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φ-fixed points of self-mappings on metric spaces with a geometric viewpoint

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Abstract. In this paper, we investigate the geometric properties of non-unique φ-fixed points. The concept of a φ-fixed point of a self-mapping $\mathcal{T}$ on a metric space X has been introduced recently. An element $x \in X$ is called a φ-fixed point of the self-mapping $\mathcal{T} : X \to X$, where $\varphi : X \to [0, \infty)$ is a given function, if x is a fixed point of $\mathcal{T}$ and $\varphi(x) = 0$. A recent open problem concerns the geometric properties of φ-fixed points, particularly the existence of a φ-fixed circle and a φ-fixed disc. In this study, we address this problem and present several solutions by employing suitable auxiliary numbers and geometric conditions. We demonstrate that a zero of a given function φ can generate a fixed circle (resp. fixed disc) contained in the fixed point set of a self-mapping $\mathcal{T}$ on a metric space. Moreover, this circle (resp. fixed disc) also lies within the set of zeros of the function φ.

1. Introduction and motivation

Let $\mathcal{T} : X \to X$ be a self-mapping on a metric space (X, d). We denote the fixed point set of $\mathcal{T}$ by $Fix(\mathcal{T})$, that is, we have

$$Fix(\mathcal{T}) = \{x \in X : \mathcal{T}x = x\} .$$

In this paper, mainly, we study on the geometric properties of the set $Fix(\mathcal{T})$ in the cases where this set is not a singleton. Recently, geometric properties of non-unique fixed points have been extensively studied by various aspects, for example, in the context of the fixed-circle problem, fixed-disc problem and so on (see, for instance, [4, 8, 13–18, 20–22] and the references therein).

In [7], a new concept of a φ-fixed point was introduced. By means of this new concept, a new problem called the φ-fixed point problem was investigated for various classes of mappings in metric spaces. Afterwards, several existence results of φ-fixed points for various classes of operators have been established (see, for example, [1–3, 5–7, 9–11, 19, 23]). Let $\mathcal{T} : X \to X$ be a self-mapping and $\varphi : X \to [0, \infty)$ be a given function. Recall that an element $x \in X$ is said to be a φ-fixed point of $\mathcal{T}$ if x is a fixed point of $\mathcal{T}$ and $\varphi(x) = 0$.

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In other words, a φ-fixed point is a fixed point of a mapping $\mathcal{T}$ such that it is also a zero of a given function φ (see [7] for basic facts of φ-fixed points). Following [7], we denote the set of all zeros of the function φ by Z_φ. Thus

$$Z_\varphi = \{x \in X : \varphi(x) = 0\}.$$

In [17], considering the non-unique fixed point cases, the concepts of a φ-fixed circle and of a φ-fixed disc were introduced as follows:

Definition 1.1. ([17])Let (X, d) be a metric space, $\mathcal{T}$ a self-mapping of X and $\varphi : X \to [0, \infty)$ a given function.

(1) A circle $C_{x_0,r} = \{x \in X : d(x, x_0) = r\}$ in X is said to be a φ-fixed circle of $\mathcal{T}$ if and only if $C_{x_0,r} \subseteq Fix(\mathcal{T}) \cap Z_\varphi$.

(2) A disc $D_{x_0,r} = \{x \in X : d(x, x_0) \leq r\}$ in X is said to be a φ-fixed disc of $\mathcal{T}$ if and only if $D_{x_0,r} \subseteq Fix(\mathcal{T}) \cap Z_\varphi$.

Example 1.2. ([17]) Let $(\mathbb{R}, d)$ be the usual metric space of real numbers. Define a self-mapping $\mathcal{T} : \mathbb{R} \to \mathbb{R}$ by

$$\mathcal{T}x = x^4 - 5x^2 + x + 4$$

and a function $\varphi : \mathbb{R} \to [0, \infty)$ by

$$\varphi(x) = |x - 1| + |x + 1| - 2. \tag{1}$$

Then we have

$$Fix(\mathcal{T}) = \{x \in \mathbb{R} : \mathcal{T}x = x\} = \{-2, -1, 1, 2\}$$

and

$$Z_\varphi = \{x \in \mathbb{R} : \varphi(x) = 0\} = [-1, 1].$$

Clearly, we obtain

$$Fix(\mathcal{T}) \cap Z_\varphi = C_{0,1} = \{-1, 1\},$$

that is, the circle $C_{0,1}$ is the φ-fixed circle of $\mathcal{T}$.

