies the condition (F) in Lemma 3.11, then $n\mapsto f_n\in WPSAP_{-\omega}(\mathbb{Z},\mathscr{B},\rho,\infty)$ whenever $f\in WPSAP_{-\omega}(\mathbb{Z},X,\rho,\infty)$.

We have the following convolution results.

Theorem 3.14. Let $\{\mathcal{S}(n)\}_{n\in\mathbb{Z}_+} \subset \mathcal{B}(X)$ be summable. If $f \in PSAP_{-\omega}(\mathbb{Z}, X, m)$, then

$$\Pi(n) := \sum_{k=-\infty}^{n} \mathcal{S}(n-k)f(k) \in PSAP_{-\omega}(\mathbb{Z}, X, m).$$

Proof. Since $\|\Pi(n)\| \le \|f\|_\infty \sum_{k=0}^{\infty} \|\mathcal{S}(n)\| = \|f\|_\infty \|\mathcal{S}\|_1 < \infty$, Π is bounded. Meanwhile,

$$\begin{aligned}
& \frac{1}{2n+1}\sum_{k=-n}^{n} \max_{s\in[k-m,k]\cap\mathbb{Z}} \|\Pi(s+\omega)+\Pi(s)\| \\
= \;& \frac{1}{2n+1}\sum_{k=-n}^{n} \max_{s\in[k-m,k]\cap\mathbb{Z}} \Big\| \sum_{i=-\infty}^{s+\omega} \mathcal{S}(s+\omega-i)f(i) + \sum_{i=-\infty}^{s} \mathcal{S}(s-i)f(i) \Big\| \\
\le \;& \frac{1}{2n+1}\sum_{k=-n}^{n} \max_{s\in[k-m,k]\cap\mathbb{Z}} \sum_{i=-\infty}^{s} \|\mathcal{S}(s-i)\| \|f(i+\omega)+f(i)\| \\
\le \;& \frac{1}{2n+1}\sum_{k=-n}^{n} \max_{s\in[k-m,k]\cap\mathbb{Z}} \sum_{i=0}^{\infty} \|\mathcal{S}(i)\| \|f(s-i+\omega)+f(s-i)\| \\
= \;& \sum_{i=0}^{\infty} \|\mathcal{S}(i)\| \Big(\frac{1}{2n+1}\sum_{k=-n}^{n} \max_{s\in[k-m,k]\cap\mathbb{Z}} \|f(s-i+\omega)+f(s-i)\| \Big)
\end{aligned}$$

It follows from Theorem 3.7(2) and the dominated convergence theorem that

$$\lim_{n\to\infty} \frac{1}{2n+1}\sum_{k=-n}^{n} \max_{s\in[k-m,k]\cap\mathbb{Z}} \|\Pi(s+\omega)+\Pi(s)\|,$$

which implies $\Pi \in PSAP_{-\omega}(\mathbb{Z}, X, m)$. □

Corollary 3.15. Let $\{\mathcal{S}(n)\}_{n\in\mathbb{Z}_+} \subset \mathcal{B}(X)$ be summable. If $f \in PSAP_{-\omega}(\mathbb{Z}, X, \infty)$, then

$$\Pi(n) := \sum_{k=-\infty}^{n} \mathcal{S}(n-k)f(k) \in PSAP_{-\omega}(\mathbb{Z}, X, \infty).$$

Corollary 3.16. Let $\{\mathcal{S}(n)\}_{n\in\mathbb{Z}_+} \subset \mathcal{B}(X)$ be summable. If $f \in WPSAP_{-\omega}(\mathbb{Z}, X, \rho, \infty)$ and $\rho \in U_T$, then

$$\Pi(n) := \sum_{k=-\infty}^{n} \mathcal{S}(n-k)f(k) \in WPSAP_{-\omega}(\mathbb{Z}, X, \rho, \infty).$$

We give some superposition theorems in the following results.

Theorem 3.17. Assume that $\mathscr{B}$ is a phase space satisfying the condition (F) in Lemma 3.11. Let $f \in l^\infty(\mathbb{Z}\times\mathscr{B}, X)$ verify the following conditions:

(A1) For any $m \in \mathbb{Z}^+$, $\lim_{n\to\infty} \frac{1}{2n+1}\sum_{k=-n}^{n} \max_{s\in[k-m,k]\cap\mathbb{Z}} \|f(s+\omega, x)+f(s,-x)\| = 0$ uniformly on any bounded set of $\mathscr{B}$;

(A2) There exists a constant $L > 0$ such that for all $x, y \in \mathscr{B}$ and $n \in \mathbb{Z}$,

$$\|f(n,x) - f(n,y)\| \le L\|x-y\|_{\mathscr{B}}.$$

Then for each $\phi \in PSAP_{-\omega}(\mathbb{Z}, X, \infty)$, $f(\cdot, \phi_\cdot) \in PSAP_{-\omega}(\mathbb{Z}, X, \infty)$.

Proof. Let $m \in \mathbb{Z}^+$ be arbitrarily given. In fact, from the condition (A2) we have

$$\begin{aligned}
&\frac{1}{2n+1}\sum_{k=-n}^{n}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|f(s+\omega,\phi_{s+\omega})+f(s,\phi_s)\| \\
\leq\ &\frac{1}{2n+1}\sum_{k=-n}^{n}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|f(s+\omega,\phi_{s+\omega})+f(s,-\phi_{s+\omega})\| \\
&+\frac{1}{2n+1}\sum_{k=-n}^{n}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|f(s,\phi_s)-f(s,-\phi_{s+\omega})\| \\
\leq\ &\frac{1}{2n+1}\sum_{k=-n}^{n}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|f(s+\omega,\phi_{s+\omega})+f(s,-\phi_{s+\omega})\| \\
&+L\frac{1}{2n+1}\sum_{k=-n}^{n}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|\phi_s+\phi_{s+\omega}\|_{\mathscr{B}}.
\end{aligned}$$

Since $\phi \in PSAP_{-\omega}(\mathbb{Z}, X, \infty)$ and Lemma 3.11, we have $\phi_s \in PSAP_{-\omega}(\mathbb{Z}, \mathscr{B}, \infty)$. Together with the condition (A1), we can see that $\frac{1}{2n+1}\sum_{k=-n}^{n}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|f(s+\omega,\phi_{s+\omega})+f(s,\phi_s)\| = 0$, i.e. for each $\phi \in PSAP_{-\omega}(\mathbb{Z}, X, \infty)$, $f(\cdot, \phi_\cdot) \in PSAP_{-\omega}(\mathbb{Z}, X, \infty)$. □

Theorem 3.18. Assume that $\rho \in U_\infty$ and $\mathscr{B}$ is a phase space satisfying the condition (F) in Lemma 3.11. Let $f \in l^\infty(\mathbb{Z}\times\mathscr{B}, X)$ meet the condition (A2) in Theorem 3.17 and the following assumption:

(A1*) For any $m \in \mathbb{Z}^+$, $\lim_{n\to\infty}\frac{1}{s(n,\rho)}\sum_{k=-n}^{n}\rho(k)\max_{s\in[k-m,k]\cap\mathbb{Z}}\|f(s+\omega,x)+f(s,-x)\| = 0$ uniformly on any bounded set of $\mathscr{B}$.

