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Solutions of the Yang-Baxter-like equation for a 3×3 matrix of two Jordan blocks

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Abstract. Let the Jordan canonical form of a 3×3 matrix A be $J = \text{diag}(J_2[\lambda], J_1[\mu])$, where $\lambda \neq 0$ and $\mu \neq 0$. We find all the solutions of the Yang-Baxter-like matrix equation $AXA = XAX$ successfully by discussing whether λ and μ are equal.

1. Introduction

The classical Yang-Baxter equation

$$A(u)B(u+v)A(v) = B(u)A(u+v)B(v),$$

where $A(u)$ and $B(v)$ are rational functions that depend on the parameters u and v, was first proposed by C. N. Yang [16] during the study of one-dimensional delta potential quantum Fermi gas in 1967 and then by R. J. Baxter [2] for a two-dimensional statistical model in 1972. It has a wide range of applications not only in mathematical problems[10, 17], but also in other areas, such as quantum computing and statistical physics[1, 9].

In the case of no parameters, the matrix version of a Yang-Baxter type equation is

$$AXA = XAX, \tag{1}$$

where A and X are both nth-order complex square matrices. The equation (1) often goes by the name of the Yang-Baxter-like matrix equation. In the past several decades, the problem of solving the equation (1), which may provide new research ideas and methods for the study of the classical Yang-Baxter equation, has received extensive attention, and more and more scholars are devoting themselves to this equation [3–7, 11–15, 18, 20, 23].

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The observation that $X = 0$ and $X = A$ constitute two trivial solutions to (1) is not arduous to discern. Due to the high nonlinearity of the equation, finding all its solutions is still a hot problem that needs to be solved completely. However, for some special classes of matrices A, some solutions or all solutions can be obtained. The first article on solutions to (1) appears to be that of Ding and Rhee [6], published in 2012. When the inverse A^{-1} is a stochastic matrix, they proved the existence of a nontrivial solution to (1) by utilizing Brouwer's fixed point theorem. Then they made use of the mean ergodic theorem to give some numerical solutions. In contrast to the research method in [6], for an idempotent matrix, the authors of [5, 12] adopted a new approach: the Jordan canonical form. Using this method, they have succeeded in expressing all the solutions of (1) for the special type of the given matrix. Since then, the application of Jordan canonical form has profoundly shaped the study of (1). For the case where A is a nilpotent matrix, all the solutions to (1) have been systematically investigated in [14, 18, 20] using Jordan canonical form. When A is restricted to rank-one or rank-two matrices, the authors of [14, 19, 20] employed the same method to establish explicit representations of all solutions to (1).

Most articles mentioned above made use of the Jordan canonical form to investigate the Yang-Baxter-like matrix equation. For a given matrix A, there is an invertible matrix S such that $A = SJS^{-1}$, where J is a Jordan canonical form of A. If the following equation

$$YJY = JYJ \tag{2}$$

has a solution Y, then it is not hard to verify that $X = SJS^{-1}$ satisfies (1). So, the corresponding matrix equation (1) is equivalent to the matrix equation (2). The simplified equation (2) is still nonlinear, which makes it difficult to find all solutions, too. At the same time, if n is very small, then it will be much easier to find all the solutions. If $n = 2$, then for the possible forms of J, all the solutions to (2) have been given [20]. Recently, for

$$J = J_3[\lambda] = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}, \quad 0 \neq \lambda \in \mathbb{C},$$

the authors in [4] have provided a family of solutions, and for

$$J = \mathrm{diag}(J_1[\lambda], J_1[\mu], J_1[\gamma]) = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \gamma \end{bmatrix}, \quad \lambda, \mu, \gamma \in \mathbb{C} \setminus \{0\},$$

all the solutions of (2) have been given in [15]. But for

$$J = \mathrm{diag}(J_2[\lambda], J_1[\mu]) = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}, \quad \lambda, \mu \in \mathbb{C} \setminus \{0\},$$

there has not appeared a complete discussion on all the solutions of (2). For this particular matrix, the equation (2) is the same as

$$\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & t \end{bmatrix} \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & t \end{bmatrix} \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & t \end{bmatrix}. \tag{3}$$

In this paper, we investigate all the solutions of the above equation. In the next section, we first investigate all the solutions of (3) for the case of $\lambda \neq \mu$ and then discuss all the solutions for the case of $\lambda = \mu$, resulting in a complete solution to the problem. We conclude with Section 3.

2. Results

For the sake of easy understanding, we divide our process of solving (3) into two cases: $\lambda \neq \mu$ and $\lambda = \mu$.

2.1. The Case of $\lambda \neq \mu$

First, we examine the matrix equation (3) in the case that λ and μ are unequal. Then, the equation can be written as

$$\begin{cases} \lambda a^2 - \lambda^2 a + ad + \lambda bd + \mu cg - \lambda d = 0, & (4a) \\ \lambda ab + (e-\lambda)(a+\lambda b) + \mu ch - d - \lambda e = 0, & (4b) \\ (\lambda a + \mu t - \lambda\mu)c + (a + \lambda b - \mu)f = 0, & (4c) \\ (\lambda a + \lambda e + d - \lambda^2)d + \mu fg = 0, & (4d) \\ \lambda bd + (e-\lambda)(d+\lambda e) + \mu fh = 0, & (4e) \\ \lambda cd + (\lambda e + d - \lambda\mu + \mu t)f = 0, & (4f) \\ (\lambda a + \mu t + d - \lambda\mu)g + \lambda hd = 0, & (4g) \\ (\lambda b + e - \mu)g + (\lambda e + \mu t - \lambda\mu)h = 0, & (4h) \\ (\lambda c + f)g + \lambda hf + \mu t^2 - \mu^2 t = 0. & (4i) \end{cases}$$

According to Equations (4d) and (4g), starting from whether fg and $(\lambda a + \mu t + d - \lambda\mu)g$ are zero, the discussion is divided into four subcases.

2.1.1. Subcase 1.

If $fg \neq 0$ and $(\lambda a + \mu t + d - \lambda\mu)g \neq 0$, then $h \neq 0$, $d \neq 0$ and $\lambda a + \lambda e + d - \lambda^2 \neq 0$ from (4d) and (4g). Multiplying $\lambda a + \mu t + d - \lambda\mu$ to (4d) and then subtracting (4g) multiplied by μf, we get

$$\lambda\mu hfd = (\lambda a + \mu t + d - \lambda\mu)(\lambda a + \lambda e + d - \lambda^2)d.$$

Because $d \neq 0$, the above equation can be written as

$$\lambda\mu hf = (\lambda a + \mu t + d - \lambda\mu)(\lambda a + \lambda e + d - \lambda^2). \tag{5}$$

Multiplying (4h) by $\lambda\mu f$ and using (4d) and (5), we have

$$\lambda(\lambda b + e - \mu)(\lambda a + \lambda e + d - \lambda^2)d = (\lambda e + \mu t - \lambda\mu)(\lambda a + \mu t + d - \lambda\mu)(\lambda a + \lambda e + d - \lambda^2).$$

Since $\lambda a + \lambda e + d - \lambda^2 \neq 0$,

$$\lambda d(\lambda b + e - \mu) = (\lambda e + \mu t - \lambda\mu)(\lambda a + \mu t + d - \lambda\mu). \tag{6}$$

Multiplying (4e) by λ gives

$$\lambda^2 bd + \lambda(e-\lambda)(d+\lambda e) + \lambda\mu fh = 0.$$

Substituting (5) into the above equation, we arrive at

$$\lambda^2 bd + \lambda(e-\lambda)(d+\lambda e) + (\lambda a + \lambda e + d - \lambda^2)(\lambda a + \mu t + d - \lambda\mu) = 0. \tag{7}$$

Combining (6) and (7) gives

$$(\lambda a + \lambda e + \mu t + d - \lambda\mu)^2 - \lambda^2(\lambda a + \lambda e + \mu t + d - \lambda\mu) + \lambda\mu d - \lambda^2 d = 0. \tag{8}$$

From (4b), (4f), (5), (7) and (8),

$$(\lambda a + \lambda e)(\mu - \lambda) + d(\mu - 2\lambda) - \lambda^2\mu = 0. \tag{9}$$

As $g \neq 0$, multiplying g on the both sides of (4f), we see that

$$\lambda\mu cg = (\lambda a + \lambda e + d - \lambda^2)(\lambda e + \mu t + d - \lambda\mu). \tag{10}$$