On the other hand, if we consider the self-mapping $\mathcal{S} : \mathbb{R} \to \mathbb{R}$ defined by $\mathcal{S}x = x^3$ together with the function $\varphi(x)$ defined in (1), we have $Fix(\mathcal{S}) = \{-1, 0, 1\}$ and hence

$$C_{0,1} \subset Fix(\mathcal{S}) \cap Z_\varphi.$$

Consequently, the circle $C_{0,1}$ is the φ-fixed circle of $\mathcal{S}$.

Example 1.3. ([17]) Consider the function $\varphi(x)$ defined in (1) together with the self-mapping $\mathcal{T} : \mathbb{R} \to \mathbb{R}$ defined by

$$\mathcal{T}x = \begin{cases} x & ; \quad |x| \leq 1 \\ x + 2 & ; \quad |x| > 1 \end{cases}.$$

Since we have $D_{0,1} = Fix(\mathcal{T}) \cap Z_\varphi$, the disc $D_{0,1} = [-1, 1]$ is a φ-fixed disc of $\mathcal{T}$. Notice that the disc $D_{0,\frac{1}{2}} = \left[-\frac{1}{2}, \frac{1}{2}\right]$ is another φ-fixed disc of $\mathcal{T}$.

Considering the above examples, the investigation of the existence and uniqueness of φ-fixed circles (resp. φ-fixed discs) seems to be an interesting problem for various classes of self-mappings. In [17], it was proposed the usage of the number $M(x,y)$ defined by

$$M(x,y)=\max\left\{d(x,y),d(x,\mathcal{T}x),d(y,\mathcal{T}y),\left[\frac{d(x,\mathcal{T}y)+d(y,\mathcal{T}x)}{1+d(x,\mathcal{T}x)+d(y,\mathcal{T}y)}\right]d(x,y)\right\}, \tag{2}$$

for all $x,y\in X$ or a modified version of it for the studies of this direction. This is because auxiliary numbers serve as essential and effective tools that facilitate the analysis of fixed-point problems. In [17], new solutions to the Rhoades' well-known problem related to discontinuity at the fixed point were investigated using the auxiliary number $M(x,y)$. Furthermore, the number $M(x,y)$ and its modified versions were used to give some fixed-point results, a common fixed-point theorem and an application to the fixed-circle problem. It was proposed that new results on the geometric properties of the φ-fixed points of a given self-mapping $\mathcal{T}$ can be investigated by the use of the number $M(x,y)$ (or a modified version of it).

In this paper, we derive several solutions to the φ-fixed circle (resp. φ-fixed disc) problem by employing suitable auxiliary numbers and geometric conditions for a self-mapping $\mathcal{T}$ on a metric space. We establish that a zero of a given function $\varphi : X\to[0,\infty)$ can generate a fixed circle (resp. fixed disc) contained in the set $Fix(\mathcal{T})\cap Z_{\varphi}$.

2. φ-fixed circles and φ-fixed discs on metric spaces

In this section, using the number $M(x,y)$ defined in (2), the numbers ρ and μ defined by

$$\rho := \inf\{d(\mathcal{T}x,x) : x\in X, x\neq \mathcal{T}x\} \tag{3}$$

and

$$\mu := \inf\left\{\sqrt{d(\mathcal{T}x,x)} : x\in X, x\neq \mathcal{T}x\right\}, \tag{4}$$

we give several φ-fixed circle (resp. φ-fixed disc) results using various geometric conditions and techniques.

In metric fixed point theory, auxiliary numbers such as the number $M(x,y)$ are introduced to refine contraction conditions beyond the classical Banach contraction principle. While Banach's theorem depends only on a single Lipschitz constant, modern generalizations often require additional parameters to capture different geometric or functional behaviors of self-maps. The introduction of the numbers ρ and μ often allows the comparison of distances between images and pre-images in different ways. In this study, these numbers play a central role in examining geometric properties of non-unique φ-fixed points. Hence, the choice of these auxiliary numbers is not arbitrary but arises naturally from the need to model the geometric and analytic behavior of the mappings more accurately.

2.1. *φ-fixed circle (resp. φ-fixed Disc) results via type 1 φ_{x_0}-contractions*

First, we define a new contraction type.

