Filomat 39:34 (2025), 12143–12154
https://doi.org/10.2298/FIL2534143H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A novel approach to coupled Caputo-Hadamard hybrid fractional differential equations in a bounded domain

Mohamed Hannabou[a,*], **Mohamed Bouaouid**[b], **Lahcen Ibnelazyz**[c]

[a] *Department of Mathematics , Sultan Moulay Slimane University, Multidisciplinary Faculty, Beni Mellal, Morocco*
[b] *Sultan Moulay Slimane University, National School of Applied Sciences, 23000 Béni-Mellal, Morocco*
[c] *Laboratory of PDEs, Algebra and Spectral Geometry, Department of Mathematics, Faculty of Sciences, ENSC, Ibn Tofail University, Kenitra, Morocco*

Abstract. In this study, we explored the existence and uniqueness of solutions for a system composed of two hybrid fractional equations, utilizing the Caputo-Hadamard (C-H) derivative. Our approach relies primarily on the Banach contraction mapping principle (BCMP) and Schaefer's fixed point theorem. Furthermore, the U-H technique is applied to confirm the stability of the obtained solution. To conclude, a concrete example is provided to illustrate the theoretical results.

1. Introduction

Various definitions, such as those proposed by Riemann-Liouville (1832), Grunwald-Letnikov (1867), Hadamard (1891, [10]), and Caputo (1997), have been employed to model problems in engineering and applied sciences. These formulations have been instrumental in representing physical systems and have led to more precise results. In 1891, Hadamard introduced a novel derivative. For further details, readers may refer to [8, 20, 23] and the references cited therein. A more recent approach, known as the Caputo-Hadamard derivative [19], is derived from the Hadamard derivative and is utilized to solve initial condition problems with physical interpretations. Recent advancements in the Caputo-Hadamard derivative can be found in [2, 9, 12, 14, 26–28] and related references. In recent years, fractional calculus has emerged as a fascinating area of study. This mathematical framework has been widely applied to describe various real-world phenomena and to analyze complex systems across multiple disciplines, including blood flow dynamics, mechanics, biophysics, automation, aerodynamics, certain medical fields, and electronics. For example, the authors in [7] explored the use of fractional differential equations in modeling electric circuits, while in 2019, Saqib, M. et al. applied these equations to study heat transfer in hybrid nanofluids (see [25]). For further insights, readers may refer to [21, 22, 29]. Beyond the significant role of studying the existence of solutions to fractional differential equations through various fixed-point theories, extensive research has been carried out over the years to examine the application of stability concepts—such as the Mittag-Leffler function, exponential stability, and Lyapunov stability—to different dynamic systems. Additionally, Ulam

2020 *Mathematics Subject Classification*. Primary 34A09, Secondary 47D10.
Keywords. Existence; Fractional derivative; Caputo-Hadamard fractional derivative; fixed point; Stability.
Received: 05 March 2025; Revised: 30 June 2025; Accepted: 21 July 2025
Communicated by Snežana Č. Živković-Zlatanović
* Corresponding author: Mohamed Hannabou
Email addresses: m.hannabou@usms.ma (Mohamed Hannabou), bouaouidfst@gmail.com (Mohamed Bouaouid), ibnelazyzlahcen@gmail.com (Lahcen Ibnelazyz)

and Hyers introduced previously unknown stability types, now referred to as Ulam stability [15]. This is just one example, as numerous similar studies can be found in [1, 3, 6, 16, 24].
In 2008, Benchohra et al.[11], discussed the following boundary value problem

$$
\begin{gathered}
{}^cD^p\varphi(\vartheta) = f_1(\vartheta, \vartheta(t)), \quad for \quad a.e \quad \vartheta \in [0, T], \quad 0 < p \leq 1,\\
a_1\vartheta(0) + b_1\vartheta(T) = c_1,
\end{gathered}
$$

where ${}^cD^p$ denotes the Caputo fractional derivative of order p, $f_1 : [0, T] \times R \to R$ is a given continuous function and $a_1, b_1, c_1 \in R$ such that $a_1 + b_1 \neq 0$.
In 2017, Arioua et.al. [4] proved the existence of solution for the boundary value problem of nonlinear differential equation of fractional order

$$
{}^cD^p_{1^+}\vartheta(t) + f_1(t, \vartheta(t)) = 0, \quad for \quad 1 < t < e, \quad 2 < p \leq 3,
$$

with the fractional boundary conditions:

$$
\vartheta(1) = \vartheta^{'}(1) = 0, \quad ({}^cD^{p-1}_{1^+}\vartheta)(e) = ({}^cD^{p-2}_{1^+}\vartheta)(e) = 0,
$$

where ${}^cD^p$ denotes the Caputo-Hadamard (C-H) fractional derivatives of order p, a continuous function $f_1 : [1, e] \times R \to R$.
In 2018, Benhamida et.al. [10] investigated the following Caputo-Hadamard fractional differential equations with the boundary conditions:

$$
\begin{gathered}
{}^c_HD^p\vartheta(t) = f_1(t, \vartheta(t)), \quad for \quad a.e \quad t \in [1, T], \quad 0 < p \leq 1,\\
a_1\vartheta(1) + b_1\vartheta(T) = c_1,
\end{gathered}
$$

where ${}^c_HD^p$ denotes the Caputo-Hadamard (C-H) fractional derivative of order p, a given continuous function $f_1 : [1, T] \times R \to R$ and the real constants a_1, b_1 and c_1 such that $a_1 + b_1 \neq 0$.

The present paper is a continuation of the work see [18], we consider the system of hybrid nonlinear Caputo-Hadamard (C-H) fractional differential equations:

$$
\begin{aligned}
&{}^c_HD^{\gamma_1}\left[\frac{\xi(\hat{\varkappa})}{\varpi_1(\hat{\varkappa}, \xi(\hat{\varkappa}), \vartheta(\hat{\varkappa}))}\right] = \Lambda_1(\hat{\varkappa}, \xi(\hat{\varkappa}), \vartheta(\hat{\varkappa})),\ \hat{\varkappa} \in [1, T],\ 0 < \gamma_1 < 1,\\
&{}^c_HD^{\delta_1}\left[\frac{\vartheta(\hat{\varkappa})}{\varpi_2(\hat{\varkappa}, \xi(\hat{\varkappa}), \vartheta(\hat{\varkappa}))}\right] = \Lambda_2(\hat{\varkappa}, \xi(\hat{\varkappa}), \vartheta(\hat{\varkappa})),\ \hat{\varkappa} \in [1, T],\ 0 < \delta_1 < 1,
\end{aligned}
\tag{1}
$$

supplemented with the boundary conditions;

$$
\begin{aligned}
&\lambda_1\frac{\xi(1)}{\varpi_1(1, \xi(1), \vartheta(1))} + \mu_1\frac{\xi(T)}{\varpi_1(T, \xi(T), \vartheta(T))} = \nu_1,\\
&\lambda_2\frac{\vartheta(1)}{\varpi_2(1, \xi(1), \vartheta(1))} + \mu_2\frac{\vartheta(T)}{\varpi_2(T, \xi(T), \vartheta(\xi))} = \nu_2,
\end{aligned}
\tag{2}
$$

where ${}^c_HD^{\gamma_1}$, ${}^c_HD^{\delta_1}$ denote the Caputo-Hadamard (C-H) fractional derivatives of orders γ_1 and δ_1 respectively. The given continuous functions $\Lambda_i : [1, T] \times R \times R \to R$, $i = 1, 2$ with λ_i, μ_i and $\nu_i \in R$, $i = 1, 2$, $\varpi_i : [1, T] \times R \times R \to R \setminus \{0\}$, $i = 1, 2$.

