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A q-spectral Polak-Ribiére-Polyak conjugate gradient method for unconstrained optimization problems with motion control application

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Abstract. Frank Hilton Jackson extended the classical concept of derivative and introduced the q-derivative, popularly known as Jackson's derivative. To solve large-scale unconstrained optimization problems, we propose a q-spectral Polak-Ribiére-Polyak (PRP) conjugate gradient method. The method can be viewed as a generalization of the spectral PRP method, replacing the classical gradient with the q-gradient vector that utilises the first-order partial q-derivatives derived from Jackson's derivative. Additionally, as the value of q approaches one, the proposed approach simplifies to the classical form. Numerical experiments are conducted and compared with existing methods to demonstrate the advantages of the proposed approach. Moreover, as an application, we solve the motion control problem.

1. Introduction

Consider the following nonlinear unconstrained optimization problem

$$\min\{\varphi(x) \mid x \in \mathbb{R}^n\}, \tag{1}$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function with gradient $g(x)$. The conjugate gradient method is one of the most efficient solvers for large-scale problems of the form (1). Its iterative sequence,

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denoted by $\{x^\ell\}$, is generated as follows:

$$x^{\ell+1} = x^\ell + \alpha^\ell d^\ell,\ \ell = 0, 1, 2 \cdots, \tag{2}$$

where α^ℓ is a positive step length computed by a particular line search, and d^ℓ is a search direction computed by

$$d^\ell = \begin{cases} -g^\ell, & \text{if } \ell = 0, \\ -g^\ell + \beta^\ell d^{\ell-1} & \text{if } \ell \geq 1, \end{cases} \tag{3}$$

where β^ℓ the conjugate gradient parameter, and $g^\ell = \nabla\varphi(x^\ell)$. The Polak–Ribiére–Polyak (PRP) method is a popular conjugate gradient method, whose conjugate gradient parameter is given by

$$\beta^\ell := (\beta^\ell)^{\text{PRP}} := \frac{(g^\ell)^T y^{\ell-1}}{\|g^{\ell-1}\|^2}, \tag{4}$$

where $\|\cdot\|$ denotes the Euclidean norm, and $y^{\ell-1} = g^\ell - g^{\ell-1}$. It is widely regarded as one of the most effective conjugate gradient methods due to its favorable practical performance on nonconvex problems and its ability to generate search directions that adapt to the local geometry of the objective function. Unlike other variants such as Fletcher–Reeves, the PRP method often exhibits faster convergence, particularly when combined with appropriate line search techniques. However, a known drawback of the classical PRP method is its potential lack of sufficient descent. This behavior can lead to stagnation or divergence. To address this, various modified versions of the PRP method have been proposed (see [33] and references therein). To ensure the global convergence of the PRP method and most conjugate gradient methods for general objective functions, the sufficient descent condition

$$(g^\ell)^T d^\ell \leq c\|g^\ell\|^2, \quad c > 0, \tag{5}$$

is typically enforced for all values of ℓ. In recent years, researchers have proposed generalizations of the PRP method that guarantee the condition (5). For instance, Zhang [33] proposed a modified PRP (MPRP) method that always produces a descent direction for the objective function, independent of the line search used. Additionally, Wan et al. [31] introduced a variant called the spectral PRP method, which ensures that the search direction is a descent direction at each iteration. More relevant contributions can be found in [11, 26–28, 32] and the references therein.

Quantum calculus (briefly, q-calculus), also known as calculus without limits, has historical roots tracing back to Euler and Jacobi in the eighteenth century, and to F.H. Jackson in the early twentieth century. Jackson made several significant contributions to the development of basic analogues, or q-analogues, particularly in the context of basic hypergeometric functions [7]. He extended the classical notions of differentiation and integration by introducing the q-derivative and q-integration [12, 13]. Since then, numerous researchers especially mathematicians and physicists have shown a deep interest in exploring and verifying the principles of quantum calculus (see [9, 15–18]). This growing interest is due in part to its wide-ranging applications in various fields, including sampling theory [1], integral inequalities [10], transform calculus [1], signal processing [2], variational calculus [5], operator theory [3], and fractional integrals and derivatives [29]. Compared to classical calculus, q-calculus offers a generalized framework that introduces a deformation parameter $q \in (0, 1)$, allowing for the construction of discrete and flexible analogues of standard differential operators. One of the key advantages of q-calculus is its ability to encode nonlocal information via the q-derivative, which has been shown to improve the exploration behavior of iterative methods in optimization. This flexibility can help avoid stagnation in flat regions or poor conditioning and offers additional degrees of freedom that can be tuned adaptively. Moreover, many classical algorithms are recovered as special cases when $q \to 1$, making q-calculus-based methods generalizations of traditional approaches.

Recently, q-calculus has been studied in various fields. In optimization, this concept was explored by Soterroni et al. [30], where the q-gradient was applied to the gradient descent method to find optimal solutions to objective functions. Similarly, in [6], q-calculus was employed in Newton's method to solve unconstrained single-objective optimization problems. This idea was further extended in the q-Newton method [23] and the q-Steepest Descent method [19] to address unconstrained multi-objective optimization problems (UMOPs). Moreover, q-analogues of several well-known methods such as BFGS [24], limited-memory BFGS [20], Fletcher–Reeves [21], Dai–Yuan [25], and steepest descent with quasi-Fejér properties [22] have also been developed to solve unconstrained optimization problems.

Quite recently, to effectively solve the unconstrained optimization problem (1), Mishra et al. [22] introduced a q-version of the descent Polak–Ribière–Polyak conjugate gradient method [33], called the q-PRP method. This method can be viewed as a generalization of the modified PRP (MPRP) method, where the classical gradient is replaced by the q-gradient vector. The q-gradient is constructed using first-order partial q-derivatives derived from Jackson's derivative, also known as the q-derivative operator. Notably, the q-PRP method reduces to the classical MPRP method as the parameter q approaches 1. Numerical results demonstrate that the q-PRP method achieves better convergence than the MPRP method on a specific class of problems with varying initial points. Building upon this advancement, further research has explored the integration of q-calculus into other conjugate gradient frameworks and quasi-Newton techniques. These efforts aim to harness the flexibility of q-derivatives in controlling descent properties and improving numerical robustness, particularly in challenging or ill-conditioned optimization landscapes. As a result, q-based methods are emerging as promising alternatives in the broader field of derivative-based optimization.