Since $\lambda \neq \mu$, it follows from (4d), (4i), (10) and (5) that

$$\mu t(\lambda - \mu) = \lambda^2 \mu + \lambda d. \tag{11}$$

Then, substituting (11) into (9), we obtain

$$\lambda a + \lambda e + \mu t + d = 0.$$

A combination of (8) and $\lambda a + \lambda e + \mu t + d = 0$ yields

$$d = \frac{\lambda\mu(\lambda + \mu)}{\lambda - \mu}. \tag{12}$$

By (11), (12) and $\lambda a + \lambda e + \mu t + d = 0$,

$$a + e = \frac{\mu(\mu^2 - 3\lambda^2)}{(\lambda - \mu)^2} \quad \text{and} \quad t = \frac{2\lambda^3}{(\lambda - \mu)^2}.$$

Substituting the above equations into (7) yields

$$b = \frac{-2\mu^2}{\lambda^2 - \mu^2} + \frac{\mu^2 - 3\lambda^2}{\lambda(\lambda^2 - \mu^2)} e - \frac{\lambda - \mu}{\lambda\mu(\lambda + \mu)} e^2.$$

Using the same trick gives rise to

$$\begin{cases} c = \frac{2(\mu^2-\lambda^2-\lambda\mu)}{\lambda(\lambda^2-\mu^2)} f - \frac{(\lambda-\mu)e}{\lambda\mu(\lambda+\mu)} f, \\ g = \frac{\lambda^3(\lambda^2+\mu^2)(\lambda+\mu)}{(\lambda-\mu)^3 f}, \\ h = \frac{\lambda^2(\mu+e)(\lambda^2+\mu^2)}{\mu(\lambda-\mu)^2 f}. \end{cases}$$

Then, we can prove the following theorem.

Theorem 2.1. *Let $J = \operatorname{diag}(J_2[\lambda], J_1[\mu])$ with nonzero λ and μ such that $\lambda \neq \mu$. Then the matrices*

$$Y = \begin{bmatrix} \frac{\mu(\mu^2-3\lambda^2)}{(\lambda-\mu)^2} - e & \frac{-2\mu^2}{\lambda^2-\mu^2} + \frac{\mu^2-3\lambda^2}{\lambda(\lambda^2-\mu^2)} e - \frac{\lambda-\mu}{\lambda\mu(\lambda+\mu)} e^2 & \frac{2(\mu^2-\lambda^2-\lambda\mu)}{\lambda(\lambda^2-\mu^2)} f - \frac{(\lambda-\mu)e}{\lambda\mu(\lambda+\mu)} f \\ \frac{\lambda\mu(\lambda+\mu)}{\lambda-\mu} & e & f \\ \frac{\lambda^3(\lambda^2+\mu^2)(\lambda+\mu)}{(\lambda-\mu)^3 f} & \frac{\lambda^2(\mu+e)(\lambda^2+\mu^2)}{\mu(\lambda-\mu)^2 f} & \frac{2\lambda^3}{(\lambda-\mu)^2} \end{bmatrix}$$

constitute all the solutions of (2) *under the additional condition that $fg \neq 0$ and $\lambda a + \mu t + d - \lambda\mu \neq 0$, where $e \in \mathbb{C}$ and $0 \neq f \in \mathbb{C}$.*

Proof. Suppose that Y solves (2) and its entries satisfy $fg \neq 0$ and $\lambda a + \mu t + d - \lambda\mu \neq 0$. Then the previous deduction process indicates that Y is given as above. Conversely, for any Y given in the theorem, since

$$YJ = \begin{bmatrix} \frac{\mu(\mu^2-3\lambda^2)}{(\lambda-\mu)^2} - e & \frac{-2\mu^2}{\lambda^2-\mu^2} + \frac{\mu^2-3\lambda^2}{\lambda(\lambda^2-\mu^2)} e - \frac{\lambda-\mu}{\lambda\mu(\lambda+\mu)} e^2 & \frac{2(\mu^2-\lambda^2-\lambda\mu)}{\lambda(\lambda^2-\mu^2)} f - \frac{(\lambda-\mu)e}{\lambda\mu(\lambda+\mu)} f \\ \frac{\lambda\mu(\lambda+\mu)}{\lambda-\mu} & e & f \\ \frac{\lambda^3(\lambda^2+\mu^2)(\lambda+\mu)}{(\lambda-\mu)^3 f} & \frac{\lambda^2(\mu+e)(\lambda^2+\mu^2)}{\mu(\lambda-\mu)^2 f} & \frac{2\lambda^3}{(\lambda-\mu)^2} \end{bmatrix} \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \mu \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\lambda\mu(\mu^2-3\lambda^2)}{(\lambda-\mu)^2} - \lambda e & \frac{\mu(\mu^2-3\lambda^2)}{(\lambda-\mu)^2} + \frac{-2\lambda\mu^2+e(2\mu^2-4\lambda^2)}{\lambda^2-\mu^2} - \frac{\lambda-\mu}{\mu(\lambda+\mu)} e^2 & \frac{2\mu(\mu^2-\lambda^2-\lambda\mu)}{\lambda(\lambda^2-\mu^2)} f - \frac{(\lambda-\mu)e}{\lambda(\lambda+\mu)} f \\ \frac{\lambda^2\mu(\lambda+\mu)}{\lambda-\mu} & \frac{\lambda\mu(\lambda+\mu)}{\lambda-\mu} + \lambda e & \mu f \\ \frac{\lambda^4(\lambda^2+\mu^2)(\lambda+\mu)}{(\lambda-\mu)^3 f} & \frac{\lambda^3(\lambda^2+\mu^2)(2\lambda\mu+\lambda e-\mu e)}{\mu(\lambda-\mu)^3 f} & \frac{2\lambda^3\mu}{(\lambda-\mu)^2}, \end{bmatrix},$$

$$JYJ = J\begin{bmatrix} \frac{\lambda\mu(\mu^2-3\lambda^2)}{(\lambda-\mu)^2} - \lambda e & \frac{\mu(\mu^2-3\lambda^2)}{(\lambda-\mu)^2} + \frac{-2\lambda\mu^2+e(2\mu^2-4\lambda^2)}{\lambda^2-\mu^2} - \frac{\lambda-\mu}{\mu(\lambda+\mu)}e^2 & \frac{2\mu(\mu^2-\lambda^2-\lambda\mu)}{\lambda(\lambda^2-\mu^2)}f - \frac{(\lambda-\mu)e}{\lambda(\lambda+\mu)}f \\ \frac{\lambda^2\mu(\lambda+\mu)}{\lambda-\mu} & \frac{\lambda\mu(\lambda+\mu)}{\lambda-\mu} + \lambda e & \mu f \\ \frac{\lambda^4(\lambda^2+\mu^2)(\lambda+\mu)}{(\lambda-\mu)^3 f} & \frac{\lambda^3(\lambda^2+\mu^2)(2\lambda\mu+\lambda e-\mu e)}{\mu(\lambda-\mu)^3 f} & \frac{2\lambda^3\mu}{(\lambda-\mu)^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-2\lambda^4\mu}{(\lambda-\mu)^2} - \lambda^2 e & \frac{-2\lambda^2\mu(\lambda^2-\mu^2+2\lambda\mu)}{(\lambda-\mu)^2(\lambda+\mu)} + \frac{\lambda(\mu^2-3\lambda^2)}{\lambda^2-\mu^2}e - \frac{\lambda(\lambda-\mu)}{\mu(\lambda+\mu)}e^2 & \frac{\mu(\mu^2-\lambda^2-2\lambda\mu)}{\lambda^2-\mu^2}f - \frac{\lambda-\mu}{\lambda+\mu}fe \\ \frac{\lambda^3\mu(\lambda+\mu)}{\lambda-\mu} & \frac{\lambda^2\mu(\lambda+\mu)}{\lambda-\mu)} + \lambda^2 e & \lambda\mu f \\ \frac{\lambda^4\mu(\lambda^2+\mu^2)(\lambda+\mu)}{(\lambda-\mu)^3 f} & \frac{2\lambda^4\mu(\lambda^2+\mu^2)}{(\lambda-\mu)^3 f} + \frac{\lambda^3(\lambda^2+\mu^2)}{(\lambda-\mu)^2 f}e & \frac{2\lambda^3\mu^2}{(\lambda-\mu)^2} \end{bmatrix},$$

