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Additive results for the inverse along an element in a ring

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Abstract. Let $\mathcal{R}$ be an associative ring. Drazin proved in [4] that if a and $w \in \mathcal{R}$ are Drazin invertible such that $aw = wa = 0$, then $a + w$ is also Drazin invertible. The same results holds for Moore-Penrose inverses in a ring with involution under the condition $aw^* = a^*w = 0$. The generalized invertibility of the sum of two elements is very useful and many authors investigated the sum under different conditions. As the inverse along an element is a generalization of the Drazin inverse and the Moore-Penrose inverses, we give some additive results of the inverse along an element of two invertible elements along elements in an associative ring. If a and w are invertible along d and c respectively, then we show under some conditions that $a + w$ is invertible along some t related to d and c. Moreover, we give the expression of the inverse $(a + w)^{\parallel t}$. As an application, we study the inverse along an element of a 2×2 block matrix. Various examples are given to illustrate our results.

1. Introduction

From now on, $\mathcal{R}$ will denote an associative unital ring whose unity is 1. Let a and x be two elements of $\mathcal{R}$. $axa = a$ says x is an *inner inverse* of a, also, $xax = x$ says x is an *outer inverse* of a. The element x if there exists is not unique in general. To force its uniqueness, we need to impose some further conditions.

An element $a \in \mathcal{R}$ is *Drazin invertible* [4], if there exists an element $x \in \mathcal{R}$ such that

$$xax = x, \ ax = xa, \ a - a^2x \ \text{ is nilpotent.}$$

If such x exists it is unique and it is called the Drazin inverse of a and is denoted by a^D. The *Drazin index* of a, denoted by $ind(a)$, is the nilpotency index of $a - a^2x$. In the case $ind(a) \leq 1$, we say that a is *group invertible* with inverse denoted by $a^\#$.

An *involution* $*$ is a bijection $x \mapsto x^*$ on $\mathcal{R}$, which satisfies the following conditions for all $a, b \in \mathcal{R}$:

i) $(a^*)^* = a$;

ii) $(ab)^* = b^*a^*$;

iii) $(a + b)^* = a^* + b^*$.

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We say that $\mathcal{R}$ is a *-ring if there is an involution on $\mathcal{R}$. If $\mathcal{R}$ is an $*$-ring then $a \in \mathcal{R}$ is *Moore-Penrose invertible* if there exists some $x \in \mathcal{R}$ such that

$$axa = a,\ xax = x,\ (ax)^* = ax,\ (xa)^* = xa.$$

Such x is unique if it exists and is denoted by a^+ (see [18, 20]).

Following Mary [13], an element $a \in \mathcal{R}$ is *invertible along* $d \in \mathcal{R}$ if there is some $x \in \mathcal{R}$ such that

$$xad = d = dax \text{ and } x \in d\mathcal{R} \cap \mathcal{R}d.$$

If such x exists it is unique and called the *inverse of* a *along* d and it is denoted by $a^{\parallel d}$. We denote by $\mathcal{R}^{\parallel d}$ the set of all invertible elements along d in $\mathcal{R}$. This inverse has the advantage that it encompasses several generalized inverses such as the Moore-Penrose inverse and the Drazin inverse:

- We have a is Drazin invertible if and only if $d = a^m$ for some $m \in \mathbb{N}$ i.e., $a^D = a^{\parallel a^m}$;
- We have a is group invertible and $ind(a) = 1$ if and only if $d = a$, i.e., $a^\# = a^{\parallel a}$;
- We have a is invertible if and only if $d = 1$, i.e., $a^{-1} = a^{\parallel 1}$.
- If $\mathcal{R}$ is a ring with involution, a is Moore-Penrose invertible if and only if a is invertible along a^*, i.e., $a^+ = a^{\parallel a^*}$.

For more details about this inverse see [13, 27].

Generalized inverses have quite important applications to difference equations or singular differential, iterative method or multibody system dynamics, Markov chains, cryptography ... etc, see for instance [2, 7, 15, 17, 21–23] and the reference therein.

The usual invertibility in a ring with unit is not in general preserved under addition, i.e., if a and w are two invertible elements in a ring, then $a + w$ is not necessary invertible. So it is natural to ask the following question: *under which condition we have*

$$(a + w)^g = a^g + w^g?$$

where a^g *is one of the generalized inverses of* a.

Drazin [4] showed that if a and $w \in \mathcal{R}$ are Drazin invertible and satisfy $aw = wa = 0$, then

$$(a + w)^D = a^D + w^D.$$

Also if A and W are square matrices then under conditions $AW^* = 0 = AW^*$ Penrose [20] proved that

$$(A + W)^+ = A^+ + W^+.$$

The sum of Drazin inverse and Moore-Penrose inverse for matrices have shown their utility across various applied mathematical contexts, particularly in numerical linear algebra, statistics, linear control theory, perturbation analysis of matrix and projection algorithms, see for instance [1–3, 11, 16, 19, 21, 23].

Several papers discuss the sum of Moore-Penrose inverse for matrices, or the addition of the Drazin inverse and its versions in rings and Banach algebras, either by introducing new conditions or by relying on matrix presentation. See [6–10, 12, 24] and the reference therein.

