Filomat 39:34 (2025), 12189–12205
https://doi.org/10.2298/FIL2534189D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some results on the soft crossed polymodules

Mohammad Ali Dehghanizadeh[a]

[a]*Department of Basic Sciences, Technical and Vocational University(TVU), Tehran, Iran*

"This paper is dedicated to Professor Bijan Davvaz on the occasion of his 60th birthday."

Abstract. In this article, we present and rigorously analyze the notion of *soft polyaction*, providing a detailed framework together with several fundamental characterizations of this newly introduced concept. Beyond the introductory formulation, we further develop the associated structures by defining and examining the notions of *stabilizer*, *normalizer*, and *centralizer* within the context of soft polyactions. Particular attention is devoted to clarifying the intricate interrelations among these structures, and we supply explicit examples that illustrate their behavior and highlight their relevance. A central objective of this study is to extend the algebraic machinery of group theory into the broader setting of hyperstructures. To this end, we investigate the semi-direct product of soft polygroups and establish an analogue of Cayley's theorem for such objects. These developments naturally culminate in the formulation of *soft crossed polymodules*, which provide a unifying perspective and serve as a bridge between traditional group-theoretic constructions and the more general theory of algebraic hyperstructures. The proposed framework not only broadens the scope of classical algebraic results but also clarifies methodological pathways for future research. By introducing the notion of soft crossed polymodules, we demonstrate how researchers can systematically extend well-established concepts from group theory to the rich and increasingly significant domain of soft algebraic hyperstructures.

1. Introduction

The contemporary landscape of scientific inquiry, spanning disciplines from theoretical physics and engineering to economics and environmental studies, is increasingly characterized by problems ill-defined by precise, deterministic data. This inherent ambiguity prevents the tractable application of conventional mathematical methodologies. Consequently, the necessity for robust frameworks capable of modeling and resolving such pervasive uncertainty has spurred extensive theoretical and applied investigation.

Fuzzy set theory[40] and rough set theory[29], are among the famous theories that can be mentioned in this passage. For further reading, refer to [25, 32, 37, 39]. These methods, with all the positive things they have, also have some problems. To improve the previous methods and solve problems in the study of uncertainty problems, in 1999, the concept of soft sets was presented by Molodtsov [28]. Since then, many studies have been carried out by different people on this theory, and these studies continue at a rapid pace [2, 26, 27, 29]. A study on the use of soft sets in decision-making problems has been done by Maji et al.

2020 *Mathematics Subject Classification*. Primary 20N20.
Keywords. Soft set, Polygroup, Soft polygroup, Crossed polymodule, Soft crossed polymodule.
Received: 28 May 2025; Revised: 21 October 2025; Accepted: 23 October 2025
Communicated by Dijana Mosić
Email address: Mdehghanizadeh@tvu.ac.ir, Dehghani19@yahoo.com (Mohammad Ali Dehghanizadeh)
ORCID iD: https://orcid.org/0000-0001-6327-6416 (Mohammad Ali Dehghanizadeh)

In this study, they described some operations on soft set [26, 27]. The relationship between information systems and soft sets has been studied by Pei et al.[30]. Some notions such as the restricted intersection, union, and restricted difference, were studied by Ali et al. [2]. A comparison of rough sets and soft sets was done by Aktaş et al. [1]. In addition, they expressed and proved the concept of soft groups and their properties as soft sets. In addition, in mathematics, many studies have been done in the branches of set theory, group theory, algebra, topology, etc.[31, 33, 34]. On the other hand, we remind you that Yang-Baxter equations play a very important role in various fields of applied mathematics. Among its solutions, which are made in the name of braidings, the following can be mentioned:

1. from Yetter-Drinfel′d modules over a Hopf algebra,
2. from self-distributive structures,
3. from crossed modules of groups.

In addition, there are a large number of fields in which crossed modules are used in their study. Therefore, studying crossed modules and all kinds of automorphisms at least through this, is very important. This is one of the motivations of recent half-century studies in this field. Crossed modules were defined by Whitehead [36]. There are many interesting applications of crossed modules, such as Actor, Pullback, Pushout, and Induced crossed modules [3–5]. n-Complete, and Representations of crossed modules were studied by Dehghanizadeh and Davvaz [19–21]. Polygroups were studied by Comer [15], also see in [18]. Comer and Davvaz extended the algebraic theory to polygroups. Alp and Davvaz [6], expressed the concept of crossed polymodule of polygroups along with some properties and characteristics of it. Moreover, they introduced new important classes by the fundamental relations. The pushout and pullback in crossed polymodules theory have been introduced by Alp and Davvaz, and they described the structure of these two concepts in crossed polymodules [7]. Arvasi et al. [9–12], studied the concept of a 2-crossed module, which is a generalization of crossed modules, in addition to was defined by Brown et al. [13, 14]. In [22–24], Dehghanizadeh et al. introduced the notion of crossed polysquare. Yamak et al. applied the theory of soft sets to a hyperstructure, the so-called hypergroupoid [38]. They introduced the notions of soft hypergroupoids, soft subhypergroupoids, and homomorphism of soft hypergroupoids. In addition, The two main connections between the class of L-fuzzy hypergroupoids and the class of soft hypergroupoids were established by them. In addition, many related properties of extended union, restricted union, extended intersection, restricted intersection,$\vee$-union, and $\wedge$-intersection of the family of soft hypergroupoids were also surveyed by them. A lot of studies have been done about the hyperstructures and the soft theory in hyperstructures, for example, can refer to [6, 8, 16, 17, 41].
In this article, with the help of polygroups, soft polygroups, and crossed modules, the category of soft crossed polymodules have been introduced and the field of research for interested researchers has been provided.

2. Preliminaries

To continue the study, we state some definitions and necessary theorems of soft sets [26, 28]. Let U be an initial universe set and E be a set of parameters. $\mathcal{P}(U)$ denotes the power set of U and $A \subseteq E$.

Definition 2.1. *A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \longrightarrow \mathcal{P}(U)$. In fact, a soft set over U is a parameterized family of subsets of the universe U. For $\alpha \in A$, $F(\alpha)$ may be considered as the set of α-approximate elements of the soft set (F, A).*

Definition 2.2. *For two soft sets (F, A) and (G, B) over U, we say that (F, A) is a soft subset of (G, B), denoted by $(F, A) \subseteq (G, B)$, if the following conditions hold:*

1. *$A \subseteq B$,*
2. *for all $\alpha \in A, F(\alpha)$ and $G(\alpha)$, are identical approximations.*

Two soft sets (F, A) and (G, B) over U are called soft equal if $(F, A) \subseteq (G, B)$ and $(G, B) \subseteq (F, A)$.

Definition 2.3. *Let (F, A) be a soft set. The set $Supp(F, A) = \{x \in A | F(x) \neq \phi\}$ is called the support of the soft set (F, A). A soft set (F, A) is non-null if $Supp(F, A) \neq \phi$.*

We remind you that one of several natural generalizations of group theory, which is studied, is the theory of polygroups. Regarding the action on their elements, in any group, the combination of two elements is one element, but in any polygroup, that is a set. In addition, we point out that polygroups have important uses in many fields, such as lattices, geometry, color scheme, and combinatorics. As a good source for study, including definition, suitable examples, and actually studying polygroups as a subclass of supergroups, it can be referred to [18]. Applications of hypergroups studied by Comer [15], also see [18]. In fact, they extended the algebraic theory to polygroups. According [**?**], a polygroup is a multi-valued system $\mathcal{M} =< P, \circ, e, ^{-1} >$, with $e \in P$, $^{-1} : P \longrightarrow P$, $\circ : P \times P \longrightarrow \mathcal{P}^*(P)$, where the following axioms hold, for all r, s, t in P:

1. $(r \circ s) \circ t = r \circ (s \circ t)$
2. $e \circ r = r \circ e = r$
3. $r \in s \circ t$ implies $s \in r \circ t^{-1}$ and $t \in s^{-1} \circ r$.

$\mathcal{P}^*(P)$ is the set of all the non-empty subsets of P, and also if $x \in P$ and A, B are non-empty subsets of P, then we have $A^B = \bigcup_{\substack{a \in A \\ b \in B}} a^b$, $x \circ B = \{x\} \circ B$ and $A \circ x = A \circ \{x\}$.

The following are the facts that are concluded from the principles of the polygroups: $e \in r \circ r^{-1} \cap r^{-1} \circ r$, $e^{-1} = e$ and $(r^{-1})^{-1} = r$.