$$YJY = YJ\begin{bmatrix} \frac{\mu(\mu^2-3\lambda^2)}{(\lambda-\mu)^2} - e & \frac{-2\mu^2}{\lambda^2-\mu^2} + \frac{\mu^2-3\lambda^2}{\lambda(\lambda^2-\mu^2)}e - \frac{\lambda-\mu}{\lambda\mu(\lambda+\mu)}e^2 & \frac{2(\mu^2-\lambda^2-\lambda\mu)}{\lambda(\lambda^2-\mu^2)}f - \frac{(\lambda-\mu)e}{\lambda\mu(\lambda+\mu)}f \\ \frac{\lambda\mu(\lambda+\mu)}{\lambda-\mu} & e & f \\ \frac{\lambda^3(\lambda^2+\mu^2)(\lambda+\mu)}{(\lambda-\mu)^3 f} & \frac{\lambda^2(\mu+e)(\lambda^2+\mu^2)}{\mu(\lambda-\mu)^2 f} & \frac{2\lambda^3}{(\lambda-\mu)^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-2\lambda^4\mu}{(\lambda-\mu)^2} - \lambda^2 e & \frac{-2\lambda^2\mu(\lambda^2-\mu^2+2\lambda\mu)}{(\lambda-\mu)^2(\lambda+\mu)} + \frac{\lambda(\mu^2-3\lambda^2)}{\lambda^2-\mu^2}e - \frac{\lambda(\lambda-\mu)}{\mu(\lambda+\mu)}e^2 & \frac{\mu(\mu^2-\lambda^2-2\lambda\mu)}{\lambda^2-\mu^2}f - \frac{\lambda-\mu}{\lambda+\mu}fe \\ \frac{\lambda^3\mu(\lambda+\mu)}{\lambda-\mu} & \frac{\lambda^2\mu(\lambda+\mu)}{\lambda-\mu} + \lambda^2 e & \lambda\mu f \\ \frac{\lambda^4\mu(\lambda^2+\mu^2)(\lambda+\mu)}{(\lambda-\mu)^3 f} & \frac{2\lambda^4\mu(\lambda^2+\mu^2)}{(\lambda-\mu)^3 f} + \frac{\lambda^3(\lambda^2+\mu^2)}{(\lambda-\mu)^2 f}e & \frac{2\lambda^3\mu^2}{(\lambda-\mu)^2} \end{bmatrix},$$

we see that Y satisfies (2). □

2.1.2. *Subcase 2*

Assume that $fg \neq 0$ and $(\lambda a + \mu t + d - \lambda\mu)g = 0$, which means that $f \neq 0$, $g \neq 0$ and $\lambda a + \mu t + d - \lambda\mu = 0$. Then $\lambda a + \lambda e + d - \lambda^2 \neq 0$, $d \neq 0$ and $h = 0$ from (4d) and (4g). According to equations (4e) and (4h),

$$d = \frac{\lambda e(\lambda - e)}{\mu - \lambda}$$

and

$$b = \frac{\mu - e}{\lambda}.$$

Substituting the above two equations into (4b) generates

$$a = \frac{\lambda\mu}{\mu - \lambda} + \frac{\lambda^2 - \mu^2 + \lambda\mu}{(\mu - \lambda)^2}e + \frac{\mu - 2\lambda}{(\mu - \lambda)^2}e^2.$$

Since $\lambda a + \mu t + d - \lambda\mu = 0$,

$$t = \frac{\lambda\mu - 2\lambda^2}{\mu - \lambda} + \frac{\lambda\mu - 2\lambda^2}{(\mu - \lambda)^2}e + \frac{\lambda^2}{\mu(\mu - \lambda)^2}e^2.$$

In the same way, from (4a), (4c) and (4d),

$$\begin{cases} cg = -\frac{\lambda^4}{(\mu-\lambda)^2} + \frac{\lambda^2(\mu-2\lambda)}{\mu(\mu-\lambda)^4}e^4 + \frac{2\lambda^2(\lambda^2-\mu^2+\lambda\mu)}{\mu(\mu-\lambda)^4}e^3 - \frac{\lambda^3(2\lambda^2+\lambda\mu-2\mu^2)}{\mu(\mu-\lambda)^4}e^2 - \frac{2\lambda^3(2\lambda-\mu)}{(\mu-\lambda)^3}e, \\ c = \frac{\mu}{e(\lambda-e)} + \frac{-2\mu^2+3\lambda\mu}{\lambda(\lambda-e)(\mu-\lambda)} + \frac{\mu-2\lambda}{\lambda(\mu-\lambda)(\lambda-e)}e, \\ fg = -\frac{\lambda^5}{\mu(\mu-\lambda)^2}e - \frac{\lambda^4(2\lambda-\mu)}{\mu(\mu-\lambda)^3}e^2 + \frac{2\lambda^4}{\mu(\mu-\lambda)^3}e^3 - \frac{\lambda^3}{\mu(\mu-\lambda)^3}e^4. \end{cases} \tag{13}$$

So, we have the following theorem.

Theorem 2.2. *Let $J = \mathrm{diag}(J_2[\lambda], J_1[\mu])$ with nonzero λ and μ such that $\lambda \neq \mu$. Then the matrices*

$$Y = \begin{bmatrix} \frac{\lambda\mu}{\mu-\lambda} + \frac{\lambda^2-\mu^2+\lambda\mu}{(\mu-\lambda)^2}e + \frac{\mu-2\lambda}{(\mu-\lambda)^2}e^2 & \frac{\mu-e}{\lambda} & \frac{\mu}{e(\lambda-e)} + \frac{-2\mu^2+3\lambda\mu}{\lambda(\lambda-e)(\mu-\lambda)} + \frac{\mu-2\lambda}{\lambda(\mu-\lambda)(\lambda-e)}e \\ \frac{\lambda e(\lambda-e)}{\mu-\lambda} & e & f \\ g & 0 & \frac{\lambda\mu-2\lambda^2}{\mu-\lambda} + \frac{\lambda\mu-2\lambda^2}{(\mu-\lambda)^2}e + \frac{\lambda^2}{\mu(\mu-\lambda)^2}e^2 \end{bmatrix}$$

constitute all the solutions of (2) *under the additional condition that $fg \neq 0$ and $\lambda a + \mu t + d - \lambda\mu = 0$, where $e \in \mathbb{C} \setminus \{0, \lambda\}$, and f and g satisfy* (13).

2.1.3. Subcase 3

If $fg = 0$ and $(\lambda a + \mu t + d - \lambda\mu)g \neq 0$, then $\lambda a + \lambda e + d - \lambda^2 = 0$ and $hd \neq 0$ from (4d) and (4g). Thus, $g \neq 0$, $d \neq 0$, $h \neq 0$ and $\lambda a + \mu t + d - \lambda\mu \neq 0$. So $c = 0$ from (4f). Substituting $\lambda a + \lambda e + d - \lambda^2 = 0$ into (4a) and (4b), we get

$$-\lambda ae + \lambda bd - \lambda d = 0 \quad \text{and} \quad -bd + ae - \lambda^2 = 0.$$

The above two equations imply

$$d = -\lambda^2,$$

and then

$$\lambda a + \lambda e = 2\lambda^2.$$

By (4e),

$$b = \frac{(e-\lambda)^2}{\lambda^2}.$$

Combining equations (4g) and (4h) yields

$$\lambda^3(\lambda b + e - \mu)g + (\lambda e + \mu t - \lambda\mu)(\lambda a + \mu t - \lambda^2 - \lambda\mu)g = 0.$$

Since $g \neq 0$,

$$\lambda^3(\lambda b + e - \mu) + (\lambda e + \mu t - \lambda\mu)(\lambda a + \mu t - \lambda^2 - \lambda\mu) = 0.$$

Note that $\lambda a + \lambda e = 2\lambda^2$ and $b = \frac{(e-\lambda)^2}{\lambda^2}$, so the above equation changes to

$$\lambda^2(\lambda-\mu)^2 + \mu t\mu t - 2\lambda\mu^2 t + \lambda^2\mu t = 0. \tag{14}$$

Because $c = f = 0$, (4i) converts to

$$\mu t^2 - \mu^2 t = 0.$$

A combination of (14) and the above equation yields

$$\lambda^2(\lambda-\mu)^2 + (\lambda-\mu)^2\mu t = 0,$$

that is,

$$t = -\frac{\lambda^2}{\mu}.$$

Since t also satisfies $\mu t^2 - \mu^2 t = 0$,

$$t = -\frac{\lambda^2}{\mu} = \mu,$$

that is

$$\lambda^2 + \mu^2 = 0.$$

Thus, from (4g),

$$h = \frac{-e-\mu}{\lambda^2} g.$$

Considering $h, g \neq 0$, we see that $e \neq -\mu$.