Then for each $\phi \in WPSAP_{-\omega}(\mathbb{Z}, X, \rho, \infty)$, $f(\cdot,\phi_\cdot) \in WPSAP_{-\omega}(\mathbb{Z}, X, \rho, \infty)$.

Proof. We can get the conclusion only by adding weighted ρ in Theorem 3.17. □

Lemma 3.19. Let $f \in l^\infty(\mathbb{Z}, X)$. Then for each $m \in \mathbb{Z}^+$, the following assertions are equivalent:

(a) $\lim_{n\to\infty}\frac{1}{2n+1}\sum_{k=-n}^{n}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|f(s+\omega)+f(s)\| = 0$, i.e. $f \in PSAP_{-\omega}(\mathbb{Z}, X, m)$.

(b) For each $\epsilon > 0$, $\lim_{n\to\infty}\frac{\text{card } M_{n,\epsilon}(f)}{2n+1} = 0$, where

$$M_{n,\epsilon}(f) = \{k \in [-n,n]\cap\mathbb{Z} : \max_{s\in[k-m,k]\cap\mathbb{Z}}\|f(s+\omega)+f(s)\| \geq \epsilon\}.$$

Proof. For each given $\epsilon > 0$, since

$$\begin{aligned}
&\frac{1}{2n+1}\sum_{k=-n}^{n}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|f(s+\omega)+f(s)\| \\
=\ &\frac{1}{2n+1}\sum_{k\in[-n,n]\setminus M_{n,\epsilon}(f)}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|f(s+\omega)+f(s)\| + \frac{1}{2n+1}\sum_{k\in M_{n,\epsilon}(f)}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|f(s+\omega)+f(s)\| \\
\geq\ &\frac{1}{2n+1}\sum_{k\in M_{n,\epsilon}(f)}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|f(s+\omega)+f(s)\| \geq \epsilon\frac{\text{card } M_{n,\epsilon}(f)}{2n+1} \geq 0,
\end{aligned}$$

we can achieve the assertion (b) by the truth of the assertion (a).

If the assertion (b) holds, then for any $\epsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that $\frac{\text{card } M_{n,\epsilon}(f)}{2n+1} < \frac{\epsilon}{2\|f\|_\infty}$, $n > N$. Thus for $n > N$,

$$\frac{1}{2n+1}\sum_{k=-n}^{n}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|f(s+\omega)+f(s)\|$$

$$= \frac{1}{2n+1}\sum_{k\in[-n,n]\setminus M_{n,\epsilon}(f)}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|f(s+\omega)+f(s)\| + \frac{1}{2n+1}\sum_{k\in M_{n,\epsilon}(f)}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|f(s+\omega)+f(s)\|$$
$$\leq \frac{1}{2n+1}(2n+1)\epsilon + 2\|f\|_\infty\frac{\text{card } M_{n,\epsilon}(f)}{2n+1} < 2\epsilon,$$

which implies $\lim_{n\to\infty}\frac{1}{2n+1}\sum_{k=-n}^{n}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|f(s+\omega)+f(s)\| = 0$, i.e. (a) holds. □

Corollary 3.20. Let $n\mapsto f_n\in l^\infty(\mathbb{Z},\mathscr{B})$. Then for each $m\in\mathbb{Z}^+$, the following assertions are equivalent:

(a) $\lim_{n\to\infty}\frac{1}{2n+1}\sum_{k=-n}^{n}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|f_{s+\omega}+f_s\|_{\mathscr{B}} = 0$, i.e. $f_n\in PSAP_{-\omega}(\mathbb{Z},\mathscr{B},m)$.

(b) For each $\epsilon>0$, $\lim_{n\to\infty}\frac{\text{card } M_{n,\epsilon}(f_n)}{2n+1} = 0$, where

$$M_{n,\epsilon}(f_n) = \{k\in[-n,n]\cap\mathbb{Z} : \max_{s\in[k-m,k]\cap\mathbb{Z}}\|f_{s+\omega}+f_s\|_{\mathscr{B}}\geq\epsilon\}.$$

Corollary 3.21. Let $\rho\in U_\infty$ and $n\mapsto f_n\in l^\infty(\mathbb{Z},\mathscr{B})$. Then for each $m\in\mathbb{Z}^+$, the following assertions are equivalent:

(a) $\lim_{n\to\infty}\frac{1}{s(n,\rho)}\sum_{k=-n}^{n}\rho(k)\max_{s\in[k-m,k]\cap\mathbb{Z}}\|f_{s+\omega}+f_s\|_{\mathscr{B}} = 0$ ($f_n\in WPSAP_{-\omega}(\mathbb{Z},\mathscr{B},\rho,m)$).

(b) For each $\epsilon>0$, $\lim_{n\to\infty}\frac{\sum_{k\in M_{n,\epsilon}(f_n)}\rho(k)}{s(n,\rho)} = 0$, where

$$M_{n,\epsilon}(f_n) = \{k\in[-n,n]\cap\mathbb{Z} : \max_{s\in[k-m,k]\cap\mathbb{Z}}\|f_{s+\omega}+f_s\|_{\mathscr{B}}\geq\epsilon\}.$$

Theorem 3.22. Let $\mathscr{B}$ be a phase space satisfying the condition (F) in Lemma 3.11. Assume that $f\in l^\infty(\mathbb{Z}\times\mathscr{B},X)$ verifies the condition (A1) in Theorem 3.17 and the following assumption:

(A3) For any $\varepsilon>0$ and any bounded subset $K\subseteq\mathscr{B}$, there exist constants $N_{\varepsilon,K}\in\mathbb{Z}^+$ and $\delta_{\varepsilon,K}>0$ such that $\|f(k,x)-f(k,y)\|\leq\varepsilon$ for all $x,y\in K$ with $\|x-y\|_{\mathscr{B}}\leq\delta_{\varepsilon,K}$ and $k\in\mathbb{Z}$ with $|k|\geq N_{\varepsilon,K}$.