Definition 2.1. Let $\mathcal{T}$ be a self-mapping on a metric space (X,d), and $\varphi : X\to[0,\infty)$ be a given function. If there exists a point $x_0\in X$ such that

$$d(\mathcal{T}x,x)>0 \Rightarrow \max\{d(x,\mathcal{T}x),\varphi(\mathcal{T}x),\varphi(x)\}\leq k\max\{d(x,x_0),\varphi(x),\varphi(x_0)\}, \tag{5}$$

for all $x\in X$ and some $k\in(0,1)$, then $\mathcal{T}$ is called a type 1 φ_{x_0}-contraction.

In the following theorems, we prove that the point x_0 and the number ρ defined in (3) produce a φ-fixed circle (resp. φ-fixed disc) under a geometric condition.

Theorem 2.2. *Let (X,d) be a metric space, the number ρ be defined as in (3) and $\mathcal{T} : X \to X$ be a type 1 φ_{x_0}-contraction with the point $x_0 \in X$ and the given function $\varphi : X \to [0,\infty)$. If $x_0 \in Z_\varphi$ and*

$$\varphi(x) \leq d(\mathcal{T}x, x) \tag{6}$$

for each $x \in C_{x_0,\rho}$, then $C_{x_0,\rho}$ is a φ-fixed circle of $\mathcal{T}$.

Proof. At first, we show that $x_0 \in Fix(\mathcal{T})$. Conversely, assume that $x_0 \neq \mathcal{T}x_0$. Then we have $d(\mathcal{T}x_0, x_0) > 0$ and using the inequality (5) together with the hypothesis $x_0 \in Z_\varphi$, we find

$$\max\{d(x_0, \mathcal{T}x_0), \varphi(\mathcal{T}x_0), \varphi(x_0)\} \leq k \max\{d(x_0, x_0), \varphi(x_0), \varphi(x_0)\} = 0,$$

and hence

$$\max\{d(x_0, \mathcal{T}x_0), \varphi(\mathcal{T}x_0)\} = 0.$$

This implies $d(x_0, \mathcal{T}x_0) = 0$, which is a contradiction with our assumption. Therefore, it should be $\mathcal{T}x_0 = x_0$, that is, $x_0 \in Fix(\mathcal{T}) \cap Z_\varphi$.

Now we have two cases.

Case 1. If $\rho = 0$, then clearly $C_{x_0,\rho} = \{x_0\} \subset Fix(\mathcal{T}) \cap Z_\varphi$ and hence, $C_{x_0,\rho}$ is a φ-fixed circle of T.

Case 2. Let $\rho > 0$. For any $x \in C_{x_0,\rho}$ with $\mathcal{T}x \neq x$, we have

$$\max\{d(x, \mathcal{T}x), \varphi(\mathcal{T}x), \varphi(x)\} \leq k \max\{d(x, x_0), \varphi(x)\}.$$

If $\max\{d(x, x_0), \varphi(x)\} = d(x, x_0) = \rho$, then by the definition of the number ρ, we get

$$\max\{d(x, \mathcal{T}x), \varphi(\mathcal{T}x), \varphi(x)\} \leq k\rho \leq kd(x, \mathcal{T}x)$$

and so $d(x, \mathcal{T}x) \leq kd(x, \mathcal{T}x)$, a contradiction by the hypothesis $k \in (0,1)$.

If $\max\{d(x, x_0), \varphi(x)\} = \varphi(x)$, we obtain

$$\max\{d(x, \mathcal{T}x), \varphi(\mathcal{T}x), \varphi(x)\} \leq k\varphi(x)$$

and hence $\varphi(x) \leq k\varphi(x)$, a contradiction.

Then, it should be $\mathcal{T}x = x$, that is, $x \in Fix(\mathcal{T})$. By (6), we have $\varphi(x) = 0$ for all $x \in C_{x_0,\rho}$. This implies $x \in Fix(\mathcal{T}) \cap Z_\varphi$ for all $x \in C_{x_0,\rho}$. Therefore, we deduce that $C_{x_0,\rho} \subset Fix(\mathcal{T}) \cap Z_\varphi$ and $C_{x_0,\rho}$ is a φ-fixed circle of $\mathcal{T}$. □

Theorem 2.3. *Let (X,d) be a metric space, the number ρ be defined as in (3) and $\mathcal{T} : X \to X$ be a type 1 φ_{x_0}-contraction with the point $x_0 \in X$ and the given function $\varphi : X \to [0,\infty)$. If $x_0 \in Z_\varphi$ and the inequality*

$$\varphi(x) \leq d(\mathcal{T}x, x)$$

is satisfied for each $x \in D_{x_0,\rho}$, then $D_{x_0,\rho}$ is a φ-fixed disc of $\mathcal{T}$.