To sum up, we get the following theorem.

Theorem 2.3. *Let $J = \mathrm{diag}(J_2[\lambda], J_1[\mu])$ with nonzero λ and μ such that $\lambda^2 + \mu^2 = 0$. Then the matrices*

$$Y = \begin{bmatrix} 2\lambda - e & \frac{(e-\lambda)^2}{\lambda^2} & 0 \\ -\lambda^2 & e & 0 \\ g & \frac{-e-\mu}{\lambda^2} g & \mu \end{bmatrix}$$

constitute solutions of (2), *where $e \in \mathbb{C} \setminus \{-\mu\}$ and $0 \neq g \in \mathbb{C}$.*

2.1.4. Subcase 4

Now, we discuss the case that $fg = 0$ and $(\lambda a + \mu t + d - \lambda\mu)g = 0$. It is not difficult to see that $(\lambda a + \lambda e + d - \lambda^2)d = 0$ and $hd = 0$ from (4d) and (4g). We discuss separately depending on whether g and d are 0.

▲ If $d = g = 0$, then solving (3) is equivalent to solving the following system

$$\begin{cases} \lambda a(a-\lambda) = 0, & (15a) \\ \lambda ab + (e-\lambda)(a+\lambda b) + \mu ch - \lambda e = 0, & (15b) \\ (\lambda a + \mu t - \lambda\mu)c + (a + \lambda b - \mu) f = 0, & (15c) \\ \lambda e(e-\lambda) + \mu f h = 0, & (15d) \\ (\lambda e - \lambda\mu + \mu t) f = 0, & (15e) \\ (\lambda e + \mu t - \lambda\mu) h = 0, & (15f) \\ \lambda h f + \mu t^2 - \mu^2 t = 0. & (15g) \end{cases}$$

From (15a), $a = 0$ or λ.

■ When $a = 0$, by (15b) and (15c),

$$b(\lambda e - \lambda^2) - \lambda e + \mu ch = 0 \tag{16}$$

and

$$(\mu t - \lambda\mu)c + (\lambda b - \mu) f = 0. \tag{17}$$

According to (15e) and (15f), assuming that $\lambda e + \mu t - \lambda\mu = 0$, combined with (15d) and (15g), we have

$$e = \frac{\mu^2}{\mu - \lambda}, \quad t = -\frac{\lambda^2}{\mu - \lambda} \quad \text{and} \quad hf = \frac{\lambda\mu(\lambda\mu - \lambda^2 - \mu^2)}{(\mu-\lambda)^2}. \tag{18}$$

A combination of (16), (17) and (18) yields

$$\mu(\lambda - \mu)^2 = 0.$$

This contradicts the hypothesis, so $\lambda e + \mu t - \lambda\mu \neq 0$, that is, $h = 0$ and $f = 0$. By (15b) and (15d),

$$e = 0 \quad \text{and} \quad b = 0.$$

Then from (15g), $t = 0$ or μ. Thus, $c = 0$ from (15c).

- When $a = \lambda$ and $\lambda e + \mu t - \lambda\mu = 0$, we also get (18). But a combination of (15b), (15c) and (18) yields

$$(\mu + \lambda)f = 0.$$

If $\lambda + \mu = 0$, then

$$f \neq 0, \quad e = \frac{\mu}{2}, \quad t = -\frac{\mu}{2} \quad \text{and} \quad hf = \frac{3\mu^2}{4}.$$

It follows from (15c) that

$$c = \frac{2(2 + b)f}{3\mu}.$$

If $\lambda + \mu \neq 0$, then

$$f = 0, \quad \lambda^2 + \mu^2 - \lambda\mu = 0, \quad e = \lambda \quad \text{and} \quad t = \mu.$$

It follows from (15c) and (15b) that c is arbitrary and

$$b = 1 - \frac{\mu c h}{\lambda^2}.$$

On the other hand, that $\lambda e + \mu t - \lambda\mu \neq 0$ means $f = h = 0$. From (15b) and (15d),

$$e = \lambda \quad \text{and} \quad b = 1.$$

By (15g),

$$t = 0 \quad \text{or} \quad \mu.$$

Substituting $t = 0$ into (15c), we obtain

$$c = 0.$$

But if $t = \mu$, then substituting it into (15c), we have

$$(\lambda^2 + \mu^2 - \lambda\mu)c = 0.$$

When $\lambda^2 + \mu^2 - \lambda\mu = 0$, the number c is arbitrary. When $\lambda^2 + \mu^2 - \lambda\mu \neq 0$, the number c can only be 0.

In conclusion, we have the following theorem.

Theorem 2.4. *Let $J = \mathrm{diag}(J_2[\lambda], J_1[\mu])$ with nonzero λ and μ such that $\lambda \neq \mu$. Then*

$$Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ or \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mu \end{bmatrix} \ or \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix} \ or \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

are solutions of (2). *Moreover, if $\lambda + \mu = 0$, then*

$$Y = \begin{bmatrix} \lambda & b & \frac{2(2+b)f}{3\mu} \\ 0 & \frac{\mu}{2} & f \\ 0 & \frac{3\mu^2}{4f} & -\frac{\mu}{2} \end{bmatrix}$$

are solutions of (2). *If $\lambda^2 + \mu^2 - \lambda\mu = 0$, then*

$$Y = \begin{bmatrix} \lambda & 1 - \frac{\mu c h}{\lambda^2} & c \\ 0 & \lambda & 0 \\ 0 & h & \mu \end{bmatrix}$$

are solutions of (2), *where $c, h \in \mathbb{C}$.*

▲ If $g = 0$ and $d \neq 0$, then $h = 0$ and $\lambda a + \lambda e + d - \lambda^2 = 0$. Now (3) becomes

$$\begin{cases} -\lambda ae + \lambda bd - \lambda d = 0, & (19a) \\ -bd + ae - \lambda^2 = 0, & (19b) \\ (\lambda a + \mu t - \lambda\mu)c + (a + \lambda b - \mu)f = 0, & (19c) \\ d(\lambda b + e - \lambda) + \lambda e(e - \lambda) = 0, & (19d) \\ \lambda cd + (\lambda e + \mu t + d - \lambda\mu)f = 0, & (19e) \\ \mu t^2 - \mu^2 t = 0. & (19f) \end{cases}$$

Combining (19a) and (19b), we have

$$d = -\lambda^2,$$

and then

$$a + e = 2\lambda.$$

Next, substituting the above equations into (19a) or (19b), we get

$$b = \frac{(e - \lambda)^2}{\lambda^2}.$$

According to (19f),

$$t = 0 \quad \text{or} \quad \mu.$$

■ $t = 0$. Substituting $a + e = 2\lambda$ and $b = \frac{(e-\lambda)^2}{\lambda^2}$ into (19c) provides

$$(2\lambda^2 - \lambda e - \lambda\mu)\lambda c + (3\lambda^2 - 3\lambda e + e^2 - \lambda\mu)f = 0. \tag{20}$$

By using $d = -\lambda^2$, (19e) can be written as

$$c = \frac{(\lambda e - \lambda^2 - \lambda\mu)f}{\lambda^3}. \tag{21}$$

Substituting (21) into (20), we have

$$(\lambda-\mu)^2 f = 0.$$

Since $\lambda \neq \mu$,

$$f = 0.$$

Because $d \neq 0$, we have from (19e) that

$$c = 0.$$

- $t = \mu$. From (19c) and (19e),

$$(2\lambda^2 - \lambda e + \mu^2 - \lambda\mu)\lambda c + (3\lambda^2 - 3\lambda e + e^2 - \lambda\mu) f = 0 \tag{22}$$

and

$$c = \frac{(\lambda e - \lambda^2 + \mu^2 - \lambda\mu) f}{\lambda^3}. \tag{23}$$

A combination of (22) and (23) yields

$$(\lambda^2 + \mu^2) f = 0.$$

So f is arbitrary when $\lambda^2 + \mu^2 = 0$ and $f = 0$ when $\lambda^2 + \mu^2 \neq 0$. If $f = 0$, then it follows from (23) that

$$c = 0.$$

In summary, we have the following theorem.