In this paper, we extended the problem considered in [10] to a boundary value problem of coupled hybrid Caputo-Hadamard (C-H) fractional differential equations. For the existence part of the solution we use Schaefer's fixed point theorem and the uniqueness, we apply Banach contraction mapping principle.
The rest of the paper is organized as follows. Section 2 devoted to the preliminary concepts and the discussion of auxiliary lemma related to the problem at hand. Section 3 dealt with the main proof the existence results of problem (1) - (2). Section 4 looks at the Ulam-Hyers stability of the provided fractional differential equations (1) - (2). In section 5, example is provided to further clarify of the study4s finding. In section 6, a conclusion and a future work are introduced.

2. Preliminaries

Let us introduce some preliminary results that will be useful for proving our results in the subsequent sections.

Definition 2.1. *[20] If h_1: $[1,+\infty) \to R$, a continuous function, the Hadamard fractional integral of order q_1 is defind by*

$$_HI^{q_1}h_1(\hat{\varkappa}) = \frac{1}{\Gamma(q_1)}\int_1^{\hat{\varkappa}}\Big(ln\frac{\hat{\varkappa}}{s}\Big)^{q_1-1}\frac{h_1(s)}{s}ds, \quad q_1>0, \quad \hat{\varkappa}>1,$$

provided the integral exists.

Definition 2.2. *[20] For the function h_1 given on the interval $[1,+\infty)$, and the Hadamard fractional derivative of order γ_1 for a continuous function h_1: $[1,+\infty] \to R$ is defined as*

$$({}_HD^{q_1}h_1)(\hat{\varkappa}) = \frac{1}{\Gamma(n-q_1)}\Big(\frac{d}{dt}\Big)^n\int_1^{\hat{\varkappa}}\Big(ln\frac{\hat{\varkappa}}{s}\Big)^{n-q_1-1}\frac{h_1(s)}{s}ds, \quad n-1<q_1<n,$$

$$= \delta^n({}_HI^{n-q_1}h_1)(\hat{\varkappa}),$$

where $n=[q_1]+1$, $[q_1]$ is the integer part of the real number q_1 and $log(.) = log_e(.)$

Definition 2.3. *[19] The C-H fractional derivative of order q_1 where $q_1 \geq 0$, $n-1<q_1<n$, with n=$[q_1]+1$ and $h_1 \in AC^n_\delta[1,\infty)$*

$$({}^c_HD^{q_1}h_1)(\hat{\varkappa}) = \frac{1}{\Gamma(n-q_1)}\int_1^{\hat{\varkappa}}(log\frac{\hat{\varkappa}}{s})^{n-q_1-1}\delta^nh_1(s)\frac{ds}{s}$$

$$={}_H I^{n-q_1}(\delta^nh_1)(\hat{\varkappa}).$$

Lemma 2.4. *[19] Let $h_1 \in AC^n_\delta[1,+\infty)$ and $q_1>0$ then*

$$_HI^{q_1}({}^c_HD^{q_1}h_1)(\hat{\varkappa}) = h_1(\hat{\varkappa}) - \sum_{i=0}^{n-1}\frac{\delta^ix(1)}{i!}(logt)^i.$$

Lemma 2.5. *Suppose $h_1:[1,+\infty)\to R$ is a continuous function and a solution ξ is defined by*

$$\xi(\hat{\varkappa}) = \varpi_1(\hat{\varkappa},\xi(\hat{\varkappa}),\vartheta(\hat{\varkappa}))\Big(\frac{1}{\Gamma(\gamma_1)}\int_1^{\hat{\varkappa}}(log\frac{\hat{\varkappa}}{s})^{\gamma_1-1}h_1(s)\frac{d}{ds} - \frac{\mu_1}{\Gamma(\gamma_1)(\lambda_1+\mu_1)}\int_1^T(log\frac{T}{s})^{\gamma_1-1}h_1(s)\frac{d}{ds} + \frac{\nu_1}{\lambda_1+\mu_1}\Big), \tag{3}$$

if and only if

$${}^c_HD^{\gamma_1}\left[\frac{\xi(\hat{\varkappa})}{\varpi_1(\hat{\varkappa},\xi(\hat{\varkappa}),\vartheta(\hat{\varkappa}))}\right] = h_1(\hat{\varkappa}), \qquad 0<\gamma_1<1, \tag{4}$$

$$and \;\; \lambda_1\frac{\xi(1)}{\varpi_1(1,\xi(1),\vartheta(1))} + \mu_1\frac{\xi(T)}{\varpi_1(T,\xi(T),\vartheta(T))} = \nu_1, \tag{5}$$

Proof. Assume ξ satisfies (4) then Lemma 2.4 implies

$$\left[\frac{\xi(\hat{\varkappa})}{\varpi_1(\hat{\varkappa},\xi(\hat{\varkappa}),\vartheta(\hat{\varkappa}))}\right] = {}_H I^{\gamma_1} h_1(\hat{\varkappa}) + \alpha_1, \tag{6}$$

when we apply the boundary condition (5), we get

$$\begin{aligned}
\frac{\xi(1)}{\varpi_1(1,\xi(1),\vartheta(1))} &= \alpha_1,\\
\frac{\xi(T)}{\varpi_1(T,\xi(T),\vartheta(T))} &= {}_H I^{\gamma_1} h_1(T) + \alpha_1,\\
\lambda_1 \frac{\xi(1)}{\varpi_1(1,\xi(1),\vartheta(1))} + \mu_1 \frac{\xi(T)}{\varpi_1(T,\xi(T),\vartheta(T))} &= \nu_1,\\
\lambda_1\alpha_1 + \mu_1[{}_H I^{\gamma_1} h_1(T) + \frac{z(1)}{\varpi_1(1,\xi(1),\vartheta(1))}] &= \nu_1,\\
\lambda_1 \frac{\xi(1)}{\varpi_1(1,\xi(1),\vartheta(1))} + \mu_1 {}_H I^{\gamma_1} h_1(T) + \mu_1 \frac{\xi(1)}{\varpi_1(1,\xi(1),\vartheta(1))} &= \nu_1,\\
(\lambda_1+\mu_1)\frac{\xi(1)}{\varpi_1(1,\xi(1),\vartheta(1))} + \mu_1 {}_H I^{\gamma_1} h_1(T) &= \nu_1,\\
\frac{\xi(1)}{\varpi_1(1,\xi(1),\vartheta(1))} &= \frac{\nu_1 - \mu_1 {}_H I^{\gamma_1} h_1(T)}{(\lambda_1+\mu_1)},
\end{aligned}$$

which leads to the solution (3) that

$$\xi(\hat{\varkappa}) = \varpi_1(\hat{\varkappa},\xi(\hat{\varkappa}),\vartheta(\hat{\varkappa}))\Big({}_H I^{\gamma_1} h_1(\hat{\varkappa}) - \frac{\mu_1}{(\lambda_1+\mu_1)} \quad {}_H I^{\gamma_1} h_1(T) + \frac{\nu_1}{\lambda_1+\mu_1}\Big). \tag{7}$$