In this work, we focus on the PRP framework because of its demonstrated efficiency in large-scale and nonconvex optimization problems, as well as its rich theoretical foundation and adaptability to modifications. The spectral PRP method [31], in particular, provides a promising direction by incorporating spectral scaling to improve the quality of search directions. These features make the PRP family an ideal candidate for extension using q-calculus, with the goal of enhancing convergence properties through a more flexible derivative formulation. To this end, we investigate whether the spectral PRP can be generalized using the q-derivative operator. The aim of this work is to provide a positive answer to this question. The main contribution of the article is the introduction of a q-spectral PRP method, referred to as q-SPRP, in which the search direction at each iteration is a q-descent direction of the objective function. Under mild conditions imposed on the q-gradient of the objective function, we establish the global convergence of the proposed method.

The remaining sections of this paper are organized as follows. In Section 2, we present the necessary definitions, preliminary results, and the proposed algorithm. The global convergence of the proposed method is analyzed in Section 3. Section 4 provides numerical simulations to illustrate the performance of the method. Finally, conclusions are drawn in Section 5.

2. Preliminaries and Algorithm

In this section, we recall some basic definitions from q-calculus that are relevant to our study.

Definition 2.1. *[4] Given $q > 0$, for $n \in \mathbb{N}$, the q-integer $[n]_q$ is defined by*

$$[n]_q := \begin{cases} \frac{1-q^n}{1-q}, & q \neq 1, \\ n, & q = 1. \end{cases}$$

Definition 2.2. *[13] Let $\varphi : \mathbb{R} \to \mathbb{R}$, the q-derivative $D_q\varphi$ of φ is given by*

$$(D_q\varphi)(x) := \frac{\varphi(x) - \varphi(qx)}{(1-q)x}, \text{ if } x \neq 0,$$

and $(D_q\varphi)(0) = \varphi'(0)$ *provided* $\varphi'(0)$ *exists. Note that if* φ *is differentiable then,*

$$\lim_{q\to 1}(D_q\varphi)(x) = \frac{\varphi(x) - \varphi(qx)}{(1-q)x} = \frac{d\varphi(x)}{dx}.$$

Definition 2.3. *The q-derivative of a function of the form* x^n *is given by*

$$D_{q,x}(x^n) := \begin{cases} \frac{1-q^n}{1-q}x^{n-1}, & q \neq 1, \\ nx^{n-1}, & q = 1. \end{cases}$$

Definition 2.4. *Let* $\varphi(x)$ *be a continuous function on* $[w, y]$*. Then there exists* $\hat{q} \in (0, 1)$ *and* $x \in (w, y)$ *such that*

$$\varphi(y) - \varphi(w) = (D_q\varphi)(x)(y - w),$$

for $q \in (\hat{q}, 1) \cup (1, \hat{q}^{-1})$*.*

Definition 2.5. *[30] Let* $\varphi : \mathbb{R}^n \to \mathbb{R}$*, the q-partial derivative of* φ *at* $x \in \mathbb{R}^n$ *respect to* x_i*, where scalar* $q \in (0, 1)$ *is given as*

$$(D_{q,x_i}f)(x) := \begin{cases} \frac{1}{(1-q)x_i}[\varphi(x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n) - \varphi(x_1, x_2, \dots, x_{i-1}, qx_i, x_{i+1}, \dots, x_n)], & x_i \neq 0, \\ \frac{\partial}{\partial x_i}\varphi(x_1, x_2, \dots x_{i-1}, 0, x_{i+1}, \dots, x_n), & x_i = 0, \\ \frac{\partial}{\partial x_i}\varphi(x_1, x_2, \dots x_{i-1}, x_i, x_{i+1}, \dots, x_n), & q = 1. \end{cases}$$

Definition 2.6. *[30] Let the parameter q be a vector, i.e.,* $q = (q_1, \dots, q_i, \dots, q_n)^T \in \mathbb{R}^n$*. The q-gradient vector of* φ *is*

$$\nabla_q\varphi(x)^T = \left[(D_{q_1,x_1}\varphi)(x), \cdots (D_{q_i,x_i}\varphi)(x) \cdots (D_{q_n,x_n}\varphi)(x)\right].$$

Remark 2.7. *[22] Let* $\{q_i^\ell\}$ *be a real sequence defined by*

$$q_i^{\ell+1} = 1 - \frac{q_i^\ell}{(\ell+1)^2},$$

for each $i = 1, \cdots, n$*, where* $\ell = 0, 1, 2, \cdots$ *and a starting number* $0 < q_i^0 < 1$*. The sequence* $\{q_i^\ell\}$ *naturally converges to* $(1, \cdots, 1)^T$ *as* $\ell \to \infty$ *[22]. Thus, the q-gradient starts to work as a classical derivative.*

Example 2.8. *Let* $\varphi : \mathbb{R}^2 \to \mathbb{R}$ *be a function defined by* $\varphi(x) = 3x_1^2 + x_1x_2^2$*. Thus, the q-gradient is:*

$$\nabla_{q^l}\varphi(x)^T = g_{q^\ell}(x^\ell) = \left[3(1 + q_1^l)x_1 + x_2^2 \quad x_1(1 + q_2^l)x_2\right].$$

In the sequel, we use the notations $g_{q^\ell}(x^\ell)$ to denote the q-gradient vector of φ at x^ℓ ($\nabla_{q^\ell}\varphi(x^\ell)$).

2.1. A q-Spectral Polak-Ribiére-Polyak Conjugate Gradient Algorithm

In this subsection, we construct our method step by step. Consider the unconstrained optimization problem defined by (1). The conjugate gradient method of Wan et al. [31] solves (1) by generating a sequence of iterate $\{x^\ell\}$ using the recursive formula (2) where d^ℓ is generated by:

$$d^\ell := \begin{cases} -g^\ell, & \ell = 0, \\ -\theta^\ell g^\ell + \beta^\ell d^{\ell-1} & \ell \geq 1, \end{cases} \tag{6}$$

where θ^ℓ and β^ℓ are specified as:

$$\theta^\ell := \frac{(d^{\ell-1})^T(y^{\ell-1})}{\|g^{\ell-1}\|^2} - \frac{(d^{\ell-1})^T(g^\ell) \cdot (g^\ell)^T(g^{\ell-1})}{\|g^\ell\|^2\|g^{\ell-1}\|^2}, \tag{7}$$

$$\beta^{\ell} := \frac{(g^{\ell})^T(y^{\ell-1})}{\|g^{\ell-1}\|^2}, \tag{8}$$

where $y^{\ell-1} = g^{\ell} - g^{\ell-1}$.

