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On solutions of a linear set-valued differential equation with a conformable fractional derivative

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Abstract. The article introduces the definition of the conformable fractional derivative and the generalized conformable fractional derivative for set-valued mapping. A linear set-valued differential equation with a conformable fractional derivative and with a generalized conformable fractional derivative is considered. Some conditions for the existence of solutions are given, and the shape of the sections at each moment of time is obtained in an analytical form.

1. Introduction

Set-valued equations are studied within the framework of an independent theory. Furthermore, they have broad applications in the fields of ordinary differential inclusions, fuzzy differential equations and inclusions and interval equations [29, 34, 36, 39, 43, 51, 60].

In 1967, M. Hukuhara introduced the notions of integral and derivative for set-valued mappings and investigated their interrelations [21]. These concepts extend the classical derivative and Riemann integral for single-valued functions to the set-valued case. However, Hukuhara's derivative has a notable limitation: for a mapping differentiable in the sense of Hukuhara, the diameter of its sections is necessarily a non-decreasing function. To overcome this drawback, alternative derivative concepts have been proposed. T.F. Bridgland [15] introduced the Huygens derivative; Yu.N. Tyurin [61], as well as H.T. Banks and M.Q. Jacobs [10], developed the π-derivative using the Radstrom embedding theorem [62]; A.V. Plotnikov proposed the T-derivative [50, 54]; Ş.E. Amrahov, A. Khastan, N. Gasilov, A.G. Fatullayev and A.V. Plotnikov, N.V. Skrypnyk formulated generalized derivatives for set-valued mappings [4, 48, 49]. Each of these derivatives has its own advantages, but also has certain disadvantages [12, 13, 16, 17, 37, 38, 43, 47, 51, 54].

In 1969, F.S. de Blasi and F. Iervolino initiated the study of differential equations involving the Hukuhara derivative [13]. Since then, numerous researchers have explored the properties of solutions to such equations [29, 34, 36, 39, 43, 46, 51, 54], as well as integro-differential equations [45, 53], higher-order equations [8, 9, 44], and differential inclusions [14, 32, 54]. In addition, studies have been carried out on differential equations

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with the π-derivative [17, 43, 55], the T-derivative [50, 54] and generalized derivatives [47–49, 52]. While these equations bear resemblance to their single-valued counterparts, their set-valued nature requires specialized analytical techniques and solution methods. Conventional approaches designed for single-valued systems often prove inadequate, necessitating the development of new or alternative methodologies. Furthermore, the set-valued context introduces unique properties that warrant further investigation.

The growing interest in single-valued equations with fractional derivatives (see [11, 27, 35, 56, 59] and references therein) and fractional-like derivatives, such as the conformable fractional derivative [5, 6, 19, 23–26, 42, 58, 59], has inspired the introduction of analogous derivatives for set-valued mappings and subsequent studies of corresponding set-valued equations. In 2003, A.N. Vityuk proposed an analog of the Riemann-Liouville fractional derivative for set-valued mappings and established the existence of solutions to the associated nonlinear equations [63, 64]. In 2019, A.A. Martynyuk introduced an analog of the conformable fractional derivative [26] for set-valued mappings and derived existence conditions along with certain solution properties for the corresponding nonlinear equations [40, 41]. Further advancements in this area were made in the works of P. Wang and J. Bi [65], P. Wang, J. Bi, and J. Bao [66], as well as T.A. Komleva, A.V. Plotnikov, and N.V. Skripnik [31].

This paper introduces the concept of a conformable fractional derivative and a generalized conformable fractional derivative for set-valued mappings, extending the derivative concepts presented in [40, 41]. Furthermore, for linear set-valued differential equations involving these conformable fractional derivatives, conditions for the existence of solutions are established, and their analytical expressions are derived.

2. Main Definitions and Notation

Let $conv(\mathbb{R}^n)$ ($n \geq 2$) be the space of nonempty convex compact subsets of $\mathbb{R}^n$ equipped with the Hausdorff metric

$$h(X, Y) = \min\{r \geq 0 : X \subset Y + B_r(0),\ Y \subset X + B_r(0)\},$$

where $X, Y \in conv(\mathbb{R}^n)$, $B_r(c) = \{x \in \mathbb{R}^n : \|x - c\| \leq r\}$.

In addition to the standard set-theoretic operations, we also consider two operations in the space $conv(\mathbb{R}^n)$: the Minkowski sum of sets and scalar multiplication of a set:

$$X + Y = \{x + y : x \in X,\ y \in Y\} \quad \text{and} \quad \lambda X = \{\lambda x : x \in X,\ \lambda \in \mathbb{R}\}.$$

The following fundamental properties hold [51, 54, 57]:

1) $(conv(\mathbb{R}^n), h)$ is a complete metric space,

2) $h(X + Z, Y + Z) = h(X, Y)$,

3) $h(\lambda X, \lambda Y) = |\lambda| h(X, Y)$ for all $X, Y, Z \in conv(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$.

It is known that the space $conv(\mathbb{R}^n)$ is not a linear space with respect to the defined operations, since, in general, there does not exist an inverse element for $X \in conv(\mathbb{R}^n)$, i.e., a set $-X$ such that $X + (-X) = \{0\}$. The inverse element exists only in the case when $X \in \mathbb{R}^n$.

The absence of an inverse element in the space $conv(\mathbb{R}^n)$ leads to the ambiguous definition of the difference of sets and the conditions for its existence.

In this paper, we will use the Hukuhara difference [21].

Definition 2.1. *[21] Let $X, Y \in conv(\mathbb{R}^n)$. A set $Z \in conv(\mathbb{R}^n)$ such that $X = Y + Z$ is called the Hukuhara difference of the sets X and Y, and it is denoted by $X \overset{H}{-} Y$.*

The main properties of the Hukuhara difference [43, 51, 54, 57] are as follows:

1) If the Hukuhara difference of two sets $X \overset{H}{-} Y$ exists, it is unique;

2) $X \overset{H}{-} X = \{0\}$ for all $X \in conv(\mathbb{R}^n)$;

3) $(X+Y)\overset{H}{-}Y = X$ for all $X, Y \in conv(\mathbb{R}^n)$.