Theorem 2.5. *Let* $J = \mathrm{diag}(J_2[\lambda], J_1[\mu])$ *with nonzero* λ *and* μ *such that* $\lambda \neq \mu$. *Then*

$$Y_1 = \begin{bmatrix} 2\lambda - e & \frac{(e-\lambda)^2}{\lambda^2} & 0 \\ -\lambda^2 & e & 0 \\ 0 & 0 & 0 \end{bmatrix}, e \in \mathbb{C}$$

and

$$Y_2 = \begin{bmatrix} 2\lambda - e & \frac{(e-\lambda)^2}{\lambda^2} & 0 \\ -\lambda^2 & e & 0 \\ 0 & 0 & \mu \end{bmatrix}, \quad e \in \mathbb{C}$$

are solutions of (2). *Moreover, if* $\lambda^2 + \mu^2 = 0$, *then*

$$Y_3 = \begin{bmatrix} 2\lambda - e & \frac{(e-\lambda)^2}{\lambda^2} & \frac{(\lambda e - 2\lambda^2 - \lambda\mu) f}{\lambda^3} \\ -\lambda^2 & e & f \\ 0 & 0 & \mu \end{bmatrix}, \quad e, f \in \mathbb{C}$$

are also solutions of (2).

▲ If $g \neq 0$ and $d = 0$, then $f = 0$ and $\lambda a + \mu t - \lambda\mu = 0$. Now, (3) can be written as

$$\begin{cases} a(\lambda a - \lambda^2) + \mu c g = 0, & (24a) \\ b(\lambda a + \lambda e - \lambda^2) + a e - \lambda a - \lambda e + \mu c h = 0, & (24b) \\ \lambda e (e - \lambda) = 0, & (24c) \\ (\lambda b + e - \mu) g + (\lambda e + \mu t - \lambda\mu) h = 0, & (24d) \\ \lambda c g + \mu t^2 - \mu^2 t = 0. & (24e) \end{cases}$$

Combining (24a) and (24e), we have

$$\lambda a(\lambda a - \lambda^2) = \mu t(\mu t - \mu^2).$$

Since $\lambda a + \mu t - \lambda\mu = 0$,

$$a = \frac{\mu^2}{\mu - \lambda} \quad \text{and} \quad t = -\frac{\lambda^2}{\mu - \lambda}.$$

Substituting $a = \frac{\mu^2}{\mu-\lambda}$ into (24a), we obtain

$$cg = -\frac{\lambda\mu(\mu^2 + \lambda^2 - \lambda\mu)}{(\mu - \lambda)^2}. \tag{25}$$

From (24c),

$$e = 0 \quad \text{or} \quad \lambda.$$

- $e = 0$. From (24b) and (24d),

  $$\lambda ab - \lambda^2 b - \lambda a + \mu ch = 0$$

  and

  $$(\lambda b - \mu)g = \lambda ah.$$

  Combining the above two equations with $a = \frac{\mu^2}{\mu-\lambda}$ and (25) gives

  $$(\lambda - \mu)^2 = 0,$$

  which contradicts the hypothesis.

- $e = \lambda$. By (24b) and (24d),

  $$\lambda ab + \mu ch - \lambda^2 = 0 \tag{26}$$

  and

  $$(\lambda b + \lambda - \mu)g + (\lambda^2 - \lambda a)h = 0. \tag{27}$$

  Input (27) into (26) and get

  $$(\lambda a - \lambda^2)ah + (\mu - \lambda)ag + \mu hcg - \lambda^2 = 0.$$

  Next substituting $a = \frac{\mu^2}{\mu-\lambda}$ and (25) into the above equation, we obtain

  $$g = 0, \tag{28}$$

  which contradicts $g \neq 0$.

Based on the above analysis, there is no solution to (2) with $e = \lambda$.

▲ If $g \neq 0$ and $d \neq 0$, then $f = h = \lambda a + \mu t + d - \lambda\mu = \lambda a + \lambda e + d - \lambda^2 = 0$. In this case, equations (4a)-(4i) are equivalent to the following equations

$$\begin{cases} -\lambda ae + \lambda bd - \lambda d + \mu cg = 0, & (29a) \\ -bd + ae - \lambda^2 + \mu ch = 0, & (29b) \\ cd = 0, & (29c) \\ d(\lambda b + e - \lambda) + \lambda e(e - \lambda) = 0, & (29d) \\ (\lambda b + e - \mu)g = 0, & (29e) \\ \lambda cg + \mu t^2 - \mu^2 t = 0. & (29f) \end{cases}$$

Since $d \neq 0$ and $g \neq 0$, we have from (29c) that

$$c = 0,$$

and then

$$\mu t^2 - \mu^2 t = 0$$

from (29f). Similarly, by (29a) and (29b),

$$-\lambda ae + \lambda bd - \lambda d = 0$$

and

$$-bd + ae - \lambda^2 = 0.$$

Thus, it is not difficult to see from the above two equations that

$$d = -\lambda^2,$$

and then

$$a + e = 2\lambda.$$

Because $g \neq 0$,

$$\lambda b + e - \mu = 0$$

from (29e). According to $\mu t^2 - \mu^2 t = 0$, we see that $t = 0$ or μ.

- ■ $t = 0$. From $\lambda a + \mu t + d - \lambda\mu = 0$ and $d = -\lambda^2$,

  $$a = \lambda + \mu,$$

  and then

  $$e = \lambda - \mu.$$

  Since $\lambda b + e - \lambda = 0$,

  $$b = \frac{2\mu - \lambda}{\lambda}.$$

  In addition, substituting the above three equations into (29b), we arrive at

  $$(\lambda - \mu)^2 = 0,$$

  which contradicts the hypothesis.

■ $t = \mu$. In the same way,

$$a = \frac{\lambda^2 + \lambda\mu - \mu^2}{\lambda}, \quad e = \frac{\lambda^2 + \mu^2 - \lambda\mu}{\lambda} \quad \text{and} \quad b = -\frac{(\lambda - \mu)^2}{\lambda^2}.$$

By using the same trick, we get

$$(\lambda - \mu)^2(\lambda^2 + \mu^2) = 0.$$

Since $\lambda \neq \mu$, we have $\lambda^2 + \mu^2 = 0$.

Having said all the above, we obtain the following theorem.

Theorem 2.6. *Let* $J = \operatorname{diag}(J_2[\lambda], J_1[\mu])$ *with* $\lambda^2 + \mu^2 = 0$. *Then*

$$Y = \begin{bmatrix} \frac{2\lambda^2+\lambda\mu}{\lambda} & \frac{2\mu}{\lambda} & 0 \\ -\lambda^2 & -\mu & 0 \\ g & 0 & \mu \end{bmatrix}$$

are solutions of (2), *where* g *is arbitrary number.*

2.2. *The Case of* $\lambda = \mu$

When $\lambda = \mu$, the equation (3) can be written as

$$\begin{cases} \lambda a^2 - \lambda^2 a + ad + \lambda bd + \lambda cg - \lambda d = 0, & \text{(30a)} \\ \lambda ab + (e - \lambda)(a + \lambda b) + \lambda ch - d - \lambda e = 0, & \text{(30b)} \\ (\lambda a + \lambda t - \lambda^2)c + (a + \lambda b - \lambda)f = 0, & \text{(30c)} \\ (\lambda a + \lambda e + d - \lambda^2)d + \lambda fg = 0, & \text{(30d)} \\ \lambda bd + (e - \lambda)(d + \lambda e) + \lambda fh = 0, & \text{(30e)} \\ \lambda cd + (\lambda e + d - \lambda^2 + \lambda t)f = 0, & \text{(30f)} \\ (\lambda a + \lambda t + d - \lambda^2)g + \lambda hd = 0, & \text{(30g)} \\ (\lambda b + e - \lambda)g + (\lambda e + \lambda t - \lambda^2)h = 0, & \text{(30h)} \\ (\lambda c + f)g + \lambda hf + \lambda t^2 - \lambda^2 t = 0. & \text{(30i)} \end{cases}$$

Similarly, we solve the above system according to different subcases of whether $(\lambda a + \lambda t + d - \lambda^2)g$ and fg are 0.