As shown by Wan et al. [31], using (6), (7) and (8) the following equality holds

$$(g^{\ell})^T(d^{\ell}) = -\|g^{\ell}\|^2, \ \forall \ell \geq 0.$$

Based on the formulae (6)-(8), we now introduce the q-spectral PRP method for solving (1). The iterative schema of the conjugate gradient method based on the q-gradient is given by

$$x^{\ell+1} = x^{\ell} + \alpha^{\ell} d^{\ell}_{q^{\ell}}. \tag{9}$$

Inspired by (6), the search direction $d^{\ell}_{q^{\ell}}$ is defined by

$$d^{\ell}_{q^{\ell}} := \begin{cases} -g^{\ell}_{q^{\ell}} & \ell = 0, \\ -\theta^{\ell}_{q^{\ell}} g^{\ell}_{q^{\ell}} + \beta^{\ell}_{q^{\ell}} d^{\ell-1}_{q^{\ell}} & \ell \geq 1, \end{cases} \tag{10}$$

$$\theta^{\ell}_{q^{\ell}} := \frac{(d^{\ell-1}_{q^{\ell-1}})^T(y^{\ell-1}_{q^{\ell}})}{\|g^{\ell-1}_{q^{\ell}}\|^2} - \frac{(d^{\ell-1}_{q^{\ell}})^T(g^{\ell}_{q^{\ell}}) \cdot (g^{\ell}_{q^{\ell}})^T(g^{\ell-1}_{q^{\ell}})}{\|g^{\ell}_{q^{\ell}}\|^2 \|g^{\ell-1}_{q^{\ell}}\|^2}, \tag{11}$$

$$\beta^{\ell}_{q^{\ell}} := \frac{(g^{\ell}_{q^{\ell}})^T(y^{\ell-1}_{q^{\ell}})}{\|g^{\ell-1}_{q^{\ell}}\|^2}. \tag{12}$$

Based on all above, we present the algorithm of the q-spectral PRP conjugate gradient method as follows:

Algorithm 1 q-Spectral PRP Conjugate Gradient Algorithm

1: **Step 0:** Given an initial point $x^0 \in \mathbb{R}^n$, constants $0 < \rho < \sigma < 1$, and $\epsilon > 0$. Set $\ell := 0$.
2: **Step 1:** If $\|g^{\ell}_{q^{\ell}}\| \leq \epsilon$, terminate the algorithm. Otherwise, compute $d^{\ell}_{q^{\ell}}$ using (10)–(12), and go to Step 2.
3: **Step 2:** Find a step length $\alpha^{\ell} > 0$ such that

$$\begin{cases} \varphi(x^{\ell}) - \varphi(x^{\ell} + \alpha^{\ell} d^{\ell}_{q^{\ell}}) \geq \rho(\alpha^{\ell})^2 \|d^{\ell}_{q^{\ell}}\|^2, \\ g_{q^{\ell}}(x^{\ell} + \alpha^{\ell} d^{\ell}_{q^{\ell}})^T d^{\ell}_{q^{\ell}} \geq -2\sigma\alpha^{\ell} \|d^{\ell}_{q^{\ell}}\|^2. \end{cases} \tag{13}$$

4: **Step 3:** Set the next iterate $x^{\ell+1} := x^{\ell} + \alpha^{\ell} d^{\ell}_{q^{\ell}}$.
5: **Step 4:** Update each $q^{\ell+1}_i := 1 - \frac{q^{\ell}_i}{(\ell+1)^2}$ for all $i = 1, \ldots, n$.
6: **Step 5:** Set $\ell := \ell + 1$ and return to **Step 1**.

3. Convergence Analysis

In this section, we provide the global convergence of Algorithm 1. We begin with the following assumptions that play a vital role in establishing the convergence proof of the proposed method.

Assumption 1.

A1. The level set $\mathbb{L} = \{x \,|\varphi(x) \leq \varphi(x^0)\}$ *at* x^0 *is bounded, namely, there exists a constant* $\hat{a} > 0$ *such that*

$$\|x\| \leq \hat{a}, \ \forall \hat{a} \in \mathbb{L}.$$

A2. In some neighborhood N of Ω, φ is continuously q-differentiable and its q-gradient is Lipschitz continuous, namely, there exists a constant $L > 0$ such that

$$\|g_q(x) - g_q(y)\| \leq L\|x - y\|, \ \forall x, y \in N. \tag{14}$$

A3. For ℓ large enough, the following inequalities hold

$$0 < (g^{\ell}_{q^{\ell}})^T (g^{\ell-1}_{q^{\ell-1}}) \leq 2(g^{\ell}_{q^{\ell}})^T (g^{\ell}_{q^{\ell}}). \tag{15}$$

Remark 3.1. *Assumptions (A1) and (A2) imply that there exists positive constants γ such that*

$$\|g_q(x)\| \leq \gamma, \ \forall x \in N. \tag{16}$$

Lemma 3.2. *If the direction $d^{\ell}_{q^{\ell}}$ is yielded by* (10)-(12), *then the following equation holds for any ℓ*

$$(g^{\ell}_{q^{\ell}})^T (d^{\ell}_{q^{\ell}}) = -\|g^{\ell}_{q^{\ell}}\|. \tag{17}$$

Proof. First, for $\ell = 0$, using (10)-(12), it is easy to see that (17) is true. For $\ell > 0$, we assume that $(d^{\ell-1}_{q^{\ell-1}})^T (g^{\ell-1}_{q^{\ell-1}}) = -\|g^{\ell-1}_{q^{\ell-1}}\|^2$ holds for $\ell - 1$. Thus, from (10)-(12), it follows that