Lemma 2.2. *[30] If $X + Y = B_1(0)$, then $X = B_\mu(z_1)$ and $Y = B_\lambda(z_2)$, where $\mu + \lambda = 1$ and $z_1 + z_2 = 0$.*

Remark 2.3. *If a set X is subtracted from the ball $B_R(a)$ in the sense of the Hukuhara difference and the difference $B_R(a)\overset{H}{-}X$ exists, then the set X is also a ball $B_r(b)$, where the radius r does not exceed R.*

We also introduce another operation: the product of a matrix with a set

$$AX = \{Ax : x \in X\},$$

where $A \in \mathbb{R}^{n\times n}$ is a real $n \times n$ matrix, and $X \in conv(\mathbb{R}^n)$.

Theorem 2.4. *[18, 22] For any matrix $A \in \mathbb{R}^{n\times n}$, there exist two orthogonal $(n \times n)$ matrices U and V such that $U^TAV = \Sigma$, where Σ is a diagonal matrix. Additionally, the matrices U and V can be chosen such that the diagonal elements of Σ satisfy the condition $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > \sigma_{r+1} = \cdots = \sigma_n = 0$, where r is the rank of matrix A. That is, if A is a non-singular matrix, then $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$.*

Therefore, the matrix A can be written as $A = U\Sigma V^T$. This decomposition is called the singular value decomposition. The columns $u_1, \ldots, u_n$ of the matrix U are called the left singular vectors, the columns $v_1, \ldots, v_n$ of the matrix V are called the right singular vectors, and the numbers $\sigma_1, \ldots, \sigma_n$ are the singular values of the matrix A.

According to [18], the set $Y = \{Ax : x \in B_1(\mathbf{0})\}$ is an r-dimensional ellipsoid, and its semi-axes are equal to the corresponding singular values of the matrix $A \in \mathbb{R}^{n\times n}$, where $r = \text{rank}(A)$.

Remark 2.5. *Clearly, if the matrix $A \in \mathbb{R}^{n\times n}$ is an orthogonal matrix, then $AB_1(0) \equiv B_1(0)$.*

We introduce the conformable fractional derivative of order α for set-valued mappings.

Let $X : [0, T] \to conv(\mathbb{R}^n)$ be a set-valued mapping, and let $k : [0, T]\times(0, 1] \to \mathbb{R}_+$ be a positive continuous function such that $k(t, 1) = 1$ for all $t \in [0, T]$.

Definition 2.6. *Let $t \in (0, T)$ and $\alpha \in (0, 1]$. If for all sufficiently small $\varepsilon > 0$, the Hukuhara differences $X(t + \varepsilon k(t, \alpha)) \overset{H}{-} X(t)$ and $X(t)\overset{H}{-}X(t - \varepsilon k(t, \alpha))$ exist, and there exists $Z \in conv(\mathbb{R}^n)$ such that the following equality holds:*

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}\left(X(t + \varepsilon k(t, \alpha)) \overset{H}{-} X(t)\right) = \lim_{\varepsilon\to 0} \varepsilon^{-1}\left(X(t)\overset{H}{-}X(t - \varepsilon k(t, \alpha))\right) = Z, \tag{1}$$

then we say that the set-valued mapping $X(\cdot)$ has a **conformable fractional derivative of order** *α at the point $t \in (0, T)$, and $D^\alpha X(t) = Z$.*

If $D^\alpha X(t)$ exists for all $t \in (0, T)$ and the limits $\lim_{t\to 0} D^\alpha X(t)$ and $\lim_{t\to T} D^\alpha X(t)$ exist, then we define $D^\alpha X(0) = \lim_{t\to 0} D^\alpha X(t)$ and $D^\alpha X(T) = \lim_{t\to T} D^\alpha X(t)$.

Definition 2.7. *If the conformable fractional derivative of order α, $D^\alpha X(t)$, exists for all $t \geq 0$, then we say that the set-valued mapping $X(\cdot)$ is α-differentiable on $\mathbb{R}_+$.*

Remark 2.8. *If $k(t, \alpha) = t^{1-\alpha}$, then equality (1) in definition 2.6 has the form*

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}\left(X\left(t + \varepsilon t^{1-\alpha}\right) \overset{H}{-} X(t)\right) = \lim_{\varepsilon\to 0} \varepsilon^{-1}\left(X(t)\overset{H}{-}X\left(t - \varepsilon t^{1-\alpha}\right)\right) = Z,$$

and we obtain a generalization of the conformable fractional derivative of order α for a single-valued function [26] for set-valued mapping, which was investigated in [40].

Remark 2.9. *If $k(t,\alpha) = e^{(\alpha-1)t}$, then equality (1) in definition 2.6 has the form*

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X\left(t+\varepsilon e^{(\alpha-1)t}\right)\overset{H}{-}X(t)\right)=\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X(t)\overset{H}{-}X\left(t-\varepsilon e^{(\alpha-1)t}\right)\right)=Z,$$

and we obtain a generalization of the conformable fractional derivative of order α for a single-valued function [23] for set-valued mapping.

Remark 2.10. *If $k(t,\alpha) = \left(t+\frac{1}{\Gamma(\alpha)}\right)^{1-\alpha}$, then equality (1) in definition 2.6 has the form*

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X\left(t+\varepsilon\left(t+\frac{1}{\Gamma(\alpha)}\right)^{1-\alpha}\right)\overset{H}{-}X(t)\right)$$

$$=\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X(t)\overset{H}{-}X\left(t-\varepsilon\left(t+\frac{1}{\Gamma(\alpha)}\right)^{1-\alpha}\right)\right)=Z,$$

where $\Gamma(\alpha)$ is the Gamma function, i.e. we obtain a generalization of the conformable fractional derivative of order α for a single-valued function [5] for set-valued mapping.

Next, we will present some properties of this conformable fractional derivative of order α.