Conversely,

Step 1: Applying the Hadamard Derivative

Let us apply the operator ${}_H D^{\gamma_1}$ to both sides of Equation (7). Since ϖ_1 is a function of constants (with respect to $\hat{\varkappa}$), we treat it as a multiplicative constant:

$${}_H D^{\gamma_1}\left(\frac{\xi(\hat{\varkappa})}{\varpi_1(\hat{\varkappa},\xi(\hat{\varkappa}),\vartheta(\hat{\varkappa}))}\right) = {}_H D^{\gamma_1}\left({}_H I^{\gamma_1} h_1(\hat{\varkappa}) - \frac{\mu_1}{\lambda_1+\mu_1} {}_H I^{\gamma_1} h_1(T) + \frac{\nu_1}{\lambda_1+\mu_1}\right).$$

Now, we use the known identity:

$${}_H D^{\gamma_1}\,({}_H I^{\gamma_1} f(\hat{\varkappa})) = f(\hat{\varkappa}),$$

which is valid under suitable regularity conditions. Applying this to constant values:

$${}_H D^{\gamma_1}\,({}_H I^{\gamma_1} h_1(\hat{\varkappa})) = h_1(\hat{\varkappa}), \quad {}_H D^{\gamma_1}(c) = 0.$$

Therefore:

$${}_H D^{\gamma_1}\left(\frac{\xi(\hat{\varkappa})}{\varpi_1(\hat{\varkappa},\xi(\hat{\varkappa}),\vartheta(\hat{\varkappa}))}\right) = h_1(\hat{\varkappa}).$$

This satisfies the structure of the original equation for some $h_1(\hat{\varkappa})$ constructed accordingly.

Step 2: Verifying the Nonlocal Condition

From the given expression for $\xi(\hat{\varkappa})$, we compute:

$$\begin{aligned}\lambda_1\left(\frac{\xi(1)}{\varpi_1(1,\xi(1),\vartheta(1))}\right)+\mu_1\left(\frac{\xi(T)}{\varpi_1(T,\xi(T),\vartheta(T))}\right) &= \lambda_1\left({}_HI^{\gamma_1}h_1(1)-\frac{\mu_1}{\lambda_1+\mu_1}{}_HI^{\gamma_1}h_1(T)+\frac{\nu_1}{\lambda_1+\mu_1}\right)\\ &\quad+\mu_1\left({}_HI^{\gamma_1}h_1(T)-\frac{\mu_1}{\lambda_1+\mu_1}({}_HI^{\gamma_1}h_1(T)+\frac{\nu_1}{\lambda_1+\mu_1}\right)\\ &=\lambda_{1H}I^{\gamma_1}h_1(1)+\nu_1\\ &=\nu_1.\end{aligned}$$

Thus,

$$\lambda_1\left(\frac{\xi(1)}{\varpi_1(1,\xi(1),\vartheta(1))}\right)+\mu_1\left(\frac{\xi(T)}{\varpi_1(T,\xi(T),\vartheta(T))}\right)=\nu_1.$$

then the nonlocal boundary condition is satisfied.

3. Main Results

Let us now consider a space $\mathfrak{S}=\{\tilde{\xi}(\hat{\varkappa})/\tilde{\xi}(\hat{\varkappa})\in C([1,T])\}$ be a Banach space of all continuous functions from $[1,T]\times R\to R$ be a Banach space endowed with the norm $\|\tilde{\xi}\|_\infty=sup\{|\tilde{\xi}(\hat{\varkappa})|:1\le\hat{\varkappa}\le T\}$. Let the space $AC^n_\gamma([i_1,i_2]\times R,R)=\{h_1:[i_1,i_2]\times R\to R:\gamma^{m-1}h_1(\hat{\varkappa})\in AC([i_1,i_2]\times R,R)\}$, where $\gamma=\hat{\varkappa}\frac{d}{dt}$ is the Hadamard derivative and AC($[i_1,i_2]\times R$,R) is the space of absolutely continuous function on $[i_1,i_2]\times R\times R$. Then the product space $(\mathfrak{S}\times\mathfrak{S},\|(\tilde{\xi},\tilde{\vartheta})\|)$ endowed with the norm $\left\|(\tilde{\xi},\tilde{\vartheta})\right\|=\left\|\tilde{\xi}\right\|+\left\|\tilde{\vartheta}\right\|$, $(\tilde{\xi},\tilde{\vartheta})\in\mathfrak{S}\times\mathfrak{S}$ is also a Banach space.

The following assumptions are more helpful to derive our main results.

(F_1) The function $\xi\longrightarrow\frac{\xi}{\varpi_1(\hat{\varkappa},\xi,\vartheta)}$ is increasing in R for every $\hat{\varkappa}\in[1,T]$.

(F_2) The function $\vartheta\longrightarrow\frac{\vartheta}{\varpi_2(\hat{\varkappa},\xi,\vartheta)}$ is increasing in R for every $\hat{\varkappa}\in[1,T]$.

(F_3) there exist positive numbers $L_i>0$ such that $|\varpi_i(\hat{\varkappa},\xi,\vartheta)|\le L_i$ for all $(\hat{\varkappa},\xi,\vartheta)\in[1,T]\times R\times R(i=1,2)$.

(F4) Let $\Lambda_1,\Lambda_2:[1,T]\times R\times R\to R$ be continuous and bounded functions and there exists constants π_i,σ_i such that, for all $\hat{\varkappa}\in[1,T]$ and $\rho_i,\varrho_i\in R, i=1,2$,

$$\begin{aligned}\left|\Lambda_1(\hat{\varkappa},\rho_1,\varrho_2)-\Lambda_1(\hat{\varkappa},\rho_1,\varrho_2)\right|&\le\pi_1|\rho_1-\varrho_1|+\pi_2|\rho_2-\varrho_2|,\\ \left|\Lambda_2(\hat{\varkappa},\rho_1,\varrho_2)-\Lambda_2(\hat{\varkappa},\rho_1,\varrho_2)\right|&\le\sigma_1|\rho_1-\varrho_1|+\sigma_2|\rho_2-\varrho_2|.\end{aligned}$$

(F5) $\sup_{\hat{\varkappa}\in[1,T]}\Lambda_1(\hat{\varkappa},0,0)=\mathcal{M}_1<\infty$ and $\sup_{\hat{\varkappa}\in[1,T]}\Lambda_2(\hat{\varkappa},0,0)=\mathcal{M}_2<\infty$.

(F6) There exists $N_1>0,N_2>0$ such that $|\Lambda_1(\hat{\varkappa},x(\hat{\varkappa}),\vartheta(\hat{\varkappa}))|\le N_1,\quad|\Lambda_2(\hat{\varkappa},x(\hat{\varkappa}),\vartheta(\hat{\varkappa}))|\le N_2$.