$$\begin{aligned}
(g^{\ell}_{q^{\ell}})^T (d^{\ell}_{q^{\ell}}) &= -\theta^{\ell}_{q^{\ell}} \|g^{\ell}_{q^{\ell}}\|^2 + \frac{(g^{\ell}_{q^{\ell}})^T (y^{\ell-1}_{q^{\ell-1}})}{\|g^{\ell-1}_{q^{\ell-1}}\|^2} (d^{\ell-1}_{q^{\ell-1}})^T (g^{\ell}_{q^{\ell}}) \\
&= -\left(\frac{(d^{\ell-1}_{q^{\ell-1}})^T (y^{\ell-1}_{q^{\ell-1}})}{\|g^{\ell-1}_{q^{\ell-1}}\|^2} - \frac{(d^{\ell-1}_{q^{\ell-1}})^T (g^{\ell}_{q^{\ell}}) \cdot (g^{\ell}_{q^{\ell}})^T (g^{\ell-1}_{q^{\ell-1}})}{\|g^{\ell}_{q^{\ell}}\|^2 \|g^{\ell-1}_{q^{\ell-1}}\|^2} \right) \|g^{\ell}_{q^{\ell}}\|^2 + \frac{(g^{\ell}_{q^{\ell}})^T (y^{\ell-1}_{q^{\ell-1}})}{\|g^{\ell-1}_{q^{\ell-1}}\|^2} (d^{\ell-1}_{q^{\ell-1}})^T (g^{\ell}_{q^{\ell}}) \\
&= -\frac{(d^{\ell-1}_{q^{\ell-1}})^T (y^{\ell-1}_{q^{\ell-1}})}{\|g^{\ell-1}_{q^{\ell-1}}\|^2} \|g^{\ell}_{q^{\ell}}\|^2 + \frac{(d^{\ell-1}_{q^{\ell-1}})^T (g^{\ell}_{q^{\ell}}) (g^{\ell}_{q^{\ell}})^T (g^{\ell-1}_{q^{\ell-1}})}{\|g^{\ell}_{q^{\ell}}\|^2 \|g^{\ell-1}_{q^{\ell-1}}\|^2} \|g^{\ell}_{q^{\ell}}\|^2 + \frac{(g^{\ell}_{q^{\ell}})^T (y^{\ell-1}_{q^{\ell-1}})}{\|g^{\ell-1}_{q^{\ell-1}}\|^2} (d^{\ell-1}_{q^{\ell-1}})^T (g^{\ell}_{q^{\ell}}) \\
&= \frac{\|g^{\ell}_{q^{\ell}}\|^2}{\|g^{\ell-1}_{q^{\ell-1}}\|^2} (-\|g^{\ell-1}_{q^{\ell-1}}\|^2) \\
&= -\|g^{\ell}_{q^{\ell}}\|^2.
\end{aligned} \tag{18}$$

Therefore, the conclusion of this lemma holds with $\ell - 1$ replaced by ℓ. □

Remark 3.3. *From Lemma 3.2, it is known that $d^{\ell}_{q^{\ell}}$ is a descent direction of φ at x^{ℓ}. Furthermore, if the exact line search is used, then $(g^{\ell}_{q^{\ell}})^T (d^{\ell-1}_{q^{\ell-1}}) = 0$, hence*

$$\theta^{\ell}_{q^{\ell}} := \frac{(d^{\ell-1}_{q^{\ell-1}})^T (y^{\ell-1}_{q^{\ell-1}})}{\|g^{\ell-1}_{q^{\ell-1}}\|^2} - \frac{(d^{\ell-1}_{q^{\ell-1}})^T (g^{\ell}_{q^{\ell}}) \cdot (g^{\ell}_{q^{\ell}})^T (g^{\ell-1}_{q^{\ell-1}})}{\|g^{\ell}_{q^{\ell}}\|^2 \|g^{\ell-1}_{q^{\ell-1}}\|^2} = \frac{(d^{\ell-1}_{q^{\ell-1}})^T (g^{\ell-1}_{q^{\ell-1}})}{\|g^{\ell-1}_{q^{\ell-1}}\|^2} = 1.$$

The next lemma shows the existence of the stepsize α^{ℓ} at each iteration.

Lemma 3.4. *Suppose that assumption A1 holds, then there exists $\alpha^{\ell} > 0$ that satisfies the two inequalities* (13).

Proof. Let $\psi(\alpha) = \varphi(x^{\ell} + \alpha d^{\ell}_{q^{\ell}}) + \rho\alpha^2 \|d^{\ell}_{q^{\ell}}\|^2$, for any $\alpha > 0$, we have

$$\lim_{\alpha \to 0^+} \frac{\psi(\alpha) - \psi(0)}{\alpha} = (g^{\ell}_{q^{\ell}})^T (d^{\ell}_{q^{\ell}}) < 0.$$

So there exists an $(\alpha^\ell)' > 0$ such that when $\alpha \in (0, (\alpha^\ell)']$, we have

$$\frac{\psi(\alpha) - \psi(0)}{\alpha} \le 0. \tag{19}$$

From assumption A1, it follows that

$$\lim_{\alpha \to +\infty} \frac{\psi(\alpha) - \psi(0)}{\alpha} = +\infty.$$

Let $\hat{\alpha}^\ell = \inf\left\{\alpha > 0 | \frac{\psi(\alpha)-\psi(0)}{\alpha} = 0\right\}$. By intermediate value theorem and (19), we know that $\hat{\alpha}^\ell > 0$ satisfies

$$\frac{\psi(\hat{\alpha}^\ell) - \psi(0)}{\hat{\alpha}^\ell} = 0. \tag{20}$$

Moreover, for each $\alpha \in (0, \hat{\alpha}^\ell]$, we have

$$\frac{\psi(\alpha) - \psi(0)}{\alpha} \le 0.$$

By mean value theorem and (20), we obtain

$$\psi'(\vartheta^\ell \hat{\alpha}^\ell) = 0, \ \ (0 < \vartheta^\ell < 1). \tag{21}$$

Therefore,

$$g_{q^\ell}(x^\ell + \vartheta^\ell \hat{\alpha}^\ell d^\ell_{q^\ell})^T (d^\ell_{q^\ell)} + 2\rho \vartheta^\ell \hat{\alpha}^\ell \|d^\ell_{q^\ell}\|^2 = 0,$$

i.e.,

$$g_{q^\ell}(x^\ell + \vartheta^\ell \hat{\alpha}^\ell d^\ell_{q^\ell})^T d^\ell_{q^\ell} + 2\rho \vartheta^\ell \hat{\alpha}^\ell \|d^\ell_{q^\ell}\|^2 \ge -2\sigma \vartheta^\ell \hat{\alpha}^\ell \|d^\ell_{q^\ell}\|^2. \tag{22}$$

Equation (21) and (22) imply that $\alpha^\ell = \vartheta^\ell \hat{\alpha}^\ell < \hat{\alpha}^\ell$ is the desired step-length. □

We know that Algorithm 1 is well defined, based on Lemma 3.2 and 3.4. In addition, the following result can be proved using Lemma 3.2, 3.4 and Assumption 1.