Lemma 2.11. *Let the set-valued mapping $X(\cdot)$ be α-differentiable at the point $t > 0$. Then*

$$D^\alpha X(t) = k(t,\alpha)D_H X(t), \tag{2}$$

where $D_H X(t)$ is the Hukuhara derivative [21].

Proof. Since the set-valued mapping $X(\cdot)$ is α-differentiable at the point $t > 0$, from (1) we have

$$D^\alpha X(t)=\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X\left(t+\varepsilon k(t,\alpha)\right)\overset{H}{-}X(t)\right).$$

Let $h = \varepsilon k(t,\alpha)$. It is clear that $\lim_{\varepsilon\to 0_+}\varepsilon k(t,\alpha) = 0_+$ for all $t > 0$. Then

$$\begin{aligned}\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X\left(t+\varepsilon k(t,\alpha)\right)\overset{H}{-}X(t)\right)&=\lim_{h\to 0}k(t,\alpha)h^{-1}\left(X\left(t+h\right)\overset{H}{-}X(t)\right)\\&=k(t,\alpha)D_H X(t),\end{aligned}$$

that is,

$$D^\alpha X(t) = k(t,\alpha)D_H X(t).$$

Similarly,

$$D^\alpha X(t)=\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X(t)\overset{H}{-}X\left(t-\varepsilon k(t,\alpha)\right)\right)=k(t,\alpha)D_H X(t).$$

Thus, the lemma is proven. □

Remark 2.12. *From Lemma 2.11, it follows that a necessary and sufficient condition for the existence of the conformable fractional derivative $D^\alpha X(t)$ for the set-valued mapping $X(\cdot)$ is the existence of the Hukuhara derivative $D_H X(t)$.*

Remark 2.13. *From Definition 2.6 and Lemma 2.11 it follows that for $\alpha = 1$ the conformable fractional derivative $D^1 X(t)$ for the set-valued mapping $X(\cdot)$ coincides with the Hukuhara derivative $D_H X(t)$.*

From Lemma 2.11 and Remark 2.12, the following observation follows.

Remark 2.14. *If the set-valued mapping $X(\cdot)$ is α-differentiable on $\mathbb{R}_+$, then the set-valued mapping $X(\cdot)$ is continuous on $\mathbb{R}_+$.*

Lemma 2.15. *Let the set-valued mapping $X(t) \equiv X$ for all $t \geq 0$. Then*

$$D^\alpha X(t) = \{0\}.$$

Proof. From Lemma 2.11, we have

$$D^\alpha X(t) = k(t,\alpha) D_H X(t).$$

Since $X(t) \equiv X$ for all $t \geq 0$ then $D_H X = \{0\}$, i.e. we get

$$D^\alpha X(t) = k(t,\alpha)\{0\} = \{0\}.$$

The lemma is proved. □

Lemma 2.16. *Let the set-valued mappings $X(\cdot)$ and $Y(\cdot)$ be α-differentiable at the point $t > 0$. Then*

$$D^\alpha \left(aX(t) + bY(t)\right) = aD^\alpha X(t) + bD^\alpha Y(t),$$

where $a, b \in \mathbb{R}_+$.

Proof. From Lemma 2.11, we have

$$D^\alpha X(t) = k(t,\alpha) D_H X(t) \quad \text{and} \quad D^\alpha Y(t) = k(t,\alpha) D_H Y(t).$$

Then

$$aD^\alpha X(t) + bD^\alpha Y(t) = ak(t,\alpha) D_H X(t) + bk(t,\alpha) D_H Y(t).$$

Similarly,

$$D^\alpha (aX(t) + bY(t)) = k(t,\alpha) D_H (aX(t) + bY(t)) = k(t,\alpha)(aD_H X(t) + bD_H Y(t)) =$$

$$= ak(t,\alpha) D_H X(t) + bk(t,\alpha) D_H Y(t).$$

Thus, the lemma is proved. □

Definition 2.17. *The fractional integral of a set-valued function $X(\cdot)$ is defined by the formula*

$$I^\alpha X(t) = \int_0^t k^{-1}(s,\alpha) X(s)\, ds, \quad t \geq 0,$$

where the integral on the right-hand side is understood in the sense of the Hukuhara integral [21].

Remark 2.18. *Clearly, if $\alpha = 1$, then $k(t,1) \equiv 1$, and $I^1 X(t)$ coincides with the Hukuhara integral, i.e.,*

$$I^1 X(t) = \int_0^t X(s)\, ds.$$

Lemma 2.19. *Let the set-valued mapping $X(\cdot)$ be continuous on $\mathbb{R}_+$, then*

$$D^\alpha I^\alpha X(t) = X(t), \quad t > 0.$$

Proof. Since the set-valued mapping $X(\cdot)$ is continuous, the fractional integral $I^\alpha X(t) = \int\limits_0^t k^{-1}(s,\alpha)X(s)\,ds$ exists for all $t \geq 0$. According to [21], the set-valued mapping $I^\alpha X(t)$ is differentiable in the sense of Hukuhara, and we have

$$D^\alpha I^\alpha X(t) = k(t,\alpha) D_H I^\alpha X(t).$$

Thus,

$$D^\alpha I^\alpha X(t) = k(t,\alpha)k^{-1}(t,\alpha)X(t) = X(t).$$

The lemma is proved. □

Lemma 2.20. *Let the set-valued mapping $X(\cdot)$ be α-differentiable on $\mathbb{R}_+$, then*

$$I^\alpha D^\alpha X(t) = X(t) \overset{H}{-} X(0), \quad t > 0. \tag{3}$$

Proof. Since the set-valued mapping $X(\cdot)$ is α-differentiable on $\mathbb{R}_+$, we have

$$D^\alpha X(t) = k(t,\alpha) D_H X(t).$$

Thus,

$$I^\alpha D^\alpha X(t) = \int\limits_0^t k^{-1}(s,\alpha)k(s,\alpha)D_H X(s)\,ds = \int\limits_0^t D_H X(s)\,ds = X(t) \overset{H}{-} X(0).$$

The lemma is proven. □

3. Linear set-valued differential equation with conformable fractional derivative

Consider the following Cauchy problem for linear set-valued differential equation with a conformable fractional derivative of order α:

$$D^\alpha X(t) = AX(t), \quad X(0) = B_1(0), \tag{4}$$

where $\alpha \in (0,1]$, $X : \mathbb{R}_+ \to conv(\mathbb{R}^2)$ is the unknown set-valued mapping, and $A \in \mathbb{R}^{2\times 2}$ is a non-degenerate matrix.