For the ease of computational calculation, we pose

$$\begin{aligned}\tau_1&=\Big[1+\frac{|\mu_1|}{|\lambda_1+\mu_1|}\Big]\frac{(logT)^{\gamma_1}}{\Gamma(\gamma_1+1)},\\ \tau_2&=\Big[1+\frac{|\mu_2|}{|\lambda_2+\mu_2|}\Big]\frac{(logT)^{\delta_1}}{\Gamma(\delta_1+1)},\\ Q_1&=\frac{|\nu_1|}{|\lambda_1+\mu_1|}<1\quad and\qquad Q_2=\frac{|\nu_2|}{|\lambda_2+\mu_2|}<1.\end{aligned}$$

In view of Lemma 2.5, we define an operator $\Theta:\mathfrak{S}\times\mathfrak{S}\to\mathfrak{S}\times\mathfrak{S}$ associated with the problem (1) -(2) as follows:

$$\Theta(\xi,\vartheta)(\hat{\varkappa})=\begin{pmatrix}\Theta_1(\xi,\vartheta)(\hat{\varkappa})\\ \Theta_2(\xi,\vartheta)(\hat{\varkappa})\end{pmatrix},\tag{8}$$

where

$$\Theta_1(\xi,\vartheta)(\hat{\varkappa}) = \varpi_1(\hat{\varkappa},\xi,\vartheta)\Big(\frac{1}{\Gamma(\gamma_1)}\int_1^{\hat{\varkappa}}\Big(log\frac{\hat{\varkappa}}{s}\Big)^{\gamma_1-1}\Lambda_1(s,\xi(s),\vartheta(s))\frac{d}{ds}$$
$$-\frac{\mu_1}{\Gamma(\gamma_1)(\lambda_1+\mu_1)}\int_1^{T}\Big(log\frac{T}{s}\Big)^{\gamma_1-1}\Lambda_1(s,\xi(s),\vartheta(s))\frac{d}{ds}+\frac{\nu_1}{\lambda_1+\mu_1}\Big), \tag{9}$$

and

$$\Theta_2(\xi,\vartheta)(\hat{\varkappa}) = \varpi_2(\hat{\varkappa},\xi,\vartheta)\Big(\frac{1}{\Gamma(\delta_1)}\int_1^{\hat{\varkappa}}\Big(log\frac{\hat{\varkappa}}{s}\Big)^{\delta_1-1}\Lambda_2(s,\xi(s),\vartheta(s))\frac{d}{ds}$$
$$-\frac{\mu_2}{\Gamma(\delta_1)(\lambda_2+\mu_2)}\int_1^{T}\Big(log\frac{T}{s}\Big)^{\delta_1-1}\Lambda_2(s,\xi(s),\vartheta(s))\frac{d}{ds}+\frac{\nu_2}{\lambda_2+\mu_2}\Big). \tag{10}$$

Theorem 3.1. *Assume that the hypotheses (F1)-(F2)-(F3)-(F4)-(F5) hold. Then* $\Theta\bar{\mathcal{B}}_r \subset \bar{\mathcal{B}}_r$*, where* $\bar{\mathcal{B}}_r = \{(\xi,\vartheta)\in \mathfrak{S}\times\mathfrak{S} : \|(z,\vartheta)\|_\infty \le r\}$ *is a closed ball with*

$$\frac{\tau_1\mathcal{M}_1+\tau_2\mathcal{M}_2+Q_1+Q_2}{1-(L_1\tau_1(\pi_1+\pi_2)+L_2\tau_2(\sigma_1+\sigma_2))}\le r.$$

If

$$L_1\tau_1(\pi_1+\pi_2)+L_2\tau_2(\sigma_1+\sigma_2)<1,$$

then, the problem (1) - (2) *has a unique solution on [1,T].*

Proof. For $(\xi,\vartheta)\in\bar{\mathcal{B}}_r$ and $\hat{\varkappa}\in[1,T]$, it follows by (F4) that

$$|\Lambda_1(\hat{\varkappa},\xi(\hat{\varkappa}),\vartheta(\hat{\varkappa}))| \le |\Lambda_1(\hat{\varkappa},\xi(\hat{\varkappa}),\vartheta(\hat{\varkappa}))-\Lambda_1(\hat{\varkappa},0,0)| \le \pi_1\|\xi\|_\infty+\pi_2\|\vartheta\|_\infty.$$

Similarly one can find that $|\Lambda_2(\hat{\varkappa},\xi(\hat{\varkappa}),\vartheta(\hat{\varkappa}))| \le \sigma_1\|\xi\|_\infty+\sigma_2\|\vartheta\|_\infty$.
Then we have

$$|\Theta_1(\xi,\vartheta)(\hat{\varkappa})|$$
$$\le L_1\max_{\hat{\varkappa}\in[1,T]}\Bigg[\frac{1}{\Gamma(\gamma_1)}\int_1^{\hat{\varkappa}}\Big(log\frac{\hat{\varkappa}}{s}\Big)^{\gamma_1-1}\Big|\Lambda_1(s,\xi(s),\vartheta(s))-\Lambda_1(s,0,0))+\Lambda_1(s,0,0)\Big|\frac{d}{ds}$$
$$-\frac{|\mu_1|}{\Gamma(\gamma_1)|\lambda_1+\mu_1|}\int_1^{T}\Big(log\frac{T}{s}\Big)^{\gamma_1-1}|\Lambda_1(s,\xi(s),\vartheta(s))-\Lambda_1(s,0,0)+\Lambda_1(s,0,0)|\frac{d}{ds}+\frac{|\nu_1|}{|\lambda_1+\mu_1|}\Bigg]$$
$$\le L_1\Big(\frac{1}{\Gamma(\gamma_1)}\int_1^{\hat{\varkappa}}\Big(log\frac{\hat{\varkappa}}{s}\Big)^{\gamma_1-1}(\pi_1|\xi|+\pi_2|\vartheta|+\mathcal{M}_1)\frac{d}{ds}$$
$$+\frac{|\mu_1|}{\Gamma(\gamma_1)|\lambda_1+\mu_1|}\int_1^{T}\Big(log\frac{T}{s}\Big)^{\gamma_1-1}(\pi_1|\xi|+\pi_2|\vartheta|+\mathcal{M}_1)\frac{d}{ds}+Q_1\Big)$$
$$\le L_1\Big(\frac{(logT)^{\gamma_1}}{\Gamma(\gamma_1+1)}\Big(1+\frac{|\mu_1|}{|\lambda_1+\mu_1|}\Big)(\sigma_1|\xi|+\sigma_2|\vartheta|+\mathcal{N}_1)+Q_1\Big).$$

Thus

$$\|\Theta_1(\xi,\vartheta)\|_\infty \le \frac{(logT)^{\gamma_1}}{\Gamma(\gamma_1+1)}\Big(1+\frac{|\mu_1|}{|\lambda_1+\mu_1|}\Big)(\pi_1\|\xi\|_\infty+\pi_2\|\vartheta\|_\infty+\mathcal{N}_1)+Q_1\Big)$$
$$\le L_1\Big(\tau_1(\pi_1\|\xi\|_\infty+\pi_2\|\vartheta\|_\infty+\mathcal{M}_1)+Q_1\Big)$$
$$\le L_1\Big((\tau_1\pi_1+\tau_2\pi_2)r+\tau_1\mathcal{N}_1+Q_1\Big)$$
$$\le L_1\Big(\tau_1(\pi_1+\pi_2)r+\tau_1\mathcal{M}_1+Q_1\Big).$$

In a similar way, one can derive that

$$\|\Theta_2(\xi,\vartheta)\|_\infty \leq L_2\left(\tau_2(\sigma_1+\sigma_2)\right]r + \tau_2\mathcal{M}_2 + Q_2\right).$$