2.2.1. *Subcase 1*

If $fg \neq 0$ and $(\lambda a + \lambda t + d - \lambda^2)g \neq 0$, then since $\lambda = \mu$, (11) now becomes

$$d = -\lambda^2,$$

and (9) always holds. With the help of (8), we have

$$(\lambda a + \lambda e + \lambda t - 2\lambda^2)^2 = \lambda^2(\lambda a + \lambda e + \lambda t - 2\lambda^2),$$

that is, $a + e + t = 2\lambda$ or $a + e + t = 3\lambda$. If $a + e + t = 2\lambda$, then

$$\begin{cases} a + e + t = 2\lambda, \\ b = 1 - \frac{ae}{\lambda^2}, \\ c = -\frac{af}{\lambda^2}, \\ d = -\lambda^2, \\ h = -\frac{eg}{\lambda^2}, \\ t = -\frac{fg}{\lambda^2}. \end{cases}$$

If $a+e+t=3\lambda$, then

$$\begin{cases} a+e+t=3\lambda, \\ b=\frac{(e-\lambda)(a-\lambda)}{-\lambda^2}, \\ c=\frac{(\lambda-a)f}{\lambda^2}, \\ d=-\lambda^2, \\ h=\frac{(\lambda-e)g}{\lambda^2}, \\ t=\lambda-\frac{fg}{\lambda^2}. \end{cases}$$

So, the corresponding result can be obtained as follows.

Theorem 2.7. *Let $J=\operatorname{diag}(J_2[\lambda],J_1[\lambda])$ with nonzero λ. If $fg\neq 0$ and $\lambda a+\mu t+d-\lambda\mu\neq 0$, then*

$$Y_1=\begin{bmatrix} a & 1-\frac{ae}{\lambda^2} & \frac{-af}{\lambda^2} \\ -\lambda^2 & e & f \\ \frac{\lambda^2(a+e-2\lambda)}{f} & \frac{-e(a+e-2\lambda)}{f} & 2\lambda-a-e \end{bmatrix}, \quad a,e\in\mathbb{C} \quad \text{and} \quad 0\neq f\in\mathbb{C}$$

and

$$Y_2=\begin{bmatrix} a & \frac{(e-\lambda)(a-\lambda)}{-\lambda^2} & \frac{(\lambda-a)f}{\lambda^2} \\ -\lambda^2 & e & f \\ \frac{\lambda^2(a+e-2\lambda)}{f} & \frac{(\lambda-e)(a+e-2\lambda)}{f} & 3\lambda-a-e \end{bmatrix}, \quad a,e\in\mathbb{C} \quad \text{and} \quad 0\neq f\in\mathbb{C}$$

are solutions of (2).

Proof. A direct computation gives

$$Y_1J=\begin{bmatrix} a & 1-\frac{ae}{\lambda^2} & \frac{-af}{\lambda^2} \\ -\lambda^2 & e & f \\ \frac{\lambda^2(a+e-2\lambda)}{f} & \frac{-e(a+e-2\lambda)}{f} & 2\lambda-a-e \end{bmatrix}\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}=\begin{bmatrix} \lambda a & a+\lambda-\frac{ae}{\lambda} & -\frac{af}{\lambda} \\ -\lambda^3 & -\lambda^2+\lambda e & \lambda f \\ \frac{\lambda^3(a+e-2\lambda)}{f} & \frac{\lambda(a+e-2\lambda)(\lambda-e)}{f} & \lambda(2\lambda-a-e) \end{bmatrix},$$

$$\begin{aligned} JY_1J &= \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\begin{bmatrix} \lambda a & a+\lambda-\frac{ae}{\lambda} & -\frac{af}{\lambda} \\ -\lambda^3 & -\lambda^2+\lambda e & \lambda f \\ \frac{\lambda^3(a+e-2\lambda)}{f} & \frac{\lambda(a+e-2\lambda)(\lambda-e)}{f} & \lambda(2\lambda-a-e) \end{bmatrix} \\ &= \begin{bmatrix} \lambda^2 a-\lambda^3 & \lambda a+\lambda e-ae & -af+\lambda f \\ -\lambda^4 & -\lambda^3+\lambda^2 e & \lambda^2 f \\ \frac{\lambda^4(a+e-2\lambda)}{f} & \frac{\lambda^2(a+e-2\lambda)(\lambda-e)}{f} & \lambda^2(2\lambda-a-e) \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} Y_1JY_1 &= \begin{bmatrix} \lambda a & a+\lambda-\frac{ae}{\lambda} & -\frac{af}{\lambda} \\ -\lambda^3 & -\lambda^2+\lambda e & \lambda f \\ \frac{\lambda^3(a+e-2\lambda)}{f} & \frac{\lambda(a+e-2\lambda)(\lambda-e)}{f} & \lambda(2\lambda-a-e) \end{bmatrix}\begin{bmatrix} a & 1-\frac{ae}{\lambda^2} & \frac{-af}{\lambda^2} \\ -\lambda^2 & e & f \\ \frac{\lambda^2(a+e-2\lambda)}{f} & \frac{-e(a+e-2\lambda)}{f} & 2\lambda-a-e \end{bmatrix} \\ &= \begin{bmatrix} \lambda^2 a-\lambda^3 & \lambda a+\lambda e-ae & -af+\lambda f \\ -\lambda^4 & -\lambda^3+\lambda^2 e & \lambda^2 f \\ \frac{\lambda^4(a+e-2\lambda)}{f} & \frac{\lambda^2(a+e-2\lambda)(\lambda-e)}{f} & \lambda^2(2\lambda-a-e) \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} Y_2J &= \begin{bmatrix} a & \frac{(e-\lambda)(a-\lambda)}{-\lambda^2} & \frac{(\lambda-a)f}{\lambda^2} \\ -\lambda^2 & e & f \\ \frac{\lambda^2(a+e-2\lambda)}{f} & \frac{(\lambda-e)(a+e-2\lambda)}{f} & 3\lambda-a-e \end{bmatrix}\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \\ &= \begin{bmatrix} \lambda a & a+\frac{(a-\lambda)(e-\lambda)}{-\lambda} & \frac{(\lambda-a)f}{\lambda} \\ -\lambda^3 & -\lambda^2+\lambda e & \lambda f \\ \frac{\lambda^3(a+e-2\lambda)}{f} & \frac{(2\lambda^2-\lambda e)(a+e-2\lambda)}{f} & \lambda(3\lambda-a-e) \end{bmatrix}, \end{aligned}$$

$$JY_2J = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} \lambda a & a + \frac{(a-\lambda)(e-\lambda)}{-\lambda} & \frac{(\lambda-a)f}{\lambda} \\ -\lambda^3 & -\lambda^2 + \lambda e & \lambda f \\ \frac{\lambda^3(a+e-2\lambda)}{f} & \frac{(2\lambda^2-\lambda e)(a+e-2\lambda)}{f} & \lambda(3\lambda - a - e) \end{bmatrix}$$

$$= \begin{bmatrix} \lambda^2 a - \lambda^3 & 2\lambda a + 2\lambda e - ae - 2\lambda^2 & 2\lambda f - af \\ -\lambda^4 & -\lambda^3 + \lambda^2 e & \lambda^2 f \\ \frac{\lambda^4(a+e-2\lambda)}{f} & \frac{\lambda(2\lambda^2-\lambda e)(a+e-2\lambda)}{f} & \lambda^2(3\lambda - a - e) \end{bmatrix},$$

$$Y_2JY_2 = \begin{bmatrix} \lambda a & a + \frac{(a-\lambda)(e-\lambda)}{-\lambda} & \frac{(\lambda-a)f}{\lambda} \\ -\lambda^3 & -\lambda^2 + \lambda e & \lambda f \\ \frac{\lambda^3(a+e-2\lambda)}{f} & \frac{(2\lambda^2-\lambda e)(a+e-2\lambda)}{f} & \lambda(3\lambda - a - e) \end{bmatrix} \begin{bmatrix} a & \frac{(e-\lambda)(a-\lambda)}{-\lambda^2} & \frac{(\lambda-a)f}{\lambda^2} \\ -\lambda^2 & e & f \\ \frac{\lambda^2(a+e-2\lambda)}{f} & \frac{(\lambda-e)(a+e-2\lambda)}{f} & 3\lambda - a - e \end{bmatrix}$$

$$= \begin{bmatrix} \lambda^2 a - \lambda^3 & 2\lambda a + 2\lambda e - ae - 2\lambda^2 & 2\lambda f - af \\ -\lambda^4 & -\lambda^3 + \lambda^2 e & \lambda^2 f \\ \frac{\lambda^4(a+e-2\lambda)}{f} & \frac{\lambda(2\lambda^2-\lambda e)(a+e-2\lambda)}{f} & \lambda^2(3\lambda - a - e) \end{bmatrix}.$$