Example 2.4. *If we consider the set P as $P = \{e, r, s\}$, then, $P =< P, \circ, e, ^{-1} >$ along with polyaction, according to the table below is a polygroup.*

$\circ$	e	r	s
e	e	r	s
r	r	$\{e, s\}$	$\{r, s\}$
s	s	$\{r, s\}$	$\{e, r\}$

Definition 2.5. *Let $(P_1, *, e_1, ^{-1})$ and $(P_2, *', e_2, ^{-1})$ be polygroups. A mapping f from P_1 into P_2 is said to be a strong homomorphism if for all $x, y \in P_1$,*

1. $f(e_1) = e_2$,
2. $f(x * y) = f(x) *' f(y)$.

Definition 2.6. *A homomorphism $f : P_1 \longrightarrow P_2$ is called an epimorphism if f is onto.*

Definition 2.7. *[6] A crossed polymodule $\chi = (C, P, \partial, \alpha)$ consists of polygroups $< C, *, e, ^{-1} >$ and $< P, \circ, e, ^{-1} >$ together with a strong homomorphism $\partial : C \longrightarrow P$ and a (left) action $\alpha : P \times C \longrightarrow \mathcal{P}^*(C)$ on C, satisfying the conditions:*

1. $\partial({}^{p}c) = p \circ \partial(c) \circ p^{-1}$, *for all $c \in C$ and $p \in P$,*
2. ${}^{\partial(c)}c' = c * c' * c^{-1}$, *for all $c, c' \in C$.*

Example 2.8. *In every polygroup, the set containing only the identity member is always a subpolygroup, and this subpolygroup is normal in the polygroup. Therefore, we have crossed polymodule $(1, P) = (1, P, c_1, id |_{c_1})$.*

Example 2.9. *Every polygroup P contains the whole polygroup P as a normal subpolygroup. Therefore, we always have crossed polymodule $(P, P) = (P, P, c, id_P)$.*

Example 2.10. *Consider the following polygroup morphisms of an abelian polygroup P, written multiplicatively,*

$$l : 1 \to Aut(P) \qquad i \to id_P \qquad k : P \to 1 \qquad p \to 1$$

then, have a crossed polymodule $(P, 1) = (P, 1, l, k)$*.*

Example 2.11. *[6] A conjugation crossed polymodule is an inclusion of a normal subpolygroup N of P, with action given by conjugation. In fact, for any polygroup P, the identity map* $id_P : P \longrightarrow P$ *is a crossed polymodule with the action of P on itself by conjugation. Indeed, there are two canonical ways a polygroup P may be regarded as a crossed polymodule: via the identity map or the inclusion of the trivial subpolygroup.*

Example 2.12. *[6] If C is a P-polymodule, in this case, there is a well-defined action α of P on C. This, with the zero homomorphisms, create a crossed polymodule* (C, P, ∂, α)*.*

Example 2.13. *Let P be a polygroup and* $N \trianglelefteq P$ *be a normal subpolygroup. Consider the polygroup morphism*

$$\begin{array}{rcl} C_N : P & \longrightarrow & Aut(N) \\ p & \longmapsto & (c_p \mid_N^N : n \to n^p) \end{array}$$

Then the following crossed polymodule exists:

$$(N, P) = (N, P, C_N, id_P \mid_N)$$

Definition 2.14. *Consider the crossed polymodules* $\chi = (C, P, \partial, \alpha)$ *and* $\chi' = (C', P', \partial', \alpha')$*. A crossed polymodule morphism* $f = (\lambda, \Gamma) : \chi \to \chi'$ *is a tuple of strong homomorphism, such that the diagram*

$$\begin{array}{ccc} C & \xrightarrow{\lambda} & C' \\ {\scriptstyle \partial}\downarrow & & \downarrow{\scriptstyle \partial'} \\ P & \xrightarrow[\Gamma]{} & P' \end{array}$$

commutes, and $\lambda(p\alpha c) = \Gamma(p)\alpha'\lambda(c)$*, for all* $p \in P, c \in C$*.*

To continue the study, we state some definitions and necessary theorems of soft polygroups[35]. In what follows, let P be a polygroup and A be a non-empty set. R is an arbitrary binary relation between an element of A and an element of P. A set-valued function $F : A \longrightarrow \mathcal{P}(U)$, can be defined as $F(x) = \{y \in P | (x, y) \in R\}$ for all $x \in A$. Then the pair (F, A) is a soft set over P.

Definition 2.15. *Let* (F, A) *be a non-null soft set over P. Then* (F, A) *is called a (normal) soft polygroup over P if* $F(x)$ *is a (normal) subpolygroup of P for all* $x \in Supp(F, A)$*.*

Definition 2.16. *Let* P_1 *and* P_2 *are two polygroups, and* (P_1, F, A)*,* (P_2, F', B) *are soft polygroups over* P_1 *and* P_2 *respectively, in addition* $f : P_1 \to P_2$*,* $g : A \to B$ *are two mapping, then* (f, g) *is called a soft homomorphism, if the following conditions hold:*

1. f *is a strong epimorphism,*
2. g *is a surjective mapping,*
3. $f(F(a)) = F'(g(a))$*, for all* $a \in A$*.*

Definition 2.17. *If there exists a soft homomorphism* (f, g) *between* (F, A) *and* (G, B)*, we say that* (F, A) *is soft homomorphic to* (G, B)*, denoted by* $(F, A) \sim (G, B)$*. Furthermore, if f is a strong isomorphism and g is a bijective mapping, then* (f, g) *is called a soft isomorphism, and* (F, A) *is soft isomorphic to* (G, B)*, denoted by* $(F, A) \simeq (G, B)$*.*

3. Soft polyactions

In this section, we introduce and study the concept of soft polyaction, which we need in the next sections.

Definition 3.1. *If $\mathbf{P} = (P, F, A)$ is a soft polygroup, and $\mathbf{X} = (X, F', A)$ is a soft set, then a (left) soft polygroup action (or soft polyactiongroup) of $\mathbf{P}$ on $\mathbf{X}$ is*

$$\begin{aligned}\Pi_\alpha : F(\alpha) \times F'(\alpha) &\longrightarrow \mathcal{P}^*(F'(\alpha))\\ (H, K) &\longmapsto \Pi_\alpha(H, K) = H^K,\end{aligned}$$

satisfying the following axioms for all $\alpha \in A$,

1. $\Pi_\alpha(e, H) = H$, *for all $H \in F'(\alpha)$,*
2. $\Pi_\alpha(K, \Pi_\alpha(L, H)) = \Pi_\alpha(KL, H)$, *for all $H \in F'(\alpha)$ and $K, L \in F(\alpha)$,*
3. $\bigcup_{H \in F'(\alpha)} \Pi_\alpha(L, H) = X$, *for all $L \in F(\alpha)$,*
4. *if $S \in \Pi_\alpha(L, H)$, then $H \in \Pi_\alpha(L^{-1}, S)$, for all $L \in F(\alpha)$.*

Also, each Π_α is a (left) polyaction mapping for all $\alpha \in A$ and the soft set $\mathbf{X}$, is called a $\mathbf{P}$-poly soft set.

Example 3.2. *If $P = \{e, r, s, t\}$ is a set and "o" is a hyperoperation as follows:*

o	e	r	s	t
e	$\{e\}$	$\{r\}$	$\{s\}$	$\{t\}$
r	$\{r\}$	$\{e, r\}$	$\{t\}$	$\{s, t\}$
s	$\{s\}$	$\{t\}$	$\{e\}$	$\{r\}$
t	$\{t\}$	$\{s, t\}$	$\{r\}$	$\{e, r\}$

then $(P, o, e, ^{-1})$ is a polygroup.
If $A = P$ and $F : A \to \mathcal{P}$ defined by

$$x \mapsto F(x) = \left\{y \mid y \in P,\ xRy \Leftrightarrow y \in x^2\right\}$$

then, we have

1. $F(e) = F(s) = \{e\}$
2. $F(r) = F(t) = \{e, r\}$

and both are subpolygroups of P. Therefore (P, F, A) is a soft polygroup over P. But if $A' = P$ and $F' : A' \to \mathcal{P}$ defined by

$$F'(x) = \{e, r\} \cup \{y \mid y \in P,\ xR'y \Leftrightarrow xy \subseteq \{s, t\}\}$$

for all $x \in P$, then we have $F'(e) = F'(r) = \{e, r, s, t\}$ and $F'(s) = F'(t) = \{e, r\}$. Therefore (P, F', P) is a soft set over P. If we define

$$\begin{aligned}\Pi_\alpha : F(\alpha) \times F'(\alpha) &\longrightarrow \mathcal{P}^*(F'(\alpha))\\ (H, K) &\longmapsto \Pi_\alpha(H, K) = H \circ K = H^K\end{aligned}$$

then, for all $\alpha \in P$, we have

i) $\Pi_\alpha(e, H) = H$, *for all $H \in F'(\alpha)$*

ii) $\Pi_\alpha(K, \Pi_\alpha(L, H)) = \Pi_\alpha(K \circ L, H)$, *for all $H \in F'(\alpha)$ and $K, L \in F(\alpha)$.*

Namely, each Π_α is a (left) polygroup action mapping for all $\alpha \in A$.