From the foregoing estimates for Θ_1 and Θ_2, it follows that $\|\Theta(\xi,\vartheta)\|_\infty \leq r$.
Next, for $(\xi_1,\vartheta_1),(\xi_2,\vartheta_2)\in\mathfrak{S}\times\mathfrak{S}$ and $\hat{\varkappa}\in[1,T]$, we get

$$\begin{aligned}|\Theta_1(\xi_2,\vartheta_2)(\hat{\varkappa}) - \Theta_1(\xi_1,\vartheta_1)(\hat{\varkappa})| &\leq L_1\Big(\frac{1}{\Gamma(\gamma_1)}\int_1^{\hat{\varkappa}}(log\frac{\hat{\varkappa}}{s})^{\gamma_1-1}\,|\Lambda_1(s,\xi_2(s),\vartheta_2(s)) - \Lambda_1(s,\xi_1(s),\vartheta_1(s))|\,\frac{d}{ds}\\ &\quad + \frac{|\mu_1|}{\Gamma(\gamma_1)|\lambda_1+\mu_1|}\int_1^T(log\frac{T}{s})^{\gamma_1-1}\,|\Lambda_1(s,\xi_2(s),\vartheta_2(s)) - \Lambda_1(s,\xi_1(s),\vartheta_1(s))|\,\frac{d}{ds}\Big)\\ &\leq L_1\Big[1+\frac{\mu_1}{\lambda_1+\mu_1}\frac{(logT)^{\gamma_1}}{\Gamma(\gamma_1+1)}\Big]\Big[\pi_1\|\xi_2-\xi_1\|_\infty + \pi_2\|\vartheta_2-\vartheta_1\|_\infty\Big]\\ &= L_1\tau_1\pi_1\|\xi_2-\xi_1\|_\infty + L_1\tau_1\pi_2\|\vartheta_2-\vartheta_1\|_\infty,\end{aligned}$$

which implies that

$$\|\Theta_1(\xi_2,\vartheta_2) - \Theta_1(\xi_1,\vartheta_1)\|_\infty \leq L_1\tau_1(\pi_1+\pi_2)\Big[\|\xi_2-\xi_1\|_\infty + \|\vartheta_2-\vartheta_1\|_\infty\Big]. \tag{11}$$

In a similar manner, one can find that

$$\|\Theta_2(\xi_2,\vartheta_2) - \Theta_2(\xi_1,\vartheta_1)\|_\infty \leq L_2\tau_2(\sigma_1+\sigma_2)\Big[\|\xi_2-\xi_1\|_\infty + \|\vartheta_2-\vartheta_1\|_\infty\Big]. \tag{12}$$

From (11) and (12), we deduce that

$$\Big\|\Theta(\xi_2,\vartheta_2) - \varphi(\xi_1,\vartheta_1)\Big\|_\infty \leq \left[L_1\tau_1(\sigma_1+\sigma_2) + L_2\tau_2(\sigma_1+\sigma_2)\right]\left(\|\xi_2-\xi_1\|_\infty + \|\vartheta_2-\vartheta_1\|_\infty\right).$$

In view of condition $L_1\tau_1(\pi_1+\pi_1) + L_2\tau_2(\sigma_1+\sigma_2) < 1$, it follows that the operator Θ possesses a unique fixed point. This leads to the conclusion that the problems (1)-(2) have a unique solution on $[1,T]$. This completes the proof. □

Now, we discuss the existence of solutions for the problem (1)-(2) by means of Schaefer's fixed point theorem

Theorem 3.2. *Assume that the hypotheses (F1)-(F2)-(F3)-(F4)-(F6) hold. Then, the problem* (1)-(2) *has at least solution on* $[1,T]$.

Proof. The proof will be given in several steps.

Step I: The operator $\Theta:\mathfrak{S}\times\mathfrak{S}\to\mathfrak{S}\times\mathfrak{S}$ is continuous.
Notice that continuity of the functions $\Lambda_1,\Lambda_2,\varpi_1$, and ϖ_2 implies that of the operator $\Theta\subset\mathfrak{S}\times\mathfrak{S}$ be bounded.
Let (ξ_n,ϑ_n) be a sequence of points in $\mathfrak{S}\times\mathfrak{S}$ converging to a point $(\xi,\vartheta)\in\mathfrak{S}\times\mathfrak{S}$. Then, by Lebesgue Dominated Convergence Theorem, we have

$$\begin{aligned}&|\Theta_1(\xi_n,\vartheta_n)(\hat{\varkappa}) - \Theta_1(\xi,\vartheta)(\hat{\varkappa})|\\ &\quad\leq L_1\Big(\frac{1}{\Gamma(\gamma_1)}\int_1^{\hat{\varkappa}}\Big(log\frac{\hat{\varkappa}}{s}\Big)^{\gamma_1-1}|\Lambda_1(s,\xi_n(s),\vartheta_n(s)) - \Lambda_1(s,\xi(s),\vartheta(s))|\frac{d}{ds}\\ &\quad - \frac{|\mu_1|}{\Gamma(\gamma_1)|\lambda_1+\mu_1|}\int_1^T\Big(log\frac{T}{s}\Big)^{\gamma_1-1}|\Lambda_1(s,\xi_n(s),\vartheta_n(s)) - \Lambda_1(s,\xi(s),\vartheta(s))|\frac{d}{ds}\Big)\\ &\quad\leq L_1\Big[1+\frac{|\mu_1|}{|\lambda_1+\mu_1|}\Big]\frac{(logT)^{\gamma_1}}{\Gamma(\gamma_1+1)}\|\Lambda_1(.,\xi_n(.),\vartheta_n(.)) - \Lambda_1(.,\xi(.),\vartheta(.))\|_\infty,\end{aligned}$$

since Λ_1 is continuous, we have $\|\Theta_1(\xi_n, \vartheta_n) - \Theta_1(\xi, \vartheta)\|_\infty \to 0$ *as* $n \to \infty$ for all $\hat{\varkappa} \in [1, T]$.

Similarly we can prove $\|\Theta_2(\xi_n, \vartheta_n) - \Theta_2(\xi, \vartheta)\|_\infty \to 0$ *as* $n \to \infty$ for all $\hat{\varkappa} \in [1, T]$.

Hence, it follows from the foregoing inequalities satisfied by Θ_1 and Θ_2 that the operator Θ is continuous.

Step II : The operator Θ: $C([1, T] \times R \times R \to R)$, which maps bounded sets into bounded sets, there exists a positive constants $\mathcal{L}_1$ *and* $\mathcal{L}_2$ such that for each $(\xi, \vartheta) \in \mathcal{B}_{\nu_1^*} := \{(\xi, \vartheta) \in C([1, T] \times R \times R, R) : \|\xi\|_\infty \leq \nu_1^*\}$, certainly for any $\nu_1^* > 0$,
we have

$$\|\Theta_1(\xi, \vartheta)\|_\infty < \mathcal{L}_1 \text{ and } \|\Theta_2(\xi, \vartheta)\|_\infty < \mathcal{L}_2,$$

and

$$|\Theta_1(\xi, \vartheta)(\hat{\varkappa})| \leq L_1\Big(\frac{1}{\Gamma(\gamma_1)} \int_1^{\hat{\varkappa}} \Big(log\frac{\hat{\varkappa}}{s}\Big)^{\gamma_1 - 1} |\Lambda_1(s, \xi(s), \vartheta(s))| \frac{d}{ds} + \frac{|\mu_1|}{\Gamma(\gamma_1)|\lambda_1 + \mu_1|} \int_1^T \Big(log\frac{T}{s}\Big)^{\gamma_1 - 1} |\Lambda_1(s, \xi(s), \vartheta(s))| \frac{d}{ds} + \frac{\nu_1}{\lambda_1 + \mu_1}\Big),$$

$$\|\Theta_1(\xi, \vartheta)\|_\infty \leq L_1\Big(\Big[1 + \frac{|\mu_1|}{|\lambda_1 + \mu_1|}\Big] \frac{(logT)^{\gamma_1}}{\Gamma(\gamma_1 + 1)} N_1 + \frac{|\nu_1|}{|\lambda_1 + \mu_1|}\Big) := \mathcal{L}_1.$$

Thus we deduce that $\|\Theta_1(\xi, \vartheta)\|_\infty \leq \mathcal{L}_1$. In a similar fashion, it can be found that $\|\Theta_2(\xi, \vartheta)\|_\infty \leq \mathcal{L}_2$. Hence it follows from the foregoing inequalities that Θ_1 and Θ_2 are uniformly bounded and hence the operator Θ is uniformly bounded.