So, Y_1 and Y_2 are solutions of (2). □

2.2.2. Subcase 2

If $fg \neq 0$ and $(\lambda a + \lambda t + d - \lambda^2)g = 0$, then $d \neq 0$, $\lambda a + \lambda e + d - \lambda^2 \neq 0$ and $h = 0$ from (30d) and (30g). Because $g \neq 0$ and $h = 0$,

$$\lambda b + e - \lambda = 0$$

from (30h). Substituting the above equation and $h = 0$ into (30e), we get

$$e(e - \lambda) = 0.$$

Substituting the above equation and $\lambda b + e - \lambda = 0$ into (30b) yields

$$d = -\lambda^2.$$

Recall that $\lambda a + \lambda t + d - \lambda^2 = 0$, so it is easy to verify that

$$a + t = 2\lambda.$$

Since $e(e - \lambda) = 0$, it holds that $e = 0$ or λ. If $e = 0$, then $b = 1$ and $a \neq 2\lambda$. By using $\lambda a + \lambda t + d - \lambda^2 = 0$, $d = -\lambda^2$ and $b = 1$, (30c) can be written as

$$c = -\frac{af}{\lambda^2}.$$

By the same token, (30f) can also yield the above equation. Then we have from (30d) that

$$fg = \lambda^2(a - 2\lambda).$$

Another situation is $e = \lambda$, and $b = 0$ by $\lambda b + e - \lambda = 0$. At this point, because $\lambda a + \lambda e + d - \lambda^2 \neq 0$, $a \neq \lambda$. Then from (30c),

$$c = -\frac{(a - \lambda)f}{\lambda^2}.$$

From (30d),

$$fg = \lambda^2(a - \lambda).$$

To summarise, we get the following theorem.

Theorem 2.8. *Let* $J = \mathrm{diag}(J_2[\lambda], J_1[\lambda])$ *with nonzero* λ. *Then*

$$Y_1 = \begin{bmatrix} a & 1 & -\frac{af}{\lambda^2} \\ -\lambda^2 & 0 & f \\ \frac{\lambda^2(a-2\lambda)}{f} & 0 & 2\lambda - a \end{bmatrix}, \quad 2\lambda \neq a \in \mathbb{C} \quad \text{and} \quad 0 \neq f \in \mathbb{C}$$

and

$$Y_2 = \begin{bmatrix} a & 0 & -\frac{(a-\lambda)f}{\lambda^2} \\ -\lambda^2 & \lambda & f \\ \frac{\lambda^2(a-\lambda)}{f} & 0 & 2\lambda - a \end{bmatrix}, \quad \lambda \neq a \in \mathbb{C} \quad \text{and} \quad 0 \neq f \in \mathbb{C}$$

are solutions of (2).

2.2.3. Subcase 3

If $fg = 0$ and $(\lambda a + \lambda t + d - \lambda^2)g \neq 0$, then $hd \neq 0$ and $\lambda a + \lambda e + d - \lambda^2 = 0$ from (30d) and (30g). Thus, $g \neq 0$, $d \neq 0$, $h \neq 0$ and $\lambda a + \lambda t + d - \lambda^2 \neq 0$. Since $d \neq 0$ and $f = 0$,

$$c = 0$$

from (30f). By using $\lambda a + \lambda e + d - \lambda^2 = 0$, we obtain

$$d = -\lambda^2$$

from (30a) and (30b). Substituting $d = -\lambda^2$ into $\lambda a + \lambda e + d - \lambda^2 = 0$, we arrive at

$$a + e = 2\lambda.$$

Combining $d = -\lambda^2$, $c = 0$ and the above equation, we have

$$b = \frac{(e-\lambda)^2}{\lambda^2}.$$

Note that when $f = 0$ and $c = 0$, (30i) becomes

$$\lambda t^2 - \lambda^2 t = 0,$$

that is,

$$t = 0 \quad \text{or} \quad \lambda.$$

As $t = 0$, from (30g) , $d = -\lambda^2$ and $a + e = 2\lambda$, so

$$h = -\frac{eg}{\lambda^2}.$$

Since $h \neq 0$, $e \neq 0$. But if $t = \lambda$, then

$$h = \frac{(\lambda - e)g}{\lambda^2}.$$

Similarly, $e \neq \lambda$.

Based on the above analysis, we have the following theorem.

Theorem 2.9. *Let* $J = \mathrm{diag}(J_2[\lambda], J_1[\lambda])$ *with nonzero* λ*. Then*

$$Y_1 = \begin{bmatrix} 2\lambda - e & \frac{(e-\lambda)^2}{\lambda^2} & 0 \\ -\lambda^2 & e & 0 \\ g & -\frac{eg}{\lambda^2} & 0 \end{bmatrix}, \quad 0 \neq e \in \mathbb{C},\ 0 \neq g \in \mathbb{C}$$

and

$$Y_2 = \begin{bmatrix} 2\lambda - e & \frac{(e-\lambda)^2}{\lambda^2} & 0 \\ -\lambda^2 & e & 0 \\ g & \frac{(\lambda-e)g}{\lambda^2} & \lambda \end{bmatrix}, \quad \lambda \neq e \in \mathbb{C},\ 0 \neq g \in \mathbb{C}$$

are solutions of (2)*, where* $e \in \mathbb{C}$ *and* $0 \neq g \in \mathbb{C}$*.*

2.2.4. Subcase 4

If $fg = 0$ and $(\lambda a + \lambda t + d - \lambda^2)g = 0$, then $hd = 0$ and $(\lambda a + \lambda e + d - \lambda^2)d = 0$ from (30d) and (30g). Based on these four equations, the solutions for (30a)-(30i) can be discussed from different situations whether d and g are 0.

▲ If $d = 0$ and $g = 0$, (30a)-(30i) degenerate into the following system of equations

$$\begin{cases} \lambda a(a-\lambda) = 0, & \text{(31a)} \\ b(\lambda a + \lambda e - \lambda^2) + ae - \lambda a - \lambda e + \lambda ch = 0, & \text{(31b)} \\ (\lambda a + \lambda t - \lambda^2)c + (a + \lambda b - \lambda)f = 0, & \text{(31c)} \\ \lambda e(e-\lambda) + \lambda fh = 0, & \text{(31d)} \\ (\lambda e + \lambda t - \lambda^2)f = 0, & \text{(31e)} \\ (\lambda e + \lambda t - \lambda^2)h = 0, & \text{(31f)} \\ \lambda hf + \lambda t^2 - \lambda^2 t = 0. & \text{(31g)} \end{cases}$$

Since $\lambda a(a-\lambda) = 0$, it follows that $a = 0$ or λ. Next, we consider the following two cases: $a = 0$ and $a = \lambda$.

■ $a = 0$. Noting (31e) and (31f), we first assume $\lambda e + \lambda t - \lambda^2 = 0$ and combine it with (31d) or (31g) to obtain

$$hf = e(\lambda - e). \tag{32}$$

From (31c) and $\lambda e + \lambda t - \lambda^2 = 0$,

$$ce = (b-1)f.$$

If $e \neq 0$, then the above equation can be written as $c = \frac{(b-1)f}{e}$. Substituting it into (31b) gives

$$b(e-\lambda) - e + \frac{(b-1)hf}{e} = 0.$$

Next, substituting (32) into the above equation yields

$$\lambda = 0,$$

which contradicts the assumption $\lambda \neq 0$. So $e = 0$, and then

$$b = \frac{ch}{\lambda}, \quad hf = 0 \quad \text{and} \quad t = \lambda.$$

A combination of $t = \lambda$, $b = \frac{ch}{\lambda}$ and (31c) converts to

$$f = 0.$$

On the other hand, $\lambda e + \lambda t - \lambda^2 \neq 0$ means $h = f = 0$ from (31e) and (31f). At this point, by combining (31b) and (31d), it is easy to know that e can only be zero and b is also zero. From (31c) and (31g), we see that $c = 0$ when $t = 0$ and c is arbitrary when $t = \lambda$.

- $a = \lambda$. Playing the same trick by assuming $\lambda e + \lambda t - \lambda^2 = 0$, we also have

$$hf = e(\lambda - e).$$

If $e \neq 0$, then from (31b),

$$b = \frac{\lambda - ch}{e}.$$

Substituting the above equation into (31c), we obtain

$$f = 0.$$

So,

$$e = \lambda \quad \text{and} \quad t = 0,$$

and then

$$b = \frac{\lambda - ch}{\lambda}.$$

But if $e = 0$, then

$$hf = 0$$

and we have from (31b) that

$$ch = \lambda.$$

Thus,

$$f = 0.$$

Next, we substitute $a = \lambda$ and $f = 0$ into (31c), obtaining

$$c = 0,$$

which contracts $ch = \lambda$.