Lemma 3.5. *Suppose Assumption 1 hold, we have*

$$\sum_{\ell \ge 0} \frac{\|g^\ell_{q^\ell}\|^4}{\|d^\ell_{q^\ell}\|^2} < \infty. \tag{23}$$

Proof. From the line search rule (13) and Assumption 1, it follows that

$$\begin{aligned}(2\sigma + L)\alpha^\ell \|d^\ell_{q^\ell}\|^2 &= 2\sigma \alpha^\ell \|d^\ell_{q^\ell}\|^2 + L\alpha^\ell \|d^\ell_{q^\ell}\|^2 \\ &\ge -(g^{\ell+1}_{q^{\ell+1}})^T (d^\ell_{q^\ell}) + (y^\ell_{q^\ell})^T (d^\ell_{q^\ell}) \\ &= -(g^\ell_{q^\ell})^T (d^\ell_{q^\ell}).\end{aligned}$$

Hence,

$$\alpha^\ell \|d^\ell_{q^\ell}\|^2 \ge \frac{1}{2\sigma + L} \left(\frac{-(g^\ell_{q^\ell})^T (d^\ell_{q^\ell})}{\|d^\ell_{q^\ell}\|^2} \right).$$

Therefore,

$$(\alpha^\ell)^2\|d^\ell_{q^\ell}\|^2 \geq \left(\frac{1}{2\sigma+L}\right)^2 \frac{-((g^\ell_{q^\ell})^T(d^\ell_{q^\ell}))^2}{\|d^\ell_{q^\ell}\|^2}.$$

From the line search procedure and Assumption A1, we have

$$\begin{aligned}\sum_{\ell=1}^{\infty}\frac{(-(g^\ell_{q^\ell})^T(d^\ell_{q^\ell}))^2}{\|d^\ell_{q^\ell}\|^2} &\leq (2\sigma+L)^2\sum_{\ell=1}^{\infty}(\alpha^\ell)^2\|d^\ell_{q^\ell}\|^2\\ &\leq \frac{(2\sigma+L)^2}{\rho}\sum_{\ell=1}^{\infty}\{\varphi(x^\ell)-\varphi(x^{\ell+1})\}\\ &< +\infty.\end{aligned}$$

It is easy to complete the proof of (23) from Lemma 17. □

The global convergence result is established as follows.

Theorem 3.6. *Under Assumption 1, we have*

$$\liminf_{\ell\to\infty}\|g^\ell_{q^\ell}\| = 0. \tag{24}$$

Proof. Suppose that (24) does not hold, i.e., there exists a $\epsilon > 0$ such that for all ℓ,

$$\|g^\ell_{q^\ell}\| \geq \epsilon, \tag{25}$$

we have from (10) that

$$\begin{aligned}\|d^\ell_{q^\ell}\|^2 &= (d^\ell_{q^\ell})^T(d^\ell_{q^\ell})\\ &= (-\theta^\ell_{q^\ell}(g^\ell_{q^\ell})^T + \beta^\ell_{q^\ell}(d^{\ell-1}_{q^{\ell-1}})^T)(-\theta^\ell_{q^\ell}g^\ell_{q^\ell} + \beta^\ell_{q^\ell}d^{\ell-1}_{q^{\ell-1}})\\ &= (\theta^\ell_{q^\ell})^2\|g^\ell_{q^\ell}\|^2 - 2\theta^\ell_{q^\ell}\beta^\ell_{q^\ell}(d^{\ell-1}_{q^{\ell-1}})^T(g^\ell_{q^\ell}) + (\beta^\ell_{q^\ell})^2\|d^{\ell-1}_{q^{\ell-1}}\|^2\\ &= (\theta^\ell_{q^\ell})^2\|g^\ell_{q^\ell}\|^2 - 2\theta^\ell_{q^\ell}\left((d^\ell_{q^\ell})^T + \theta^\ell_{q^\ell}(g^\ell_{q^\ell})^T\right)g^\ell_{q^\ell} + (\beta^\ell_{q^\ell})^2\|d^{\ell-1}_{q^{\ell-1}}\|^2\\ &= (\theta^\ell_{q^\ell})^2\|g^\ell_{q^\ell}\| - 2\left(\theta^\ell{}_{q^\ell}\right)^2\|g^\ell_{q^\ell}\|^2 + (\beta^\ell_{q^\ell})^2\|d^{\ell-1}_{q^{\ell-1}}\|^2\\ &= (\beta^\ell_{q^\ell})^2\|d^{\ell-1}_{q^{\ell-1}}\|^2 - 2\theta^\ell_{q^\ell}(d^\ell_{q^\ell})^T(g^\ell_{q^\ell}) - \left(\theta^\ell{}_{q^\ell}\right)^2\|g^\ell_{q^\ell}\|^2.\end{aligned}$$

Dividing by $\|g^\ell_{q^\ell}\|^4$ in both sides of the above equality and from (12), (15), (17) and (25) we obtain

$$\begin{aligned}\frac{\|d^\ell_{q^\ell}\|^2}{\|g^\ell_{q^\ell}\|^4} &= \frac{(\beta^\ell_{q^\ell})^2\|d^{\ell-1}_{q^{\ell-1}}\|^2 - 2\theta^\ell_{q^\ell}(d^\ell_{q^\ell})^T(g^\ell_{q^\ell}) - (\theta^\ell_{q^\ell})^2\|g^\ell_{q^\ell}\|^2}{\|g^\ell_{q^\ell}\|^4}\\ &= \frac{\left((g^\ell_{q^\ell})^T(g^\ell_{q^\ell}) - (g^\ell_{q^\ell})^T(g^{\ell-1}_{q^{\ell-1}})\right)^2}{\|g^{\ell-1}_{q^{\ell-1}}\|^4}\frac{\|d^{\ell-1}_{q^{\ell-1}}\|^2}{\|g^\ell_{q^\ell}\|^4} - \frac{(\theta^\ell_{q^\ell}-1)^2}{\|g^\ell_{q^\ell}\|^2} + \frac{1}{\|g^\ell_{q^\ell}\|^2}\\ &= \frac{\|d^{\ell-1}_{q^{\ell-1}}\|^2}{\|g^{\ell-1}_{q^{\ell-1}}\|^4} - \frac{\|d^{\ell-1}_{q^{\ell-1}}\|^2}{\|g^{\ell-1}_{q^{\ell-1}}\|^4}\frac{\left(2(g^\ell_{q^\ell})^T(g^\ell_{q^\ell}) - (g^\ell_{q^\ell})^T(g^{\ell-1}_{q^{\ell-1}})\right)(g^\ell_{q^\ell})^T(g^{\ell-1}_{q^{\ell-1}})}{\|g^\ell_{q^\ell}\|^4} - \frac{(\theta^\ell_{q^\ell}-1)^2}{\|g^\ell_{q^\ell}\|^2} + \frac{1}{\|g^\ell_{q^\ell}\|^2}\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\|d^{\ell-1}_{q^{\ell-1}}\|^2}{\|g^{\ell-1}_{q^{\ell-1}}\|^4} + \frac{1}{\|g^{\ell}_{q^{\ell}}\|^2} \\
&\leq \sum_{i=0}^{\ell-1} \frac{1}{\|g^{i}_{q^{i}}\|^2} \\
&\leq \frac{\kappa}{\epsilon^2}.
\end{aligned}
$$

