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Partial metrics that are coming from polynomials

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Abstract. Partial metrics were introduced to model partially defined information, which appears in computer science. There are distance functions where the distance from a point to itself need not be equal to zero. In this paper, we introduce a class of partial metrics that is closed under addition, multiplication, supremum, and infimum. This class of partial metrics induces a topology on $\mathbb{R}$ that is finer than the upper topology. We also introduce intrinsic partial metrics and prove that these partial metrics are associated with functions derived from the supremum and infimum of polynomials. Moreover, we show that the supremum of infimum of polynomials is itself a polynomial if and only if it is equal to one of the original polynomials, and then we extend this result to rational functions.

1. Introduction and preliminaries

Metric spaces are the essential tools in mathematics for studying distance, convergence, and continuity. However, metrics axioms can be too restrict for certain situations. For example symmetric condition can be too restrictive when one deals with one sided roads or the zero self-distance condition where elements may be partially defined, incomplete, or have some internal structure that is not fully resolved. This comes up in computer science where objects such as programs, processes, or data types may exist in varying stages of completion or refinement. To overcome this issue Steve Matthews, a computer scientist, came up with idea of partial metrics in [12] to deal with the reality of having only partial information and to model the gaining of partial knowledge about ideal objects through a computer program. In the theory of partial metrics the smaller the self-distance the more complete or defined the object is.

The theory of partial metric spaces not only extends many classical results of metric spaces, such as Banach's fixed point theorem (see [5]), but also provides new tools to analyze structures where traditional metrics fail to apply such as asymmetric topology. Asymmetric topologies that arise in domain theory which are a part of computer logic. The structures of domain theory are continuous posets, and each continuous poset has a partial metric on it valued to a value lattice that gives rise to all three topologies studied there: Scott topology, lower topology, and the join of these two, Lawson topology; see [9]. Each

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partial metric has a natural completion which turns out to be the order theoretic round ideal completion, that yields a continuous poset, see [10]. To learn more about partial metrics and their development see [1], [3], [9], [11], [12], [14], [16], and [19].

Partial metric spaces also offer a fixed point theorem. Beyond computer science, partial metrics have applications in biological and information systems, where elements have partial information, see [18].

Not only many of the desirable properties of metric spaces carries over to partial metrics but also some new structure that deals with the nature of partiality, approximation, and information.

As seen below in the definition of partial metrics it is not required that the distance of a point from itself be zero.

Definition 1.1. A *partial metric* is a function $p\colon X \times X \to [0, \infty)$ satisfying the following weakening of identity, and other metric conditions, including symmetry:
ssd: For every $x, y \in X$, $p(x, y) \geq p(x, x)$,
psy: For every $x, y \in X$, $p(x, y) = p(y, x)$,
ptr: For every $x, y, z \in X$, $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$,
pt0: For every $x, y \in X$, $x = y$ if $p(x, y) = p(x, x) = p(y, y)$.

The partial metric axioms can be described as following:
ssd: small self-distance (no element is closer to another element than to itself),
psy: symmetry,
ptr: triangularity, and
pt0: separation.

Each axiom boils down to one of the metric axioms when $p(x, x) = 0$ is assumed; for example, pt0 becomes $d(x, y) = 0 \Rightarrow x = y$ in the metric case and implies that the associated topology is Hausdorff.

It is straightforward to see that a partial metric $p : X \times X \to \mathbb{R}$ is a metric if and only if $p(x, x) = 0$ for every $x \in X$.

A *pseudopartial metric* is a function $p\colon X \times X \to [0, \infty)$ satisfying all of the axioms of partial metrics except the last one.

Example 1.2. Consider $p : [0, \infty) \times [0, \infty) \to \mathbb{R}$ with $p(x, y) = (x \vee y)^n$, where $x \vee y = \max\{x, y\}$ and n is a positive integer. Analogously, $(x \wedge y) = \min\{x, y\}$.

We prove that p is a partial metric. While proving, we use the fact that $p(x, x) = x^n$ as well as the property below:

$$p(x, y) = p(x \vee y, x \vee y). \tag{1}$$

ssd: For every $x, y \in X$, $p(x, y) = (\max\{x, y\})^n \geq x^n = p(x, x)$,
psy: For every $x, y \in X$, $p(x, y) = p(x \vee y, x \vee y) = p(y \vee x, y \vee x) = p(y, x)$,
ptr: For every $x, y, z \in X$, $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$ because:

We consider three cases: when y is the smallest of x, y, and z, when y is the largest of x, y, and z, or when y is in the middle. Either $x \leq z$ or $z \leq x$. Without loss of generality we can assume that $x \leq z$ and investigate the three cases. One can easily verify that it is the same when $z \leq x$.

If y is the smallest of x, y, and z then $p(x, z) = (x \vee z)^n = z^n$, $p(x, y) = (x \vee y)^n = x^n$, $p(y, z) = (y \vee z)^n = z^n$, and $p(y, y) = (y \vee y)^n = y^n$. Thus, $p(x, y) + p(y, z) - p(y, y) = x^n + z^n - y^n = z^n + (x^n - y^n) \geq z^n + 0 = z^n = p(x, z)$.

If y is the largest of x, y, and z then $p(x, z) = (x \vee z)^n = z^n$, $p(x, y) = (x \vee y)^n = y^n$, $p(y, z) = (y \vee z)^n = y^n$, and $p(y, y) = (y \vee y)^n = y^n$. Thus, $p(x, y) + p(y, z) - p(y, y) = y^n + y^n - y^n = y^n \geq z^n = p(x, z)$.

If y is in the middle then $x \leq y \leq z$ and so, $p(x, z) = (x \vee z)^n = z^n$, $p(x, y) = (x \vee y)^n = y^n$, $p(y, z) = (y \vee z)^n = z^n$, and $p(y, y) = (y \vee y)^n = y^n$. Thus, $p(x, y) + p(y, z) - p(y, y) = y^n + z^n - y^n = z^n = p(x, z)$.

pt0: For every $x, y \in X$ if $p(x, y) = p(x, x) = p(y, y)$, then $(x \vee y)^n = x^n = y^n$. Thus, $(x \vee y)^n = x^n = y^n$. Since $x, y \geq 0$ we $x^n = y^n$ implies $x = y$.

For a pseudopartial metric p on the set X, $a \in X$, and $r > 0$, define
$B_r(a) = \{x \in X : p(x, a) - p(x, x) < r\}$ and $B^*_r(a) = \{x \in X : p(x, a) - p(a, a) < r\}$. Then define,

$\tau_p = \{U \subseteq X : a \in U \Rightarrow \exists r > 0\ B_r(a) \subseteq U\}$ and
$\tau_p^* = \{U \subseteq X : a \in U \Rightarrow \exists r > 0\ B_r^*(a) \subseteq U\}$.

Then both τ_p and τ_p^* form topologies on the set X and τ_p is called the induced topology by p. Both topologies τ_p and τ_p^* are T_0 if and only if p is a partial metric; for details see [9].