As the inverse along an element is a generalization of the Drazin, the group inverse and the Moore-Penrose inverses, then motivated by papers cited above, we give some additive results of the inverse along an element of two invertible elements along elements in an associative ring. If a and w are invertible along d and c respectively, then we show under some conditions that $a + w$ is invertible along some t related to d and c. Moreover, we give the expression of the inverse $(a + w)^{\parallel t}$. As an application, we study the inverse along an element of a 2×2 block matrix. Many examples are given to illustrate our results.

2. The inverse along an element for the sum of two elements

We start by the main result which generalizes the Drazin's result and the Penrose's result for the sum of two Drazin (Moore-Penrose) invertible elements.

Theorem 2.1. *Let $a \in \mathcal{R}^{\|d}$ and $w \in \mathcal{R}^{\|c}$ with $d \neq c$. If $ac = ca = 0$ and $wd = dw = 0$ then $a + w \in \mathcal{R}^{\|(d+c)}$ and*

$$(a + w)^{\|(d+c)} = a^{\|d} + w^{\|c}.$$

Proof. Since $a \in \mathcal{R}^{\|d}$ then $a^{\|d} \in d\mathcal{R} \cap \mathcal{R}d$ and $a^{\|d}ad = d = daa^{\|d}$. Also $w \in \mathcal{R}^{\|c}$ gives $w^{\|c} \in c\mathcal{R} \cap \mathcal{R}c$ and $w^{\|c}wc = c = cww^{\|c}$.
Set $y_1 = a^{\|d}$ and $y_2 = w^{\|c}$. Then

$$\begin{cases} y_1 = dt = xd \text{ for some } t, x \in \mathcal{R} \text{ and } & y_1ad = d = day_1. \\ y_2 = ct' = x'c \text{ for some } t', x' \in \mathcal{R} \text{ and } & y_2wc = c = cwy_2. \end{cases}$$

Now set $y = y_1 + y_2$. We will show that $y = (a + w)^{\|d+c}$.

(*i*) $y \in (d + c)\mathcal{R} \cap \mathcal{R}(d + c)$. Indeed:

$$\begin{aligned}(xy_1a + x'y_2w)(d + c) &= xy_1ad + xy_1ac + x'y_2wd + x'y_2wc \\ &= xd + 0 + 0 + x'c \\ &= y_1 + y_2 \\ &= y.\end{aligned}$$

Hence $y \in \mathcal{R}(d + c)$. Also

$$\begin{aligned}(d + c)(ay_1t + wy_2t') &= day_1t + dwy_2t' + cay_1t + cwy_2t' \\ &= dt + 0 + 0 + ct' \\ &= y_1 + y_2 \\ &= y.\end{aligned}$$

Thus $y \in (d + c)\mathcal{R}$. Consequently, we obtain that $y \in (d + c)\mathcal{R} \cap \mathcal{R}(d + c)$.

(*ii*) We have

$$\begin{aligned}y(a + w)(d + c) &= yad + yac + ywd + ywc \\ &= yad + ywc = y_1ad + y_1wc + y_2ad + y_2wc \\ &= d + xdwc + x'cad + c \\ &= d + 0 + 0 + c \\ &= d + c.\end{aligned}$$

(*iii*) We also have

$$\begin{aligned}(d + c)(a + w)y &= day + dwy + cay + cwy \\ &= day + cwy = day_1 + cwy_1 + day_2 + cwy_2 \\ &= d + cwdt + dact' + c \\ &= d + 0 + 0 + c = d + c.\end{aligned}$$

From (*i*), (*ii*) and (*iii*) we conclude that $y = a^{\|d} + w^{\|c}$ is the inverse of $a + w$ along $d + c$. □

Example 2.2. Let $\mathcal{R} = \mathcal{M}_2(\mathbb{Q})$ be the ring of 2×2 matrices over $\mathbb{Q}$ and let $a, w, d, c \in \mathcal{R}$ be given by

$$a = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad d = \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}.$$

It is easy to see that a is invertible along d with $a^{\|d} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}$ and w is invertible along c with $w^{\|c} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$. Also, we have $ac = ca = 0 = wd = dw$. Applying Theorem 2.1, we get that $a + w = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ is invertible along $d + c = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}^{\|\begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}$.

By induction we obtain the following corollary.

Corollary 2.3. *Let $a_1, \dots, a_n \in \mathcal{R}$ and $d_1, \dots, d_n \in \mathcal{R}$ be such that $a_i \in \mathcal{R}^{\|d_i}$, $a_i d_j = d_j a_i = 0$ for $i \neq j$. Then the sum $a_1 + \cdots + a_n$ is invertible along $d_1 + \cdots + d_n$ with inverse*

$$(a_1 + \cdots + a_n)^{\|(d_1+\cdots+d_n)} = a_1^{\|d_1} + \cdots + a_n^{\|d_n}.$$

As an immediate consequence of the previous theorem, we retrieve the result obtained in [4, Corollary 1]

Corollary 2.4. *Let $a, w \in \mathcal{R}$. If a and w are Drazin invertible such that $aw = wa = 0$, then $a + w$ is also Drazin invertible.*

Proof. Assume that a and w are Drazin invertible. Then a is invertible along a^m and w is invertible along w^n with $a^D = a^{\|a^m}$ and $w^D = w^{\|w^n}$, where $m = ind(a)$ and $n = ind(w)$. Let $k = \max(m, n)$. Set $d = a^k$ and $c = w^k$. Since $aw = wa = 0$, then $ac = ca = 0 = wd = dw$. Now by Theorem 2.1, $a + w \in \mathcal{R}^{\|(d+c)} = \mathcal{R}^{\|(a^k+w^k)} = \mathcal{R}^{\|(a+w)^k}$. Therefore $a + w$ is Drazin invertible and $(a + w)^D = a^D + w^D$. □

We retrieve also the result of Penrose [20].