Then for each $\phi\in PSAP_{-\omega}(\mathbb{Z},X,\infty)$, $\mathbb{F}(\cdot) := f(\cdot,\phi_\cdot)\in PSAP_{-\omega}(\mathbb{Z},X,\infty)$.

Proof. Let $m\in\mathbb{Z}^+$. For each $\phi\in PSAP_{-\omega}(\mathbb{Z},X,\infty)$, we can get $\phi_n\in PSAP_{-\omega}(\mathbb{Z},\mathscr{B},\infty)$ from Lemma 3.11. Let $K=\overline{\{\phi_n : n\in\mathbb{Z}\}}$. By the fact $\phi_n\in PSAP_{-\omega}(\mathbb{Z},\mathscr{B},\infty)$ and Corollary 3.20, for any $\varepsilon>0$ there exists $N_\varepsilon^1\in\mathbb{Z}^+$ such that for $n>N_\varepsilon^1$, $\frac{\text{card } M_{n,\varepsilon}(\phi_n)}{2n+1}\leq\frac{\varepsilon}{4\|\mathbb{F}\|_\infty}$. From the condition (A1), for this ε, there exists $N_\varepsilon^2\in\mathbb{Z}^+$ verifying $\frac{1}{2n+1}\sum_{k=-n}^{n}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|f(s+\omega,\phi_{s+\omega})+f(s,-\phi_{s+\omega})\|\leq\frac{\varepsilon}{4}$ for $n>N_\varepsilon^2$. From the condition (A3), for above $\varepsilon>0$ there exists $\delta_{\varepsilon,K}=\varepsilon$ and $N_{\varepsilon,K} := N_\varepsilon^3\in\mathbb{Z}^+$ such that $\|f(s,\phi_s)-f(s,-\phi_{s+\omega})\|\leq\frac{\varepsilon}{4}$ whenever $\|\phi_s+\phi_{s+\omega}\|_{\mathscr{B}}\leq\varepsilon$ and $|s|>N_\varepsilon^3$. Thus we have for $n>N_\varepsilon=\max\{N_\varepsilon^1,N_\varepsilon^2,N_\varepsilon^3\}$

$$\frac{1}{2n+1}\sum_{k=-n}^{n}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|f(s+\omega,\phi_{s+\omega})+f(s,\phi_s)\|$$
$$\leq \frac{1}{2n+1}\sum_{k=-n}^{n}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|f(s+\omega,\phi_{s+\omega})+f(s,-\phi_{s+\omega})\|$$
$$+\frac{1}{2n+1}\sum_{k=-n}^{n}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|f(s,\phi_s)-f(s,-\phi_{s+\omega})\|$$
$$\leq \frac{\varepsilon}{4}+\frac{1}{2n+1}\sum_{k\in[-n,n]\setminus M_{n,\varepsilon}(\phi_n)}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|f(s,\phi_s)-f(s,-\phi_{s+\omega})\|$$
$$+\frac{1}{2n+1}\sum_{k\in M_{n,\varepsilon}(\phi_n)}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|f(s,\phi_s)-f(s,-\phi_{s+\omega})\|$$
$$\leq \frac{\varepsilon}{4}+\frac{\varepsilon}{4}+2\|\mathbb{F}\|_\infty\frac{\text{card } M_{n,\varepsilon}(\phi_n)}{2n+1} < \varepsilon,$$

which implies $\lim_{n\to\infty}\frac{1}{2n+1}\sum_{k=-n}^{n}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|f(s+\omega,\phi_{s+\omega})+f(s,\phi_s)\|=0$, i.e. for each given $\phi\in PSAP_{-\omega}(\mathbb{Z},X,\infty)$, $\mathbb{F}(\cdot)=f(\cdot,\phi_\cdot)\in PSAP_{-\omega}(\mathbb{Z},X,\infty)$. □

Theorem 3.23. Assume that $\rho\in U_\infty$ and $\mathscr{B}$ is a phase space and satisfies the condition (F) in Lemma 3.11. Let $f\in l^\infty(\mathbb{Z}\times\mathscr{B},X)$ satisfy conditions (A1*) and (A3) in Theorems 3.18-3.22. Then for each $\phi\in WPSAP_{-\omega}(\mathbb{Z},X,\rho,\infty)$, $\mathbb{F}(\cdot):=f(\cdot,\phi_\cdot)\in WPSAP_{-\omega}(\mathbb{Z},X,\rho,\infty)$.

Proof. The proof can be similarly conducted by adding weighted ρ in Theorem 3.22. □

The following theorems are based upon sets $\Sigma(n,m)$ and $\Sigma(n,\rho,m)$ defined by equations (2)-(3).

Theorem 3.24. Assume that $\mathscr{B}$ is a phase space verifying the condition (F) in Lemma 3.11. Let $f\in l^\infty(\mathbb{Z}\times\mathscr{B},X)$ satisfy the condition (A1) in Theorem 3.17 and suppose further that

(A4) There exists a function $\mathcal{L}(\cdot)\in\Sigma(n,m)$ such that for any $\varepsilon>0$, there is a constant $\delta>0$ satisfying $\|f(k,x)-f(k,y)\|\le\mathcal{L}(k)\varepsilon$ for all $x,y\in K$ any bounded subset of $\mathscr{B}$ with $\|x-y\|_{\mathscr{B}}\le\delta$ and $k\in\mathbb{Z}$.

Then for each $\phi\in PSAP_{-\omega}(\mathbb{Z},X,\infty)$, $\mathbb{F}(\cdot):=f(\cdot,\phi_\cdot)\in PSAP_{-\omega}(\mathbb{Z},X,\infty)$.