Note that $p(x, y) = \max\{x, y\}$ is not a partial metric on the real numbers as $p(x, y)$ can be negative when both x and y are negative. However, several variants of the concept of a partial metric have already been studied in the literature. For example, O'Neill in [15], allowed partial metrics to admit negative values. In other words, he defined a partial metric on a set X to be a function $p : X \times X \to \mathbb{R}$ which satisfies the axioms of partial metrics above. So, by O'Neill's definition of partial metrics, $\vee$ is a partial metric on $\mathbb{R}$. This partial metric plays an important role as a partial metric as the Euclidean metric plays as a metric.

So, if one allows negative distance then $(\max\{x, y\})^n$ will be a partial metric only when n is odd. In this case, $p(-x, -x) = -p(x, x)$ for every $x \in \mathbb{R}$.

In [7], by defining and characterizing intrinsic partial metrics, it was shown that partial metrics coming from the supremum are the most natural partial metrics on a lattice ordered group. We will postpone the discussion of intrinsic partial metrics on the ring $\mathbb{R}$ towards the end of this paper as the focus of this paper is on the partial metric on the subsets of $\mathbb{R}$.

Below, we provide an example of partial metrics involving computing:

Example 1.3. The n-place binary expansion of a number x, gives us an interval of width 2^{-n}. In general, to study the theory of approximating a number by smaller and smaller intervals containing it, we study the poset of closed real intervals, under the partial order $\supseteq$ and there we define $p([a, b], [c, d]) = (d \vee b) - (a \wedge c)$. It is straightforward to see that p is a partial metric which helps to study the situation. In this case, $p([a, b], [a, b]) = b - a$ provides a measure of how well $[a, b]$ "knows" x.

In section two we will discuss partial metrics that satisfy Equation 1 and classify these partial metrics on the subsets of $\mathbb{R}$.

In section three we introduce max-intrinsic (pseudo)partial metric producer functions and prove that the topology induced by their associated (pseudo)partial metric equals to the upper subspace topology if and only if is continuous from the left at every number in the domain.

In the last section we discuss intrinsic partial metrics on the rings of $\mathbb{R}$. We prove that the intrinsic partial metrics on $\mathbb{R}$ are the supremum of infimum of polynomials.

2. Max-intrinsic partial metrics

Definition 2.1. Let $X \subseteq \mathbb{R}$. A function $p : X \times X \to \mathbb{R}$ is called a *max-intrinsic* on X, if $p(x, y) = p(x \vee y, x \vee y)$ for every $x, y \in X$ and

$$x, -x \in X \Rightarrow p(-x, -x) = -p(x, x)$$

Example 2.2. Let $X \subseteq \mathbb{R}$. The function $e : X \times X \to \mathbb{R}$ defined by $e(x, y) = \max\{x, y\}$ is a max-intrinsic function on X.

It is obvious that if $p : X \times X \to \mathbb{R}$ is a max-intrinsic map, then $p(x, y) = p(y, x)$. However, it is not necessarily a partial metric. For example define $p : \{0, 1\} \to [0, \infty)$ by $p(0, 1) = p(1, 0) = p(1, 1) = 1$ and $p(0, 0) = 2$. Then p is a max-intrinsic map. However, p does not satisfy the *ptr* property of partial metrics as $p(0, 0) \nleq p(0, 1) + p(1, 0) - p(1, 1)$.

Definition 2.3. Let $X \subseteq \mathbb{R}$. A max-intrinsic function $p : X \times X \to \mathbb{R}$ is called a

- *max-intrinsic partial metric* on X, if it satisfies *ssd*, *psy*, *ptr*, and *pt0* of Definition 1.1.
- *max-intrinsic pseudopartial metric* on X, if it satisfies *ssd*, *psy*, and *ptr* of Definition 1.1.

Suppose $X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$ is a function. Define $p_f : X \times X \to \mathbb{R}$ by $p_f(x, y) = f(x \vee y)$.

Lemma 2.4. *Let $X \subseteq [0, \infty)$. Then,*

- *If $f : X \to \mathbb{R}$ is an increasing function, then p_f is a max-intrinsic pseudopartial metric.*
- *If $f : X \to \mathbb{R}$ is a strictly increasing function, then p_f is a max-intrinsic partial metric.*

Proof. First, note that if f is increasing, then p_f is a max-intrinsic map because for every $x, y \in X$, we have $p_f(x, y) = f(x \vee y) = f((x \vee y) \vee (x \vee y)) = p_f((x \vee y), (x \vee y))$. Next for the first part suppose $f : X \to \mathbb{R}$ is an increasing function. Then, for every $x, y \in X$ we have, $p_f(x, x) = f(x \vee x) = f(x) \leq f(x \vee y) = p_f(x, y)$. It is obvious that p_f is symmetric. By the same method of Example 1.2 and using the fact that f is increasing it can be shown that $p_f(x, z) + p_f(y, y) \leq p_f(x, y) + p_f(y, z)$ and we will leave it to the reader. Thus, p_f is a max-intrinsic pseudopartial metric.

For the second part, it is enough to prove if f is strictly increasing, then p_f has the last property of partial metrics. Note that if $p_f(x, y) = p_f(x, x) = p_f(y, y)$, then $f(x \vee y) = f(x) = f(y)$. Thus, $x = y$ as f is strictly increasing. Therefore, p_f is a max-intrinsic partial metric. □

Suppose $X \subseteq [0, \infty)$ and $p : X \times X \to \mathbb{R}$ is a max-intrinsic function. Define $f_p : X \to \mathbb{R}$ by $f_p(x) = p(x, x)$.

Lemma 2.5. *Let $X \subseteq [0, \infty)$. Then,*

- *If $p : X \times X \to \mathbb{R}$ is a max-intrinsic pseudopartial metric, then f_p is an increasing map.*
- *If $p : X \times X \to \mathbb{R}$ is a max-intrinsic partial metric, then f_p is an strictly increasing map.*

Proof. Suppose $P : X \times X \to \mathbb{R}$ is a max-intrinsic pseudopartial metric. Then $x \leq y$ implies $x \vee y = y$. Thus, $f_p(x) = p(x, x) \leq p(x, y) = p(x \vee y, x \vee y) = p(y, y) = f_p(y)$ and therefore, f_p is an increasing function.

For the second part, it is enough to prove that $x < y$ implies $f_p(x) < f_p(y)$. Suppose, for the sake of contradiction, that $f_p(x) \not< f_p(y)$. Since f is increasing, this implies $f_p(x) = f_p(y)$. Therefore, $p(x, x) = f_p(x) = f_p(y) = p(y, y) = p(x \vee y, x \vee y) = p(x, y)$. The last property of partial metrics then implies $x = y$, which is a contradiction. □

We can summarize Lemma 2.4 and Lemma 2.5 with the following theorem:

Theorem 2.6. *Let $X \subseteq [0, \infty)$, $f : X \to \mathbb{R}$ be a function, and $e : X \times X \to \mathbb{R}$ be the partial metric $e(x, y) = x \vee y$. Then, $f \circ e$ is a partial metric if and only if f is a strictly increasing map.*

Proof. Note that by Lemma 2.4, if f is a strictly increasing map, then $p_f = f \circ e$ is a partial metric.

For the converse, suppose $f \circ e$ is a partial metric. We will prove that $x < y$ implies $f(x) < f(y)$. Note that $x < y$ implies $y = x \vee y$ and so, $(f \circ e)(x, y) = (f \circ e)(y, y)$. Since $f \circ e$ is a partial metric, in this case we have:

- $(f \circ e)(y, y) = (f \circ e)(x, y) \geq (f \circ e)(x, x)$ and
- $(f \circ e)(x, y) = (f \circ e)(y, y) = (f \circ e)(x, x)$ implies $x = y$.