Corollary 2.5. *Let $\mathcal{R}$ be a *-ring. If a and $w \in \mathcal{R}$ are Moore-Penrose invertible such that $aw^* = w^*a = 0$, then their sum $a + w$ is also Moore-Penrose invertible; and $(a + w)^+ = a^+ + w^+$.*

Proof. We have a and w are Moore-Penrose invertible is equivalent to $a^{\|a^*}$ and $w^{\|w^*}$ exist. Set $d = a^*$ and $c = w^*$, we have $ac = ca = 0$ and hence $(ac)^* = (ca)^* = 0$ i.e., $c^*a^* = wa^* = a^*w = 0 = dw = wd$. According to Theorem 2.1,

$$a + w \in \mathcal{R}^{\|a^*+w^*} = R^{\|(a+w)^*},$$

which is equivalent to $a + w$ is Moore-Penrose invertible and

$$(a + w)^+ = a^+ + w^+.$$

□

Remark 2.6. *In the Theorem 2.1, we assume that $d \neq c$. In the case $d = c$ and under the same conditions as in Theorem 2.1, it is not difficult to see that automatically $d = 0$. Which is a trivial case, since every element is invertible along 0.*

3. The inverse along an element for 2 × 2 block matrices

For a 2×2 matrix over a ring, the inverse along a 2×2 matrix were studied in [14, 26]. In this section, we applied the obtained results in Section 2 to investigate the inverse along an element for 2×2 matrices over a ring.

Lemma 3.1. *Let $\mathcal{M}_2(\mathcal{R})$ be the ring of 2×2 matrices over the ring $\mathcal{R}$. If $a \in \mathcal{R}^{\|d}$ and $w \in \mathcal{R}^{\|c}$ then we have $A \in \mathcal{M}_2(\mathcal{R})^{\|D}$ and $W \in \mathcal{M}_2(\mathcal{R})^{\|C}$ where*

$$A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, W = \begin{pmatrix} 0 & 0 \\ 0 & w \end{pmatrix}, D = \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}.$$

In this case, we have

$$A^{\|D} = \begin{pmatrix} a^{\|d} & 0 \\ 0 & 0 \end{pmatrix} \text{ and } W^{\|C} = \begin{pmatrix} 0 & 0 \\ 0 & w^{\|c} \end{pmatrix}.$$

Proof. Suppose that $a \in \mathcal{R}^{\|d}$. Then $a^{\|d} = xd = dy$ for some $x, y \in \mathcal{R}$. Set $Y = \begin{pmatrix} a^{\|d} & 0 \\ 0 & 0 \end{pmatrix}$. We have

- $YAD = \begin{pmatrix} a^{\|d} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a^{\|d}ad & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix} = D$

- $DAY = \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a^{\|d} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} daa^{\|d} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix} = D$

- $\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} xd & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a^{\|d} & 0 \\ 0 & 0 \end{pmatrix} = Y \Rightarrow Y \in \mathcal{M}_2(\mathcal{R})D.$

- $\begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} dy & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a^{\|d} & 0 \\ 0 & 0 \end{pmatrix} = Y \Rightarrow Y \in D\mathcal{M}_2(\mathcal{R}).$

  Hence we obtain that Y is the inverse of A along D.

  Similarly, we show that W is invertible along C and $W^{\|C} = \begin{pmatrix} 0 & 0 \\ 0 & w^{\|c} \end{pmatrix}$. □

Theorem 3.2. *Let $\mathcal{M}_2(\mathcal{R})$ the ring of 2×2 matrices over the ring $\mathcal{R}$. If $a \in \mathcal{R}^{\|d}$ and $w \in \mathcal{R}^{\|c}$ then $\begin{pmatrix} a & 0 \\ 0 & w \end{pmatrix} \in \mathcal{M}_2(\mathcal{R})^{\left\| \begin{pmatrix} d & 0 \\ 0 & c \end{pmatrix} \right.}$ and*

$$\begin{pmatrix} a & 0 \\ 0 & w \end{pmatrix}^{\left\| \begin{pmatrix} d & 0 \\ 0 & c \end{pmatrix} \right.} = \begin{pmatrix} a^{\|d} & 0 \\ 0 & w^{\|c} \end{pmatrix}.$$

If $d = c$, we get

$$\begin{pmatrix} a & 0 \\ 0 & w \end{pmatrix}^{\left\| \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \right.} = \begin{pmatrix} a^{\|d} & 0 \\ 0 & w^{\|d} \end{pmatrix}.$$

Proof. Set

$$A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, D = \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}, W = \begin{pmatrix} 0 & 0 \\ 0 & w \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}.$$