Proof. The proof follows from the proof of Theorem 2.2. □

Unless otherwise stated throughout the paper we present necessary illustrative examples by defining new self-mappings of the usual metric spaces $(\mathbb{R}, d)$ and $(\mathbb{C}, d)$. The following example illustrates Theorem 2.2 and Theorem 2.3.

Example 2.4. Define a self-mapping $\mathcal{T} : \mathbb{R} \to \mathbb{R}$ by

$$\mathcal{T}x = \begin{cases} \frac{x}{2} & , \quad x > 2 \\ x & ; \quad x \leq 2 \end{cases}$$

and consider the function $\varphi : \mathbb{R} \to [0,\infty)$ defined by

$$\varphi(x) = \begin{cases} \frac{x}{4} & ; \quad x > 0 \\ 0 & ; \quad x \leq 0 \end{cases}.$$

It is easy to show that $\mathcal{T}$ is a type 1 φ_{x_0}-contraction with the point $x_0 = -1$ and $k = \frac{1}{2}$. Indeed, we obtain

$$\max\left\{\left|x - \frac{x}{2}\right|, \frac{x}{8}, \frac{x}{4}\right\} = \frac{x}{2} \leq \frac{1}{2}\max\left\{|x+1|, \frac{x}{4}, 0\right\} = \frac{x+1}{2} = \frac{x}{2} + \frac{1}{2}$$

for each $x > 2$, and we have

$$\begin{aligned} \rho &= \inf\{d(\mathcal{T}x, x) : x \in X, x \neq \mathcal{T}x\} \\ &= \inf\left\{\left|x - \frac{x}{2}\right| = \frac{x}{2} : x > 2\right\} \\ &= 1. \end{aligned}$$

That is, all conditions of Theorem 2.2 are satisfied by $\mathcal{T}$. Notice that we have

$$Fix(\mathcal{T}) \cap Z_\varphi = (-\infty, 0],$$

and we find

$$C_{-1,1} = \{-2, 0\} \subset Fix(\mathcal{T}) \cap Z_\varphi.$$

Then by definition, $C_{-1,1}$ is a φ-fixed circle of the self-mapping $\mathcal{T}$.

It is obvious that $\mathcal{T}$ also satisfies all hypotheses of Theorem 2.3 and $D_{-1,1}$ is a φ-fixed disc of $\mathcal{T}$.

Remark 2.5. Notice that the condition $\varphi(x) \leq d(\mathcal{T}x, x)$ is crucial for both Theorem 2.2 and Theorem 2.3. For any $x_0 \in (-1, 0]$, this condition is not satisfied for the points $x = 1 + x_0 \in C_{x_0,1}$ and $x \in D_{x_0,1}$ with $x > 0$. In fact, it is easy to check that $\mathcal{T}$ is also a type 1 φ_{x_0}-contraction with each of the points $x_0 \in (-1, 0]$.

In the following, we define a new contraction type using the auxiliary number $M(x, y)$ defined in (2).

Definition 2.6. Let $\mathcal{T}$ be a self-mapping of a metric space (X, d), and $\varphi : X \to [0,\infty)$ be a given function. If there exists a point $x_0 \in X$ such that

$$d(\mathcal{T}x, x) > 0 \Rightarrow \max\{d(x, \mathcal{T}x), \varphi(\mathcal{T}x), \varphi(x)\} \leq k \max\{M(x, x_0), \varphi(x), \varphi(x_0)\}, \tag{7}$$

for each $x \in X$ and some $k \in \left(0, \frac{1}{2}\right)$, then $\mathcal{T}$ is called a generalized type 1 φ_{x_0}-contraction.