The above inequality implies that

$$\sum_{\ell\geq 1} \frac{\|g^{\ell}_{q^{\ell}}\|^4}{\|d^{\ell}_{q^{\ell}}\|^2} \geq \epsilon^2 \sum_{\ell\geq 1} \frac{1}{\ell} = +\infty,$$

which contradicts with (23). Thus, the result (24) holds. □

4. Numerical Experiments

4.1. Test on unconstrained optimization problem

We now present numerical experiments for Algorithm 1. A total of 30 test problems from [14] were considered, and 37 experiments were conducted using 37 different starting points. All computations were performed on a laptop with an Intel(R) Core(TM) i3-4005U CPU @ 1.70 GHz and 4 GB RAM, using R version 3.6.1.

The stopping condition used was:

$$\left\|g^{\ell}_{q}(x^{\ell})\right\| \leq 10^{-6},$$

with a maximum iteration limit of 400. To compare the performance of the proposed method across different starting points, we employed the performance profile technique introduced by Dolan and Moré [8]. Figure 1 shows the performance profile of Algorithm 1. It clearly indicates that the proposed q-SPRP method outperforms the existing method across the tested problems.

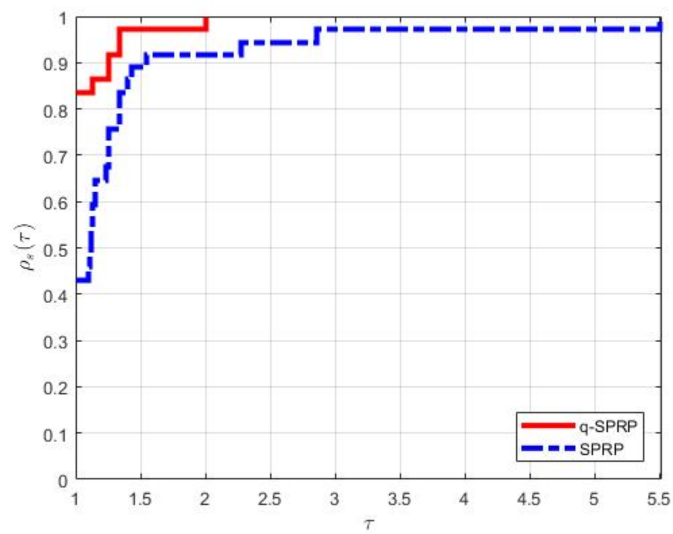


Figure 1: Performance profile based on number of iterations using Table 1 and Table 2.

Table 1: Numerical Results

Serial	*q*-Spectral PRP Algorithm				
Number	Test Problem	Starting Point	x^*	$f(x^*)$	*it*
1	Rosenbrock	(3,4)ˆT	(0.9993397 0.9986580)ˆT	4.84E-07	16
2	Rosenbrock	(4,3)ˆT	(1.065881 1.136169)ˆT	0.004340773	13
3	SPHERE	(1,2,3)ˆT	(4.438117e-14 9.803269e-14 -1.129652e-13)ˆT	2.43E-26	3
4	SPHERE	(3 ,2, 1)ˆT	(-1.293611e-06 -1.037781e-06 -7.819502e-07)ˆT	3.36E-12	3
5	ACKLEY	(0.2,0.2)ˆT	(-4.105524e-07 -4.105524e-07)ˆT	1.64E-06	3
6	ACKLEY 2	(-6,-6)ˆT	(-0.00000000003559351 -0.00000000003559351)ˆT	-200	2
7	ACKLEY 2	(-4, -5)ˆT	(1.633527e-09 , 2.246892e-09)ˆT	-200	2
8	Beale	(3,2)ˆT	(3.0005905 , 0.5001345)ˆT	1.45E-10	11
9	Beale	(4 , 1)ˆT	(3.0095993 , 0.5024505)ˆT	1.47E-05	9
10	Bohachevsky	(0.2 , 0.2)ˆT	(-5.841102e-07 , 3.085369e-08)ˆT	4.92E-12	5
11	Bohachevsky	(0.1 , 0.1)ˆT	(2.276130e-06 -1.151592e-07)ˆT	7.47E-11	5
12	Booth	(40 , 5)ˆT	(0.9999947 , 3.0000062)ˆT	6.888418e-11	4
13	Booth	(60 , 80)ˆT	(0.9994183 , 3.0008450)ˆT	1.33E-06	6
14	DROP-WAVE	(0.1 , 0.2)ˆT	(4.767480e-14 , 2.085834e-14)ˆT	-1	3
15	Colville	(1, 1, 1, 0.8)ˆT	(1.0394380 , 1.0802997 , 0.9614360 , 0.9241323)ˆT	0.00568415	7
16	Colville	(0.8, 1,1,1)ˆT	(0.9768628 , 0.9543990, 1.0241951, 1.0488717)ˆT	0.00212309	11
17	Csendes	(-4 , -5)ˆT	(-0.03013482 , -0.01924908)ˆT	8.146356e-10	7
18	Csendes	(0.4 , 1)ˆT	(-0.01171701 , 0.01905952)ˆT	1.41E-10	5
19	Cube	(-7 , 10)ˆT	(0.9805592 , 0.9422543)ˆT	0.000408162	20
20	Cube	(1 , -6)ˆT	(0.9397829 , 0.8228094)ˆT	0.008808971	13
21	Deckkers-Aarts	(10 , 50)ˆT	(9.597778e-07 , 1.494765e+01)ˆT	-24776.51	8
22	Deckkers-Aarts	(9, 40)ˆT	(-1.078035e-06 1.494759e+01)ˆT	-24776.51	10
23	Dixon Price	(7 , 4)ˆT	(1.001180 , -0.707635)ˆT	1.59E-06	8
24	Easom	(2.5 , 2.1)ˆT	(3.142261 , 3.141810)ˆT	-1	4
25	Egg Crate	(-1.3 , -1.6)ˆT	(-5.302490e-07 , -4.244362e-07)ˆT	1.20E-11	6
26	Exponential	(-3 , -1)ˆT	(4.306445e-08 , 1.072428e-08)ˆT	-1	4
27	Freudenstein Roth	(4 , 5)ˆT	(4.981870 , 4.000912)ˆT	0.001151386	7
28	Six Hump Camel	(7 , 1)ˆT	(-0.08978278 , 0.71276481)ˆT	-1.031628	9
29	Three Hump Camel	(0.6 , 0.7)ˆT	(-2.648394e-07 , -5.719217e-07)ˆT	6.188417e-13	6
30	Sum Squares	(4 , 70)ˆT	(-1.110432e-05 , 1.337182e-05)ˆT	4.81E-10	4
31	GRAMACY & LEE	1	0.9490034	-0.5266035	3
32	Rotated Ellipse 2	(100 , 1.4)	(-1.438029e-07 , -1.555622e-07)ˆT	2.25E-14	4
33	Zakharov	(-6 , -5)	(2.640608e-07 , -3.044188e-07)ˆT	1.92E-13	8
34	Zirilli	(3 , 7)ˆT	(-1.046510e+00 , -2.308637e-05)ˆT	-0.352386	6
35	Zett1	(8 , 4)ˆT	(-2.990219e-02 , -2.078441e-08)ˆT	-0.003791237	9
36	Wayburn Seader 3	(5 , 6)ˆT	(5.147307 , 6.839722)ˆT	19.10588	5
37	Wayburn Seader 2	(1 , 1)ˆT	(0.4604724 , 1.0073360)ˆT	19.20677	7