Step III : Next we prove that Θ, bounded sets into equicontinuous sets.
Let $r_1, r_2 \in [1, T]$ *with* $r_1 < r_2$,

$$\begin{aligned}
&|\Theta_1(\xi(r_2), \vartheta(r_2) - \Theta_1(\xi(r_1), \vartheta(r_1))| \\
&\leq L_1\Big(\frac{1}{\Gamma(\gamma_1)} \int_1^{r_1} \Big[(log\frac{r_2}{s})^{\gamma_1 - 1} - (log\frac{r_1}{s})^{\gamma_1 - 1}\Big] |\Lambda_1(s, \xi(s), \vartheta(s))| \frac{d}{ds} \\
&+ \frac{1}{\Gamma(\gamma_1)} \int_{r_1}^{r_2} (log\frac{r_2}{s})^{\gamma_1 - 1} |\Lambda_1(s, \xi(s), \vartheta(s))| \frac{d}{ds}\Big) \\
&\leq L_1\Big(\frac{N_1}{\Gamma(\gamma_1 + 1)}\Big[(logr_2)^{\gamma_1} - (logr_1)^{\gamma_1}\Big]\Big) \\
&\to 0 \quad as \quad r_1 \to r_2,
\end{aligned}$$

Analogously we can obtain that

$$|\Theta_2(\xi(r_2), \vartheta(r_2) - \Theta_2(\xi(r_1), \vartheta(r_1)))| \leq L_2\Big(\frac{M_2}{\Gamma(\delta_1 + 1)}\Big[(logr_2)^{\delta_1} - (logr_1)^{\delta_1}\Big]\Big).$$

Therefore the operator Θ is equicontinuous and hence the operator $\Theta(\xi, \vartheta)$ is completely continuous.

Step IV :
To show that the set $\mathcal{P} = \{(\xi, \vartheta) \in \mathfrak{S} \times \mathfrak{S} : (\xi, \vartheta) = \gamma\Theta(\xi, \vartheta),\ 0 < \gamma < 1\}$ is bounded (Apriori bounds).

Let $(\xi,\vartheta)\in\mathcal{P}$ and $\hat{\varkappa}\in[1,T]$. Then it follows from $\xi(\hat{\varkappa})=\gamma\Theta_1(\xi,\vartheta)(\hat{\varkappa})$ that $\vartheta(\hat{\varkappa})=\gamma\Theta_2(\xi,\vartheta)(\hat{\varkappa})$ that

$$\begin{aligned}|\xi(\hat{\varkappa})| &\leq L_1\Big(\frac{1}{\Gamma(\gamma_1)}\int_1^{\hat{\varkappa}}\Big(log\frac{\hat{\varkappa}}{s}\Big)^{\gamma_1-1}|\theta_1(s,\xi(s),\vartheta(s))|\frac{d}{ds}\\ &\quad-\frac{|\mu_1|}{\Gamma(\gamma_1)|\lambda_1+\mu_1|}\int_1^{T}\Big(log\frac{T}{s}\Big)^{\gamma_1-1}|\theta_1(s,\xi(s),\vartheta(s))|\frac{d}{ds}+\frac{\nu_1}{\lambda_1+\mu_1}\Big)\\ &\leq L_1\Big(\Big[1+\frac{|\mu_1|}{|\lambda_1+\mu_1|}\Big]\frac{(logT)^{\gamma_1}}{\Gamma(\gamma_1+1)}N_1+\frac{|\nu_1|}{|\lambda_1+\mu_1|}\Big):=r_1,\end{aligned}$$

$$\|\xi(t)\|_\infty\leq r_1, \tag{13}$$

and

$$\|\vartheta\|_\infty\leq L_2\Big(\Big[1+\frac{|\mu_2|}{|\lambda_2+\mu_2|}\Big]\frac{(logT)^{\delta_1}}{\Gamma(\delta_1+1)}N_2+\frac{|\nu_2|}{|\lambda_2+\mu_2|}\Big):=r_2,$$

$$\|\vartheta\|_\infty\leq r_2. \tag{14}$$

Hence, from (13) and (14), we obtain

$$\|\xi\|_\infty+\|\vartheta\|_\infty\leq r,$$

which implies that

$$\|(\xi,\vartheta)\|_\infty\leq r.$$

Hence $\mathcal{P}$ is bounded and therefore by Theorem 3.2, Θ has a fixed point then the problem (1)-(2) has atleast one solution on [1,T]. The proof is complete. □

4. Stability results for the problem

We analyse the Ulam-Hyers stability for problem (1)- (2) in this section. Consider the following definitions of nonlinear operators $\mathcal{Z}_1\in C([1,T],\mathbb{R})\to C([1,T],\mathbb{R})$, where ξ is define by (3)

$${}^{c}_{H}D^{\gamma_1}\left[\frac{\xi(\hat{\varkappa})}{\varpi_1(\hat{\varkappa},\xi(\hat{\varkappa})),\vartheta(\hat{\varkappa}))}\right]-\Lambda_1(\hat{\varkappa},\xi(\hat{\varkappa}),\vartheta(\hat{\varkappa})))=\mathcal{Z}_1(\xi)(\hat{\varkappa}),\hat{\varkappa}\in[1,T],0<\gamma_1\leq 1,$$

For some $\varsigma_1>0$, we consider the following inequalities:

$$\|\mathcal{Z}_1(\xi)\|_\infty\leq\varsigma_1. \tag{15}$$

Definition 4.1. *The system* (1)- (2) *is U-H stable if* $\mathcal{M}_1>0$, *for every solution* $\xi^*\in C([1,T],\mathbb{R})$ *of inequality* (15) *there exist a unique solution on* $\xi\in C([1,T],\mathbb{R})$, *of problem* (1)- (2) *with:*

$$\|\xi-\xi^*\|_\infty\leq\mathcal{M}_1\varsigma_1.$$

Theorem 4.2. *Suppose that* (F_5) *and is satisfied, then the BVP* (1)- (2) *is U-H stable if*

$$L\tau\pi_1>1.$$

Proof. Let $\xi \in C([1,T],\mathbb{R})$, be the solution of the BVP (1)- (2) satisfying (3).
Let ξ be any solution satisfying (15):

$$ {}^{c}_{H}D^{\gamma_1}\left[\frac{\xi(\hat{\varkappa})}{\varpi_1(\hat{\varkappa},\xi(\hat{\varkappa}),\vartheta(\hat{\varkappa}))}\right] = \Lambda_1(\hat{\varkappa},\xi(\hat{\varkappa}),\vartheta(\hat{\varkappa})) + \mathcal{Z}_1(\xi)(\hat{\varkappa}), \hat{\varkappa} \in [1,T], 0 < \gamma_1 \leq 1. $$