In the following, we prove a φ-fixed circle theorem using the number μ defined in (4). We note that

$$\begin{aligned} M(x_0, x_0) &= \max\left\{d(x_0,x_0), d(x_0,\mathcal{T}x_0), d(x_0,\mathcal{T}x_0), \left[\frac{d(x_0,\mathcal{T}x_0) + d(x_0,\mathcal{T}x_0)}{1 + d(x_0,\mathcal{T}x_0) + d(x_0,\mathcal{T}x_0)}\right] d(x_0,x_0)\right\} \\ &= d(x_0, \mathcal{T}x_0). \end{aligned}$$

Theorem 2.7. *Let (X, d) be a metric space, the number μ be defined as in (4) and $\mathcal{T} : X \to X$ be a generalized type 1 φ_{x_0}-contraction with the point $x_0 \in X$ and the given function $\varphi : X \to [0,\infty)$. If $x_0 \in Z_\varphi$ and the inequalities*

$$d(\mathcal{T}x, x_0) \leq \mu, \tag{8}$$

$$\varphi(x) \leq d(\mathcal{T}x, x)$$

hold for each $x \in C_{x_0,\mu}$, then $C_{x_0,\mu}$ is a φ-fixed circle of $\mathcal{T}$.

Proof. If $x_0 \neq \mathcal{T}x_0$, then we have $d(\mathcal{T}x_0, x_0) > 0$, and using (7) and the hypothesis $x_0 \in Z_\varphi$, we find

$$\begin{aligned} \max\{d(x_0,\mathcal{T}x_0), \varphi(\mathcal{T}x_0), \varphi(x_0)\} &\leq k\max\{M(x_0,x_0), \varphi(x_0), \varphi(x_0)\} \\ &\leq kM(x_0,x_0) \\ &= kd(x_0,\mathcal{T}x_0), \end{aligned}$$

and hence

$$\max\{d(x_0,\mathcal{T}x_0), \varphi(\mathcal{T}x_0)\} \leq kd(x_0,\mathcal{T}x_0).$$

This implies $d(x_0,\mathcal{T}x_0) \leq kd(x_0,\mathcal{T}x_0)$, which is a contradiction since $k \in \left(0,\frac{1}{2}\right)$. It should be $\mathcal{T}x_0 = x_0$, that is, we get $x_0 \in Fix(\mathcal{T})$. Therefore, we have $x_0 \in Fix(\mathcal{T}) \cap Z_\varphi$.

If $\mu = 0$, then clearly $C_{x_0,\mu} = \{x_0\} \subset Fix(\mathcal{T}) \cap Z_\varphi$ and hence, $C_{x_0,\mu}$ is a φ-fixed circle of $\mathcal{T}$.

Now, let $\mu > 0$. For any $x \in C_{x_0,\mu}$ with $\mathcal{T}x \neq x$, we have

$$\max\{d(x,\mathcal{T}x), \varphi(\mathcal{T}x), \varphi(x)\} \leq k\max\{M(x,x_0), \varphi(x)\}.$$

If $\max\{M(x,x_0), \varphi(x)\} = \varphi(x)$, we obtain

$$\max\{d(x,\mathcal{T}x), \varphi(\mathcal{T}x), \varphi(x)\} \leq k\varphi(x)$$

and hence $\varphi(x) \leq k\varphi(x)$, a contradiction by the hypothesis $k \in \left(0,\frac{1}{2}\right)$.

Let $\max\{M(x,x_0), \varphi(x)\} = M(x,x_0)$. We find

$$\begin{aligned} M(x,x_0) &= \max\left\{d(x,x_0), d(x,\mathcal{T}x), d(x_0,\mathcal{T}x_0), \left[\frac{d(x,\mathcal{T}x_0)+d(x_0,\mathcal{T}x)}{1+d(x,\mathcal{T}x)+d(x_0,\mathcal{T}x_0)}\right]d(x,x_0)\right\} \\ &\leq \max\left\{\mu, d(x,\mathcal{T}x), 0, \left[\frac{\mu+\mu}{1+d(x,\mathcal{T}x)}\right]\mu\right\} \\ &= \max\left\{\mu, d(x,\mathcal{T}x), 0, \frac{2\mu^2}{1+d(x,\mathcal{T}x)}\right\}. \end{aligned}$$