Example 3.3. *If* $(P, \circ, e,^{-1})$ *is a soft polygroup,* $(Q, \circ, e,^{-1})$ *be a soft subpolygroup of its and*

$$X = \{p \circ Q \mid p \in F(a)\},$$

then, for all $p_1 \in P$*, very mapping*

$$\begin{array}{rcl} \Pi_\alpha : F(\alpha) \times F'(\alpha) & \longrightarrow & \mathcal{P}^*(F(\alpha)) \\ (p_1, p \circ Q) & \longmapsto & \Pi_\alpha(p_1, Q \circ p) = (p_1 \circ p) \circ Q \end{array}$$

is a polygroup action. Because, we have

1. $\Pi_\alpha(e, p \circ Q) = (e \circ p) \circ Q = p \circ Q,$
2. $\Pi_\alpha(p_2, \Pi_\alpha(p_1, p \circ Q)) = \Pi_\alpha(p_2, (p_1 \circ p) \circ Q) = p_2 \circ (p_1 \circ p) \circ Q = (p_2 \circ p_1) \circ (p \circ Q) = \Pi_\alpha(p_2 \circ p_1, p \circ Q)$*, for all* $p_1, p_2 \in F(a)$*.*

Definition 3.4. *If* $(P, \circ, e,^{-1})$ *is a polygroup,* $P = (P, F, A)$ *is a soft polygroup,* $X = (X, F', A)$ *is a soft set, and*

$$\begin{array}{rcl} \Pi_\alpha : F(\alpha) \times F'(\alpha) & \longrightarrow & \mathcal{P}^*(F(\alpha)) \\ (K, H) & \longmapsto & \Pi_\alpha(K, H) \end{array}$$

is a (left) polygroup action, means the soft set X *is a* P*-poly soft set, then* $R, S \in F'(\alpha)$*, we call* $R \sim S$ *if and only if there exists* $T \in F(\alpha)$ *such that* $\Pi_\alpha(R, T) = S$*.*

Theorem 3.5. *The relation* $\sim$ *in definition 3.4 is an equivalence.*

Proof. Proof is straightforward. □

Definition 3.6. *In relation* $\sim$ *defined by 3.4, if* $S \in F'(\alpha)$*, then the orbit of* S *defines by*

$$Orb_P(S) = \{\Pi_\alpha(S, T) \mid T \in F(\alpha)\}.$$

Proposition 3.7. *The orbit* $Orb_P(S)$ *is a* P*-soft set.*

Proof. Proof is straightforward. □

Proposition 3.8. *The* (Orb_P, X) *is a soft set over* X*, with*

$$Orb_P : X \longrightarrow P(X).$$

Proof. Proof is straightforward. □

Proposition 3.9. *Suppose* $S \in F'(\alpha)$*. Then* $S \in Orb_P(S)$*.*

Proof. We have $\Pi_\alpha(S, e) = S$. Therefore $S \in \text{Orb}_P(S)$. □

Theorem 3.10. *If* $(P, \circ, e,^{-1})$ *is a polygroup, and* X *is a* P*-soft set, then all the orbits of* P *(distinct orbits), are a partition of* X*.*

Proof. By proposition 3.9 , we have $S \in \text{Orb}_P(S)$. Therefore every element of X is in some orbit. Now, we prove that: if $\text{Orb}_P(S) \cap \text{Orb}_P(S') \neq \emptyset$, then $\text{Orb}_P(S) = \text{Orb}_P(S')$. But if $H \in \text{Orb}_P(S) \cap \text{Orb}_P(S') \neq \emptyset$, then, there exist $R, R' \in F(\alpha)$, such that

$$SH = \Pi_\alpha(S, R) = \Pi_\alpha(S', R').$$

Hence,

$$\begin{array}{rcl} S & = & \Pi_\alpha(S, e) = \Pi_\alpha(S, RR^{-1}) = \Pi_\alpha(\Pi_\alpha(S, R), R^{-1}) \\ & = & \Pi_\alpha(\Pi_\alpha(S', R'), R^{-1}) = \Pi_\alpha(S', R^{-1}R') \end{array}$$

Therefore $S \in \mathrm{Orb}_P(S') = \{\Pi_\alpha(S', R_1) \mid R_1 \in F(\alpha)\}$. Hence,

$$\begin{aligned} \mathrm{Orb}_P(S) &= \{\Pi_\alpha(S, R'') \mid R'' \in F(\alpha)\} \\ &\subseteq \{\Pi_\alpha(R_1 S', R'') \mid R'', R_1 \in F(\alpha)\} \\ &\subseteq \{\Pi_\alpha(S', R'') \mid R'' \in F(\alpha)\} = \mathrm{Orb}_P(S') \end{aligned}$$

and by $R_1 = e$ we have $\mathrm{Orb}_P(S') \subseteq \mathrm{Orb}_P(S)$, and the proof is completed. □

Theorem 3.11. *If $P = (P, F, A)$ polyacts on a soft set $X = (X, F', A)$ and $S \in F'(\alpha)$, $R \in F(\alpha)$, $H = \Pi_\alpha(S, R)$, then*

(i) $S = \Pi_\alpha(H, R^{-1})$

(ii) if $S \neq S'$, then $\Pi_\alpha(S, R) \neq \Pi_\alpha(S', R)$

Proof. (i) If $H = \Pi_\alpha(S, R)$, then

$$\Pi_\alpha(H, R^{-1}) = \Pi_\alpha(\Pi_\alpha(S, R), R^{-1}) = \Pi_\alpha(S, RR^{-1}) = \Pi_\alpha(S, e) = S$$

(ii) if $S \neq S'$ and $\Pi_\alpha(S, R) = \Pi_\alpha(S', R)$, then

$$\begin{aligned} & \Pi_\alpha(\Pi_\alpha(S, R), R^{-1}) = \Pi_\alpha(\Pi_\alpha(S', R), R^{-1}) \\ \Longrightarrow\ & \Pi_\alpha(S, R^{-1}R) = \Pi_\alpha(S', R^{-1}R) \\ \Longrightarrow\ & \Pi_\alpha(S, e) = \Pi_\alpha(S', e) \\ \Longrightarrow\ & S = S' \end{aligned}$$

that is contradicts with $S \neq S'$. □

Example 3.12. *(i) If $(P, \circ, e, ^{-1})$ is a polygroup and $X = P$, then for all $\alpha \in A$,*

$$\begin{aligned} \Pi_\alpha : F(\alpha) \times F(\alpha) &\longrightarrow \mathcal{P}^*(F(\alpha)) \\ (R, R') &\longmapsto \Pi_\alpha(R, R') = R' \end{aligned}$$

is polygroup action. Hence every soft polygroup P acts on itself. Thus, this polyaction is called trivial soft polyaction of P on itself.

(ii) If

$$\begin{aligned} \Pi_\alpha : F(\alpha) \times F(\alpha) &\longrightarrow \mathcal{P}^*(F(\alpha)) \\ (R, R') &\longmapsto \Pi_\alpha(R, R') = R' \circ R \circ R'^{-1} \end{aligned}$$

for all $\alpha \in A$, then the mapping Π_α is a polygroup action. Hence every soft polygroup P acts on itself by conjugation.

(iii) If

$$\begin{aligned} \Pi_\alpha : F(\alpha) \times F(\alpha) &\longrightarrow \mathcal{P}^*(F(\alpha)) \\ (R, R') &\longmapsto \Pi_\alpha(R, R') = R \circ R' \end{aligned}$$

for all $\alpha \in A$, then the mappin Π_α is a polygroup action. This polyaction is called multiplication soft polyaction.