Table 2: Numerical Results

Serial Number	Spectral PRP Algorithm				
	Test Problem	Starting Point	x^*	$f(x^*)$	it
1	Rosenbrock	(3,4)ˆT	(0.9999998 0.9999995)ˆT	4.84E-07	20
2	Rosenbrock	(4,3)ˆT	(0.9999194 , 0.9998484)ˆT	1.56E-08	16
3	SPHERE	(1,2,3)ˆT	(-4.275638e-07 -3.976857e-07 -3.678076e-07)ˆT	4.76E-13	3
4	SPHERE	(3 ,2, 1)ˆT	(-1.125211e-13 ,9.832413e-14 ,-3.757411e-14)ˆT	2.37E-26	3
5	ACKLEY	(0.2,0.2)ˆT	(3.764868e-11 , 3.764868e-11)ˆT	1.51E-10	4
6	ACKLEY 2	(-6,-6)ˆT	(3.630772e-08 , 3.630772e-08)ˆT	-200	2
7	ACKLEY 2	(-4, -5)ˆT	(-4.263692e-07 , 3.132365e-08)ˆT	-200	11
8	Beale	(3,2)ˆT	(3.0000951 , 0.5000238)ˆT	3.41E-09	12
9	Beale	(4 , 1)ˆT	(3.0000004 , 0.5000001)ˆT	2.77E-14	10
10	Bohachevsky	(0.2 , 0.2)ˆT	(3.324145e-13 , -2.460865e-14)ˆT	0	5
11	Bohachevsky	(0.1 , 0.1)ˆT	(1.990839e-06 , -3.959075e-07)ˆT	6.20E-11	5
12	Booth	(40 , 5)ˆT	(1 , 3)ˆT	2.55E-17	3
13	Booth	(60 , 80)ˆT	(1 , 3)ˆT	8.79E-17	3
14	DROP-WAVE	(0.1 , 0.2)ˆT	(-5.062833e-07 , -5.107584e-07)ˆT	-1	4
15	Colville	(1, 1, 1, 0.8)ˆT	(0.9995939 , 0.9991930 , 1.0004239 , 1.0008541)ˆT	6.49E-07	20
16	Colville	(0.8, 1,1,1)ˆT	(0.9997061 , 0.9994101 , 1.0002997 , 1.0005954)ˆT	3.19E-07	25
17	Csendes	(-4 , -5)ˆT	(-0.03055821 , -0.01731213)ˆT	8.71E-10	8
18	Csendes	(0.4 , 1)ˆT	(0.01957772 , 0.01183933)ˆT	1.60E-10	4
19	Cube	(-7 , 10)ˆT	(0.9999933 , 0.9999799)ˆT	4.51E-11	22
20	Cube	(1 , -6)ˆT	(1 , 1)ˆT	9.29E-17	20
21	Deckkers-Aarts	(10 , 50)ˆT	(4.652559e-08 , 1.494511e+01)ˆT	-24776.52	9
22	Deckkers-Aarts	(9, 40)ˆT	(3.694906e-07 1.494511e+01)ˆT	-24776.52	8
23	Dixon Price	(7 , 4)ˆT	(1.0000020 , -0.7071099)ˆT	9.87E-11	9
24	Easom	(2.5 , 2.1)ˆT	(3.141593 , 3.141593)ˆT	-1	5
25	Egg Crate	(-1.3 , -1.6)ˆT	(-8.137993e-07 , -8.800207e-08)ˆT	1.74204E-11	6.00E+00
26	Exponential	(-3 , -1)ˆT	(1.527478e-10 , -4.751579e-10)ˆT	-1	3
27	Freudenstein Roth	(4 , 5)ˆT	(5.000009 , 4.000000)ˆT	1.18699E-10	8.00E+00
28	Six Hump Camel	(7 , 1)ˆT	(-0.08984676 , 0.71266325)ˆT	-1.031639	8
29	Three Hump Camel	(0.6 , 0.7)ˆT	(-1.088238e-10 1.920156e-10)ˆT	3.97E-20	6.00E+00
30	Sum Squares	(4 , 70)ˆT	(-4.084082e-06 , -4.088583e-06)ˆT	4.83E-10	4
31	GRAMACY & LEE	1	-0.5266048	0.9489337	3.00E+00
32	Rotated Ellipse 2	(100 , 1.4)	(-1.249927e-06 , -1.367064e-06)ˆT	1.72E-12	5
33	Zakharov	(-6 , -5)	(1.359022e-15 , -5.823407e-16)ˆT	2.20E-30	8
34	Zirilli	(3 , 7)ˆT	(-1.046681e+00 , 3.696484e-12)ˆT	-0.3523861	8
35	Zett1	(8 , 4)ˆT	(-2.989599e-02 , -6.894129e-08)ˆT	-0.003791237	10
36	Wayburn Seader 3	(5 , 6)ˆT	(5.146885 , 6.839598)ˆT	19.10588	7
37	Wayburn Seader 2	(1 , 1)ˆT	(0.4248603 , 0.9999999)ˆT	1.68E-14	10