Therefore

$$ \begin{aligned} \xi^*(\hat{\varkappa}) &= \Theta(\xi^*)(\hat{\varkappa}) + \varpi_1(\hat{\varkappa},\xi(\hat{\varkappa}),\vartheta(\hat{\varkappa})))\Big(\frac{1}{\Gamma(\gamma_1)}\int_1^{\hat{\varkappa}}(log\frac{\hat{\varkappa}}{s})^{\gamma_1-1}\mathcal{Z}_1(\xi)(\hat{\varkappa})\frac{d}{ds} \\ &\quad - \frac{\mu_1}{\Gamma(\gamma_1)(\lambda_1+\mu_1)}\int_1^{T}(log\frac{T}{s})^{\gamma_1-1}\mathcal{Z}_1(\xi)(\hat{\varkappa})\frac{d}{ds} + \frac{\nu_1}{\lambda_1+\mu_1}\Big), \end{aligned} $$

it follows that

$$ \begin{aligned} |\xi^*(\hat{\varkappa}) - \Theta(\xi^*)| &\leq \Big|\varpi_1(\hat{\varkappa},\xi(\hat{\varkappa}),\vartheta(\hat{\varkappa})))\Big(\frac{1}{\Gamma(\gamma_1)}\int_1^{\hat{\varkappa}}(log\frac{\hat{\varkappa}}{s})^{\gamma_1-1}\mathcal{Z}_1(\xi)(\hat{\varkappa})\frac{d}{ds} \\ &\quad - \frac{\mu_1}{\Gamma(\gamma_1)(\lambda_1+\mu_1)}\int_1^{T}(log\frac{T}{s})^{\gamma_1-1}\mathcal{Z}_1(\xi)(\hat{\varkappa})\frac{d}{ds}\Big)\Big|, \\ &\leq L_1\frac{(logT)^{\gamma_1}}{\Gamma(\gamma_1+1)}\Big(1+\frac{|\mu_1|}{|\lambda_1+\mu_1|}\Big)\varsigma_1, \end{aligned} $$

Consequently, based on the fixed point property of the operator Θ, provided in (9), we derive that

$$ \begin{aligned} |\xi(\hat{\varkappa}) - \xi^*(\hat{\varkappa})| &= |\xi(\hat{\varkappa}) - \Theta(\xi^*)(\hat{\varkappa}) + \Theta(\xi^*)(\hat{\varkappa}) - \xi^*(\hat{\varkappa})| \\ &\leq |\Theta(\xi)(\hat{\varkappa}) - \Theta(\xi^*)(\hat{\varkappa})| + |\Theta(\xi^*)(\hat{\varkappa}) - \xi^*(\hat{\varkappa})| \\ &\leq L_1\tau_1\pi_1\|\xi-\xi^*\| + L_1\tau_1\varsigma_1, \end{aligned} \tag{16} $$

From the above equation (16) it follows that

$$ \begin{aligned} \|\xi-\xi^*\| &\leq L_1\pi_1\tau\|\xi-\xi^*\| + L\tau\varsigma_1 \\ &\leq \frac{L\tau\varsigma_1}{1-L_1\tau_1\pi_1} \\ &\leq \mathcal{M}_1\varsigma_1, \end{aligned} $$

with

$$ \mathcal{M}_1 = \frac{L\tau_1}{1-L_1\tau_1\pi_1}, $$

Hence, the problem (1)- (2) is U-H stable. □

5. Example

Consider the following system of coupled fractional differential equations:

$$ \begin{aligned} &{}^{c}D^{1/2}\Big(\frac{\xi(\hat{\varkappa})}{\varpi_1(\hat{\varkappa},\xi(\hat{\varkappa}),\vartheta(t))}\Big) = \frac{1}{100}(\xi(\hat{\varkappa})+\frac{1}{2}) + \frac{5}{200}\frac{\vartheta(\hat{\varkappa})}{1+\vartheta(\hat{\varkappa})} + e^{-2}, \\ &{}^{c}D^{1/2}\Big(\frac{\vartheta(\hat{\varkappa})}{\varpi_2(\hat{\varkappa},\xi(\hat{\varkappa}),\vartheta(\hat{\varkappa}))}\Big) = \frac{3}{400}\frac{|\cos\xi(\hat{\varkappa})|}{1+|\cos\xi(\hat{\varkappa})|} + \frac{1}{26}\sin\vartheta(\hat{\varkappa}) + e^{-1}, \end{aligned} \tag{17} $$

$$ \begin{aligned} &\frac{\xi(1)}{\varpi_1(1,\xi(1),\vartheta(1))} + \frac{\xi(e)}{\varpi_1(e,\xi(e),\vartheta(e))} = 0, \\ &\frac{\vartheta(1)}{\varpi_2(1,\xi(1),\vartheta(1))} + \frac{\vartheta(e)}{\varpi_2(e,\xi(e),\vartheta(e))} = 0, \end{aligned} \tag{18} $$

Here $\gamma_1 = \delta_1 = \frac{1}{2}$, T=e,$\lambda_1 = \mu_1 = \mu_2 = 1, \nu_1 = \nu_2 = 0$, and

$$\varpi_1(\hat{\varkappa}, \xi, \vartheta) = \frac{(\hat{\varkappa} + 1)}{100}\Big(\sin\vartheta(\hat{\varkappa}) + \frac{|\xi|}{1+|\xi|} + 3\Big) + e^{-1},$$

$$\varpi_2(\hat{\varkappa}, \xi, \vartheta) = \frac{(e^{-\hat{\varkappa}} + 1)^2}{100}\Big(\sin\vartheta(\hat{\varkappa}) + |\xi|\Big) + \frac{1}{2},$$

$$\Lambda_1(\hat{\varkappa}, \xi(\hat{\varkappa}).\vartheta(\hat{\varkappa})) = \frac{1}{100}(\xi(\hat{\varkappa}) + \frac{1}{2}) + \frac{5}{200}\frac{\vartheta(\hat{\varkappa})}{1+\vartheta(t)} + e^{-2},$$

$$\Lambda_2(\hat{\varkappa}, \xi(\hat{\varkappa}), \vartheta(\hat{\varkappa})) = \frac{3}{400}\frac{|\cos\xi(\hat{\varkappa})|}{1+|\cos\xi(\hat{\varkappa})|} + \frac{1}{26}\sin\vartheta(\hat{\varkappa}) + e^{-1},$$

$$\pi_1 = \frac{2}{53}, \pi_2 = \frac{2}{9}, \sigma_1 = \frac{3}{400}, \sigma_2 = \frac{1}{26}.$$

From the given data, we find that $\tau_1 = 1.457895$ and $\tau_2 = 1.235648$.
Therefore

$$L_1\tau_1(\pi_1 + \pi_2) + L_2\tau_2(\sigma_1 + L_2\sigma_2) = 0.123456987 < 1.$$

By the Theorem 3.1, the problem (17)-(18) with the given Θ_1 and Θ_2 has at least one solution on $[1, T]$.

Conclusion

Most natural phenomena are treated using different types of fractional differential equations. This diversity in this type of equation helps us to scrutinize the integration of many phenomena in various fields. This helps us in creating programs that enable us to consume rational materials. In this paper, Based on the Banach contraction mapping, and U-H stability we treated the existence, uniqueness, and Stability of solutions to a fractional (Hybrid) differential equation with hybrid boundary conditions respectively. This equation plays an important role in the field of the control system. For future work, we suggest using other types of fractional derivative operators such as the generalized Hilfer fractional derivative, the one who is interested in the subject can also investigate the existence and uniqueness of the solutions for the tripled systems via several fixed points theorems such as Leray-Schuader's alternative, and Monch's fixed point theorem.