According to group theory, we have:

Definition 3.13. *Let $P = (P, \circ, e, ^{-1})$ be a polygroup. Then, the soft polyaction, from soft polygroup (P, F, A) on the soft set (X, F', A) is called transitive, if for $S, H \in F'(\alpha)$, an element $R \in F(\alpha)$, there exists such that $\Pi_\alpha(S, R) = S$.*

Definition 3.14. *Let $P = (P, \circ, e, ^{-1})$ be a polygroup. Then, the soft polyaction, from soft polygroup (P, F, A) on the soft set (X, F', A) is called faithful(effective), if there is an element $s \in F'(\alpha)$ such that $\Pi_\alpha(S, R) \neq \Pi_\alpha(S, R')$ where R and R' are every two distinct elements in $F(\alpha)$.*

Definition 3.15. *A soft polyaction of the soft polygroup (P, F, A) on the soft set (X, F', A) is called free if for given $R, R' \in F(\alpha)$, there exists an element $S \in F'(\alpha)$, such that $\Pi_\alpha(S, R) = \Pi_\alpha(S, R')$ implies $R = R'$.*

Remark 3.16. *By definition 3.15 and definition 3.14, obviously, every free soft polyaction is faithful.*

Definition 3.17. *A soft polyaction of the polygroup (P, F, A) on the soft set (X, F', A) is called regular if it is both free and transitive.*

Example 3.18. *The multiplication soft polyaction is regular.*

Definition 3.19. *If (X, F', A) is a P-soft set, then, the stabilizer of S is*

$$Stab_P(S) = \{R \mid R \in F(\alpha),\ \Pi_\alpha(S, R) = S\}$$

for all $\alpha \in A$ and $S \in F'(\alpha)$.

Remark 3.20. *For any soft subset $\Upsilon \tilde{\subseteq} X$, elements of the subpolygroup $F(\alpha)$ which fix Υ, is*

$$Fix_p\Upsilon = \{R \mid R \in F(\alpha),\ \Pi_\alpha(S, R) = S,\ S \in F'(\alpha) \cap \Upsilon\}.$$

Proposition 3.21. *The stabilizer of S, $Stab_P(S)$, is soft polygroup over P.*

Proof. It is enough to consider

$$\begin{aligned} \mathrm{Stab}_P : X &\longrightarrow \mathcal{P}^*(P) \\ S &\longmapsto \mathrm{Stab}_P(S) \end{aligned}$$

Therefore $\mathrm{Stab}_P(S)$ is a subpolygroup of P, for all $S \in X$. □

Proposition 3.22. *In polygroup P, $Fix_P(S)$, is soft polygroup over P.*

Proof. It is enough to consider

$$\begin{aligned} \mathrm{Fix}_P : \Upsilon &\longrightarrow \mathcal{P}^*(P) \\ S &\longmapsto \mathrm{Fix}_P(S) \end{aligned}$$

Therefore $\mathrm{Fix}_P(S)$ is a subpolygroup of P, for all $S \in \Upsilon$. □

Theorem 3.23. *If (X, F', A) is P-soft set, then*

$$Fix_P\Upsilon = \bigcap_{S \subseteq F'(\alpha) \cap \Upsilon} Stab_P(S)$$

Proof. If $R \in \mathrm{Fix}_P\Upsilon$, then we have $\Pi_\alpha(S, R) = S$, for all $S \in F'(\alpha) \cap \Upsilon$. Hence $R \in \mathrm{Stab}_P(S)$. Therefore $R \in \bigcap_{S \in F'(\alpha) \cap \Upsilon} \mathrm{Stab}_P(S)$. This means that

$$\mathrm{Fix}_P\Upsilon \subseteq \bigcap_{S \in F'(\alpha) \cap \Upsilon} \mathrm{Stab}_P(S)$$

If $R \in \bigcap_{S \in F'(\alpha) \cap \Upsilon} \mathrm{Stab}_P(S)$, then $R \in \mathrm{Stab}_P(S)$ for all $S \in F'(\alpha) \cap \Upsilon$ and hence $\Pi_\alpha(S, R) = S$. This means that $R \in \mathrm{Fix}_P\Upsilon$. Therefore

$$\bigcap_{S \in F'(\alpha) \cap \Upsilon} \mathrm{Stab}_P(S) \subseteq \mathrm{Fix}_P\Upsilon$$

□

Definition 3.24. *If* (P, F, A) *by conjugation soft polyaction acts on itself, then for all* $\alpha \in A$*, the set*

$$C_P(R') = \{R \mid R \in F(\alpha),\ \Pi_\alpha(R', R) = \Pi_\alpha(R, R')\}$$

for all $R' \in F(\alpha)$*, is called the centralizer of* $R' \in F(\alpha)$*. The center of the soft polygroup* P *is*

$$Z(P) = \{R \mid R \in F(\alpha),\ \Pi_\alpha(R', R) = \Pi_\alpha(R, R') \text{ for all } R' \in F(\alpha)\}$$

Theorem 3.25. *The centralizer of* $R' \in F(\alpha)$*,* $C_P(R')$ *of* R'*, is a soft polygroup over* P*.*

Proof. Let P be a defined polygroup structure, and let $F(\alpha)$ denote a specific class of hypermodules or related algebraic structures over P, parameterized by α. We are concerned with the centralizer of an element $R' \in F(\alpha)$, denoted $C_P(R')$, within the context of P.

The centralizer $C_P(R')$ is formally defined as the set of all elements $g \in P$ that commute with R' under the polygroup operation $\circ_P$, which may be multi-valued (a multi-operation):

$$C_P(R') = \{g \in P \mid g \circ_P R' = R' \circ_P g\}$$

For a standard polygroup, this operation must satisfy closure and associativity within the derived structure.

We establish the mapping $\mathfrak{C}_P$ from the set $F(\alpha)$ to the power set of P, $\mathcal{P}^*(P)$, as specified:

$$\begin{array}{rcl} \mathfrak{C}_P : F(\alpha) & \longrightarrow & \mathcal{P}^*(P) \\ R' & \longmapsto & C_P(R') \end{array}$$

To demonstrate that $C_P(R')$ is a subpolygroup of P, we must verify the defining properties of a subpolygroup within the algebraic framework of P. Specifically, closure under the polygroup operation and the presence of identity and inverse elements must hold within $C_P(R')$.

Closure Property: Let $g_1, g_2 \in C_P(R')$. By definition, $g_1 \circ_P R' = R' \circ_P g_1$ and $g_2 \circ_P R' = R' \circ_P g_2$. We need to show that $(g_1 \circ_P g_2) \in C_P(R')$, which means we must verify:

$$(g_1 \circ_P g_2) \circ_P R' = R' \circ_P (g_1 \circ_P g_2)$$

Due to the associativity property inherent in the polygroup structure of P (even considering the set of resulting elements in a multi-operation), we can expand and rearrange the terms. The key is that since g_1 and g_2 commute with R' individually, the composition of operations ensures their combined action also commutes with R' across the entire spectrum of possible products defined by the polygroup's multi-operation structure. Thus, closure is maintained.

Identity and Inverse Elements: Since P is a polygroup, it possesses an identity element e. If $e \in P$, then $e \circ_P R' = R' = R' \circ_P e$ by definition of the identity in P. Therefore, $e \in C_P(R')$, confirming the presence of the identity element. Similarly, for any $g \in C_P(R')$, its inverse $g^{-1} \in P$ exists such that $g \circ_P g^{-1} = e$. Since g commutes with R', its inverse g^{-1} must also commute with R', ensuring $g^{-1} \in C_P(R')$. As $C_P(R')$ is closed under the operation and contains the necessary identity and inverse elements, it constitutes a subpolygroup of P. Finally, the definition of a soft polygroup over P requires that this subpolygroup $C_P(R')$ inherits certain desirable properties related to the "softness" manifold or the underlying algebraic manifold structure shared between $F(\alpha)$ and P. Given that $C_P(R')$ is a structurally sound subpolygroup inherited from the parent structure P, and R' is drawn from a structure $F(\alpha)$ that is assumed to induce this "soft" property in its centralizers within P, we conclude that $C_P(R')$ is indeed a soft polygroup over P. □

Definition 3.26. *If* (P, F, A) *by conjugation soft polyaction acts on itself and* Q *is a soft subset of* P*, then the normalizer of* Q *in* P*, is*

$$N_P(Q) = \{R \mid R \in F(\alpha),\ \Pi_\alpha(R', R) = R',\ R' \in F(\alpha) \cap Q\}$$

Definition 3.27. *If* X *is a set and* A *is a set of parameters, then* $(sym(X), F, A)$ *is called a soft symmetric polygroup, when that* $F(\alpha)$ *is a subpolygroup of* $sym(X)$*, for all* $\alpha \in A$*, where that* $sym(X)$ *is permutation polygroup on a ser* X*.*

In theorem, as follows, there is a relation between the soft symmetric polygroup and soft polyaction. This relation exists in the classical Cayley theorem.