Proof. Let $m\in\mathbb{Z}^+$. For each $\phi\in PSAP_{-\omega}(\mathbb{Z},X,\infty)$, we can get $\phi_n\in PSAP_{-\omega}(\mathbb{Z},\mathscr{B},\infty)$ from Lemma 3.11. Let $K=\overline{\{\phi_n:n\in\mathbb{Z}\}}$. From the condition (A4), for any $\varepsilon>0$ there exists $\delta>0$ such that $\|f(k,\phi_k)-f(k,-\phi_{k+\omega})\|\le\mathcal{L}(k)\varepsilon$ whenever $\|\phi_k+\phi_{k+\omega}\|_{\mathscr{B}}\le\delta$ for all $k\in\mathbb{Z}$. From the condition (A1), for above $\varepsilon>0$ there exists $N_\varepsilon^1\in\mathbb{Z}^+$ such that $\frac{1}{2n+1}\sum_{k=-n}^{n}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|f(s+\omega,\phi_{s+\omega})+f(s,-\phi_{s+\omega})\|\le\frac{\varepsilon}{2}$ for $n>N_\varepsilon^1$. By Corollary 3.20 and $\phi_n\in PSAP_{-\omega}(\mathbb{Z},\mathscr{B},m)$, for above $\varepsilon>0$ there exists $N_\varepsilon^2\in\mathbb{Z}^+$ such that $\frac{\text{card }M_{n,\delta}(\phi_n)}{2n+1}\le\frac{\varepsilon}{4\|\mathbb{F}\|_\infty}$ for $n>N_\varepsilon^2$. Since $\mathcal{L}(\cdot)\in\Sigma(n,m)$, there exist $N^3\in\mathbb{Z}^+$ and $M>0$ such that $\frac{1}{2n+1}\sum_{k=-n}^{n}\max_{s\in[k-m,k]\cap\mathbb{Z}}\mathcal{L}(s)\le M$ for $n>N^3$. For $n>N=\max\{N_\varepsilon^1,N_\varepsilon^2,N^3\}$, we have

$$\begin{aligned}
&\frac{1}{2n+1}\sum_{k=-n}^{n}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|f(s+\omega,\phi_{s+\omega})+f(s,\phi_s)\|\\
\le\;&\frac{1}{2n+1}\sum_{k=-n}^{n}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|f(s+\omega,\phi_{s+\omega})+f(s,-\phi_{s+\omega})\|\\
&+\frac{1}{2n+1}\sum_{k=-n}^{n}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|f(s,\phi_s)-f(s,-\phi_{s+\omega})\|\\
\le\;&\frac{1}{2n+1}\sum_{k=-n}^{n}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|f(s+\omega,\phi_{s+\omega})+f(s,-\phi_{s+\omega})\|\\
&+\frac{1}{2n+1}\sum_{k\in[-n,n]\setminus M_{n,\delta}(\phi_n)}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|f(s,\phi_s)-f(s,-\phi_{s+\omega})\|\\
&+\frac{1}{2n+1}\sum_{k\in M_{n,\delta}(\phi_n)}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|f(s,\phi_s)-f(s,-\phi_{s+\omega})\|\\
\le\;&\frac{\varepsilon}{2}+\Big(\frac{1}{2n+1}\sum_{k=-n}^{n}\max_{s\in[k-m,k]\cap\mathbb{Z}}\mathcal{L}(s)\Big)\varepsilon+2\|\mathbb{F}\|_\infty\frac{\text{card }M_{n,\delta}(\phi_n)}{2n+1}\\
\le\;&\frac{\varepsilon}{2}+M\varepsilon+\frac{\varepsilon}{2}=(M+1)\varepsilon,
\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\frac{1}{2n+1}\sum_{k=-n}^{n}\max_{s\in[k-m,k]\cap\mathbb{Z}}\|f(s+\omega,\phi_{s+\omega})+f(s,\phi_s)\|=0,$$

i.e. $\mathbb{F}(\cdot):=f(\cdot,\phi_\cdot)\in PSAP_{-\omega}(\mathbb{Z},X,\infty)$ for each $\phi\in PSAP_{-\omega}(\mathbb{Z},X,\infty)$. □

Theorem 3.25. Assume that $\rho \in U_\infty$ and $\mathscr{B}$ is a phase space satisfying the condition (F) in Lemma 3.11. Let $f \in l^\infty(\mathbb{Z} \times \mathscr{B}, X)$ verify the condition (A1*) in Theorem 3.18 and the following assumption:
(A4*) There exists a function $\mathcal{L}(\cdot) \in \Sigma(n, \rho, m)$ such that for any $\varepsilon > 0$, there is a constant $\delta > 0$ satisfying $\|f(k, x) - f(k, y)\| \leq \mathcal{L}(k)\varepsilon$ for all $x, y \in K$ any bounded subset of $\mathscr{B}$ with $\|x - y\|_{\mathscr{B}} \leq \delta$ and $k \in \mathbb{Z}$.
Then for each $\phi \in WPSAP_{-\omega}(\mathbb{Z}, X, \rho, \infty)$, $\mathbb{F}(\cdot) := f(\cdot, \phi_\cdot) \in WPSAP_{-\omega}(\mathbb{Z}, X, \rho, \infty)$.

Proof. The conclusion can be similarly proved by adding weighted ρ in Theorem 3.24. □

Theorem 3.26. Assume that $\mathscr{B}$ is a phase space satisfying the condition (F) in Lemma 3.11. Let $f \in l^\infty(\mathbb{Z} \times \mathscr{B}, X)$ verify the condition (A1) in Theorem 3.17 and suppose further that

(A5) There exists a function $\mathcal{L}(\cdot) \in \Sigma(n, m)$ such that for all x, $y \in \mathscr{B}$ and $k \in \mathbb{Z}$, $\|f(k, x) - f(k, y)\| \leq \mathcal{L}(k)\|x - y\|_{\mathscr{B}}$.

Then for each $\phi \in PSAP_{-\omega}(\mathbb{Z}, X, \infty)$, $f(\cdot, \phi_\cdot) \in PSAP_{-\omega}(\mathbb{Z}, X, \infty)$.

Proof. We can prove the result through replacing δ in Theorem 3.24 by ε. □

Theorem 3.27. Assume that $\rho \in U_\infty$ and $\mathscr{B}$ is a phase space satisfying the condition (F) in Lemma 3.11. Let $f \in l^\infty(\mathbb{Z} \times \mathscr{B}, X)$ verify the condition (A1*) in Theorem 3.18 and the following assumption:

(A5*) There exists a function $\mathcal{L}(\cdot) \in \Sigma(n, m)$ such that for all x, $y \in \mathscr{B}$ and $k \in \mathbb{Z}$, $\|f(k, x) - f(k, y)\| \leq \mathcal{L}(k)\|x - y\|_{\mathscr{B}}$.