As $a \in \mathcal{R}^{\|d}$ and $w \in \mathcal{R}^{\|c}$, it follows from Lemma 3.1 that $A \in \mathcal{M}_2(\mathcal{R})^{\|D}$ and $W \in \mathcal{M}_2(\mathcal{R})^{\|C}$ with

$$A^{\|D} = \begin{pmatrix} a^{\|d} & 0 \\ 0 & 0 \end{pmatrix} \text{ and } W^{\|C} = \begin{pmatrix} 0 & 0 \\ 0 & w^{\|c} \end{pmatrix}.$$

Furthermore, we have $AC = CA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = WD = DW$. Now if we apply Theorem 2.1 for A and W, we obtain $A + W = \begin{pmatrix} a & 0 \\ 0 & w \end{pmatrix}$ is invertible along $D + C = \begin{pmatrix} d & 0 \\ 0 & c \end{pmatrix}$ and

$$(A + W)^{\|D+C} = A^{\|D} + W^{\|C} = \begin{pmatrix} a^{\|d} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & w^{\|c} \end{pmatrix} = \begin{pmatrix} a^{\|d} & 0 \\ 0 & w^{\|c} \end{pmatrix}.$$

□

Theorem 3.3. *Let $\mathcal{R}$ be a ring and $\mathcal{M}_2(\mathcal{R})$ be the ring of 2×2 matrices over $\mathcal{R}$. If $z \in \mathcal{R}$, $a \in \mathcal{R}^{\|d}$ and $w \in \mathcal{R}^{\|c}$ then, $X = \begin{pmatrix} a & z \\ 0 & w \end{pmatrix}$ is invertible along $T = \begin{pmatrix} d & dc \\ 0 & c \end{pmatrix}$ and $X^{\|T} = \begin{pmatrix} a^{\|d} & -a^{\|d}zw^{\|c} \\ 0 & w^{\|c} \end{pmatrix}$.*

Proof. We will prove that $Y = \begin{pmatrix} a^{\|d} & -a^{\|d}zw^{\|c} \\ 0 & w^{\|c} \end{pmatrix}$ is the inverse of X along T.

(i) We have

$$\begin{aligned} YXT &= \begin{pmatrix} a^{\|d} & -a^{\|d}zw^{\|c} \\ 0 & w^{\|c} \end{pmatrix}\begin{pmatrix} a & z \\ 0 & w \end{pmatrix}\begin{pmatrix} d & dc \\ 0 & c \end{pmatrix} \\ &= \begin{pmatrix} a^{\|d}a & a^{\|d}z - a^{\|d}zw^{\|c}w \\ 0 & w^{\|c}w \end{pmatrix}\begin{pmatrix} d & dc \\ 0 & c \end{pmatrix} \\ &= \begin{pmatrix} a^{\|d}ad & a^{\|d}adc + a^{\|d}zc - a^{\|d}zw^{\|c}wc \\ 0 & w^{\|c}wc \end{pmatrix} \\ &= \begin{pmatrix} d & dc \\ 0 & c \end{pmatrix} = T. \end{aligned}$$

(ii) Also

$$\begin{aligned} TXY &= \begin{pmatrix} d & dc \\ 0 & c \end{pmatrix}\begin{pmatrix} a & z \\ 0 & w \end{pmatrix}\begin{pmatrix} a^{\|d} & -a^{\|d}zw^{\|c} \\ 0 & w^{\|c} \end{pmatrix} \\ &= \begin{pmatrix} da & dz + dcw \\ 0 & cw \end{pmatrix}\begin{pmatrix} a^{\|d} & -a^{\|d}zw^{\|c} \\ 0 & w^{\|c} \end{pmatrix} \\ &= \begin{pmatrix} daa^{\|d} & -daa^{\|d}zw^{\|c} + dzw^{\|c} + dcww^{\|c} \\ 0 & cww^{\|c} \end{pmatrix} \\ &= \begin{pmatrix} d & -dzw^{\|c} + dzw^{\|c} + dc \\ 0 & c \end{pmatrix} = \begin{pmatrix} d & dc \\ 0 & c \end{pmatrix} = T. \end{aligned}$$

(iii) As $a \in \mathcal{R}^{\|d}, w \in \mathcal{R}^{\|c}$ then $a^{\|d} = dx = yd$ and $w^{\|c} = cx' = y'c$, for some $x, y, x', y' \in \mathcal{R}$.
Let

$$V = \begin{pmatrix} x & -w^{\|c} - xzw^{\|c} \\ 0 & x' \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} y & -a^{\|d} - a^{\|d}zy' \\ 0 & y' \end{pmatrix}.$$

Then

$$\begin{aligned} TV &= \begin{pmatrix} d & dc \\ 0 & c \end{pmatrix}\begin{pmatrix} x & -w^{\|c} - xzw^{\|c} \\ 0 & x' \end{pmatrix} \\ &= \begin{pmatrix} dx & -dw^{\|c} - dxzw^{\|c} + dcx' \\ 0 & cx' \end{pmatrix} \\ &= \begin{pmatrix} a^{\|d} & -dw^{\|c} - a^{\|d}zw^{\|c} + dw^{\|c} \\ 0 & w^{\|c} \end{pmatrix} \\ &= \begin{pmatrix} a^{\|d} & -a^{\|d}zw^{\|c} \\ 0 & w^{\|c} \end{pmatrix} = Y. \end{aligned}$$