Definition 3.1. *A set-valued mapping $X : \mathbb{R}_+ \to conv(\mathbb{R}^2)$ is called a solution to the Cauchy problem (4) if it is continuous and satisfies the differential equation (4) for all $t \geq 0$ and $X(0) = B_1(0)$.*

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a, b, c, d \in \mathbb{R}$ such that $ad - bc \neq 0$.

It is easy to obtain that the singular values of the matrix A are given by

$$\sigma_1 = \sqrt{\frac{a^2+b^2+c^2+d^2+\sqrt{\delta}}{2}}, \quad \sigma_2 = \sqrt{\frac{a^2+b^2+c^2+d^2-\sqrt{\delta}}{2}},$$

where $\delta = \left(a^2+b^2+c^2+d^2\right)^2 - 4\,(ad-bc)^2$.

Obviously,

$$\delta = \left(a^2+b^2+c^2+d^2\right)^2 - 4\,(ad-bc)^2 = \left(a^2-d^2\right)^2 + \left(b^2-c^2\right)^2 + 2(ab+cd)^2 + 2(ac+bd)^2,$$

so $\delta \geq 0$.

Thus, if $d = a$ and $c = -b$, or $d = -a$ and $b = c$, that is, if

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} a & b \\ b & -a \end{pmatrix},$$

then $\delta = 0$ and $\sigma_1 = \sigma_2 = \sigma = \sqrt{a^2 + b^2}$. In other cases, $\delta \neq 0$.

Theorem 3.2. *If the matrix A is such that $\delta = 0$, then the Cauchy problem (4) has the following solution:*

$$X(t) = e^{\sigma \int_0^t k^{-1}(s,\alpha)\,ds} B_1(\mathbf{0}),$$

where $t \geq 0$.

Proof. The proof that $X(\cdot)$ is a solution of the Cauchy problem (4) will be done by directly substituting the set-valued mapping

$$X(t) = e^{\sigma \int_0^t k^{-1}(s,\alpha)\,ds} B_1(\mathbf{0})$$

into the differential equation (4) and verifying the identity:

$$D^\alpha \left(e^{\sigma \int_0^t k^{-1}(s,\alpha)\,ds} B_1(\mathbf{0}) \right) \equiv A e^{\sigma \int_0^t k^{-1}(s,\alpha)\,ds} B_1(\mathbf{0}).$$

Since $\sigma > 0$ and $k(t, \alpha) > 0$ for all $t \geq 0$ and $\alpha \in (0, 1]$, then the function $f(t, \alpha) = e^{\sigma \int_0^t k^{-1}(s,\alpha)\,ds}$ is an increasing function of t for any fixed α. Clearly, the function $f(t, \alpha)$ is differentiable with respect to t on $\mathbb{R}_+$. From [43, 51], since $X(t) = f(t, \alpha)B_1(\mathbf{0})$, the Hukuhara derivative $D_H X(t)$ exists for all $t \geq 0$ and is given by

$$D_H X(t) = \frac{\partial f(t, \alpha)}{\partial t} B_1(\mathbf{0}) = \sigma k^{-1}(t, \alpha) e^{\sigma \int_0^t k^{-1}(s,\alpha)\,ds} B_1(\mathbf{0}).$$

Since $D^\alpha X(t) = k(t, \alpha) D_H X(t)$, we obtain

$$D^\alpha X(t) = k(t, \alpha) \sigma k^{-1}(t, \alpha) e^{\sigma \int_0^t k^{-1}(s,\alpha)\,ds} B_1(\mathbf{0}) = \sigma e^{\sigma \int_0^t k^{-1}(s,\alpha)\,ds} B_1(\mathbf{0}).$$

Since the matrix A is such that the singular values $\sigma_1 = \sigma_2 = \sigma$, then the singular decomposition of matrix A is $A = U\Sigma V^T$, where U, V are orthogonal matrices and $\Sigma = \sigma I$, with I being the identity matrix. It is also known that $V^T B_r(\mathbf{0}) = B_r(\mathbf{0})$ and $U B_r(\mathbf{0}) = B_r(\mathbf{0})$ for all $r > 0$.

Thus,

$$A e^{\sigma \int_0^t k^{-1}(s,\alpha)\,ds} B_1(\mathbf{0}) = U\Sigma V^T e^{\sigma \int_0^t k^{-1}(s,\alpha)\,ds} B_1(\mathbf{0}) = U\sigma I V^T e^{\sigma \int_0^t k^{-1}(s,\alpha)\,ds} B_1(\mathbf{0})$$

$$= \sigma U I V^T e^{\sigma \int_0^t k^{-1}(s,\alpha)\,ds} B_1(\mathbf{0}) = \sigma e^{\sigma \int_0^t k^{-1}(s,\alpha)\,ds} U I V^T B_1(\mathbf{0}) = \sigma e^{\sigma \int_0^t k^{-1}(s,\alpha)\,ds} B_1(\mathbf{0}).$$

Therefore,

$$D^\alpha \left(e^{\sigma \int_0^t k^{-1}(s,\alpha)\,ds} B_1(\mathbf{0}) \right) \equiv \sigma e^{\sigma \int_0^t k^{-1}(s,\alpha)\,ds} B_1(\mathbf{0}) \equiv A e^{\sigma \int_0^t k^{-1}(s,\alpha)\,ds} B_1(\mathbf{0}),$$

and thus $X(\cdot)$ is a solution of the differential equation (4).

The theorem is proven. □

Further, we consider the case when the matrix A is such that $\delta \neq 0$.