Therefore, if $x < y$, it follows that $(f \circ e)(y, y) = (f \circ e)(x, y) > (f \circ e)(x, x)$ which implies $f(y) > f(x)$. Thus, we have shown that $x < y$ implies $f(x) < f(y)$, completing the proof. □

In [13] a function $f : [0, \infty) \to [0, \infty)$ is called a *partial metric preserving function* if, for every partial metric $p : X \times X \to [0, \infty)$, the function $f \circ p$ is also a partial metric.

There, they prove that a function $f : [0, \infty) \to [0, \infty)$ is partial metric preserving if and only if f is strictly increasing and concave. So, it makes sense that we obtain a larger class of functions as we consider a single partial metric e.

The sum, supremum, and infimum of (strictly) increasing functions is an (strictly) increasing function which makes the set of (strictly) increasing functions a lattice semigroup; see Definition 2.11. =

Lemma 2.7. *Let $X \subseteq [0,\infty)$ and $p_1, p_2 : X \times X \to [0,\infty)$ be max-intrinsic partial metric (pseudopartial metric) on X. Then, $p_1 + p_2$, $p_1 \vee p_2$, and $p_1 \wedge p_2$ are max-intrinsic partial metric (pseudopartial metric).*

Proof. Suppose $p_1, p_2 : X \times X \to \mathbb{R}$ are max-intrinsic pseudopartial metric. It is straightforward to verify that $p_1 + p_2$ is a max-intrinsic pseudopartial metric and both $p_1 \vee p_2$ and $p_1 \wedge p_2$ satisfy *ssd*, *psy*.

Next we show that $p_1 \vee p_2$ and $p_1 \wedge p_2$ satisfy *ptr*. First we prove $(p_1 \vee p_2)(x,z) + (p_1 \vee p_2)(y,y) \leq (p_1 \vee p_2)(x,y) + (p_1 \vee p_2)(y,z)$, for every $x,y,z \in X$. Without loss of generality assume $x \leq z$. Then,

$$\begin{aligned}
&(p_1 \vee p_2)(x,z) + (p_1 \vee p_2)(y,y)\\
&= (p_1(x,z) \vee p_2(x,z)) + (p_1(y,y) \vee p_2(y,y))\\
&= (p_1(x \vee z, x \vee z) \vee p_2(x \vee z, x \vee z)) + (p_1(y,y) \vee p_2(y,y))\\
&= (p_1(z,z) \vee p_2(z,z)) + (p_1(y,y) \vee p_2(y,y))\\
&\leq (p_1(y,z) \vee p_2(y,z)) + (p_1(x,y) \vee p_2(x,y))\\
&= (p_1 \vee p_2)(x,y) + (p_1 \vee p_2)(y,z).
\end{aligned}$$

All of the above equalities and the inequity hold for $p_1 \wedge p_2$ as well.

Finally we prove if p_1 and p_2 satisfy *pt0*, then $p_1 + p_2$, $p_1 \vee p_2$, and $p_1 \wedge p_2$ satisfy *pt0*. We prove it for $p_1 \vee p_2$ and $p_1 + p_2$ and will leave the proof for $p_1 \wedge p_2$ to the reader.

For $p_1 + p_2$: suppose $(p_1 + p_2)(x,y) = (p_1 + p_2)(x,x) = (p_1 + p_2)(y,y)$. Thus, $p_1(x,y) + p_2(x,y) - (p_1(x,x) + p_2(x,x)) = 0$ or $(p_1(x,y) - p_1(x,x)) + (p_2(x,y) - p_2(x,x)) = 0$. Since $p_1(x,y) - p_1(x,x), p_2(x,y) - p_2(x,x) \geq 0$, we have $p_1(x,y) = p_1(x,x)$ and $p_2(x,y) = p_2(x,x)$. Similarly, we can show $p_1(x,y) = p_1(y,y)$ and $p_2(x,y) = p_2(y,y)$. Thus, $p_1(x,y) = p_1(x,x) = p_1(y,y)$. Since p_1 is a partial metric, $x = y$.

For $p_1 \vee p_2$: assume $(p_1 \vee p_2)(x,y) = (p_1 \vee p_2)(x,x) = (p_1 \vee p_2)(y,y)$ for every $x,y \in X$. We prove $x = y$. Without loss of generality assume $x \leq y$. Either $p_1(x,x) \leq p_2(x,x)$ or $p_2(x,x) \leq p_1(x,x)$. Without loss of generality assume $p_1(x,x) \leq p_2(x,x)$. Then, $p_2(x,y) = p_2(x \vee y, x \vee y) = p_2(y,y) \leq (p_1 \vee p_2)(y,y) = (p_1 \vee p_2)(x,x) = p_2(x,x) \leq p_2(x,y)$. Thus, $p_2(x,y) = p_2(y,y) = p_2(x,x)$ which implies $x = y$ as p_2 satisfy *pt0*. □

The set of pseudopartial metrics and the set of partial metrics are closed under addition but not under supremum or infimum as it can be seen in the following examples.

Example 2.8. Let d_1 be the Euclidean metric on $\{0,1,2\}$ and d_2 to be the constant function 1 on $\{0,1,2\} \times \{0,1,2\}$. Then, both d_1 and d_2 are pseudopartial metrics however $(d_1 \vee d_2)$ is not a pseudopartial metric because
$(d_1 \vee d_2)(0,2) + (d_1 \vee d_2)(1,1) = 2 + 1 \not\leq 1 + 1 = (d_1 \vee d_2)(0,1) + (d_1 \vee d_2)(1,2)$.

Example 2.9. Let d_1 and d_2 be the following metrics on $\{0,1,2\}$:
$d_1(2,0) = d_1(0,2) = d_1(1,2) = d_1(2,1) = 3$, $d_1(1,0) = d_1(0,1) = 1$ and
$d_2(2,0) = d_2(0,2) = d_2(0,1) = d_2(1,0) = 3$, $d_2(2,1) = d_2(1,2) = 1$.
One can verify that $(d_1 \wedge d_2)$ is not a pseudopartial metric because
$(d_1 \wedge d_2)(0,2) + (d_1 \wedge d_2)(1,1) = 3 + 0 \not\leq 1 + 1 = (d_1 \wedge d_2)(0,1) + (d_1 \wedge d_2)(1,2)$

Example 2.10. Define partial metrics p_1 and p_2 on $\{0,1\}$ as follows:
$p_1(0,0) = p_1(0,1) = p_1(1,0) = 1$, $p_1(1,1) = 0$ and
$p_2(1,1) = p_2(0,1) = p_2(1,0) = 1$ and $p_2(0,0) = 0$.
Then neither $p_1 \vee p_2$ nor $p_1 \wedge p_2$ is a partial metric though both are pseudopartial metrics.

After the following definitions, we are ready to prove the relationship between this subclass of partial metrics and a subclass of real valued functions.

We bring the following definitions from [2].