Hence $Y \in T\mathcal{M}_2(\mathcal{R})$. Also

$$\begin{aligned} UT &= \begin{pmatrix} y & -a^{\|d} - a^{\|d}zy' \\ 0 & y' \end{pmatrix}\begin{pmatrix} d & dc \\ 0 & c \end{pmatrix} \\ &= \begin{pmatrix} yd & ydc - a^{\|d}c - a^{\|d}zy'c \\ 0 & y'c \end{pmatrix} \\ &= \begin{pmatrix} a^{\|d} & a^{\|d}c - a^{\|d}c - a^{\|d}zw^{\|c} \\ 0 & w^{\|c} \end{pmatrix} \\ &= \begin{pmatrix} a^{\|d} & -a^{\|d}zw^{\|c} \\ 0 & w^{\|c} \end{pmatrix} = Y. \end{aligned}$$

Thus $Y \in \mathcal{M}_2(\mathcal{R})T$. Therefore, we conclude that X is invertible along T and

$$X^{\|T} = \begin{pmatrix} a & z \\ 0 & w \end{pmatrix}^{\|\begin{pmatrix} d & dc \\ 0 & c \end{pmatrix}} = \begin{pmatrix} a^{\|d} & -a^{\|d}zw^{\|c} \\ 0 & w^{\|c} \end{pmatrix}.$$

□

With same argument we obtain the following result for lower triangular matrices.

Corollary 3.4. *Let $\mathcal{R}$ be a ring and $\mathcal{M}_2(\mathcal{R})$ be the ring of 2×2 matrices over $\mathcal{R}$. If $z \in \mathcal{R}$, $a \in \mathcal{R}^{\|d}$ and $w \in \mathcal{R}^{\|c}$ then, $X = \begin{pmatrix} a & 0 \\ z & w \end{pmatrix}$ is invertible along $T = \begin{pmatrix} d & 0 \\ cd & c \end{pmatrix}$ and $X^{\|T} = \begin{pmatrix} a^{\|d} & 0 \\ -w^{\|c}za^{\|d} & w^{\|c} \end{pmatrix}$.*

Remark 3.5. *In the previous theorem, if we take $z = 0$ then we get $\begin{pmatrix} a & 0 \\ 0 & w \end{pmatrix}$ is invertible along $\begin{pmatrix} d & dc \\ 0 & c \end{pmatrix}$ with inverse $\begin{pmatrix} a^{\|d} & 0 \\ 0 & w^{\|c} \end{pmatrix}$. Also by Theorem 3.2, $\begin{pmatrix} a & 0 \\ 0 & w \end{pmatrix}$ is invertible along $\begin{pmatrix} d & 0 \\ 0 & c \end{pmatrix}$ with the same inverse $\begin{pmatrix} a^{\|d} & 0 \\ 0 & w^{\|c} \end{pmatrix}$.*

Example 3.6. Let $\mathcal{R} = \mathbb{Q}$. We have 2 is invertible along 3 with $2^{\|3} = \frac{1}{2}$, and -5 is invertible along -4 with $-5^{\|-4} = -\frac{1}{5}$. By Theorem 3.3, we have for all $z \in \mathbb{Q}$, $\begin{pmatrix} 2 & z \\ 0 & -5 \end{pmatrix}$ is invertible along $\begin{pmatrix} 3 & -12 \\ 0 & -4 \end{pmatrix}$ and

$$\begin{pmatrix} 2 & z \\ 0 & -5 \end{pmatrix}^{\|\begin{pmatrix} 3 & -12 \\ 0 & -4 \end{pmatrix}} = \begin{pmatrix} \frac{1}{2} & \frac{1}{10}z \\ 0 & -\frac{1}{5} \end{pmatrix}.$$

One may expect that the converse of the previous theorem holds. The following theorem answers this question in a manner.

Theorem 3.7. *Let $\mathcal{UT}_2(\mathcal{R})$ be the ring of 2×2 upper triangular matrices over the ring $\mathcal{R}$ and let a, w, d, c and $z \in \mathcal{R}$. Then we have $a \in \mathcal{R}^{\|d}$ and $w \in \mathcal{R}^{\|c}$ if and only if $X = \begin{pmatrix} a & z \\ 0 & w \end{pmatrix}$ is invertible along $T = \begin{pmatrix} d & dc \\ 0 & c \end{pmatrix}$.*

In this case, $X^{\|T} = \begin{pmatrix} a^{\|d} & -a^{\|d}zw^{\|c} \\ 0 & w^{\|c} \end{pmatrix}$.

Proof. Assume that $X = \begin{pmatrix} a & z \\ 0 & w \end{pmatrix}$ is invertible along $T = \begin{pmatrix} d & dc \\ 0 & c \end{pmatrix}$ with $X^{\|T} = Y = \begin{pmatrix} \alpha & -\alpha z\beta \\ 0 & \beta \end{pmatrix}$. Then

$$YXT = T = TXY \text{ and } Y = T\mathcal{UT}_2(\mathcal{R}) \cap \mathcal{UT}_2(\mathcal{R})T.$$

Firstly,

$$TXY = \begin{pmatrix} d a\alpha & -d a\alpha z\beta + dz\beta + dcw\beta \\ 0 & cw\beta \end{pmatrix}$$

and

$$YTX = \begin{pmatrix} \alpha ad & \alpha adc + \alpha zc - \alpha z\beta wc \\ 0 & \beta wc \end{pmatrix}.$$

By matrices identification, we get that $\alpha ad = da\alpha = d$ and $cw\beta = \beta wc = c$.