Theorem 3.3. *If the matrix A is symmetric and $d \neq -a$, then the differential equation (4) has the following solution*

$$X(t) = Ue^{\int_0^t k^{-1}(s,\alpha)ds\,\Sigma} B_1(\mathbf{0}), \quad t \geq 0,$$

where $\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$, *and* $\sigma_{1,2} = \left| \frac{a+d\pm\sqrt{(a-d)^2+4b^2}}{2} \right|$.

Proof. Since the matrix A is symmetric and $d \neq -a$, it has the following form:

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}.$$

It is known that the eigenvalues of a symmetric matrix A are real and, in our case ($\delta \neq 0$), distinct and non-zero. Let's consider all possible cases for the eigenvalues of matrix A: 1) the eigenvalues of matrix A are positive, 2) the eigenvalues of matrix A have different signs, and 3) the eigenvalues of matrix A are negative.

1) The eigenvalues $\lambda_{1,2} = \frac{a+d\pm\sqrt{D}}{2}$ of matrix A are positive (i.e., A is positive definite), where $D = (a-d)^2 + 4b^2$. In this case, the singular decomposition coincides with the spectral decomposition, i.e., $\sigma_1 = \lambda_1$, $\sigma_2 = \lambda_2$, and $U\Lambda U^T = U\Sigma U^T$, where

$$\Lambda = \Sigma = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad U = \begin{pmatrix} \frac{b}{\sqrt{(\lambda_1-a)^2+b^2}} & \frac{\lambda_2-d}{\sqrt{(\lambda_2-d)^2+b^2}} \\ \frac{\lambda_1-a}{\sqrt{(\lambda_1-a)^2+b^2}} & \frac{b}{\sqrt{(\lambda_2-d)^2+b^2}} \end{pmatrix}.$$

2) The eigenvalues $\lambda_{1,2}$ of matrix A have different signs (i.e., A is indefinite) and $|\lambda_1| > |\lambda_2|$. In this case, the singular decomposition is such that $\sigma_1 = |\lambda_1|$, $\sigma_2 = |\lambda_2|$, and

$$U\Sigma W^T = U|\Lambda|DU^T,$$

where $W^T = DU^T$ and $D = \begin{pmatrix} \frac{\lambda_1}{|\lambda_1|} & 0 \\ 0 & \frac{\lambda_2}{|\lambda_2|} \end{pmatrix}$.

3) The eigenvalues $\lambda_{1,2}$ of matrix A are negative (i.e., A is negative definite) and $|\lambda_1| > |\lambda_2|$. In this case, the singular decomposition is such that $\sigma_1 = |\lambda_1|$, $\sigma_2 = |\lambda_2|$ and $U\Sigma W^T = U|\Lambda|DU^T$, where $W^T = DU^T$ and $D = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

Thus, in the general case, the singular decomposition of matrix A has the form $A = U\Sigma DU^T$, where $\Sigma = |\Lambda|$.

To prove that $X(\cdot)$ is a solution to the differential equation (4), we proceed with direct substitution of the set-valued mapping

$$X(t) = Ue^{\int_0^t k^{-1}(s,\alpha)ds\,\Sigma} B_1(\mathbf{0})$$

into the differential equation (4) and verify the identity:

$$D^\alpha \left(Ue^{\int_0^t k^{-1}(s,\alpha)ds\,\Sigma} B_1(\mathbf{0}) \right) \equiv AUe^{\int_0^t k^{-1}(s,\alpha)ds\,\Sigma} B_1(\mathbf{0}). \tag{5}$$

By Definition 2.6 and because $B_1(\mathbf{0})$ is a centrally symmetric body and $(-1)B_1(\mathbf{0}) = B_1(\mathbf{0})$, we have

$$\lim_{\varepsilon\to 0_+} \varepsilon^{-1} \left(X(t+\varepsilon k(t,\alpha)) \overset{H}{-} X(t) \right) = \lim_{\varepsilon\to 0_+} \varepsilon^{-1} \left(Ue^{\int_0^{t+\varepsilon k(t,\alpha)} k^{-1}(s,\alpha)ds\,\Sigma} B_1(\mathbf{0}) \overset{H}{-} Ue^{\int_0^t k^{-1}(s,\alpha)ds\,\Sigma} B_1(\mathbf{0}) \right) =$$

$$= U \lim_{\varepsilon \to 0_+} \varepsilon^{-1} \left(e^{\int_0^{t+\varepsilon k(t,\alpha)} k^{-1}(s,\alpha) ds\, \Sigma} - e^{\int_0^t k^{-1}(s,\alpha) ds\, \Sigma} \right) B_1(\mathbf{0}) = U k(t,\alpha) \Sigma k^{-1}(t,\alpha) e^{\int_0^t k^{-1}(s,\alpha) ds\, \Sigma} B_1(\mathbf{0}) = U \Sigma e^{\int_0^t k^{-1}(s,\alpha) ds\, \Sigma} B_1(\mathbf{0}).$$

Similarly,

$$\lim_{\varepsilon \to 0_+} \varepsilon^{-1} \left(X(t) \overset{H}{-} X(t - \varepsilon k(t,\alpha)) \right) = U \lim_{\varepsilon \to 0_+} \varepsilon^{-1} \left(e^{\int_0^t k^{-1}(s,\alpha) ds\, \Sigma} - e^{\int_0^{t-\varepsilon k(t,\alpha)} k^{-1}(s,\alpha) ds\, \Sigma} \right) B_1(\mathbf{0}) =$$

$$= U \Sigma e^{\int_0^t k^{-1}(s,\alpha) ds\, \Sigma} B_1(\mathbf{0}).$$

Thus,

$$D^\alpha X(t) = D^\alpha \left(e^{\int_0^t k^{-1}(s,\alpha) ds\, \Sigma} B_1(\mathbf{0}) \right) = U \Sigma e^{\int_0^t k^{-1}(s,\alpha) ds\, \Sigma} B_1(\mathbf{0}).$$