Recall that *po-semigroup* is a poset $(M, \leq)$ with an associative operation $+$ such that for every $a,x,y \in M$, $x \leq y$ implies $a + x \leq a + y$ and $x + a \leq y + a$. A po-semigroup with identity 0, such that for every x, $0 + x = x + 0 = x$ is called a po-monoid.

A po-semigroup is called *abelian* if $+$ is commutative.

We found that ℓ-semigroups have been defined differently in different articles for example see [2] and [4] and we work on abelian structures. So, we bring the following definition for clarification.

Definition 2.11. The algebraic structure $(M, \vee, \wedge, +)$ is an (abelian) lattice ordered semigroup (*(abelian) ℓ-semigroup*) if

- $(M, \vee, \wedge)$ is a lattice,
- $(M, +)$ is an (abelian) semigroup,
- $x + (y \vee z) = (x + y) \vee (x + z)$ and $x + (y \wedge z) = (x + y) \wedge (x + z)$

An (abelian) ℓ-semigroup with identity 0, such that for every x, $0 + x = x + 0 = x$ is called an *(abelian) ℓ-monoid*.

One can see that the set of max-intrinsic partial metrics on a set as well as the set of strictly increasing maps on the set is an abelian ℓ-semigroup.

Theorem 2.12. *If $X \subseteq [0, \infty)$, then the abelian ℓ-semigroup of max-intrinsic partial metric on X is isomorphic with the abelian ℓ-semigroup of strictly increasing maps on X.*

Proof. Suppose $M(X)$ is the set of max-intrinsic partial metric on X and $S(X)$ is the set of strictly increasing maps on X. Define $\alpha : M(X) \to S(X)$ by $\alpha(p) = f_p$ and $\beta : S(X) \to M(X)$ by $\beta(f) = p_f$. We prove:

(a) both α and β preserve $+$, $\vee$, and $\wedge$,
(b) both $\alpha \circ \beta$ and $\beta \circ \alpha$ are the identity map.

For (a): We prove it only for + as the proof for $\vee$ and $\wedge$ is similar. Let $d, k \in M(X)$ and Let $g, h \in S(X)$. Then,
$\alpha(d+k)(x) = f_{d+k}(x) = (d+k)(x, x) = d(x, x) + k(x, x) = f_d(x) + f_k(x) = \alpha(d)(x) + \alpha(k)(x) = (\alpha(d) + \alpha(k))(x)$ and
$\beta(g+h)(x, y) = p_{g+h}(x, y) = (g+h)(x \vee y) = g(x \vee y) + h(x \vee y) = p_g(x, y) + p_h(x, y) = \beta(g)(x, y) + \beta(h)(x, y) = (\beta(g) + \beta(h))(x, y)$.
For (b): $((\alpha \circ \beta)(g))(x) = (\alpha(\beta(g))(x) = \beta(g)(x, x) = g(x \vee x) = g(x)$ and
$((\beta \circ \alpha)(k))(x, y) = (\beta(f_k))(x, y) = p_{f_k}(x, y) = f_k(x \vee y) = k(x \vee y, x \vee y) = k(x, y)$. □

Note that for every $X \subseteq R$ the zero function from X to $\mathbb{R}$ is an increasing functions and the zero function from $X \times X$ to $\mathbb{R}$ is a pseudopartial metric. Thus, we have

Corollary 2.13. *If $X \subseteq [0, \infty)$, then the abelian ℓ-monoid of max-intrinsic pseudopartial metric on X is isomorphic with the abelian ℓ-monoid of the increasing maps on X.*

Definition 2.14. Let $X \subseteq \mathbb{R}$. Then a map $f : X \to \mathbb{R}$ is called *weakly odd* if

$$x, -x \in X \Rightarrow f(-x) = -f(x)$$

Remark 2.15. Note that the supremum or infimum of two weakly odd maps is not necessarily odd. For example $f \vee g$ where $f(x) = x$ and $g(x) = x^3$ is not an odd function.

Thus, we have the following corollaries:

Corollary 2.16. *If $X \subseteq \mathbb{R}$, then there is a one to one po-semigroup correspondence between the set of max-intrinsic partial metric on X and the set of weakly odd strictly increasing maps on X.*

Corollary 2.17. *If $X \subseteq \mathbb{R}$, then there is a one to one po-monoid correspondence between the set of max-intrinsic pseudopartial metric on X and the set of weakly odd increasing maps on X.*

Remark 2.18. All of the results in this section are valid if we replace $X \subseteq \mathbb{R}$ by $X \subseteq G$, where G is an abelian lattice ordered group. They also can be generalized for a $\vee$-semilattice. However, we try to stay within the set of real numbers.

3. Max-intrinsic (pseudo)partial metric producer functions

In this section, we want to characterize those polynomials that correspond with a max-intrinsic partial metric or a max-intrinsic pseudopartial metric on $\mathbb{R}$.

Definition 3.1. Let $X \subseteq \mathbb{R}$. Then a map $\lambda : X \to \mathbb{R}$ is called *max-intrinsic partial metric producer* for X (*max-intrinsic pseudopartial metric producer*) if $p_\lambda : X \times X \to \mathbb{R}$ defined by $p_\lambda(x, y) = \lambda(x \vee y)$ is a max-intrinsic partial metric (max-intrinsic pseudopartial metric) on X.

By Theorem 2.12 any max-intrinsic partial metric producer function of $X \subseteq \mathbb{R}$ must be strictly increasing. Thus, in case this function is differentiable, its derivative must be positive. On the other hand by Corollary 2.13 any max-intrinsic pseudopartial metric producer function of $X \subseteq \mathbb{R}$ must be increasing. Hence, in case this function is differentiable, its derivative must be non-negative. In case that $[0, \infty) \subsetneq X$ in both partial metric and pseudopartial metric cases such a function must also be weakly odd. The next theorem puts all of these properties together for polynomials:

Theorem 3.2. *For the polynomial $\lambda(x) = \Sigma_{l=1}^{t} a_l x^l$ the following are equivalent:*

(a) *λ is a max-intrinsic partial metric producer function,*
(b) *λ is a max-intrinsic pseudopartial metric producer function,*
(c) *λ is odd and strictly increasing on $\mathbb{R}$,*
(d) *λ is odd and increasing on $\mathbb{R}$,*
(e) *$\lambda(x) = \Sigma_{l=1}^{t} a_{2l-1} x^{2l-1}$, where $\lambda'(x) \geq 0$ for every $x \in \mathbb{R}$.*

Proof. By Corollary 2.16 (a) and (c) are equivalent and by Corollary 2.17 (b) and (d) are equivalent. Since a polynomial is increasing if and only if it is strictly increasing, (c) and (d) are equivalent.

So, the proof is complete if we prove (a) implies (e) as it is straightforward to prove that (e) implies (d). Suppose $\lambda(x) = \Sigma_{l=1}^{m} a_l x^l$. Then, λ is an odd function. Thus, $\lambda(x) + \lambda(-x) = 0$ for every x, $\Sigma_{l=1}^{m} (a_l + (-1)^l a_l) x^l = 0$. When l is odd $a_l + (-1)^l a_l = 0$. So, $\Sigma_{l=1}^{[\frac{m+1}{2}]} 2a_{2l} x^{2l} = 0$. Therefore, $a_{2l} = 0$ for $l = 1, \cdots, [\frac{m+1}{2}]$.