Then for each $\phi \in WPSAP_{-\omega}(\mathbb{Z}, X, \rho, \infty)$, $\mathbb{F}(\cdot) := f(\cdot, \phi_\cdot) \in WPSAP_{-\omega}(\mathbb{Z}, X, \rho, \infty)$.

Proof. The proof can be conducted by a slight modification of Theorem 3.25. □

Remark 3.28. (1) It is obvious that $\nu \in \Sigma(n, m)$ if $\nu(\cdot) \equiv L > 0$ or $\nu : \mathbb{Z} \to \mathbb{R}_+$ is q-th summable for $1 \leq q < \infty$. Thus $\nu \in \Sigma(n, m)$ if $\nu : \mathbb{Z} \to \mathbb{R}_+$ is q-th summable for $1 \leq q \leq \infty$ by letting $\nu(\cdot) \equiv L > 0$ whenever $q = \infty$.
(2) We can get $\nu \in \Sigma(n, \rho, m)$ if $\nu(\cdot) \equiv L > 0$ or $(\rho\nu)(\cdot) : \mathbb{Z} \to \mathbb{R}_+$ is summable with $(\rho\nu)(k) := \rho(k)\nu(k)$ for each $k \in \mathbb{Z}$.
(3) If $\nu(\cdot) : \mathbb{Z} \to \mathbb{R}_+$ verifies that $(\rho\nu)(\cdot) : \mathbb{Z} \to \mathbb{R}_+$ is q-th summable for $1 < q < \infty$ with $\lim_{n\to\infty} \frac{(2n+1)^{\frac{q-1}{q}}}{s(n,\rho)} < \infty$, then $\nu \in \Sigma(n, \rho, m)$.
(4) We can similarly present and discuss notions of pseudo ω-antiperiodic sequences [18] with infinite delay as aforementioned techniques.

4. Existence results

In this section, we consider existence results for pseudo S-asymptotically ω-antiperiodic solutions to the Volterra difference equation (1).

For a given $\lambda \in \mathbb{C}$, we denote $S(\lambda, k) \in \mathbb{C}$ by the solution of the difference equation

$$\begin{cases} S(\lambda, k+1) = \lambda \sum_{j=0}^{k} a(k-j)S(\lambda, j), \ k \in \mathbb{Z}_+, \\ S(\lambda, 0) = 1. \end{cases}$$

Let $\Omega_S := \left\{\lambda \in \mathbb{C} : \sum_{k=0}^{\infty} |S(\lambda, k)| < \infty\right\}$. For any $\lambda \in \Omega_S$ and $p \in l^\infty(\mathbb{Z}, X)$, it follows from [12] that the expression $u(n+1) = \sum_{k=-\infty}^{n} S(\lambda, n-k)p(k)$ formulates the solution to the following linear equation $u(n+1) = \lambda \sum_{j=-\infty}^{n} a(n-j)u(j) + p(n)$, $n \in \mathbb{Z}$. Thus for $\lambda \in \Omega_S$, if $k \mapsto S(\lambda, n-k)p(k, u_k)$ is summable on $\mathbb{Z}_+$ for each $n \in \mathbb{Z}$, then a sequence defined by $u(n+1) = \sum_{k=-\infty}^{n} S(\lambda, n-k)p(k, u_k)$ is a solution to Eq. (1) (see also [8]).

We always assume that $\mathscr{B}$ is a phase space satisfying the condition (F) in Lemma 3.11 in the following hypotheses:

(Hp1) For all $(k,u) \in \mathbb{Z}\times\mathscr{B}$ and $m \in \mathbb{Z}^+$, $\lim_{n\to\infty} \frac{1}{2n+1}\sum_{k=-n}^{n} \max_{s\in[k-m,k]\cap\mathbb{Z}} \|p(s+\omega,u)+p(s,-u)\| = 0$ uniformly on any bounded set of $\mathscr{B}$.

(Hp2) There exists a function $L_p : \mathbb{R}_+ \to \mathbb{R}_+$ such that for each $r \geq 0$ and all $u, v \in \mathscr{B}$ with $\|u\|_{\mathscr{B}} \leq r$, $\|v\|_{\mathscr{B}} \leq r$, $\|p(k,u) - p(k,v)\| \leq L_p(r)\|u - v\|_{\mathscr{B}}$, $k \in \mathbb{Z}$.

(Hp3) There exists a summable function $\mathcal{L}_p(\cdot) : \mathbb{Z} \to \mathbb{R}_+$ such that for all $u, v \in \mathscr{B}$ and $k \in \mathbb{Z}$, $\|p(k,u)-p(k,v)\| \leq \mathcal{L}_p(k)\|u - v\|_{\mathscr{B}}$.

Lemma 4.1. Let $\lambda \in \Omega_S$. Assume further that conditions (Hp1)-(Hp2) hold. Then for each $u \in PSAP_{-\omega}(\mathbb{Z}, X, \infty)$, $P(n) := \sum_{k=-\infty}^{n} S(\lambda, n-k)p(k, u_k) \in PSAP_{-\omega}(\mathbb{Z}, X, \infty)$.

Proof. For each given $u \in PSAP_{-\omega}(\mathbb{Z}, X, \infty)$, we can get $n \mapsto u_n \in PSAP_{-\omega}(\mathbb{Z}, \mathscr{B}, \infty)$ from Lemma 3.11. It is bounded and there exists a bounded subset K of $\mathscr{B}$ such that $u_n \in K$ for all $n \in \mathbb{Z}$. It follows from the condition (Hp2) that $p(k, u_\cdot)$ is uniformly continuous on the bounded subset K uniformly for $k \in \mathbb{Z}$. Thus we have from Theorem 3.22 together with conditions (Hp1)-(Hp2) that $p(\cdot, u_\cdot) \in PSAP_{-\omega}(\mathbb{Z}, X, \infty)$ if $u \in PSAP_{-\omega}(\mathbb{Z}, X, \infty)$. It further follows from Corollary 3.15 that $P(n) \in PSAP_{-\omega}(\mathbb{Z}, X, \infty)$ for each $u \in PSAP_{-\omega}(\mathbb{Z}, X, \infty)$. □

Lemma 4.2. Suppose that $\lambda \in \Omega_S$ and conditions (Hp1) and (Hp3) are satisfied. Then $P(n) := \sum_{k=-\infty}^{n} S(\lambda, n-k)p(k, u_k) \in PSAP_{-\omega}(\mathbb{Z}, X, \infty)$ whenever $u \in PSAP_{-\omega}(\mathbb{Z}, X, \infty)$.