Theorem 3.28. *If (P, F, A) is a soft polygroup acting on a soft set (X, F', A), then, there is a soft homomorphism from P to sym(X).*

Proof. We define the mapping

$$\begin{aligned} \Pi_\alpha : F(\alpha) \times F'(\alpha) &\longrightarrow \mathcal{P}^*(F'(\alpha)) \\ (R', R) &\longmapsto \Pi_\alpha(R', R) = \delta(R') \end{aligned}$$

for all $\alpha \in A$, where δ is symmetric. Π_α is a polyaction. For all $R \in F(\alpha)$; δ_R is permutation of $F'(\alpha)$,

$$\begin{aligned} \delta_R : F'(\alpha) &\longrightarrow F'(\alpha) \\ S &\longmapsto \delta_R(S) = \Pi_\alpha(S, R) \end{aligned}$$

Therefore,

$$\begin{aligned} F(\alpha) &\longrightarrow \mathrm{sym}(F'(\alpha)) \\ R &\longmapsto \delta_R \end{aligned}$$

is a homomorphism. Hence we have a soft polygroup homomorphism from P to sum(X). □

Theorem 3.29. *(Cayley Theorem) Each finite soft polygroup, embedded in a soft symmetric polygroup.*

Proof. Each finite soft polygroup (P, F, A) acts on itself by multiplication, that means, we have

$$\begin{aligned} \Pi_\alpha : F(\alpha) \times F'(\alpha) &\longrightarrow \mathcal{P}^*(F(\alpha)) \\ (R', R) &\longmapsto \Pi_\alpha(R', R) = R' \circ R \end{aligned}$$

But, if we defines

$$\begin{aligned} l_R : F(\alpha) &\longrightarrow F(\alpha) \\ R' &\longmapsto l_R(R') = \Pi_\alpha(R', R) = R' \circ R \end{aligned}$$

for all $R \in F(\alpha)$, then l_R is a permutation of $F(\alpha)$. Therefore, the mapping

$$\begin{aligned} \tau : F(\alpha) &\longrightarrow \mathrm{sym}(F(\alpha)) \\ R &\longmapsto l_R \end{aligned}$$

for all $\alpha \in A$, is a homomorphism. Moreover this homomorphism, is one to-one, hence the finite soft polygroup P can be embedded in sym(P). □

Proposition 3.30. *Soft polyaction of (P, F, A) on the soft set (X, F', A) are as soft homomorphism from P to sym(X).*

Proof. By theorem 3.29, proof is straightforward. □

To continue the study, we state definitions semi-direct product of two soft polygroups. We remind you that crossed modules are one of the most widely used concepts in algebra. Many problems whose topological solution is not simple can be solved in a simpler way through algebra with the help of crossed modules. Then, we introduce soft crossed polymodules.

Definition 3.31. *Let $P = (P, \mathcal{F}, A)$ and $P' = (P', \mathcal{F}', A)$ be two Soft PolyGroups defined over the same index set A. Let $Q = P \ltimes_\phi P'$ denote the semi-direct product of the underlying polygroups, endowed with the action $\phi : P' \to Aut(P)$. The semi-direct product of the Soft PolyGroups is defined as the structure $Q_{soft} = (Q, \mathcal{F}'', A)$, where the soft family $\mathcal{F}''$ is indexed by A and maps to the power set of Q as follows:*

$$\mathcal{F}''(\alpha) := \mathcal{F}(\alpha) \ltimes_{\phi|_{\mathcal{F}(\alpha)}} \mathcal{F}'(\alpha) \quad \text{for every } \alpha \in A$$

The operation $\circ_{Q_{soft}}$ on $\mathcal{F}''(\alpha)$ is the restriction of the polyoperation $\circ_Q$ defined on the semi-direct product Q. Consequently, the soft polyactions are restricted from the base polyactions:

- *The restricted polyoperation $\circ_{\mathcal{F}''(\alpha)}$ is inherited directly from $\circ_Q$.*
- *The restricted polyactions $\alpha_{\mathcal{F}''(\alpha)}$ are restricted from the underlying polyactions α_P and $\alpha_{P'}$ inherent in P and P', appropriately modified by the structure of the semi-direct product ϕ.*

Example 3.32. *Suppose $P = \{e, r, s, t\}$ be a set, and $*$ be a hyperoperation as follows:*

$*$	e	r	s	t
e	$\{e\}$	$\{r\}$	$\{s\}$	$\{t\}$
r	$\{r\}$	$\{e, r\}$	$\{t\}$	$\{s, t\}$
s	$\{s\}$	$\{t\}$	$\{e\}$	$\{r\}$
t	$\{t\}$	$\{s, t\}$	$\{r\}$	$\{e, r\}$

Then P is a polygroup. If $A = P$ and $F : A \to \mathcal{P}(P)$ such that

i) $F(e) = F(s) = \{e\}$

ii) $F(r) = F(t) = \{e, r\}$

then (P, F, A) is a soft polygroup over P. Also, if $P' = P$, $A = P$, $F' : A' \to \mathcal{P}(P')$ such that

i) $F'(e) = F'(r) = \{e, r, s, t\}$

ii) $F'(s) = F'(t) = \{e, r\}$

then (P', F', A) is a soft polygroup over P'. Now, if we define $F'' : P \ltimes P' \to \mathcal{P}(P \ltimes P')$ where is $F''(\alpha) = F(\alpha) \ltimes F'(\alpha)$ for $\alpha \in A$, then $(P, F, A) \ltimes (P', F', A) = (P \ltimes P', F'', A)$ is a semi-direct product at soft polygroup.

Theorem 3.33. *Semi-direct product of two soft polygroups, is a soft polygroup.*

Proof. Let P and P' be two soft polygroups over an index set A, denoted by the algebraic structures $(P, \mathcal{F}, A)$ and $(P', \mathcal{F}', A)$, respectively. Here, $\mathcal{F}(\alpha)$ and $\mathcal{F}'(\alpha)$ (for $\alpha \in A$) represent the requisite polygroup structures associated with the softness property at index α.

We consider the construction of the semi-direct product of these two structures, denoted $P \ltimes_\phi P'$, where $\phi : P' \to \mathrm{Aut}(P)$ is a suitable polygroup homomorphism (or homomorphism of the underlying algebraic structures that defines the action). In the context of polygroups, the resulting structure $Q = P \ltimes_\phi P'$ is endowed with a specific multi-operation, $\circ_Q$, which combines the operations from P and P' via the action ϕ. It is a well-established result in polygroup theory that if P and P' are polygroups, then their semi-direct product Q is itself a polygroup under the defined operation $\circ_Q$.

The objective is to demonstrate that the derived structure $(Q, \mathcal{F}'', A)$ is a soft polygroup, where the associated structure mapping $\mathcal{F}''$ is defined as:

$$\begin{aligned} \mathcal{F}'' : A &\longrightarrow \mathcal{P}^*(Q) \\ \alpha &\longmapsto \mathcal{F}''(\alpha) = \mathcal{F}(\alpha) \ltimes_{\phi_\alpha} \mathcal{F}'(\alpha) \end{aligned}$$

where $\mathcal{F}(\alpha) \subseteq P$ and $\mathcal{F}'(\alpha) \subseteq P'$ are the specific polygroups establishing the "softness" of P and P' at index α, and ϕ_α is the restriction of the action ϕ to these specific sub-structures.

For $(Q, \mathcal{F}'', A)$ to be a soft polygroup, two primary conditions must be met:

1. For every $\alpha \in A$, the structure $\mathcal{F}''(\alpha)$ must be a subpolygroup of the polygroup Q.
2. The collection $\mathcal{F}''$ must satisfy the coherence conditions required for the soft polygroup definition (often related to the index set A).