On the other side, let $Y = TS = NT$ with $S = \begin{pmatrix} s_1 & s_2 \\ 0 & s_4 \end{pmatrix} \in \mathcal{UT}_2(\mathcal{R})$ and $N = \begin{pmatrix} n_1 & n_2 \\ 0 & n_4 \end{pmatrix} \in \mathcal{UT}_2(\mathcal{R})$. Then

$$TS = \begin{pmatrix} ds_1 & ds_2 + dc_s 4 \\ 0 & cs_4 \end{pmatrix} = Y = \begin{pmatrix} \alpha & -\alpha z\beta \\ 0 & \beta \end{pmatrix}.$$

Thus, $\alpha \in d\mathcal{R}$ and $\beta \in c\mathcal{R}$. Furthermore,

$$NT = \begin{pmatrix} n_1 d & n_1 dc + n_2 c \\ 0 & n_4 c \end{pmatrix} = Y = \begin{pmatrix} \alpha & -\alpha z\beta \\ 0 & \beta \end{pmatrix}$$

which implies that $\alpha \in \mathcal{R}d$ and $\beta \in \mathcal{R}c$. Hence $\alpha \in d\mathcal{R} \cap \mathcal{R}d$ and $\beta \in c\mathcal{R} \cap \mathcal{R}c$. Finally, we conclude that $a \in \mathcal{R}^{\parallel d}$ and $w \in \mathcal{R}^{\parallel c}$ with $a^{\parallel d} = \alpha$ and $w^{\parallel c} = \beta$.

The only if is obtained by Theorem 3.3. □

Corollary 3.8. *Let $\mathcal{LT}_2(\mathcal{R})$ be the ring of 2×2 lower triangular matrices over the ring $\mathcal{R}$ and let a, w, d, c and $z \in \mathcal{R}$. Then we have $a \in \mathcal{R}^{\parallel d}$ and $w \in \mathcal{R}^{\parallel c}$ if and only if $X = \begin{pmatrix} a & 0 \\ z & w \end{pmatrix}$ is invertible along $T = \begin{pmatrix} d & 0 \\ cd & c \end{pmatrix}$.*

In this case, $X^{\parallel T} = \begin{pmatrix} a^{\parallel d} & 0 \\ -w^{\parallel c} z a^{\parallel d} & w^{\parallel c} \end{pmatrix}$.

Theorem 3.9. *If $z, t \in \mathcal{R}$, $a \in \mathcal{R}^{\parallel d}$ and $w \in \mathcal{R}^{\parallel c}$ such that $ct = 0 = td$, then $\begin{pmatrix} a & z \\ t & w \end{pmatrix}$ is invertible along $\begin{pmatrix} d & dc \\ 0 & c \end{pmatrix}$ and*

$$\begin{pmatrix} a & z \\ t & w \end{pmatrix}^{\parallel \begin{pmatrix} d & dc \\ 0 & c \end{pmatrix}} = \begin{pmatrix} a^{\parallel d} & -a^{\parallel d} z w^{\parallel c} \\ 0 & w^{\parallel c} \end{pmatrix}.$$

Proof. We have $\begin{pmatrix} a & z \\ t & w \end{pmatrix} = Q_1 + Q_2$, with $Q_1 = \begin{pmatrix} a & z \\ 0 & w \end{pmatrix}$ and $Q_2 = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}$. By Theorem 3.3, Q_1 is invertible along $\begin{pmatrix} d & dc \\ 0 & c \end{pmatrix}$ and

$$Q_1^{\parallel \begin{pmatrix} d & dc \\ 0 & c \end{pmatrix}} = \begin{pmatrix} a^{\parallel d} & -a^{\parallel d} z w^{\parallel c} \\ 0 & w^{\parallel c} \end{pmatrix}.$$

Furthermore, Q_2 is invertible along $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ with $Q_2^{\parallel \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Also $\begin{pmatrix} d & dc \\ 0 & c \end{pmatrix}\begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}\begin{pmatrix} d & dc \\ 0 & c \end{pmatrix}$, since $td = 0 = ct$. Applying Theorem 2.1, we obtain that $Q_1 + Q_2$ is invertible along $\begin{pmatrix} d & dc \\ 0 & c \end{pmatrix}$ and

$$(Q_1 + Q_2)^{\parallel \begin{pmatrix} d & dc \\ 0 & c \end{pmatrix}} = \begin{pmatrix} a & z \\ t & w \end{pmatrix}^{\parallel \begin{pmatrix} d & dc \\ 0 & c \end{pmatrix}} = \begin{pmatrix} a^{\parallel d} & -a^{\parallel d} z w^{\parallel c} \\ 0 & w^{\parallel c} \end{pmatrix}.$$

□

Corollary 3.10. *If $z, t \in \mathcal{R}$, $a \in \mathcal{R}^{\|d}$ and $w \in \mathcal{R}^{\|c}$ such that $zc = 0 = dz$, then $\begin{pmatrix} a & z \\ t & w \end{pmatrix}$ is invertible along $\begin{pmatrix} d & 0 \\ cd & c \end{pmatrix}$ and*

$$\begin{pmatrix} a & z \\ t & w \end{pmatrix}^{\|\begin{pmatrix} d & 0 \\ cd & c \end{pmatrix}} = \begin{pmatrix} a^{\|d} & 0 \\ -w^{\|c}ta^{\|d} & w^{\|c} \end{pmatrix}.$$

If we take $t = 0$ in Corollary 3.10, we obtain the following corollary.