Therefore, $\lambda(x) = \Sigma_{l=1}^{t} a_{2l-1} (x)^{2l-1}$, where $t = [\frac{m+1}{2}]$. The derivative of λ by part (d) must be non-negative. □

Recall that the *upper topology* (*lower topology*) on $\mathbb{R}$ is $\{(a, \infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ ($\{(-\infty, a) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$). We denote this topology by $\mathcal{U}$ ($\mathcal{L}$). We denote the upper (lower) subspace topology on X by $\mathcal{U}_X$ ($\mathcal{L}_X$).

Theorem 3.3. *If $X \subseteq \mathbb{R}$ and $\lambda : X \to \mathbb{R}$ is max-intrinsic partial metric producer then τ_{p_λ} is finer than the upper subspace topology on X.*

Proof. Suppose $\lambda : X \to \mathbb{R}$ is a max-intrinsic pseudopartial metric producer map. Consider $T = (a, \infty) \cap X \in \mathcal{U}_X$, we prove $T \in \tau_{p_\lambda}$. If $T = \emptyset$ then $T \in \tau_{p_\lambda}$. So assume $T \neq \emptyset$ and let $x \in T$. We show that there is an $\epsilon > 0$ such that $B_\epsilon(x) \subseteq T$.

Either $(a, x) \cap X \neq \emptyset$ or $(a, x) \cap X = \emptyset$. If $(a, x) \cap X \neq \emptyset$, then there is $z \in (a, x) \cap X$. Let $0 < \epsilon < \lambda(x) - \lambda(z)$. Now, $B_\epsilon(x) = \{y \in X : p_\lambda(x, y) - p_\lambda(y, y) < \epsilon\} = \{y \in X : \lambda(x \vee y) - \lambda(y \vee y) < \epsilon\}$. We prove $\{y \in X : \lambda(x \vee y) - \lambda(y \vee y) < \epsilon\} \subseteq T$.

Let $\lambda(x \vee y) - \lambda(y \vee y) < \epsilon$. Then either $y \leq x$ or $y \geq x$. It is obvious that if $y \geq x$ then $y > z$ and therefore, $y \in (z, \infty) \cap X \subseteq T$. If $y \leq x$, then $\lambda(x \vee y) = \lambda(x)$. Therefore, $\lambda(x) - \lambda(y) = \lambda(x \vee y) - \lambda(y \vee y) < \epsilon$ implies $\lambda(x) - \lambda(y) < \epsilon < \lambda(x) - \lambda(z)$ or $\lambda(z) < \lambda(y)$. By Theorem 3.2 λ is increasing and so, $z < y$. Thus, $y \in (z, \infty) \cap X \subseteq (a, \infty) \cap X = T$. Consequently, $\mathcal{U}_X \subseteq \tau_{p_\lambda}$.

For the case $(a, x) \cap X = \emptyset$: consider $A = \{y \in X : y < x\}$. Either $A \neq \emptyset$ or $A = \emptyset$. If $A \neq \emptyset$, then $y \leq a$ for every $y \in A$. Let $h = \bigvee A$. Then it is straightforward to see that $B_\epsilon(x) \subseteq T$ for $\epsilon < \lambda(x) - \lambda(h)$ and we leave it to the reader. Thus, for this case also, $T \in \tau_{p_\lambda}$. On the other hand if $A = \emptyset$, then $X = T$. Now, $B_1(x) = \{y \in X : p_\lambda(x, y) - p_\lambda(y, y) < 1\} = \{y \in X : \lambda(x \vee y) - \lambda(y \vee y) < 1\} = \{y \in X : \lambda(y) - \lambda(y) < 1\} = X \subseteq X = T$. Thus, $T \in \tau_{p_\lambda}$. Consequently, $\mathcal{U}_X \subseteq \tau_{p_\lambda}$. □

The following example shows that the inclusion in Theorem 3.3 can be proper.

Example 3.4. Define $\lambda : [0,\infty) \to [0,\infty)$ by $\lambda(x) = \begin{cases} x, & \text{if } 0 \le x < 1 \\ 4x & \text{if } x \ge 1 \end{cases}$.

Then $[1,\infty) \in \tau_{p_\lambda}$ as $B_1(1) = \{y \in X : p_\lambda(1,y) - p_\lambda(y,y) < 1\} = \{y \in X : \lambda(1 \vee y) - \lambda(y \vee y) < 1\} = \{y \in X : \lambda(1 \vee y) - \lambda(y) < 1\}$. Note that if $y < 1$ then, $1 \vee y = 1$. Therefore, $\lambda(y) = y$ and $\lambda(1 \vee y) = \lambda(1) = 4$. Consequently, $\lambda(1 \vee y) - \lambda(y) = 4 - y > 4 - 1 = 3$ and so, $y \notin B_1(1)$. Since $y < 1$ implies $y \notin B_1(1)$, $B_1(1) = [1,\infty)$. Thus, $[1,\infty) \in \tau_{p_\lambda}$ which is not in the upper subspace topology.

Theorem 3.5. *Let $X \subseteq \mathbb{R}$ and $\lambda : X \to \mathbb{R}$ be a max-intrinsic pseudopartial metric producer map. Then, τ_{p_λ} equals to the upper subspace topology on X if and only if λ is continuous from the left at every $a \in X$.*

Proof. Suppose $\lambda : X \to \mathbb{R}$ is a max-intrinsic pseudopartial metric producer map such that τ_{p_λ} equals to the upper subspace topology on X. We prove that λ is continuous from the left at every $a \in X$. So, we show:

$\forall \epsilon > 0, \exists \delta > 0$ such that $0 < a - x < \delta \Rightarrow |\lambda(a) - \lambda(x)| < \epsilon$.

By the way of contradiction assume that there is a $b \in X$ such that λ is not continuous from the left at b. We prove that then $[b,\infty) \cap X \in \tau_{p_\lambda} \setminus \mathcal{U}_X$.

Since λ is not continuous from the left at b, there is an $\epsilon_0 > 0$ that for every $\delta > 0$ we can find $x_\delta \in X$ such that $0 < b - x_\delta < \delta$ and $|\lambda(b) - \lambda(x_\delta)| > \epsilon_0$.

We prove that $[b,\infty) \cap X \in \tau_{p_\lambda} \setminus \mathcal{U}_X$. First we prove $[b,\infty) \cap X \in \tau_{p_\lambda}$. Let $z \in [b,\infty) \cap X$. We prove $B_{\epsilon_0}(z) \subseteq [b,\infty) \cap X$. It is enough to show $y \in B_{\epsilon_0}(z) \Rightarrow y \ge b$. Note that:

$y \in B_{\epsilon_0}(z) \Leftrightarrow p_\lambda(z,y) - p_\lambda(y,y) < \epsilon_0 \Leftrightarrow \lambda(z \vee y) - \lambda(y \vee y) < \epsilon_0 \Leftrightarrow \lambda(z \vee y) - \lambda(y) < \epsilon_0$.