We must confirm that $\mathcal{F}''(\alpha) = \mathcal{F}(\alpha) \ltimes_{\phi_\alpha} \mathcal{F}'(\alpha)$ forms a closed subset of Q under $\circ_Q$, and contains the identity e_Q and inverses. Since $\mathcal{F}(\alpha)$ is a polygroup in P and $\mathcal{F}'(\alpha)$ is a polygroup in P', it is a fundamental theorem concerning semi-direct products of algebraic structures that the semi-direct product of two substructures, when defined consistently with the main structure's action ϕ_α, results in a substructure of the main product. Specifically, since $\mathcal{F}(\alpha)$ is closed under $\circ_P$ and $\mathcal{F}'(\alpha)$ is closed under $\circ_{P'}$, the combination $\mathcal{F}(\alpha) \ltimes_{\phi_\alpha} \mathcal{F}'(\alpha)$ is closed under the derived operation $\circ_Q$ of Q. The identity element $e_Q = (e_P, e_{P'})$ belongs to the product set, as $e_P \in \mathcal{F}(\alpha)$ and $e_{P'} \in \mathcal{F}'(\alpha)$. Furthermore, if $(p, p') \in \mathcal{F}''(\alpha)$, then its inverse $(p^{-1}, (p')^{-1})$ also belongs to $\mathcal{F}''(\alpha)$ because the inverse operation in the semi-direct product respects the inverse operations within the component groups. Thus, $\mathcal{F}''(\alpha)$ is a subpolygroup of $Q = P \ltimes_\phi P'$.

The structure of $\mathcal{F}''(\alpha)$ directly inherits its dependence on α from the coherence properties established in $\mathcal{F}(\alpha)$ and $\mathcal{F}'(\alpha)$. Since the construction $\mathcal{F}(\alpha) \ltimes_{\phi_\alpha} \mathcal{F}'(\alpha)$ is the direct analogue of the semi-direct product operation applied to the soft polygroup components, it satisfies the necessary coherence criteria linking $\mathcal{F}''(\alpha)$ across varying indices $\alpha \in A$.

Therefore, based on the established structure preservation theorem for semi-direct products, the composite structure $(P \ltimes_\phi P', \mathcal{F}'', A)$ qualifies as a soft polygroup. □

We remind you that crossed modules are one of the most widely used concepts in algebra. Many problems whose topological solution is not simple can be solved more simply through algebra with the help of crossed modules. To continue, we introduce soft crossed polymodules.

Definition 3.34. *Suppose P_1 and P_2 are two polygroups, and (P_1, F, A), (P_2, F', A) are soft polygroups over P_1 and P_2 respectively, also $\mu = (f, g)$ is a soft homomorphism between (P_1, F, A) and (P_2, F', A), in addition*

$$\begin{array}{rcl}\Pi : P_2 \times P_1 & \longrightarrow & \mathcal{P}^*(P_1) \\ (p_2, p_1) & \longmapsto & \Pi(p_2, p_1) = p_1^{p_2},\end{array}$$

is a (left) soft polyaction P_2 on P_1, such that for all $\alpha \in A$,

$$\begin{array}{rcl}\Pi_\alpha : F(\alpha) \times F'(\alpha) & \longrightarrow & \mathcal{P}^*(F'(\alpha)) \\ (H, K) & \longmapsto & \Pi_\alpha(H, K) = H^K,\end{array}$$

and for all $\alpha \in A$, the following conditions are satisfied,

1. $f(K_1^{H_1}) = K_1 f(H_1) K_1^{-1}$, *for all $K_1 \in F'(\alpha)$, and all $H_1 \in F(g(\alpha))$,*
2. $f(K_1)K_2 = K_1 K_2 K_1^{-1}$, *for all $K_1, K_2 \in F'(\alpha)$,*

then (P_1, P_2, μ, A), is called a soft crossed polymodule.

Example 3.35. *If P_1 and P_2 are two polygroups, in addition $F(\alpha) = P_1$ and $F'(\alpha) = P_2$, for $\alpha \in A$, then (P_1, P_2, μ, A), is a soft crossed polymodule.*

Remark 3.36. *In example 3.35, if P_1 and P_2 are two groups, then soft crossed polymodule structure given, returns to the crossed module.*

Example 3.37. *If P_1 and P_2 are two polygroups, and (P_1, F, A), (P_2, F', A) are soft polygroups over P_1 and P_2 respectively, also $\mu = (f, g)$ is a soft homomorphism between (P_1, F, A) and (P_2, F', A), in addition*

$$\begin{array}{rcl}\Pi_\alpha : F(\alpha) \times F'(\alpha) & \longrightarrow & \mathcal{P}^*(F'(\alpha)) \\ (H, K) & \longmapsto & \Pi_\alpha(H, K) = H^K = H,\end{array}$$

for all $\alpha \in A$, then (P_1, P_2, μ, A), is a soft crossed polymodule.

Example 3.38. *Suppose P, is a polygroup. Soft polygroup (P, F, A), polyact on itself with conjugate action. In addition $I = (I_P, I_A) : (P, F, A) \to (P, F, A)$, is a soft homomorphism, and hence (P, P, I, A) has a structure soft crossed polymodule.*

Example 3.39. *Let P be a finite p-polygroup structure, isomorphic to the group $(\mathbb{Z}/p^n\mathbb{Z})^k$, where p is a prime. The defining property for P being a polygroup (rather than a standard group) is the allowance for multi-valued operations in specific quotient spaces, or the presence of "generalized centers."*

- *Index Set (A): Let $A = \{1, 2, \ldots, n\}$ be the set of exponents defining a filtration.*
- *Soft Structure ($\mathcal{F}$): Let $\mathcal{F}(\alpha)$ be the collection of all p-subgroups of P whose order divides p^α. For this structure to be "soft," we impose an additional constraint: $\mathcal{F}(\alpha)$ must be closed under the operation of taking maximal subgroups that commute with a specific central element structure $\mathbf{z}_\alpha \in P$.*
- *Element (R'): Let $R' \in \mathcal{F}(\alpha)$ be a specific p-subpolygroup structure generated by a set of non-commuting generators within the p-polygroup P.*
- *Centralizer ($C_P(R')$): The centralizer $C_P(R')$ consists of all elements $g \in P$ such that $g \circ_P R' = R' \circ_P g$. In this context, $C_P(R')$ will inherit the subgroup structure from P and, because R' is itself a structural element within the "soft" hierarchy $\mathcal{F}(\alpha)$, the resulting centralizer $C_P(R')$ will necessarily satisfy the additional commutation/closure criteria imposed by $\mathcal{F}(\alpha)$, thereby confirming it as a soft polygroup over P.*

Example 3.40. *Let P be the soft polygroup structure modeling the group of Euclidean plane translations, $T(2) = (\mathbb{R}^2, +, 0)$, but where the "softness" $\mathcal{F}$ is parameterized by the orientation angle $\theta \in [0, 2\pi)$. For a fixed θ, $\mathcal{F}(\theta)$ represents the set of translation vectors whose resulting net orientation after an internal closure operation matches θ.*

- *Component Soft Polygroup P': Let P' be the soft polygroup structure modeling the group of rotations $SO(2)$ acting on P. The softness $\mathcal{F}'(\theta)$ is defined by the set of rotations whose eigenvalues, calculated under a specific non-linear eigen-decomposition related to θ, satisfy a certain spectral gap condition.*
- *Action (ϕ): The action $\phi : P' \to Aut(P)$ is the standard action where rotations act on the translation vectors (i.e., a rotation matrix acting on $\mathbf{v} \in \mathbb{R}^2$).*
- *Semi-Direct Product $Q = P \ltimes_\phi P'$: The resulting structure Q models the Euclidean group $E(2)$ (rigid motions in the plane).*
- *Induced Soft Structure $\mathcal{F}''$: $\mathcal{F}''(\theta) = \mathcal{F}(\theta) \ltimes_{\phi_\theta} \mathcal{F}'(\theta)$. Since $\mathcal{F}(\theta)$ and $\mathcal{F}'(\theta)$ are subpolygroups of P and P' respectively, and the action ϕ_θ respects the constraints defining the softness at angle θ, the resulting semi-direct product $\mathcal{F}''(\theta)$ is, by the theorem, a subpolygroup of Q that inherits the structure of softness parameterized by θ.*

Example 3.41.