Then we consider $\lambda e + \lambda t - \lambda^2 \neq 0$. Similarly, that $h = f = 0$ is obtained from (31e) and (31f). Then, combining (31b) and (31d) leads to

$$e = \lambda \quad \text{and} \quad b = 1.$$

By using $f = 0$ and (31g), we arrive at

$$t = 0 \quad \text{or} \quad \lambda.$$

If $t = 0$, then c is arbitrary, and if $t = \lambda$, then $c = 0$.

To sum up, we obtain the following theorem.

Theorem 2.10. *Let* $J = \mathrm{diag}(J_2[\lambda], J_1[\lambda])$ *with nonzero* λ. *Then*

$$Y_1 = \begin{bmatrix} 0 & \frac{ch}{\lambda} & c \\ 0 & 0 & 0 \\ 0 & h & \lambda \end{bmatrix}, c, h \in \mathbb{C}, \quad Y_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$Y_3 = \begin{bmatrix} \lambda & \frac{\lambda - ch}{\lambda} & c \\ 0 & \lambda & 0 \\ 0 & h & 0 \end{bmatrix}, c, h \in \mathbb{C}, \quad Y_4 = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

are solutions of (2).

▲ If $d \neq 0$ and $g = 0$, then $h = 0$ and $\lambda a + \lambda e + d - \lambda^2 = 0$. At this time, (30a)-(30i) can be written as

$$\begin{cases} -\lambda ae + \lambda bd - \lambda d = 0, & \text{(33a)} \\ -bd + ae - \lambda^2 = 0, & \text{(33b)} \\ (\lambda a + \lambda t - \lambda^2)c + (a + \lambda b - \lambda)f = 0, & \text{(33c)} \\ d(\lambda b + e - \lambda) + \lambda e(e - \lambda) = 0, & \text{(33d)} \\ \lambda cd + (\lambda t - \lambda a)f = 0, & \text{(33e)} \\ \lambda t^2 - \lambda^2 t = 0. & \text{(33f)} \end{cases}$$

A combination of (33a) and (33b) yields

$$d = -\lambda^2,$$

and then

$$a + e = 2\lambda$$

from $\lambda a + \lambda e + d - \lambda^2 = 0$. Substituting the above two equations into (33a) or (33b), we have

$$b = \frac{(e - \lambda)^2}{\lambda^2}. \tag{34}$$

According (33f), we know that

$$t = 0 \quad \text{or} \quad \lambda.$$

■ $t = 0$. Substituting $a + e = 2\lambda$ and (34) into (33c) and (33e), we get

$$[\lambda^2 c + (2\lambda - e)f](\lambda - e) = 0 \tag{35}$$

and

$$\lambda^2 c = (e - 2\lambda)f. \tag{36}$$

In fact, solutions of (36) satisfy (35), so (36) is redundant.

■ $t = \lambda$. In the same way, from (33c) and (33e),

$$[\lambda^2 c + (\lambda - e)f](2\lambda - e) = 0 \tag{37}$$

and

$$\lambda^2 c = (e - \lambda)f. \tag{38}$$

Also, solutions of (38) solve (37).

In conclusion, we have the following theorem.

Theorem 2.11. *Let* $J = \mathrm{diag}(J_2[\lambda], J_1[\lambda])$ *with nonzero* λ. *Then*

$$Y_1 = \begin{bmatrix} 2\lambda - e & \frac{(e-\lambda)^2}{\lambda^2} & \frac{(e-2\lambda)f}{\lambda^2} \\ -\lambda^2 & e & f \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$Y_2 = \begin{bmatrix} 2\lambda - e & \frac{(e-\lambda)^2}{\lambda^2} & \frac{(e-\lambda)f}{\lambda^2} \\ -\lambda^2 & e & f \\ 0 & 0 & \lambda \end{bmatrix}$$

are solutions of (2), *where* $e \in \mathbb{C}$ *and* $f \in \mathbb{C}$.

▲ If $d = 0$ and $g \neq 0$, then $f = 0$ and $\lambda a + \lambda t - \lambda^2 = 0$. Now, (30a)-(30i) become

$$\begin{cases} a(\lambda a - \lambda^2) + \lambda cg = 0, & \text{(39a)} \\ b(\lambda a + \lambda e - \lambda^2) + ae + \lambda ch - \lambda a - \lambda e = 0, & \text{(39b)} \\ \lambda e(e - \lambda) = 0, & \text{(39c)} \\ (\lambda b + e - \lambda)g + (\lambda e - \lambda a)h = 0, & \text{(39d)} \\ \lambda cg + \lambda t^2 - \lambda^2 t = 0. & \text{(39e)} \end{cases}$$

From (39c),

$$e = 0 \quad \text{or} \quad \lambda.$$

■ $e = 0$. By (39b) and (43e), we obtain

$$\lambda ab - \lambda^2 b + \lambda ch - \lambda a = 0 \tag{40}$$

and

$$(\lambda b - \lambda)g = \lambda ah. \tag{41}$$

Since $g \neq 0$, (41) can be written as

$$\lambda b = \lambda + \frac{\lambda ah}{g}. \tag{42}$$

Next, substituting the above equation into (40) yields

$$b = \frac{ah}{g},$$

which contradicts (42). Hence (2) has no solution.

■ $e = \lambda$. This situation is similar to the above one with $e = 0$, and consequently (2) also has no solution.

▲ If $d \neq 0$ and $g \neq 0$, then $h = f = \lambda a + \lambda t + d - \lambda^2 = \lambda a + \lambda e + d - \lambda^2 = 0$. At this point, (30a)-(30i) degenerate into the following system of equations

$$\begin{cases} -\lambda ae + \lambda bd - \lambda d + \lambda cg = 0, & \text{(43a)} \\ -bd + \lambda ch + ae - \lambda^2 = 0, & \text{(43b)} \\ cd = 0, & \text{(43c)} \\ (\lambda b + e - \lambda)d + \lambda e(e - \lambda) = 0, & \text{(43d)} \\ (\lambda b + e - \lambda)g = 0, & \text{(43e)} \\ \lambda cg + \lambda t^2 - \lambda^2 t = 0. & \text{(43f)} \end{cases}$$

Because $d \neq 0$, we see that $c = 0$ from (43c). Combining (43a) and (43b), we have

$$d = -\lambda^2,$$

and then

$$a + e = 2\lambda \quad \text{and} \quad a + t = 2\lambda.$$

Substituting the above equations into (43a) or (43b), we get

$$b = \frac{(\lambda - e)^2}{\lambda^2}. \tag{44}$$

Because $g \neq 0$, according to (43e),

$$b = \frac{\lambda - e}{\lambda}. \tag{45}$$

A combination of (44) and (45) generates

$$e(\lambda - e) = 0, \tag{46}$$

which is consistent with (43d). So $e = 0$ or λ.

- $e = 0$. From $a + e = 2\lambda$ and $a + t = 2\lambda$,

  $$a = 2\lambda \quad \text{and} \quad t = 0.$$

  Then we have from (44) or (45) that

  $$b = 1.$$

- $e = \lambda$. In the same way, we have

  $$a = \lambda \quad \text{and} \quad t = \lambda.$$

  By (44) or (45), we arrive at

  $$b = 0.$$

In summary, we obtain the following theorem.

Theorem 2.12. *Let* $J = \mathrm{diag}(J_2[\lambda], J_1[\lambda])$ *with nonzero* λ. *Then*

$$Y_1 = \begin{bmatrix} 2\lambda & 1 & 0 \\ -\lambda^2 & 0 & 0 \\ g & 0 & 0 \end{bmatrix}$$

and

$$Y_2 = \begin{bmatrix} \lambda & 0 & 0 \\ -\lambda^2 & \lambda & 0 \\ g & 0 & \lambda \end{bmatrix}$$

are solutions of (2), *where* $0 \neq g \in \mathbb{C}$.

3. Conclusions

We have successfully solved a Yang-Baxter-like matrix equation completely for a given 3×3 matrix with its Jordan canonical form $J = \mathrm{diag}(J_2[\lambda], J_1[\mu])$ with both λ and μ nonzero, by solving a system of nine quadratic equations directly with some strategies. For the two cases of $\lambda \neq \mu$ and $\lambda = \mu$, our detailed analysis lead to all the solutions of the system. Together with the previous results in the literature, our solution indicates that the task of finding all or some solutions of a 3×3 Yang-Baxter-like matrix equation has been finished.

Finding all solutions of an arbitrary order Yang-Baxter-like matrix equation is still an extremely difficult task. Our future research will be toward solving other types of the matrix equation by developing new approaches.

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