Next we prove $y \in B_{\epsilon_0}(z)$ implies $b \le y$. For contrary, assume $y_0 \in B_{\epsilon_0}(z)$ such that $y_0 < b < z$. Then, $(y_0,\infty) \cap X \subseteq B_{\epsilon_0}(z)$ since λ is increasing. Now by our assumption, for $\delta = b - y_0 > 0$ there exists $x_\delta \in X$ such that $0 < b - x_\delta < \delta = b - y_0$ such that $|\lambda(b) - \lambda(x_\delta)| > \epsilon_0$. Therefore, $x_\delta \in (y_0,\infty) \cap X \subseteq B_{\epsilon_0}(z)$. Thus, $\epsilon_0 > \lambda(z \vee x_\delta) - \lambda(x_\delta) = \lambda(z) - \lambda(x_\delta) \ge \lambda(b) - \lambda(x_\delta) > \epsilon_0$, a contradiction. Thus, $B_{\epsilon_0}(z) \subseteq [b,\infty) \cap X$ and therefore, $[b,\infty) \cap X \in \tau_{p_\lambda}$.

In order to prove $[b,\infty) \cap X \notin \mathcal{U}_X$, it is enough to show that $(b - r,\infty) \cap X \nsubseteq [b,\infty) \cap X$ for every $r > 0$. This follows from our assumption: for $\epsilon_0 > 0$ we have that for all $\delta > 0$ there exists $x_\delta \in X$ such that $0 < b - x_\delta < \delta$ satisfying $|\lambda(b) - \lambda(x_\delta)| > \epsilon_0$. For every fixed r let $\delta = \frac{r}{2}$. Then, $b > x_\delta > b - \delta > b - r$ and so, $x_\delta \in (b - r,\infty) \cap X \nsubseteq [b,\infty) \cap X$. Therefore, $[b,\infty) \cap X \notin \mathcal{U}_X$ and so, $[b,\infty) \cap X \in \tau_{p_\lambda} \setminus \mathcal{U}_X$. Consequently, $\tau_{p_\lambda} \ne \mathcal{U}_X$ which is a contradiction. Thus, λ is continuous from the left at every $a \in X$.

Conversely, assume λ is continuous from the left at every $a \in X$. We prove $\tau_{p_\lambda} = \mathcal{U}_X$. By Theorem 3.3 it is enough to prove that $\tau_{p_\lambda} \subseteq \mathcal{U}_X$. It is enough to prove that $b \in V \in \tau_{p_\lambda}$ then there is an $r > 0$ such that $(b - r,\infty) \cap X \subseteq V$. Since $b \in V \in \tau_{p_\lambda}$, there is an $\epsilon > 0$ such that $B_\epsilon(b) \subseteq V$. Note that,

$B_\epsilon(b) = \{y \in X : p_\lambda(b,y) - p_\lambda(y,y) < \epsilon\} = \{y \in X : \lambda(b \vee y) - \lambda(y \vee y) < \epsilon\} = \{y \in X : \lambda(b \vee y) - \lambda(y) < \epsilon\}$.

On the other hand since λ is continuous from the left at $b \in X$:

$\forall \epsilon > 0, \exists \delta_\epsilon > 0$ such that $0 < b - x < \delta_\epsilon \Rightarrow |\lambda(b) - \lambda(x)| < \epsilon$.

The proof is complete if we show $(b - \delta_{\epsilon_0},\infty) \subseteq B_\epsilon(b) \subseteq V$. Note that if $x \in (b - \delta_{\epsilon_0},\infty)$ either $x < b$ or $x > b$.

If $x < b$ then $b - \delta_{\epsilon_0} < x < b$ implies, $0 < b - x < \delta_{\epsilon_0}$. The continuity condition and λ being increasing implies $\lambda(b) - \lambda(x) < \epsilon_0$. Now, $x < b$ implies $b \vee x = b$ and so $\lambda(b) - \lambda(x) < \epsilon_0$ is equivalent to $\lambda(b \vee x) - \lambda(x) < \epsilon_0$. Thus, $x \in B_\epsilon(b)$. On the other hand if $x > b$ then obviously, $x \in B_\epsilon(b)$. Consequently, $(b - \delta_{\epsilon_0},\infty) \subseteq B_\epsilon(b) \subseteq V$. Therefore, $V \in \mathcal{U}_X$ and the proof is complete. □

We leave to the reader the validity of the next corollary.

Corollary 3.6. *Let $X \subseteq \mathbb{R}$ and $\lambda : X \to \mathbb{R}$ be a max-intrinsic pseudopartial metric producer map. Then $\tau_{p_\lambda} = \mathcal{U}_X$ and $\tau^*_{p_\lambda} = \mathcal{L}_X$ if and only if λ is continuous at every $a \in X$.*

4. Intrinsic partial metrics for lattice ordered rings

Charles Holland in [6] defined an intrinsic metric on a lattice ordered group G to be an intrinsic right invariant function $d : G^2 \to G$ which is symmetric. He showed that d is an intrinsic metric if and only if for some integer n, $d(x, y) = n|x - y|$ for each $x, y \in G$.

Intrinsic metrics on non-abelian lattice-ordered groups do not satisfy the triangle inequality property of metrics; the triangle inequality holds for $d(x, y) = n|x - y|$, where n is a positive integer if and only if G is abelian (see [8] for $n = 1$ and [6] for any n). In other words $d(x, y) \leq d(x, z) + d(z, y)$ if and only if G is abelian.

In [7] intrinsic partial metrics were defined and characterized and results relating commutativity to their key properties, such as the triangle inequality, were obtained: it was shown that if $(n\vee)(x, y) = n(x \vee y)$ defines a partial metric on an ℓ-group G, then the identity $na + nb = nb + na$ holds for every $a, b \in G$ (this property is sometimes called the commutativity of the nth power). The converse is true if n is prime; more clearly, if n is a prime number and $na + nb = nb + na$ for every $a, b \in G$, then $n\vee$ is a partial metric on G.

Here we try to define and characterize intrinsic partial metrics for the lattice ordered ring R.

A *lattice ordered group* (or *ℓ-group*) is a group G which is also a lattice under a partial order $\leq$, in which for all $a, b, x, y \in G$, $x \leq y \Rightarrow a + x + b \leq a + y + b$.

We bring the following definitions for lattice ordered rings from [17].

A lattice ordered ring (*ℓ-ring*) is a ring R whose additive group is an ℓ-group and the product of positive elements is positive.

For every a in the ℓ-ring R, the *positive part* of a is $a \vee 0$ and it is denoted by a^+, and the *negative part* of a is $-a \vee 0$ and it is denoted by a^-. We have $a = a^+ - a^-$.

The *absolute value* of a is denoted by $|a|$ and is defined $|a| = a^+ + a^-$ and has the following properties:

(i) $|x| = |-x|$

(ii) $|x| \geq 0$

(iii) $|x| = 0$ if and only if $x = 0$.

Also (iv) $|x + y| \leq |x| + |y|$.

For $a, b \in R$ we have $|ab| \leq |a||b|$.

If R is a lattice ordered ring, a function $f : R^n \to R$ is *intrinsic* if there is a word in the language of lattice-ordered rings such that for each $(r_1, ..., r_n) \in R^n$, $f(r_1, ..., r_n)$ is given by that word. For example, $f(x, y, z, t) = (3x - y \vee (z \wedge 0))t$ is such a function, and in general, intrinsic functions for ℓ-rings are defined using $0, +, -, \vee, \wedge$, and product.