If $M(x,x_0) = d(x,\mathcal{T}x)$ then we get

$$\max\{d(x,\mathcal{T}x), \varphi(\mathcal{T}x), \varphi(x)\} \leq kd(x,\mathcal{T}x)$$

and so $d(x,\mathcal{T}x) \leq kd(x,\mathcal{T}x)$, a contradiction by the hypothesis $k \in \left(0,\frac{1}{2}\right)$. If $M(x,x_0) = \frac{2\mu^2}{1+d(x,\mathcal{T}x)}$ then by the definition of the number μ we have

$$\begin{aligned} \max\{d(x,\mathcal{T}x), \varphi(\mathcal{T}x), \varphi(x)\} &\leq k\frac{2\mu^2}{1+d(x,\mathcal{T}x)} \\ &\leq \frac{2k\left(\sqrt{d(x,\mathcal{T}x)}\right)^2}{1+d(x,\mathcal{T}x)} \\ &< 2kd(x,\mathcal{T}x), \end{aligned}$$

and hence $d(x,\mathcal{T}x) < 2k(x,\mathcal{T}x)$, a contradiction by the hypothesis $k \in \left(0,\frac{1}{2}\right)$. Then, it should be $\mathcal{T}x = x$, that is, $x \in Fix(\mathcal{T})$ in all of the above cases. By the hypothesis $\varphi(x) \leq d(\mathcal{T}x, x)$, we have $\varphi(x) = 0$ for all $x \in C_{x_0,\mu}$. This implies $x \in Fix(\mathcal{T}) \cap Z_\varphi$ for all $x \in C_{x_0,\mu}$. Thus, $C_{x_0,\mu} \subset Fix(\mathcal{T}) \cap Z_\varphi$, that is, $C_{x_0,\mu}$ is a φ-fixed circle of $\mathcal{T}$. □

Theorem 2.8. *Let (X,d) be a metric space, μ defined as in (4), $\mathcal{T} : X \to X$ a type 1 generalized φ_{x_0}-contraction with the point $x_0 \in X$ and the given function $\varphi : X \to [0,\infty)$. If $x_0 \in Z_\varphi$ and the inequalities*

$$d(\mathcal{T}x, x_0) \leq \mu,$$

$$\varphi(x) \leq d(\mathcal{T}x, x)$$

hold for all $x \in D_{x_0,\mu}$, then $D_{x_0,\mu}$ is a φ-fixed disc of $\mathcal{T}$.

Proof. The proof is obtained in a similar way to that of Theorem 2.7. □

Now, we present two illustrative examples.

Example 2.9. Consider the self-mapping $\mathcal{T} : \mathbb{R} \to \mathbb{R}$ defined by

$$\mathcal{T}x = \begin{cases} x & ; \quad x \geq -3 \\ x-1 & ; \quad x < -3 \end{cases}$$

and the function $\varphi : \mathbb{R} \to [0, \infty)$ defined by

$$\varphi(x) = \begin{cases} |x| & ; \quad x < -1 \\ 0 & ; \quad x \geq -1 \end{cases} . \tag{9}$$

We have

$$\begin{aligned} \mu &= \inf\left\{ \sqrt{d(\mathcal{T}x, x)} : x \in X, x \neq \mathcal{T}x \right\} \\ &= \inf\left\{ \sqrt{|x-1-x|} : x < -3 \right\} \\ &= 1. \end{aligned}$$

It is easy to show that $\mathcal{T}$ is not a type 1 φ_{x_0}-contraction with the point $x_0 = 0$ and any $k \in (0,1)$. Indeed, we obtain

$$\max\{d(x, \mathcal{T}x), \varphi(\mathcal{T}x), \varphi(x)\} = \max\{|x-1-x|, |x-1|, |x|\} = |x-1|$$

and

$$\max\{d(x,0), \varphi(x), \varphi(0)\} = \max\{|x-0|, |x|, 0\} = |x|,$$

for all $x < -3$. The inequality $|x-1| \leq k|x|$ leads a contradiction for any $k \in (0,1)$. Hence, (5) is not satisfied by $\mathcal{T}$.

We have

$$M(x,0) = \max\left\{ |x|, 1, 0, \left[\frac{|x| + |x-1|}{1+1} \right] |x| \right\} = \frac{|x| + |x-1|}{2} |x|,$$

for each $x < -3$. If we choose $k = \frac{3}{7}$, then (7) is satisfied and so $\mathcal{T}$ is a generalized type 1 φ_{x_0}-contraction with the point $x_0 = 0$.

We have

$$Fix(\mathcal{T}) \cap Z_{\varphi} = [-3, \infty) \cap [-1, \infty) = [-1, \infty).$$

Therefore,

$$C_{0,1} = \{-1, 1\} \subset Fix(\mathcal{T}) \cap Z_{\varphi},$$

and hence, $C_{0,1}$ is a φ-fixed circle of $\mathcal{T}$.