Since the singular decomposition of the symmetric matrix A has the form $A = U\Sigma D U^T$, we get

$$A U e^{\int_0^t k^{-1}(s,\alpha) ds\, \Sigma} B_1(\mathbf{0}) = U \Sigma D U^T U e^{\int_0^t k^{-1}(s,\alpha) ds\, \Sigma} B_1(\mathbf{0}) = U \Sigma e^{\int_0^t k^{-1}(s,\alpha) ds\, \Sigma} B_1(\mathbf{0}).$$

It is evident that the identity (5) holds, and therefore, $X(\cdot)$ is a solution to the differential equation (4). The theorem is proved. □

4. Linear set-valued differential equation with generalized conformable fractional derivative

Definition 4.1. *We will say that a set-valued mapping $X(\cdot)$ has a* **generalized conformable fractional derivative of order** *α $D_g^\alpha X(t) \in conv(\mathbb{R}^n)$ at the point $t \in (0,T)$ if for all sufficiently small $\varepsilon > 0$ the corresponding Hukuhara differences exist and at least one of the following equalities holds:*

i) $\lim_{\varepsilon \to 0} \varepsilon^{-1} \left(X(t + \varepsilon k(t,\alpha)) \overset{H}{-} X(t) \right) = \lim_{\varepsilon \to 0} \varepsilon^{-1} \left(X(t) \overset{H}{-} X(t - \varepsilon k(t,\alpha)) \right) = D_g^\alpha X(t)$,

ii) $\lim_{\varepsilon \to 0} \varepsilon^{-1} \left(X(t) \overset{H}{-} X(t + \varepsilon k(t,\alpha)) \right) = \lim_{\varepsilon \to 0} \varepsilon^{-1} \left(X(t - \varepsilon k(t,\alpha)) \overset{H}{-} X(t) \right) = D_g^\alpha X(t)$,

iii) $\lim_{\varepsilon \to 0} \varepsilon^{-1} \left(X(t + \varepsilon k(t,\alpha)) \overset{H}{-} X(t) \right) = \lim_{\varepsilon \to 0} \varepsilon^{-1} \left(X(t - \varepsilon k(t,\alpha)) \overset{H}{-} X(t) \right) = D_g^\alpha X(t)$,

iv) $\lim_{\varepsilon \to 0} \varepsilon^{-1} \left(X(t) \overset{H}{-} X(t + \varepsilon k(t,\alpha)) \right) = \lim_{\varepsilon \to 0} \varepsilon^{-1} \left(X(t) \overset{H}{-} X(t - \varepsilon k(t,\alpha)) \right) = D_g^\alpha X(t)$.

Definition 4.2. *If the generalized conformable fractional derivative of order α, $D_g^\alpha X(t)$, exists for all $t \geq 0$, then we say that the set-valued mapping $X(\cdot)$ is generalized–differentiable of order α on $\mathbb{R}_+$.*

Remark 4.3. *Consequently, if the set-valued function $X(\cdot)$ is α-differentiable on the interval $t > 0$, then the set-valued function $X(\cdot)$ is generalized-differentiable of order α on the interval $t > 0$.*

Lemma 4.4. *If the set-valued mapping $X(\cdot)$ is generalized-differentiable of order α on the interval $t > 0$, then*

$$D_g^\alpha X(t) = k(t,\alpha) D_g X(t),$$

where $D_g X(t)$ is the generalized derivative [48].

Proof. If the set-valued mapping $X(\cdot)$ is generalized-differentiable of order α on the interval $t > 0$, then the following can be obtained using the definition from Lemma 4.1. It is evident that it follows directly from the definition that

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} \left(X(t + \varepsilon k(t,\alpha)) \overset{H}{-} X(t) \right) = \lim_{\varepsilon \to 0} \varepsilon^{-1} \left(X(t) \overset{H}{-} X(t - \varepsilon k(t,\alpha)) \right) = D_g^\alpha X(t).$$

Let $\theta = \varepsilon k(t,\alpha)$, then we have

$$D_g^{\alpha}X(t) = \lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X\left(t+\varepsilon k(t,\alpha)\right)\overset{H}{-}X(t)\right) = \lim_{\theta\to 0}k(t,\alpha)\theta^{-1}\left(X\left(t+\theta\right)\overset{H}{-}X(t)\right) =$$

$$= k(t,\alpha)\lim_{\theta\to 0}\theta^{-1}\left(X\left(t+\theta\right)\overset{H}{-}X(t)\right) = k(t,\alpha)D_gX(t).$$

Similarly,

$$D_g^{\alpha}X(t) = \lim_{\varepsilon\to 0}\varepsilon^{-1}\left(X(t)\overset{H}{-}X\left(t-\varepsilon k(t,\alpha)\right)\right) = \lim_{\theta\to 0}k(t,\alpha)\theta^{-1}\left(X(t)\overset{H}{-}X\left(t-\theta\right)\right) =$$

$$= k(t,\alpha)\lim_{\theta\to 0}\theta^{-1}\left(X(t)\overset{H}{-}X\left(t-\theta\right)\right) = k(t,\alpha)D_gX(t).$$

Thus, we have proven the lemma. □

Remark 4.5. *It follows from Lemma 4.4 that a generalized conformable fractional derivative $D_g^{\alpha}X(t)$ for a set-valued mapping $X(\cdot)$ exists if and only if a generalized derivative $D_gX(t)$ exists.*
Moreover, if $\alpha = 1$, then $D_g^1X(t) = D_gX(t)$.

Consider the following Cauchy problem for linear set-valued differential equation with a generalized conformable fractional derivative of order α:

$$D_g^{\alpha}X(t) = AX(t), \quad X(0) = B_1(\mathbf{0}), \tag{6}$$

where $X : \mathbb{R}_+ \to conv(\mathbb{R}^2)$ is a set-valued mapping and $A \in \mathbb{R}^{2\times 2}$ is a constant matrix.