Definition 4.1. An *intrinsic partial metric* on an ℓ-ring R is an intrinsic function $p\colon R \times R \to R$ such that for each $x, y \in R$, $p(x, y) = p(x \vee y, x \vee y)$ and $p(-x, -x) = -p(x, x)$.

The proof of the next theorem is straightforward and we leave it to the reader.

Theorem 4.2. *If R is a lattice ordered ring, then $p(x, y) = \Sigma_{l=1}^{t} n_{2l-1}(x \vee y)^{2l-1}$, where $n_1, n_2, \cdots, n_t$ are positive integers is an intrinsic partial metric on R.*

Notice that for every $x \in R$, $< x >$, the lattice ordered ring generated by x, is a commutative ring and for every $y \in < x >$ we have $y = \Sigma_{l=1}^{t} \bigvee_{i\in I_l} \bigwedge_{j\in J_l} k_l(i, j)x^l$ for some finite index sets I_l, J_l, where each $k_l(i, j)$ is an integer and $l = 1, 2, \cdots, t$.

We are now ready to discuss intrinsic partial metric on $\mathbb{R}$:

Suppose p is an intrinsic partial metric on $\mathbb{R}$. Define $H(x) = p(x, x)$. Thus $H(x) = -H(-x)$ for every $x \in \mathbb{R}$. Notice that $H(x) \in < x >$, where $< x >$ is the lattice ordered ring generated by x. Thus, since + distributes over $\vee$ and $\wedge$ and also the two operations $\vee$ and $\wedge$ distribute over each other, $H(x) = \bigwedge_{i\in I} \bigvee_{j\in J} p_{(i,j)}(x)$ for some finite index sets I, J, where by definition of intrinsic each $p_{(i,j)}$ is a particular polynomial with integer coefficients and no constant. Thus, $H(x)$ will be the infimum of supremum of polynomials. Therefore, $p(x, y) = p(x \vee y, x \vee y) = H(x \vee y, x \vee y) = \bigwedge_{i\in I} \bigvee_{j\in J} p_{(i,j)}(x \vee y)$.

While we were hoping that $p(x, y)$ to be a single polynomial, we found the example below which led us to our main theorem of this section.

We recall that if $f, g : X \to \mathbb{R}$ then $f \vee g$ and $f \wedge g$ are defined such that for every $x \in X$, $(f \vee g)(x) = f(x) \vee g(x)$ and $(f \wedge g)(x) = f(x) \wedge g(x)$.

Example 4.3. Let $q(x) = (x \vee -x^2) \wedge x^2$. Then, one can easily verify that,

$$q(x) = \begin{cases} x, & \text{if } x \leq -1; \\ -x^2, & \text{if } -1 \leq x \leq 0 \\ x^2, & \text{if } 0 \leq x \leq 1 \\ x, & \text{if } x \geq 1 \end{cases}.$$

As it is seen q is not a single polynomial even though it is an odd and strictly increasing.

Theorem 4.4. *Let $P(x) = \bigwedge_{i\in I} \bigvee_{j\in J} p_{ij}(x)$ where I, J are finite index sets and $p_{ij}(x)$'s are polynomials on $\mathbb{R}$. (For two polynomials p, q, $(p \bigvee q) = \max\{p, q\}$, $(p \bigwedge q) = \min\{p, q\}$). Then, $P(x)$ is a polynomial if and only if for some $i_0 \in I$, $j_0 \in J$, $P(x) = p_{i_0 j_0}(x)$.*

Proof. Obviously, if $P(x) = p_{i_0 j_0}(x)$ for some $i_0 \in I$, $j_0 \in J$, then $P(x)$ is a polynomial.

For the converse, assume $P(x)$ is a polynomial. We will prove that $P(x) = p_{i_0 j_0}(x)$ for some $i_0 \in I$, $j_0 \in J$. For every $i \in I$ define $g_i(x) = \bigvee_{j\in J} p_{ij}(x)$ and without loss of generality, assume for each $i \in I$, if $j, j' \in J$, $j \neq j'$, $p_{ij} \neq p_{ij'}$ (i.e., p_{ij} and $p_{ij'}$ are distinct polynomials.) For each $i \in I$, set

$$S_i = \{x \in \mathbb{R} : p_{ij}(x) = p_{ij'}(x), j, j' \in J, j \neq j'\}.$$

S_i is the finite set of common intersections of p_{ij} and $p_{ij'}$, possibly empty.

Let

$$S = \bigcup_{i\in I} S_i.$$

Since J is finite and any two polynomials p_{ij} and $p_{ij'}$, if not identical, can have at most a finite number of intersections, it follows that each S_i, if not empty, is finite. Also since I is finite, S, if not empty, is finite. The case when S is empty can be argued easily. So assume S consists of the points $x_1 < x_2 < \cdots < x_n$.

We prove that for every $i_0 \in I$ the set of polynomials $\{p_{i_0 j} : j \in J\}$ with respect to $\leq$ relation forms a chain on the interval $(-\infty, x_1)$. Otherwise, there are $j_1, j_2 \in J$ such that $p_{i_0 j_1} \nleq p_{i_0 j_2}$ and $p_{i_0 j_2} \nleq p_{i_0 j_1}$. Then, there are $a, b \in (-\infty, x_1)$ such that $p_{i_0 j_1}(a) > p_{i_0 j_2}(a)$ and $p_{i_0 j_2}(b) > p_{i_0 j_1}(b)$. Without loss of generality assume $a < b$. Thus, if $h(x) = p_{i_0 j_1}(x) - p_{i_0 j_2}(x)$, $h(a) > 0$ and $h(b) < 0$. Then, by the Intermediate value theorem, there is a $c \in (a, b) \subseteq (-\infty, x_1)$ such that $h(c) = 0$. Equivalently, $p_{i_0 j_1}(c) = p_{i_0 j_2}(c)$. So, $c \in S_{i_0}$ which is impossible because $c < x_1$. Thus, $\{p_{i_0 j} : j \in J\}$ with respect to $\leq$ relation forms a chain on the interval $(-\infty, x_1)$.

For each $i \in I$ as the set of polynomials $\{p_{ij} : j \in J\}$ with respect to $\leq$ relation forms a chain on the interval $(-\infty, x_1)$, it follows that $g_i(x) = \bigvee_{j\in J} p_{ij}(x) = p_{ij_i}(x)$ for some $j_i \in J$. Thus, $P(x) = \bigwedge_{i\in I} p_{ij_i}(x)$ on $(-\infty, x_1)$. Without loss of generality assume $p_{ij_i} \neq p_{i' j_{i'}}$ on $(-\infty, x_1)$ for each $i, i' \in I$ with $i \neq i'$.

Set $T = \{x \in (-\infty, x_1) : p_{ij_i}(x) = p_{i' j_{i'}}(x), i, i' \in I, i \neq i'\}$. Since every p_{ij_i}, $i \in I$ is a polynomial, I is finite, and $p_{ij_i}(x) \neq p_{i' j_{i'}}(x)$ on $(-\infty, x_1)$ for each $i, i' \in I$ with $i \neq i'$, T must be finite.

If $T \neq \emptyset$, let w be the smallest element of T. Equivalently, $w = \bigwedge T$. If $T = \emptyset$, set $\bigwedge T = x_1$. We prove that in either case, the set of polynomials $\{p_{ij_i} : i \in I\}$ with respect to the relation $\leq$ forms a chain on the interval $(-\infty, \bigwedge T)$.