4.2. Two-arm Robotic Motion Control

To demonstrate the applicability of Algorithm 1, we apply Algorithm 1 in solving the motion control of a two-joint planar robotic manipulator (TJ-PRM) [34]. Let $\theta^\ell \in \mathbb{R}^2$ be a joint angle vector and $r^\ell \in \mathbb{R}^2$ be an effector position vector. Given the discrete-time kinematics equation of TJ-PRM at the position level

$$\varphi(\theta^\ell) = r^\ell \tag{26}$$

where φ is a kinematic mapping with well known structure specified as

$$\varphi(\theta) = \begin{bmatrix} l_1c_1 + l_2c_2 \\ l_1s_1 + l_2s_2 \end{bmatrix}, \tag{27}$$

where $c_1 = \cos(\theta_1)$, $s_1 = \sin(\theta_1)$, $c_2 = \cos(\theta_1 + \theta_2)$, $s_2 = \sin(\theta_1 + \theta_2)$, and l_1, l_2 are the length of the rod.

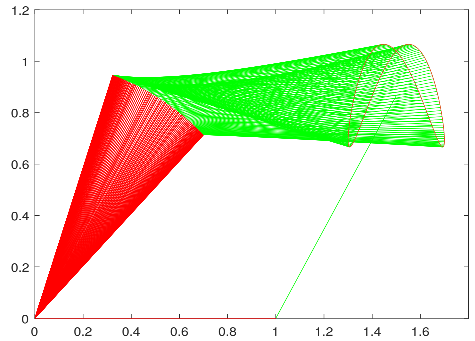
In this study, we consider the special case (when $q = 1$) and apply our method to solve (26). To solve, (26), we solve the following optimization problem defined at each time instant $t^\ell \in [0, t^\varphi]$ (where t^φ is the end of task duration) as follows:

$$\min_{\theta^\ell \in \mathbb{R}^2} \frac{1}{2}\|\varphi(\theta^\ell) - r^\ell\|^2. \tag{28}$$

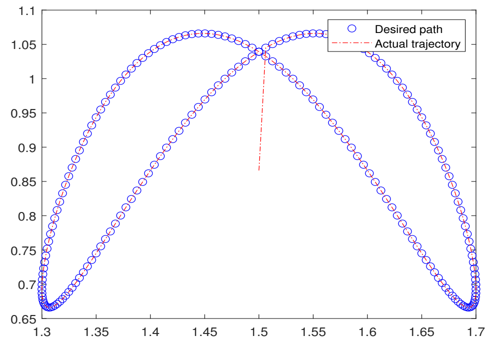
Similar to [34], in this study, the end effector in this experiment is controlled to track a Lissajous curve, which is expressed as

$$r^{\ell} = \begin{bmatrix} 1.5 + 0.2\sin(\pi t_k/5) \\ \sqrt{3}/2 + 0.2\sin(2\pi t_k/5 + \pi/3) \end{bmatrix}.$$

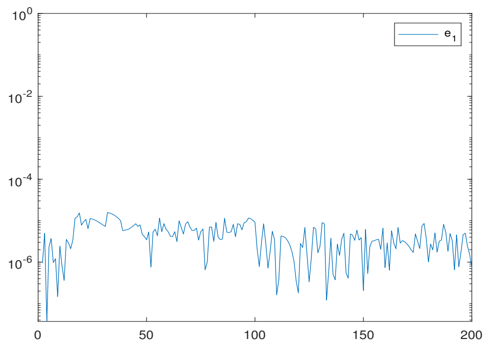
For the implementation of Algorithm (1), the starting point θ^0 is set as $\theta^0 = [0, \frac{\pi}{3}]^\top$ and the control parameters ρ and σ are set as 0.6 and 0.0001 respectively. Note that the task duration $[0, t^{\varphi}]$ is divided into 200 equal parts with t^{φ} set as ten seconds. Figure 2 shows the experimental results of Algorithm 1 in solving (28). Observing figure 2, it is easy to visualize that that Algorithm 1 successfully completes the given task.



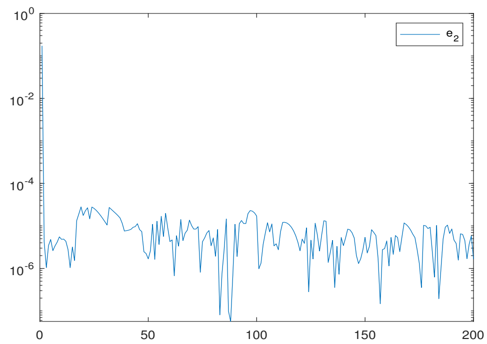
(a) Manipulator trajectories



(b) End-effector trajectory and desired path



(c) Tracking errors on the horizontal x-axis



(d) Tracking errors on the vertical y-axis

Figure 2: Numerical results generated by Algorithm 1

5. Conclusion

In this paper, we proposed a q-spectral PRP conjugate gradient method for solving unconstrained optimization problems. The method extends the classical spectral PRP approach by incorporating the q-derivative operator, leading to a flexible framework that reduces to the classical case as the parameter q approaches one. We established the global convergence of the proposed method under mild assumptions using Wolfe-type line search conditions. To evaluate the practical performance of the algorithm, we conducted numerical experiments on a set of benchmark problems. The results demonstrate that the proposed q-SPRP method outperforms the classical method in terms of convergence behavior across various initial conditions. Additionally, we applied the method to a motion control problem, further confirming its applicability to real-world scenarios.

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Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

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