Definition 4.6. *A set-valued mapping $X : \mathbb{R}_+ \to conv(\mathbb{R}^2)$ is called a solution to the Cauchy problem (6) if it is continuous and satisfies the differential equation (6) for all $t \geq 0$ and $X(0) = B_1(0)$.*

Remark 4.7. *According to Remark 4.3, it follows that if the set-valued mapping $X(t)$ is a solution of the Cauchy problem (6), then it is also a solution of the Cauchy problem (4).*

Remark 4.8. *In the works [30, 33, 48] it is described the set-valued differential equation with generalized derivative*

$$D_gX(t) = AX(t), \quad X(0) = B_1(\mathbf{0}) \tag{7}$$

and there are noted the following observations:

1) *Cauchy problem (7) admits multiple solutions, among which one or two are referred to as *basic solutions* (their diameters are monotone functions), while the remaining are classified as *mixed solutions* (their diameters are not monotone functions). Notably, the first basic solution, $X_1(t)$, satisfies the condition that $diam(X_1(t))$ is a non-decreasing function and also solves the corresponding differential equation involving the Hukuhara derivative. The second basic solution, $X_2(t)$, is characterized by the condition that $diam(X_2(t))$ is a decreasing function.*

2) *If the singular values of the matrix A satisfy $\sigma_1 = \sigma_2 = \sigma$, then Cauchy problem (7) admits two basic solutions, $X_1(t)$ and $X_2(t)$, whose cross-sections at each moment in time t are the circles $B_{e^{\sigma t}}(0)$ and $B_{e^{-\sigma t}}(0)$, respectively. Conversely, if the singular values of the matrix A satisfy $\sigma_1 \neq \sigma_2$, then differential equation (7) has only one basic solution, $X_1(t)$, whose cross-section at each moment in time t is an ellipse with semi-axes $e^{\sigma_1 t}$ and $e^{\sigma_2 t}$.*

Further we will obtain similar results.

Theorem 4.9. *If the matrix A is such that $\delta = 0$, then the Cauchy problem (6) has two basic solutions, $X_1(\cdot)$ and $X_2(\cdot)$, which are given by:*

$$X_1(t) = e^{\sigma \int_0^t k^{-1}(s,\alpha)\,ds} B_1(\mathbf{0}) \quad and \quad X_2(t) = e^{-\sigma \int_0^t k^{-1}(s,\alpha)\,ds} B_1(\mathbf{0}),$$

for $t \geq 0$, where $\sigma = \sqrt{a^2 + b^2}$.

Proof. By Theorem 3.2 it follows that $X_1(t)$ is a solution to the Cauchy problem (4) and has non-decreasing function $diam(X(t))$. Thus, by Remark 4.3, $X_1(t)$ is the first basic solution to the Cauchy problem (6).

We will prove that $X_2(\cdot)$ is the second basic solution of the Cauchy problem (6) by substituting the set-valued mapping

$$X_2(t) = e^{-\sigma \int_0^t k^{-1}(s,\alpha)\,ds} B_1(\mathbf{0})$$

into differential equation (6) and checking that the identity holds:

$$D_g^{\alpha}\left(e^{-\sigma \int_0^t k^{-1}(s,\alpha)\,ds} B_1(\mathbf{0})\right) \equiv A e^{-\sigma \int_0^t k^{-1}(s,\alpha)\,ds} B_1(\mathbf{0}).$$

Since $\sigma > 0$ and $k(t,\alpha) > 0$ for all $t \geq 0$ and $\alpha \in (0,1]$, then the function $f(t,\alpha) = e^{-\sigma \int_0^t k^{-1}(s,\alpha)\,ds}$ is a decreasing function in t for any fixed α. Moreover, $f(t,\alpha)$ is a differentiable function with respect to t on $\mathbb{R}_+$.

From [48], we have that $X_2(t) = f(t,\alpha)B_1(\mathbf{0})$, and

$$D_g X_2(t) = \frac{\partial f(t,\alpha)}{\partial t} B_1(\mathbf{0}) = \sigma k^{-1}(t,\alpha) e^{-\sigma \int_0^t k^{-1}(s,\alpha)\,ds} B_1(\mathbf{0}).$$

Thus, we can deduce that $D^{\alpha}X_2(t) = k(t,\alpha)D_g X_2(t)$, which gives:

$$D^{\alpha}X_2(t) = k(t,\alpha)\sigma k^{-1}(t,\alpha) e^{-\sigma \int_0^t k^{-1}(s,\alpha)\,ds} B_1(\mathbf{0}) = \sigma e^{-\sigma \int_0^t k^{-1}(s,\alpha)\,ds} B_1(\mathbf{0}).$$

Since, $A = U\Sigma V^T$, where U, V are orthogonal matrices, $\Sigma = \sigma I$ and $\sigma = \sqrt{a^2 + b^2}$, we have the following:

$$A e^{-\sigma \int_0^t k^{-1}(s,\alpha)\,ds} B_1(\mathbf{0}) = U\Sigma V^T e^{-\sigma \int_0^t k^{-1}(s,\alpha)\,ds} B_1(\mathbf{0})$$

$$= U\sigma I V^T e^{-\sigma \int_0^t k^{-1}(s,\alpha)\,ds} B_1(\mathbf{0}) = \sigma e^{-\sigma \int_0^t k^{-1}(s,\alpha)\,ds} U I V^T B_1(\mathbf{0}).$$

Thus, we obtain:

$$A e^{-\sigma \int_0^t k^{-1}(s,\alpha)\,ds} B_1(\mathbf{0}) = \sigma e^{-\sigma \int_0^t k^{-1}(s,\alpha)\,ds} B_1(\mathbf{0}).$$

Finally, we conclude that:

$$D_g^{\alpha}X_2(t) \equiv \sigma e^{-\sigma \int_0^t k^{-1}(s,\alpha)\,ds} B_1(\mathbf{0}) \equiv AX_2(t),$$

which means that $X_2(\cdot)$ is a solution of the generalized differential equation (6). □

Theorem 4.10. *If the matrix A be symmetric matrix and $a \neq -d$, then the Cauchy problem (6) has only the first basic solution*

$$X_1(t) = U e^{\int_0^t k^{-1}(s,\alpha)\,ds \Sigma} B_1(\mathbf{0}), \quad t \geq 0,$$

where $\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$ and $\sigma_{1,2} = \left|\frac{a+d \pm \sqrt{(a-d)^2 + 4b^2}}{2}\right|$.