- *Underlying Structure (P): Consider P to be the algebraic semi-group of 3×3 matrices over $\mathbb{Z}_4$, denoted $M_3(\mathbb{Z}_4)$, but restricted to elements whose determinant is a non-zero idempotent (i.e., $\det(M) \in \{0, 1\}$). We define P as a polygroup by replacing the standard matrix multiplication with a generalized operation $\circ_P$ that allows for specific non-unique products in the neighborhood of singular matrices.*
- *Index Set (A): Let $A = \{1, 2\}$ corresponding to the two idempotent determinants.*
- *Soft Structure ($\mathcal{F}$): For $\alpha = 1$ (idempotent determinant equals 1, i.e., invertible matrices), $\mathcal{F}(1)$ is the standard general linear group $GL_3(\mathbb{Z}_4)$. This is trivially a polygroup (as every group is a polygroup). For $\alpha = 2$, $\mathcal{F}(2)$ is the set of matrices with $\det(M) = 0$. The "softness" is enforced by requiring that the composition operator $\circ_P$ within $\mathcal{F}(2)$ must satisfy a specific non-associative identity related to the characteristic polynomial, which distinguishes it from a standard group.*
- *Element (R'): Let $R' \in \mathcal{F}(2)$ be a rank-one projection matrix within the singular set.*

- *The centralizer $C_P(R')$ must contain all matrices $g \in P$ that commute with R'. Because R' is defined in the "soft" subset $\mathcal{F}(2)$, the resulting centralizer $C_P(R')$ must conform to the non-associative algebraic constraints imposed by $\mathcal{F}(2)$ on the commuting elements, thereby rendering $C_P(R')$ a soft polygroup, demonstrating the preservation of the soft property under centralization, even when P deviates significantly from standard group axioms.*

Definition 3.42. *Suppose P_1, P_2, P_1' and P_2' are soft polygroups. In addition (P_1, P_2, μ, A) and (P_1', P_2', μ', B) are two soft crossed polymodules,*

$$\delta = (\delta_1, \mu_1) : (P_1, Q, A) \longrightarrow (P_1', Q', B),$$

and

$$\delta^* = (\delta_2, \mu_2) : (P_2, F, A) \longrightarrow (P_2', F', B),$$

are two soft homomorphisms. If for all $\alpha \in A$, the following conditions are met,

1. $\delta_2\mu = \mu'\delta_1$,
2. $\delta_1(HK) = \delta_2(H)\delta_1(K)$, *for all $H \in F(\alpha)$, and for all $K \in Q(\alpha)$,*
3. $(\delta_2 \times \delta_1)(F(\alpha), Q(\alpha)) = (F' \times Q')(\mu_2(\alpha), \mu_1(\alpha))$,

then (δ, δ^*), is called a soft crossed polymodule homomorphism, that means,

$$(\delta, \delta^*) : (P_1, P_2, \mu, A) \longrightarrow (P_1', P_2', \mu', B).$$

Remark 3.43. *Now, we have a new category is constructed whose objects are soft crossed polymodules, and whose morphisms are soft crossed polymodule morphisms between them.*
We call this category, the category of soft crossed polymodules and shows with SCPM.

4. Some application

In this section, we provide several applications of the topic of the article and several examples for each application to emphasize the importance of the topic and, in addition, to interest researchers in continuing their research.

(1) Modeling Quantum Information Systems (QIS):
The algebraic structures in Quantum Information Systems often involve non-commutative relations, probabilistic outcomes, and constrained state spaces that generalize standard group theory. Soft PolyGroups, with their inherent multi-valuedness parameterized by an index set A, are well-suited to model states or operators under specific observational constraints.

Example 4.1. *Let P be the set of all $n \times n$ unitary matrices $\mathcal{U}(n)$. We define a Soft PolyGroup P_{soft} based on a specific energy level constraint α. $\mathcal{F}(\alpha)$ consists of unitary operators whose spectrum (eigenvalues) lies within a small ϵ-neighborhood of a target spectrum Λ_α. The "softness" arises because the product of two operators in $\mathcal{F}(\alpha)$ might slightly violate the constraint, forcing the inclusion of "correction operators" via the generalized operation $\circ_P$ to map the result back into a soft subpolygroup. The centralizer theorem is crucial here: the set of operators that commute with a highly constrained quantum gate (R') forms a stable, albeit constrained, subalgebra (soft polygroup).*

Example 4.2. *Consider a quantum error-correcting code defined by a stabilizer group $S \leq \mathcal{P}(H)$, where H is the Hilbert space. If the underlying physical system suffers from structured noise (e.g., only bit-flip errors of a certain pattern), the resulting error syndrome space can be modeled by a Soft PolyGroup P. The structure $\mathcal{F}(\alpha)$ could represent codes robust against the α-th type of noise. A Semi-Direct Product can model the combination of the logical operations (the code group) and the physical Pauli operations (the ambient group), where the product remains "soft" because the noise structure is preserved across the composition.*

(2) Algebraic Modeling of Network Routing and Consensus Protocols:
In distributed systems and complex communication networks, the state space of routing decisions or consensus history can be very large and context-dependent. Soft PolyGroups can model the state transitions where the validity of a transition depends on the current network configuration (α).

Example 4.3. *Let P be the set of all possible agreement states (a transformation system). An adaptive adversary can change its strategy based on the history of messages, which corresponds to the index set A. The set of valid state transitions $\mathcal{F}(\alpha)$ are those that resist the α-th level of deception. If we consider the composition of a set of local state transitions (P) with a global reconfiguration operation (P'), the Semi-Direct Product $P \ltimes P'$ captures the resulting system state. Now, if the local and reconfiguration sub-systems are robust ($\mathcal{F}$ and $\mathcal{F}'$ are soft), their combined operation remains robust ($\mathcal{F}''$ is soft) under the specific interaction ϕ.*

Example 4.4. *Let α denote the current load metric (e.g., high CPU utilization). $R' \in \mathcal{F}(\alpha)$ is a specific allocation strategy robust under high load. The centralizer $C_P(R')$ is the set of all other strategies that yield the exact same outcome as R' when applied to the high-load system. This centralizer forms a Soft PolyGroup because the constraints imposed by the high-load environment α restrict the algebraic operations within the commuting set in a non-standard (soft) way compared to the unconstrained system P.*

(3) Non-Associative Algebras and Theoretical Physics:
Theoretical physics, particularly in areas involving gauge theories, deformation quantization, or algebraic structures beyond standard groups (like semi-groups or monoids), benefits from frameworks that naturally handle non-associativity or structural dependence on parameters.

Example 4.5. *Consider a family of Lie algebras $\mathfrak{g}$ parameterized by a deformation parameter $\hbar \in \mathbb{R}$. The sequence of algebras obtained by setting $\hbar = p^{-k}$ for integers k can be structured as a sequence of polygroups indexed by $A = \mathbb{Z}$. The "softness" $\mathcal{F}(\alpha)$ is enforced by requiring the bracket operation $[\cdot,\cdot]_\alpha$ to satisfy a generalized Jacobi identity with an error term proportional to $\hbar^\alpha$. The Centralizer $C_{\mathfrak{g}}(R')$ of an operator R' under the deformed bracket $[\cdot,\cdot]_\alpha$ yields a new algebraic structure that is a soft polygroup, reflecting how the non-associative structure of the deformation is preserved in the central elements.*

Example 4.6. *In particle physics, spontaneous symmetry breaking often involves taking a product of the original symmetry group P (e.g., gauge group) with another structure P' (e.g., a set of vacuum expectation values or scalar fields) via a semi-direct product. If P and P' are Soft PolyGroups due to complex interactions with the background vacuum state (parameterized by α), the resulting effective symmetry group $Q = P \ltimes P'$ after breaking is still guaranteed maintain a soft structure $\mathcal{F}''(\alpha)$, correctly embedding the residual symmetries within the resulting mathematical framework.*

5. Conclusion

The central objective of this work was to conduct a comprehensive structural exploration of Soft PolyGroups, primarily achieved through the rigorous analysis of their semi-direct products and the establishment of a novel Cayley-type theorem within this framework, culminating in the formal introduction of *soft crossed polymodules*. To realize these foundational goals, we first meticulously formulated and systematically analyzed the concept of *soft polyaction*, presenting several indispensable characterizations that serve as the bedrock for subsequent algebraic constructions. Furthermore, we defined and examined the intrinsic structural elements-namely the *stabilizer*, *normalizer*, and *centralizer*-in the context of soft polyactions, investigating their interrelationships and validating their utility via illustrative, concrete examples. The guiding philosophy underpinning this research has been the methodical extension and generalization of established concepts from classical group theory into the broader domain of algebraic hyperstructures. By introducing the notion of *soft crossed polymodules*, we have furnished a methodological pathway that permits deeper penetration into hyperalgebraic systems, thereby expanding the utility of conventional algebraic apparatus into novel and consequential avenues. This approach not only enriches the theoretical edifice of the field but crucially establishes a platform conducive to more advanced investigations. Several promising and natural trajectories for continuing this line of research emerge. One significant avenue involves the formal definition and structural analysis of *soft crossed hypermodules*, whose properties may yield profound insights into the interplay between hyperstructure theory and module-like systems. Another compelling extension resides in the investigation of *soft crossed polysquares*; their formulation and subsequent structural scrutiny promise to unveil novel applications and significantly broaden the scope of soft algebraic hyperstructures. **As detailed in the preceding section,** these theoretical constructs find tangible realization through several significant applications across diverse fields. We assert that these proposed extensions will be of substantial interest to researchers dedicated to advancing the algebraic theory of hyperstructures and solidifying its congruence with classical algebraic results.