Corollary 3.11. *Let $\mathcal{M}_2(\mathcal{R})$ be the ring of 2×2 matrices over the ring $\mathcal{R}$. If $t \in \mathcal{R}$, $a \in \mathcal{R}^{\|d}$ and $w \in \mathcal{R}^{\|c}$ such that $zc = 0 = dz$, then $\begin{pmatrix} a & z \\ 0 & w \end{pmatrix}$ is invertible along $\begin{pmatrix} d & 0 \\ cd & c \end{pmatrix}$ and*

$$\begin{pmatrix} a & z \\ 0 & w \end{pmatrix}^{\|\begin{pmatrix} d & 0 \\ cd & c \end{pmatrix}} = \begin{pmatrix} a^{\|d} & 0 \\ 0 & w^{\|c} \end{pmatrix}.$$

The following corollary is also an immediate consequence of Theorem 3.9.

Corollary 3.12. *Let $c, d \in \mathcal{R}$ and $a, w \in \mathcal{M}_2(\mathcal{R})$ such that $a = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ and $w = \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix}$. Assume that the following conditions hold:*
i) $a_1w_1 + a_2w_3$ is invertible along d;
ii) $a_3w_2 + a_4w_4$ is invertible along c;
iii) $c(a_3w_1 + a_4w_3) = 0 = (a_3w_1 + a_4w_3)d$.
Then $aw = \begin{pmatrix} a_1w_1 + a_2w_3 & a_1w_2 + a_2w_4 \\ a_3w_1 + a_4w_3 & a_3w_2 + a_4w_4 \end{pmatrix}$ is invertible along $\begin{pmatrix} d & dc \\ 0 & c \end{pmatrix}$, with inverse

$$\begin{pmatrix} (a_1w_1 + a_2w_3)^{\|d} & -(a_1w_1 + a_2w_3)^{\|d}(a_1w_2 + a_2w_4)(a_3w_2 + a_4w_4)^{\|c} \\ 0 & (a_3w_2 + a_4w_4)^{\|c} \end{pmatrix}.$$

Example 3.13. Let $\mathcal{R} = \mathcal{M}_2(\mathbb{Q})$ and $a, d, w, c, z, t \in \mathcal{R}$ such that

$$a = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}; \quad d = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}; \quad w = \begin{pmatrix} 0 & 0 \\ 0 & -5 \end{pmatrix}; \quad c = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

$$t = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}; \quad z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \text{ with } z_1, z_2, z_3, z_4 \in \mathbb{Q} \text{ arbitrary.}$$

We have a is invertible along d with $a^{\|d} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{pmatrix}$ and w is invertible along c with $w^{\|c} = \begin{pmatrix} 0 & 0 \\ \frac{-1}{5} & \frac{-1}{5} \end{pmatrix}$. Also we can check that $ct = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = td$. Then by Theorem 3.9, $\begin{pmatrix} a & z \\ t & w \end{pmatrix}$ is invertible along $\begin{pmatrix} d & dc \\ 0 & c \end{pmatrix}$ with inverse

$$\begin{pmatrix} a & z \\ t & w \end{pmatrix}^{\|\begin{pmatrix} d & dc \\ 0 & c \end{pmatrix}} = \begin{pmatrix} a^{\|d} & -a^{\|d}zw^{\|c} \\ 0 & w^{\|c} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{z_2+z_4}{10} & \frac{z_2+z_4}{10} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{5} & \frac{-1}{5} \end{pmatrix}.$$

Theorem 3.14. *Let $\mathcal{M}_2(\mathcal{R})$ be the ring of 2×2 matrices over the ring $\mathcal{R}$. If $t \in \mathcal{R}$, $a \in \mathcal{R}^{\|d}$ and $w \in \mathcal{R}^{\|c}$ such that $ct = 0 = td$, then $\begin{pmatrix} a & 0 \\ t & w \end{pmatrix}$ is invertible along $\begin{pmatrix} d & 0 \\ 0 & c \end{pmatrix}$ and*

$$\begin{pmatrix} a & 0 \\ t & w \end{pmatrix}^{\|\begin{pmatrix} d & 0 \\ 0 & c \end{pmatrix}} = \begin{pmatrix} a^{\|d} & 0 \\ 0 & w^{\|c} \end{pmatrix}.$$