Proof. It follows from Theorem 3.26 and Remark 3.28(1) together with conditions (Hp1) and (Hp4) that $p(\cdot, u_\cdot) \in PSAP_{-\omega}(\mathbb{Z}, X, \infty)$ for each $u \in PSAP_{-\omega}(\mathbb{Z}, X, \infty)$. Thus we have from Corollary 3.15 that $P(n) \in PSAP_{-\omega}(\mathbb{Z}, X, \infty)$ whenever $u \in PSAP_{-\omega}(\mathbb{Z}, X, \infty)$. □

In what follows, we suppose that $\lambda \in \Omega_S$ with $|S(\lambda,\cdot)|_1 = \sum_{k=0}^{\infty} |S(\lambda,k)|$ and H is a constant in the Remark 2.3.

Theorem 4.3. If conditions (Hp1)-(Hp2) are satisfied, then Eq. (1) admits a solution $u \in PSAP_{-\omega}(\mathbb{Z}, X, \infty)$ whenever

$$\sup_{r>0}[r - HrL_p(Hr)|S(\lambda,\cdot)|_1] > |S(\lambda,\cdot)|_1 \sup_{k\in\mathbb{Z}} \|p(k,0)\|. \tag{4}$$

Proof. We define the operator $\mathcal{P} : PSAP_{-\omega}(\mathbb{Z}, X, \infty) \to PSAP_{-\omega}(\mathbb{Z}, X, \infty)$ by

$$(\mathcal{P}u)(n) = \sum_{k=-\infty}^{n-1} S(\lambda, n-1-k)p(k, u_k). \tag{5}$$

It is known from Lemma 4.1 that $\mathcal{P}$ is well-defined for each $u \in PSAP_{-\omega}(\mathbb{Z}, X, \infty)$. By the condition (4), there exists a constant $r > 0$ such that

$$r - HrL_p(Hr)|S(\lambda,\cdot)|_1 > |S(\lambda,\cdot)|_1 \sup_{k\in\mathbb{Z}} \|p(k,0)\|. \tag{6}$$

Let $\mathbb{B} = \{u \in PSAP_{-\omega}(\mathbb{Z}, X, \infty) : \|u\|_\infty \leq r\}$, then $\mathbb{B}$ is a closed subset of $PSAP_{-\omega}(\mathbb{Z}, X, \infty)$. We claim that $\mathcal{P}(\mathbb{B}) \subseteq \mathbb{B}$. In fact, for any $u \in \mathbb{B}$ and all $n \in \mathbb{Z}$, we have

$$\begin{aligned}\|(\mathcal{P}u)(n)\| &\leq \sum_{k=-\infty}^{n-1} |S(\lambda, n-1-k)|[\|p(k,0)\| + \|p(k,u_k) - p(k,0)\|] \\ &\leq \Big[\sup_{k\in\mathbb{Z}} \|p(k,0)\| + HrL_p(Hr)\Big]|S(\lambda,\cdot)|_1,\end{aligned}$$

which implies from (6) that $\|\mathcal{P}u\|_\infty \leq r$ and thus $\mathcal{P}(\mathbb{B}) \subseteq \mathbb{B}$.

It follows from (6) that $r - HrL_p(Hr)|S(\lambda,\cdot)|_1 > 0$, i.e. $HL_p(Hr)|S(\lambda,\cdot)|_1 < 1$. For each $u, v \in \mathbb{B}$ and all $n \in \mathbb{Z}$, we have

$$\begin{aligned}\|(\mathcal{P}u)(n) - (\mathcal{P}v)(n)\| &\leq \sum_{k=-\infty}^{n-1} |S(\lambda, n-1-k)|\|p(k,u_k) - p(k,v_k)\| \\ &\leq HL_p(Hr)|S(\lambda,\cdot)|_1\|u-v\|_\infty,\end{aligned}$$

which implies $\|\mathcal{P}u - \mathcal{P}v\|_\infty \leq HL_p(Hr)|S(\lambda,\cdot)|_1\|u-v\|_\infty$. Thus $\mathcal{P}$ is a contraction on $\mathbb{B}$ and has a unique fixed point $u \in \mathbb{B}$, which is also a solution $u \in PSAP_{-\omega}(\mathbb{Z}, X, \infty)$ to Eq. (1). □

Corollary 4.4. If conditions (Hp1)-(Hp2) hold with $L_p(\cdot) \equiv L > 0$ and $HL|S(\lambda,\cdot)|_1 < 1$, then Eq. (1) has a unique solution $u \in PSAP_{-\omega}(\mathbb{Z}, X, \infty)$.

Proof. Since $1 - HL|S(\lambda,\cdot)|_1 > 0$ and $|S(\lambda,\cdot)|_1 \sup_{k\in\mathbb{Z}} \|p(k,0)\| < \infty$, there exists a $r* > 0$ satisfying $r* > \frac{|S(\lambda,\cdot)|_1 \sup_{k\in\mathbb{Z}} \|p(k,0)\|}{1-HL|S(\lambda,\cdot)|_1}$. Thus for all $r > r*$, we have $r - HrL|S(\lambda,\cdot)|_1 > |S(\lambda,\cdot)|_1 \sup_{k\in\mathbb{Z}} \|p(k,0)\|$, i.e. (6) holds for all $r > r*$. We can infer from the proof of Theorem 4.3 that Eq. (1) admits a unique solution $u \in PSAP_{-\omega}(\mathbb{Z}, X, \infty)$. □

Theorem 4.5. Assume that conditions (Hp1) and (Hp3) hold. Then Eq. (1) admits a unique solution $u \in PSAP_{-\omega}(\mathbb{Z}, X, \infty)$.