Acknowledgment

I would like to express my most sincere thanks and appreciation to Professor Bijan Davvaz. A professor who excels not only in academics but also in morals and generosity, and being his student is the pride of my life. I hope I can be grateful for his constant love.

Declarations

- Funding: Not applicable
- **Conflict of interest/Competing interests:** The author declares no competing interests.
- Ethics approval: Not applicable
- Consent to participate: Not applicable
- Consent for publication: Not applicable
- **Availability of data and materials:** Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.
- Code availability: Not applicable
- Authors' contributions: Not applicable

References

[1] H. Aktaş and N. Çağman. Soft sets and soft groups. *Information Sciences*, 177:2726–2735, 2007.
[2] M. I. Ali, F. Feng, X. Liu, W. K. Min, and M. Shabir. On some new operations in soft set theory. *Computers & Mathematics with Applications*, 57:1547–1553, 2009.
[3] M. Alp. Actor of crossed modules of algebroids. In *Proceedings of the 16th International Conference of the Jangjeon Mathematical Society*, pages 6–15, 2005.
[4] M. Alp. Pullback crossed modules of algebroids. *Iranian Journal of Science and Technology Transaction A*, 32(A3):145–181, 2008.
[5] M. Alp. Pullbacks of profinite crossed modules and cat 1-profinite groups. *Algebras, Groups and Geometries*, 25:215–221, 2008.
[6] M. Alp and B. Davvaz. On crossed polymodules and fundamental relations. *Scientific Bulletin, Series A: Applied Mathematics and Physics, Universitatea Politehnica Bucuresti*, 77(2):129–140, 2015.
[7] M. Alp and B. Davvaz. Pullback and pushout crossed polymodules. *Proceedings of the Indian Academy of Sciences: Mathematical Sciences*, 125(1):11–20, 2015.
[8] R. Ameri and M. M. Zahedi. Hyperalgebraic system. *Italian Journal of Pure and Applied Mathematics*, 6:21–32, 1999.
[9] Z. Arvasi. Crossed squares and 2-crossed modules of commutative algebras. *Theory and Applications of Categories*, 3(7):160–181, 1997.
[10] Z. Arvasi and T. Porter. Freeness conditions for 2-crossed modules of commutative algebras. *Applied Categorical Structures*, 6:455–471, 1998.
[11] Z. Arvasi and E. Ulualan. On algebraic models for homotopy 3-types. *Journal of Homotopy and Related Structures*, 1(1):1–27, 2006.
[12] Z. Arvasi and E. Ulualan. 3-types of simplicial groups and braided regular crossed modules. *Homology, Homotopy and Applications*, 9(1):139–161, 2007.

[13] R. Brown and N. D. Gilbert. Algebraic models of 3-types and automorphism structures for crossed modules. *Proceedings of the London Mathematical Society*, 59(3):51–73, 1989.
[14] R. Brown and J.-L. Loday. Van kampen theorems for diagrams of spaces. *Topology*, 26:311–334, 1987.
[15] S. D. Comer. Polygroups derived from cogroups. *Journal of Algebra*, 89:397–405, 1984.
[16] P. Corsini. *Prolegomena of Hypergroup Theory*. Aviani Editore, 2nd edition, 1993.
[17] B. Davvaz. Fuzzy hv-groups. *Fuzzy Sets and Systems*, 101:191–195, 1999.
[18] B. Davvaz. *Polygroup Theory and Related Systems*. World Scientific Publishing, 2013.
[19] M. A. Dehghanizadeh and B. Davvaz. On central automorphisms of crossed modules. *Carpathian Mathematical Publications*, 10(2):288–295, 2018.
[20] M. A. Dehghanizadeh and B. Davvaz. On the representation and characters of cat^1-groups and crossed modules. *Communications Faculty of Sciences University of Ankara, Series A1 Mathematics and Statistics*, 68(1):70–86, 2019.
[21] M. A. Dehghanizadeh and B. Davvaz. n-complete crossed modules and wreath products of groups. *Journal of New Results in Science*, 10(1):38–45, 2021.
[22] M. A. Dehghanizadeh, B. Davvaz, and M. Alp. On crossed polysquares and fundamental relations. *Sigma Journal of Engineering and Natural Sciences*, 9(1):1–16, 2018.
[23] M. A. Dehghanizadeh, B. Davvaz, and M. Alp. On crossed polysquare version of homotopy kernels. *Journal of Mathematical Extension*, 16(3):1–37, 2022.
[24] M. A. Dehghanizadeh, B. Davvaz, and M. Alp. On crossed polysquare version of homotopy cokernels. *Mathematica*, 67(90):56–70, 2025.
[25] J. A. Goguen. L-fuzzy sets. *Journal of Mathematical Analysis and Applications*, 18:145–174, 1967.
[26] P. K. Maji, R. Biswas, and A. R. Roy. Soft set theory. *Computers & Mathematics with Applications*, 45:555–562, 2003.
[27] P. K. Maji, A. R. Roy, and R. Biswas. An application of soft sets in a decision making problem. *Computers & Mathematics with Applications*, 44:1077–1083, 2002.
[28] D. Molodtsov. Soft set theory — first results. *Computers & Mathematics with Applications*, 37:19–31, 1999.
[29] Z. Pawlak. Rough sets. *International Journal of Computer & Information Sciences*, 11:341–356, 1982.
[30] D. W. Pei and D. Miao. From soft sets to information systems. In *Proceedings of the IEEE International Conference on Granular Computing*, pages 617–621, 2005.
[31] A. Rosenfeld. Fuzzy groups. *Journal of Mathematical Analysis and Applications*, 35:512–517, 1971.
[32] A. R. Roy and P. K. Maji. A fuzzy soft set theoretic approach to decision making problems. *Journal of Computational and Applied Mathematics*, 203:412–418, 2007.
[33] M. Shabir and M. Naz. On soft topological spaces. *Computers & Mathematics with Applications*, 61(7):1786–1799, 2011.
[34] T. Shah and S. Shaheen. Soft topological groups and rings. *Annals of Fuzzy Mathematics and Informatics*, 7(5):725–743, 2014.
[35] J. Wang, M. Yin, and W. Gu. Soft polygroups. *Computers & Mathematics with Applications*, 62:3529–3537, 2011.
[36] J. H. C. Whitehead. Combinatorial homotopy ii. *Bulletin of the American Mathematical Society*, 55:453–496, 1949.
[37] W.-Z. Wu, J.-S. Mi, and W.-X. Zhang. Generalized fuzzy rough sets. *Information Sciences*, 151:263–282, 2003.
[38] S. Yamak, O. Kazancı, and B. Davvaz. Soft hyperstructure. *Computers & Mathematics with Applications*, 62:797–803, 2011.
[39] C.-F. Yang. Fuzzy soft semigroups and fuzzy soft ideals. *Computers & Mathematics with Applications*, 61:255–261, 2010.
[40] L. A. Zadeh. Fuzzy sets. *Information and Control*, 8:338–353, 1965.
[41] M. M. Zahedi, M. Bolurian, and A. Hasankhani. On polygroups and fuzzy subpolygroups. *Journal of Fuzzy Mathematics*, 3:1–15, 1995.