Proof. Set $\begin{pmatrix} a & 0 \\ t & w \end{pmatrix} = Q_1 + Q_2$, where $Q_1 = \begin{pmatrix} a & 0 \\ 0 & w \end{pmatrix}$ and $Q_2 = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}$. We have, Q_2 is invertible along $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ with inverse $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Also by Theorem 3.2, Q_1 is invertible along $\begin{pmatrix} d & 0 \\ 0 & c \end{pmatrix}$ with $\begin{pmatrix} a^{\|d} & 0 \\ 0 & c^{\|w} \end{pmatrix}$ as inverse. Now since

$$Q_2\begin{pmatrix} d & 0 \\ 0 & c \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & c \end{pmatrix}Q_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

it follows from Theorem 2.1 that $\begin{pmatrix} a & 0 \\ t & w \end{pmatrix} = Q_1 + Q_2$ is invertible along $\begin{pmatrix} d & 0 \\ 0 & c \end{pmatrix}$ with $\begin{pmatrix} a & 0 \\ t & w \end{pmatrix}^{\left\|\begin{pmatrix} d & 0 \\ 0 & c \end{pmatrix}\right.} = \begin{pmatrix} a^{\|d} & 0 \\ 0 & w^{\|c} \end{pmatrix}$. □

Lemma 3.15. *Let $a, d \in \mathcal{R}$ be such that a^2 is invertible along d. Then a is invertible along ad and $a^{\|ad} = a(a^2)^{\|d}$. Also a is invertible along da and $a^{\|da} = (a^2)^{\|d}a$.*

Proof. Assume that $a^2 \in \mathcal{R}^{\|d}$. Let $x = (a^2)^{\|d}$. Then $xa^2d = d = da^2x$ and $x \in d\mathcal{R} \cap \mathcal{R}d$. Hence, $(ax)a(ad) = ad = (ad)a(ax)$ and $ax \in ad\mathcal{R} \cap a\mathcal{R}d \subseteq ad\mathcal{R} \cap \mathcal{R}da^2x \subseteq ad\mathcal{R} \cap \mathcal{R}ax$. Therefore $a \in \mathcal{R}^{\|ad}$ and $a^{\|ad} = ax$.

With the same argument, we show that a is invertible along da and $a^{\|da} = (a^2)^{\|d}a$. □

Theorem 3.16. *Let $a, w, z, t \in \mathcal{R}$ be such that $a \in \mathcal{R}^{\|d}$, $w \in \mathcal{R}^{\|c}$ and $zt \in \mathcal{R}^{\|u}$. Assume that the following conditions hold:*

i) $uzt = ztu$;
ii) $td = ct = 0$;
iii) $zc = dz = 0$;
iv) $wtu = tua = 0$;
v) $ztuzw = aztuz = 0$.

Then $\begin{pmatrix} a & z \\ t & w \end{pmatrix}$ is invertible along $\begin{pmatrix} d & ztuz \\ tu & c \end{pmatrix}$ and

$$\begin{pmatrix} a & z \\ t & w \end{pmatrix}^{\left\|\begin{pmatrix} d & ztuz \\ tu & c \end{pmatrix}\right.} = \begin{pmatrix} a^{\|d} & z(tz)^{\|tuz} \\ t(zt)^{\|u} & w^{\|c} \end{pmatrix}.$$

Proof. Let $\begin{pmatrix} a & z \\ t & w \end{pmatrix} = Q_1 + Q_2$, where $Q_1 = \begin{pmatrix} a & 0 \\ 0 & w \end{pmatrix}$ and $Q_2 = \begin{pmatrix} 0 & z \\ t & 0 \end{pmatrix}$. We have $Q_2^2 = \begin{pmatrix} zt & 0 \\ 0 & tz \end{pmatrix}$. Since zt is invertible along u, then by virtue of Theorem 6.1 in [5], tz is invertible along tuz and $(tz)^{\|tuz} = t((zt)^{\|u})^2z$. Thus it follows from Theorem 3.2 that $Q_2^2 = \begin{pmatrix} zt & 0 \\ 0 & tz \end{pmatrix}$ is invertible along $\begin{pmatrix} u & 0 \\ 0 & tuz \end{pmatrix}$ and

$$\begin{pmatrix} zt & 0 \\ 0 & tz \end{pmatrix}^{\left\|\begin{pmatrix} u & 0 \\ 0 & tuz \end{pmatrix}\right.} = \begin{pmatrix} (zt)^{\|u} & 0 \\ 0 & (tz)^{\|tuz} \end{pmatrix}.$$

Moreover, it follows from Lemma 3.15 that Q_2 is invertible along $\begin{pmatrix} 0 & ztuz \\ tu & 0 \end{pmatrix}$ with inverse $\begin{pmatrix} 0 & z(tz)^{\|tuz} \\ t(zt)^{\|u} & 0 \end{pmatrix}$.

Since

$$Q_1\begin{pmatrix} 0 & ztuz \\ tu & 0 \end{pmatrix} = \begin{pmatrix} 0 & ztuz \\ tu & 0 \end{pmatrix}Q_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$Q_2\begin{pmatrix} d & 0 \\ 0 & c \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & c \end{pmatrix}Q_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

then applying Theorem 2.1 to Q_1 and Q_2 we obtain $\begin{pmatrix} a & z \\ t & w \end{pmatrix}$ is invertible along $\begin{pmatrix} d & ztuz \\ tu & c \end{pmatrix}$ and

$$\begin{pmatrix} a & z \\ t & w \end{pmatrix}^{\|\begin{pmatrix} d & ztuz \\ tu & c \end{pmatrix}} = \begin{pmatrix} a^{\|d} & z(tz)^{\|tuz} \\ t(zt)^{\|u} & w^{\|c} \end{pmatrix}.$$

□

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