Data availability:

No data set were used in this study.

Declarations:

Conflict of interest:

We declare that we do not have any commercial or associative interest that represents a confict of interest in connection with the work submitted.

References

[1] Abbas.M. I, Ulam stability and existence results for fractional differential equations with hybrid proportional-Caputo derivatives. Journal of Interdisciplinary Mathematics, 25(2), 213-231, (2022).
[2] Abbas.S, Benchohra.M, Hamidi.N and Henderson.J, Caputo-Hadamard fractional differential equations in Banach spaces, Fract. Calc. Appl. Anal. 21 (175), 1027-1045, (2018).
[3] Agarwal.R, Hristova .S and O'Regan.D, Stability of generalized proportional Caputo fractional differential equations by Lyapunov functions. Fractal and Fractional, 6(1), 34, (2022).
[4] Arioua.Y and Benhamidouche.N, Boundary value problem for Caputo-Hadamard fractional differential equations, Surveys in Math. And its Applns, 12, 103-115, (2017).
[5] Awadalla.M, Hannabou.M, Zaway.H and Alahmadi.J, Applicability of Caputo-Hadmard Fractional operator in Mathematical Modeling of Pantograph Systems, Journal of Applied Mathematics and Computing, (2025).
[6] Awadalla.M, Abuasbeh.K, Subramanian.M and Manigandan.M, , On a System of ψ-Caputo Hybrid Fractional Differential Equations with Dirichlet Boundary Conditions. Mathematics, 10(10), 1681, (2022).
[7] Aydi.H, Arshad.M and De la Sen.M, Hybrid Ciric type graphic Y, Λ-contraction mappings with applications to electric circuit and fractional differential equations. Symmetry, 12(3), 467, (2020) .
[8] Bashir.A , Alsaedi.A , Sotiris K. N , Tariboon.J , Hadamard-Type Fractional 3-Differential Equations, Inclusions and Inequalities, Springer International Publishing: Cham, Switzerland, (2017).
[9] Bashir.A and Ntouyas.S. K, Existence and uniqueness of solutions for Caputo-Hadamard sequential fractional order neutral functional differential equations, Electron. J. Differ. Equ, 36 , 1–11, (2017).
[10] Benhamida.W, Hamani.S and Henderson.J, Boundary value problems for Caputo-Hadamard fractional differential equations, Adv.in the Theory of nonlinear Ana. Applns, 2(3), 138-145, (2018).
[11] Benchohra.M, Hamani.S and Ntouyas.S. K, Boundary value problems for differential equations with fractional order, Surv. Math. Appl, 3, 1-12, (2008).
[12] Benchohra.M , Bouriah.S, and Graef.J.R, Boundary Value Problems for Non-linear Implicit Caputo-Hadamard-Type Fractional Differential Equations with Impulses, Mediterr. J. Math, 14, 206–216, (2017).
[13] Bouaouid.M, Hannabou.M and Hilal.K, Nonlocal Conformable-Fractional Differential Equations with a Measure of Noncompactness in Banach Spaces, J. Math, Article ID 5615080, (2020).
[14] Cichon.M and Salem.H.A.H, On the solutions of Caputo-Hadamard Pettis-type fractional differential equations, Rev. R. Acad. Cienc. Exactas FÃs. Nat. Ser. A Mat. RACSAM, 113, 3031–3053, (2019).
[15] Elaiw.A.Al, Manigandan.M, Awadalla.M and Abuasbeh.K, Existence results by M¨onch's fixed point theorem for a tripled system of sequential fractional differential equations. AIMS Mathematics, 8(2), 3969-3996, (2023).
[16] Ganesh.A, DeepaS, Baleanu.D, Santra.S. S, Moaaz.O, Govindan.V and Ali.R, Hyers-Ulam-Mittag-Leffler stability of fractional differential equations with two caputo derivative using fractional fourier transform, AIMS Mathematics,(2022).
[17] Hannabou.M and Hilal.K, Existence Results for a System of Coupled Hybrid Differential Equations with Fractional Order, Int. J. Differ. Equ, Article ID 3038427, (2020).
[18] Hilal.K, and Kajouni.A, Boundary value problems for hybrid differential equations with fractional order, Advances in Difference Equations, (2015).
[19] Jarad.J, Abdeljawad.T, and Baleanu.D, Caputo-type modification of the Hadamard fractional derivatives, Adv. Differ. Equ, (2012).
[20] Kilbas.A. A, Srivastava.H.M and Trujillo.J.J, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, 204; Elsevier Science B.V.: Amsterdam, The Netherlands, (2006).
[21] Miller.K.S, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York, 1993.
[22] Podlubny.I, Fractional Differential Equations, Academic Press, New York, 1993.
[23] Podlubny.I, Fractional Differential Equations, Academic Press: San Diego, CA, USA, 1999.
[24] Shahmohammadi.M.A, Mirfatah.S. M, Salehipour.H, Azhari.M, and Civalek.Ö, Free vibration and stability of hybrid nanocomposite-reinforced shallow toroidal shells using an extended closed-form formula based on the Galerkin method. Mechanics of Advanced Materials and Structures, 29(26), 5284-5300, (2022).
[25] Saqib.M, Khan.I and Shafie.S, Application of fractional differential equations to heat transfer in hybrid nanofluid: modeling and solution via integral transforms. Advances in Difference Equations, (1), 1-18, (2019).
[26] Wang.G, Ren.X, Zhang.L and Bachir. Ahmad, Explicit iteration and unique positive solution for a Caputo-Hadamard fractional turbulent flow model, IEEE Access, 7, 109833–109839, (2019).
[27] Yukunthorn.W, Bachir. A, Ntouyas.S. K and Tariboon.J, On Caputo-Hadamard type fractional impulsive hybrid systems with nonlinear fractional integral conditions, Nonlinear Anal. Hybrid Syst, 19, 77–92, (2016).
[28] Zhang.X, On impulsive partial differential equations with Caputo-Hadamard fractional derivatives, Adv. Differ. Equ, (2016).
[29] Zhou.Y, Basic theory of fractional differential equations , Xiangtan University, China, 2014.
[30] M. Echchehira, M. Hannabou, M. Atraoui & M. Bouaouid, *On an image denoising model based on Atangana–Baleanu fractional derivative*, Physica A: Statistical Mechanics and its Applications, 130633, (2025).
[31] M. Hannabou, M. Echchehira, M. Atraoui & M. Bouaouid, *Existence and uniqueness of boundary value problems involving nonlinear hybrid differential equations with Caputo–Fabrizio fractional derivative*, Proyecciones (Antofagasta), 44(3), 516–527, (2025).
[32] Y. Assebbane, M. Echchehira, M. Atraoui, M. Hannabou & M. Bouaouid, *Fractional power series methods for solving fractional differential equations*, Kragujevac Journal of Mathematics, 51(1), 95–111, (2027).
[33] Y. Assebbane, M. Hannabou, M. Atraoui & M. Bouaouid, *Existence and uniqueness of boundary value problems for nonlinear hybrid differential equations with ABC-fractional derivative*, Filomat, 39(11), 3759–3768, (2025).