Proof. From theorem 4.7 the Cauchy problem (4) has a solution

$$X_1(t) = U e^{\int_0^t k^{-1}(s,\alpha)\,ds \Sigma} B_1(\mathbf{0}),$$

which is the first basic solution for the Cauchy problem (6).

Now suppose that a second basic solution $X_2(\cdot)$ for the Cauchy problem (6) exists. Then $X_2(t)$ must satisfy the following equation:

$$X_2(t) + A\int_0^t k^{-1}(s,\alpha)X_2(s)\,ds = B_1(\mathbf{0}).$$

Let us fix some arbitrary $T > 0$. Then

$$X_2(T) + A\int_0^T k^{-1}(s,\alpha)X_2(s)\,ds = B_1(\mathbf{0}).$$

Hence,

$$B_1(\mathbf{0})\overset{H}{-}X_2(T) = A\int_0^T k^{-1}(s,\alpha)X_2(s)\,ds.$$

Since $B_1(\mathbf{0})$ is the ball and the difference of Hukuhara $B_1(\mathbf{0})\overset{H}{-}X_2(T)$ exists, then $X_2(T)$ is the ball, i.e. $X_2(T) \equiv B_{r(T)}(\mathbf{0})$, where $0 \le r(T) \le 1$. Therefore, for all $t \ge 0$, we have:

$$X_2(t) = B_{r(t)}(\mathbf{0}).$$

Further,

$$\int_0^T k^{-1}(s,\alpha)X_2(s)\,ds = \int_0^T k^{-1}(s,\alpha)B_{r(s)}(\mathbf{0})\,ds = \int_0^T k^{-1}(s,\alpha)r(s)B_1(\mathbf{0})\,ds$$
$$= R(T)B_1(\mathbf{0}) = B_{R(T)}(\mathbf{0}),$$

where $R(T) = \int_0^T k^{-1}(s,\alpha)r(s)\,ds$.

Thus, we obtain the equation:

$$B_{r(T)}(\mathbf{0}) + AB_{R(T)}(\mathbf{0}) = B_1(\mathbf{0}). \tag{8}$$

Since matrix A has different singular values, then $AB_{R(T)}(\mathbf{0})$ is an ellipse. Therefore, $B_{r(T)}(\mathbf{0}) + AB_{R(T)}(\mathbf{0})$ cannot be a ball. Therefore, equation (8) does not hold, and we complete the proof. □

Remark 4.11. *If $k(t,\alpha) = t^{1-\alpha}$, then in this case $D^\alpha X(t) = t^{1-\alpha}D_H X(t)$ and $D_g^\alpha X(t) = t^{1-\alpha}D_g X(t)$ and the analytical formulas for the solutions will look as follows:*

Theorem 3.2 - $X(t) = e^{\alpha^{-1}\sigma t^\alpha}B_1(\mathbf{0})$;
Theorem 3.3 - $X(t) = Ue^{\alpha^{-1}t^\alpha\Sigma}B_1(\mathbf{0})$;
Theorem 4.9 - $X_1(t) = e^{\alpha^{-1}\sigma t^\alpha}B_1(\mathbf{0})$, $\quad X_2(t) = e^{-\alpha^{-1}\sigma t^\alpha}B_1(\mathbf{0})$;
Theorem 4.10 - $X_1(t) = Ue^{\alpha^{-1}t^\alpha\Sigma}B_1(\mathbf{0})$.

Remark 4.12. *If $k(t,\alpha) = e^{(\alpha-1)t}$, then in this case $D^\alpha X(t) = e^{\alpha-1t}D_H X(t)$ and $D_g^\alpha X(t) = e^{\alpha-1t}D_g X(t)$ and the analytical formulas for the solutions will look as follows:*

Theorem 3.2 - $X(t) = e^{\frac{\sigma}{1-\alpha}e^{(\alpha-1)t}}B_1(\mathbf{0})$;
Theorem 3.3 - $X(t) = Ue^{\frac{1}{1-\alpha}e^{(\alpha-1)t}\Sigma}B_1(\mathbf{0})$;
Theorem 4.9 - $X_1(t) = e^{\frac{\sigma}{1-\alpha}e^{(\alpha-1)t}}B_1(\mathbf{0})$, $\quad X_2(t) = e^{\frac{\sigma}{\alpha-1}e^{(\alpha-1)t}}B_1(\mathbf{0})$;
Theorem 4.10 - $X_1(t) = Ue^{\frac{1}{1-\alpha}e^{(\alpha-1)t}\Sigma}B_1(\mathbf{0})$.

Remark 4.13. *If $k(t,\alpha) = \left(t + \frac{1}{\Gamma(\alpha)}\right)^{1-\alpha}$, then in this case $D^\alpha X(t) = \left(1 + \frac{1}{\Gamma(\alpha)}\right)^{1-\alpha} D_H X(t)$ and $D_g^\alpha X(t) = \left(1 + \frac{1}{\Gamma(\alpha)}\right)^{1-\alpha} D_g X(t)$ and the analytical formulas for the solutions will look as follows:*

Theorem 3.2 - $X(t) = e^{\frac{\sigma}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^\alpha}B_1(\mathbf{0})$;
Theorem 3.3 - $X(t) = Ue^{\frac{1}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^\alpha\Sigma}B_1(\mathbf{0})$;
Theorem 4.9 - $X_1(t) = e^{\frac{\sigma}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^\alpha}B_1(\mathbf{0})$, $\quad X_2(t) = e^{-\frac{\sigma}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^\alpha}B_1(\mathbf{0})$;
Theorem 4.10 - $X_1(t) = Ue^{\frac{1}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^\alpha\Sigma}B_1(\mathbf{0})$.

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