Proof. Consider the operator $\mathcal{P}$ defined by (5). It is known from Lemma 4.2 that $(\mathcal{P}u)(n) \in PSAP_{-\omega}(\mathbb{Z}, X, \infty)$ for each $u \in PSAP_{-\omega}(\mathbb{Z}, X, \infty)$. Thus the operator $\mathcal{P}$ is well-defined. Since $\lambda \in \Omega_S$, we have $\sup_{n\in\mathbb{Z}} |S(\lambda, n)| \leq C^*$ for a suitable constant $C^* > 0$. For each $u, v \in PSAP_{-\omega}(\mathbb{Z}, X, \infty)$ and all $n \in \mathbb{Z}$, we have

$$\begin{aligned}\|(\mathcal{P}u)(n) - (\mathcal{P}v)(n)\| &\leq \sum_{k=-\infty}^{n-1} |S(\lambda, n-1-k)|\|p(k,u_k) - p(k,v_k)\| \\ &\leq HC^*\Big(\sum_{k=-\infty}^{n-1} \mathcal{L}_p(k)\Big)\|u-v\|_\infty.\end{aligned}$$

Generally, from [11, Lemma 3.2.] and technique of mathematical induction, we have

$$\begin{aligned}&\|(\mathcal{P}^m u)(n) - (\mathcal{P}^m v)(n)\| \\ \leq\ & \sum_{k=-\infty}^{n-1} |S(\lambda, n-1-k)|\|p(k,(\mathcal{P}^{m-1}u)_k) - p(k,(\mathcal{P}^{m-1}v)_k)\| \\ \leq\ & \frac{(HC^*)^m}{(m-1)!}\Big[\sum_{k=-\infty}^{n-1} \mathcal{L}_p(k)\Big(\sum_{j=-\infty}^{k-1} \mathcal{L}_p(j)\Big)^{m-1}\Big]\|u-v\|_\infty \\ \leq\ & \frac{(HC^*)^m}{m!}\Big(\sum_{k=-\infty}^{n-1} \mathcal{L}_p(k)\Big)^m \|u-v\|_\infty \leq \frac{(HC^*\|\mathcal{L}_p\|_1)^m}{m!}\|u-v\|_\infty,\end{aligned}$$

which implies that $\|\mathcal{P}^m u - \mathcal{P}^m v\|_\infty \leq \frac{(HC^*\|\mathcal{L}_p\|_1)^m}{m!}\|u-v\|_\infty$. Since $\frac{(HC^*\|\mathcal{L}_p\|_1)^m}{m!} < 1$ for sufficient large $m \in \mathbb{Z}^+$, we conclude from the Banach fixed point theorem that $\mathcal{P}$ admits a unique fixed point $u \in PSAP_{-\omega}(\mathbb{Z}, X, \infty)$ which is a solution to Eq. (1). □

Example 4.6. We give a simple example to illustrate our main results. Assume that $a(k) = a^k$ with $|a| < 1$ for $a \in \mathbb{C}$ and $\lambda \in \mathcal{D}(-a, 1) := \{z \in \mathbb{C} : |z+a| < 1\}$. It is known from [12, Example 3.3] that $S(\lambda, k) = \lambda(\lambda+a)^{k-1}, k \geq 1$ and $\mathcal{D}(-a, 1) \subseteq \Omega_S$. We consider the following difference equation:

$$u(n+1) = \lambda \sum_{k=-\infty}^{n} a^{n-k}u(k) + p(n, u_n), n \in \mathbb{Z}. \tag{7}$$

Let $p(k,u_k) := \eta\frac{\sin(u_k+1)}{e^{|k|}}(\eta > 0)$, we have for $m,\omega \in \mathbb{Z}^+$

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{2n+1}\sum_{k=-n}^{n}\max_{s\in[k-m,k]\cap\mathbb{Z}}\left\|\eta\frac{\sin(u_s+1)}{e^{|s+\omega|}}+\eta\frac{\sin(-u_s+1)}{e^{|s|}}\right\|\\ \le\ &\lim_{n\to\infty}\frac{\eta(e^{\omega}+1)}{2n+1}\sum_{k=-n}^{n}\max_{s\in[k-m,k]\cap\mathbb{Z}}\frac{1}{e^{|s|}}=0,\end{aligned}$$

which shows the condition (Hp1) holds. Moreover for each $u_k, v_k \in \mathscr{B}$ and $k \in \mathbb{Z}$,

$$\|p(k,u_k)-p(k,v_k)\| = \|\eta\frac{\sin(u_k+1)}{e^{|k|}}-\eta\frac{\sin(v_k+1)}{e^{|k|}}\| \le \frac{\eta}{e^{|k|}}\|u_k-v_k\|_{\mathscr{B}} \le \eta\|u_k-v_k\|_{\mathscr{B}},$$

which shows the condition (Hp2) holds. Hence for η small enough, we can get Eq. (7) has a unique solution $u \in PSAP_{-\omega}(\mathbb{Z}, X, \infty)$ via Corollary 4.4 .

Finally we give numerical simulations for solutions of Eq. (7). Let $f(n) := \frac{-n+4}{8}, n \in [-4, 12] \cap \mathbb{Z}$ be a function with 16 as the antiperiodic length. In Figure 1, we let $a = \frac{1}{2}, \lambda = \frac{1}{3}$ and $p_1(n, u_n) := f(n) + \frac{\sin(u(n-100)+1)}{1+n^2}, n \in \mathbb{Z}$, thus $u(n+1) = \sum_{k=-\infty}^{n} \frac{1}{3} \times \left(\frac{5}{6}\right)^{n-k-1} p_1(k, u_k)$. In Figure 2, we let $a = \frac{2}{5}, \lambda = \frac{2}{5}$ and $p_2(n, u_n) := f(n)\cos\frac{n\pi}{8} + \frac{\sin(u(n-100)+1)}{1+n^2}, n \in \mathbb{Z}$, thus $u(n+1) = \sum_{k=-\infty}^{n} \frac{2}{5} \times \left(\frac{4}{5}\right)^{n-k-1} p_2(k, u_k)$. The notation $\ast$ denotes values of u at n.