Otherwise, there are $i, i' \in I$ such that $p_{ij_i} \nleq p_{i' j_{i'}}$ and $p_{i' j_{i'}} \nleq p_{ij_i}$. Then, there are $a, b \in (-\infty, \bigwedge T)$ such that $p_{ij_i}(a) > p_{i' j_{i'}}(a)$ and $p_{i' j_{i'}}(b) > p_{ij_i}(b)$. Without loss of generality assume $a < b$. Thus, $k(a) > 0$ and $k(b) < 0$, where $k(x) = p_{ij_i}(x) - p_{i' j_{i'}}(x)$. Thus, by the Intermediate value theorem, there is a $c \in (a, b) \subseteq (-\infty, \bigwedge T)$ such that $k(c) = 0$. Equivalently, $p_{ij_i}(c) = p_{i' j_{i'}}(c)$. So, $c \in T$ which is impossible because $c < \bigwedge T$.

Suppose on the interval $(-\infty, \bigwedge T)$ we have $\min\{p_{ij_i} : i \in I\} = p_{kj_k}$ Then, $P(x) = p_{kj_k}(x)$ on $(-\infty, \bigwedge T)$. However, by the assumption, P is a polynomial on $\mathbb{R}$. Since two distinct polynomials cannot take equal values on a non-empty open interval, it follows that $P(x) = p_{kj_k}(x)$ on $\mathbb{R}$. Hence the proof is complete. □

Remark 4.5. Algorithmically, to determine if $P(x)$ is a polynomial, we first determine the set S. Next, for each pair of distinct indices $r, s \in I$ we determine the intersections of $g_r(x)$ and $g_s(x)$. If $\overline{S}$ is the union of S and these intersections, by sorting its elements we determine distinct intervals, where over each of them $P(x)$ is a polynomial. In each of these intervals, including the two infinite intervals, we select an interior point (e.g., midpoint of the finite intervals and points away by one unit from the finite endpoint of the infinite intervals). This results in a finite set of points W. Finally, $P(x)$ is a polynomial if and only if there exists $i_0 \in I$, $j_0 \in J$, such that $P(w) = p_{i_0 j_0}(w)$ for all $w \in W$.

Example 4.6. Consider $P(x) = \Big((.5x^2) \vee (x^3 - x)\Big) \wedge \Big((.6(x^2-1)) \vee (-.6(x^2-1))\Big)$. In this example $I = J = \{1, 2\}$. The Figure 4 shows the corresponding sets S and $\overline{S}$.

The red curves in the figure represent the graphs of $.5x^2$ and $x^3 - x$, while the black curves represent the graphs of $.6(x^2-1)$ and $-.6(x^2-1)$. From these, the graphs of $g_1(x) = \Big((.5x^2) \vee (x^3 - x)\Big)$ and $g_2(x) = \Big((.6(x^2-1)) \vee (-.6(x^2-1))\Big)$ can be traced. The set S consists of the union of red and black dots on the x-axis. The green points on the x-axis represent the set of intersections of $g_1(x)$ and $g_2(x)$. Once $\overline{S}$, the set of red, black, and green points, are determined, we sort them to determine intervals over $P(x)$ is polynomial. By selecting a point in the interior of each of the intervals, we obtain a set W such that $P(x)$ is a polynomial if and only if for some $i_0 \in I$, $j_0 \in J$, $P(w) = p_{i_0 j_0}(w)$ for all $w \in W$.

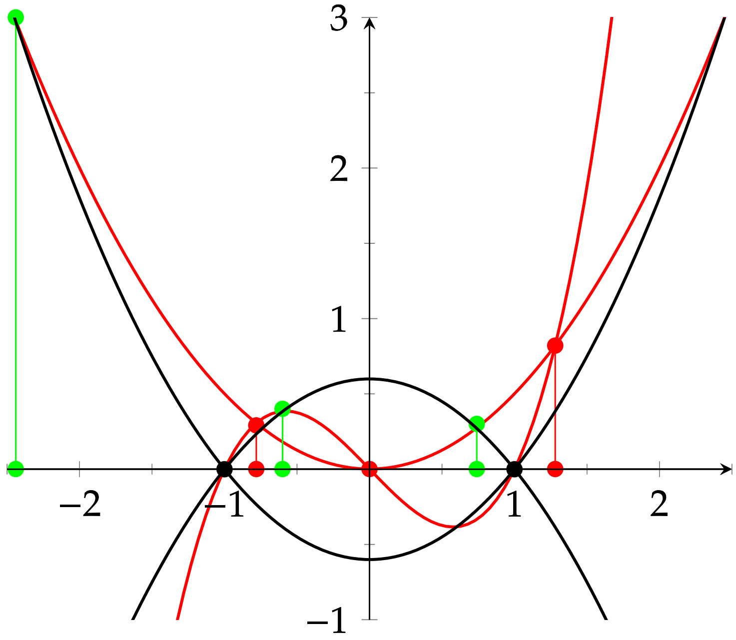

Figure 1: An example identifying intervals where $P(x)$ is a polynomial.

We conclude this section by extending Theorem 4.4 for rational functions.

Theorem 4.7. *Suppose for each $i \in I$, $j \in J$, I, J finite index sets, $r_{ij}(x) = p_{ij}(x)/q_{ij}(x)$, where $p_{ij}(x)$ and $q_{ij}(x)$ are real polynomials and relatively prime. Then $R(x) = \bigwedge_{i\in I} \bigvee_{j\in J} r_{ij}(x)$ is a rational function if and only if there exists $i_0 \in I$, $j_0 \in J$ such that $R(x) = r_{i_0 j_0}(x)$.*

Proof. Suppose $R(x) = \bigwedge_{i\in I} \bigvee_{j\in J} r_{ij}(x)$, is a rational function. We prove there exists $i_0 \in I$, $j_0 \in J$ such that $R(x) = r_{i_0 j_0}(x)$. Without loss of generality assume for each $i \in I$, $j, j' \in J$, $j' \neq j'$ implies $r_{ij}(x) \neq r_{ij'}(x)$.

Set $Q(x) = \prod_{i\in I, J\in J} q_{ij}^2(x)$. Since $Q(x) \geq 0$, we have $R(x)Q(x) = \bigwedge_{i\in I} \bigvee_{j\in J} r_{ij}(x)Q(x)$ Note that for each $i \in I$, $j \in J$, $Q(x)r_{ij}(x)$ is a polynomial.

Assume $R(x) = P(x)/S(x)$, where $P(x)$ and $S(x)$ are real polynomials and relatively prime. Then, $R(x)Q(x)S(x)^2 = \bigwedge_{i\in I} \bigvee_{j\in J} r_{ij}(x)Q(x)S(x)^2$. Now, $R(x)Q(x)S(x)^2$ is a polynomial which is the infimum of supremum of polynomials $Q(x)r_{ij}(x)S(x)^2$. Thus, by Theorem 4.4, there is an $i_0 \in I$ and $j_0 \in J$ such that $R(x)Q(x)S(x)^2 = Q(x)r_{i_0 j_0}(x)S(x)^2$. Consequently, $R(x) = r_{i_0 j_0}(x)$. = The converse is trivial. □

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