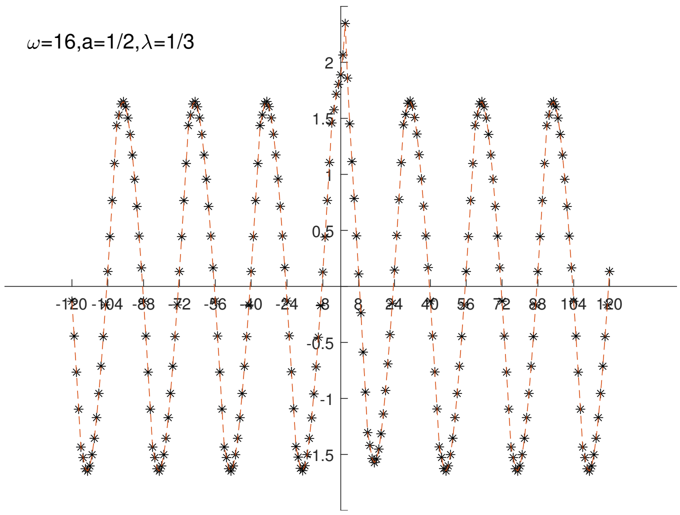

Figure 1: Solution $u(n)$ for p_1 on $[-120, 120]$.

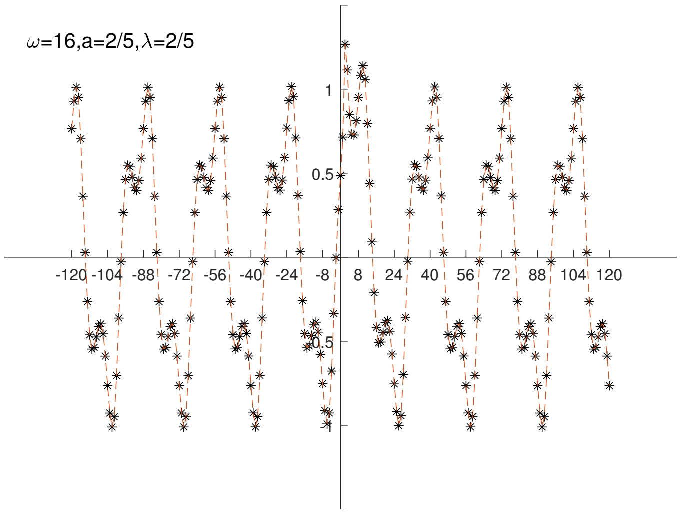

Figure 2: Solution $u(n)$ for p_2 on $[-120, 120]$.

References

[1] R. P. Agarwal, C. Cuevas, M. Frasson, *Semilinear functional difference equations with infinite delay*, Math. Comput. Model. **55** (2012), 1083-1105.

[2] N. S. Al-Islam, S. M. Alsulami, T. Diagana, *Existence of weighted pseudo anti-periodic solutions to some non-autonomous differential equations*, Appl. Math. Comput. **218** (2012), 6536-6648.

[3] E. Alvarez, C. Lizama, R. Ponce, *Weighted pseudo antiperiodic solutions for fractional integro-differential equations in Banach spaces*, Appl. Math. Comput. **259** (2015), 164-172.

[4] F. Bernardo, C. Cuevas, H. Soto, *Qualitative theory for Volterra difference equations*, Math. Meth. Appl. Sci. **41** (2018), 5423-5458.

[5] D. Brindle, G. M. N'Guérékata, *S-asymptotically ω-periodic sequential solutions to differene equations*, Nonlinear Stud. **26** (2019), 1-12.

[6] L. D. Campo, M. Pinto, C. Vidal, *Almost and asymptotically almost periodic solutions of abstract retarded functional difference equations in phase space*, J. Differ. Equ. Appl. **17** (2011), 915-934.

[7] J. Cao, Q. Yang, Z. Huang, *Existence of anti-periodic mild solutions for a class of semilinear fractional differential equations*, Commun. Nonlinear Sci. Numer. Simulat. **17** (2012), 277-283.

[8] A. Castro, C. Cuevas, F. Dantas, H. Soto, *About the behavior of solutions for Volterra difference equations with infinite delay*, J. Comput. Appl. Math. **255** (2014), 44-59.

[9] Y. K. Chang, P. Lü, *Weighted pseudo asymptotically antiperiodic sequential solutions to semilinear difference equations*, J. Differ. Equ. Appl. **27** (2021), 1482-1506.

[10] S. K. Choi, N. Koo, *Almost periodic solutions of nonlinear discrete Volterra equations with unbounded delay*, Adv. Differ. Equ. **2008** (2008), Art. ID 692713, 15pp.

[11] C. Cuevas, C. Lizama, *Semilinear evolution equation of second order via maximal regularity*, Adv. Differ. Equ. **2008** (2008), Art. ID 316207, 20pp.

[12] C. Cuevas, H. R. Henríquez, *C. Lizama, On the existence of almost automorphic solutions of Volterra difference equations*, J. Differ. Equ. Appl. **18** (2012), 1931-1946.

[13] S. Elaydi, *An Introduction to Difference Equations*, Springer, New York, 2005.

[14] Y. Hino, S. Murakami, T. Naito, *Functional Differential Equations with Infinite Delay*, Springer, 2006.

[15] M. Kostić, *Selected Topics in Almost Periodicity*, W. de Gruyter, Berlin, 2022.

[16] M. Kostić, *Almost Periodic Type Solutions to: Integro-Differential-Difference Equations*, W. de Gruyter, Berlin, 2025.

[17] Y. Li, J. Qin, B. Li, *Anti-periodic solutions for quaternion-valued high-order Hopfield neural networks with time-varying delays*, Neural Process Lett. **49** (2019), 1217-1237.

[18] P. Lü, Y. K. Chang, *Pseudo antiperiodic solutions to Volterra difference equations*, Mediterr. J. Math. **20** (2023), Art. 36, 23pp.

[19] S. Murakami, *Representation of solutions of linear functional difference equations in phase space*, Nonlinear Anal. TMA **30** (1997), 1153-1164.

[20] H. Matsunaga, S. Murakami, *Asymptotic behavior of solutions of functional difference equations*, J. Math. Anal. Appl. **305** (2005), 391-410.

[21] G. M. N'Guérékata, V. Valmorin, *Antiperiodic solutions of semilinear integrodifferential equations in Banach spaces*, Appl. Math. Comput. **218** (2012), 11118-11124.

[22] Y. Song, *Asymptotically almost periodic solutions of nonlinear Volterra difference equations with unbounded delay*, J. Differ. Equ. Appl. **14** (2008), 971-986.

[23] Z. Xia, *Discrete weighted pseudo asymptotic periodicity of second order difference equations*, Discrete Dyn. Nat. Soc. **2014** (2014), 1-8.