$\mathcal{T}$ also satisfies all conditions of Theorem 2.8, and $D_{0,1}$ is a φ-fixed disc of $\mathcal{T}$.

Example 2.10. Consider the self-mapping $\mathcal{T} : \mathbb{R} \to \mathbb{R}$ defined by

$$\mathcal{T}x = \begin{cases} 2x & ; \quad x < -1 \\ x & ; \quad x \geq -1 \end{cases}$$

together with the function $\varphi : \mathbb{R} \to [0, \infty)$ defined in (9). We have

$$\begin{aligned} \rho &= \inf\{d(\mathcal{T}x, x) : x \in X, x \neq \mathcal{T}x\} \\ &= \inf\{|2x - x| : x < -1\} = \inf\{|x| : x < -1\} = 1 \end{aligned}$$

and

$$\begin{aligned}\mu &= \inf\left\{\sqrt{d(\mathcal{T}x,x)} : x\in X, x\neq \mathcal{T}x\right\}\\ &= \inf\left\{\sqrt{|2x-x|} : x<-1\right\} = \inf\left\{\sqrt{|x|} : x<-1\right\}\\ &= 1.\end{aligned}$$

We show that $\mathcal{T}$ is not a type 1 φ_{x_0}-contraction (resp. generalized type 1 φ_{x_0}-contraction) with the point $x_0 = 0$. Indeed, for any $x < -1$, we have $d(x,\mathcal{T}x) > 1$ and

$$\max\{d(x,\mathcal{T}x), \varphi(\mathcal{T}x), \varphi(x)\} = \max\{|x|, 2|x|, |x|\} = 2|x|, \tag{10}$$

$$\max\{d(x,x_0), \varphi(x), \varphi(x_0)\} = \max\{|x|, |x|, 0\} = |x|, \tag{11}$$

$$\max\{M(x,x_0), \varphi(x), \varphi(x_0)\} = \max\left\{\frac{3|x|^2}{1+|x|}, |x|, 0\right\} = \frac{3|x|^2}{1+|x|}. \tag{12}$$

Considering (10) and (11), we see that $\mathcal{T}$ can not be a type 1 φ_{x_0}-contraction with the point $x_0 = 0$ for any $k \in (0,1)$. Also, considering (10) and (12), we see that $\mathcal{T}$ can not be a generalized type 1 φ_{x_0}-contraction with the point $x_0 = 0$ for any $k \in \left(0, \frac{1}{2}\right)$. Observe that $C_{0,1}$ is a φ-fixed circle and $D_{-1,1}$ is a φ-fixed disc of $\mathcal{T}$.

Remark 2.11. Example 2.10 shows that the converse statements of Theorem 2.2 and Theorem 2.3 (resp. Theorem 2.7 and Theorem 2.8) are not true in general.

2.2. φ-Fixed Circle (resp. φ-Fixed Disc) Results via Type 2 φ_{x_0}-Contractions

We define a new type of a φ_{x_0}-contraction.

Definition 2.12. Let $\mathcal{T}$ be a self-mapping of a metric space (X,d), and $\varphi : X \to [0,\infty)$ be a given function. If there exists a point $x_0 \in X$ such that

$$d(\mathcal{T}x,x) > 0 \Rightarrow \max\{d(x,\mathcal{T}x), \varphi(\mathcal{T}x)\} + \varphi(x) \leq k\max\{d(x,x_0), \varphi(x)\} + \varphi(x_0), \tag{13}$$

for each $x \in X$ and some $k \in (0,1)$, then $\mathcal{T}$ is called a type 2 φ_{x_0}-contraction.

Theorem 2.13. *Let (X,d) be a metric space, the number ρ be defined as in (3), and $\mathcal{T} : X \to X$ be a type 2 φ_{x_0}-contraction with the point $x_0 \in X$ and the given function $\varphi : X \to [0,\infty)$. If $x_0 \in Z_\varphi$ and*

$$\varphi(x) \leq d(\mathcal{T}x,x)$$

for each $x \in C_{x_0,\rho}$, then $C_{x_0,\rho}$ is a φ-fixed circle of $\mathcal{T}$.

Proof. To show $x_0 \in Fix(\mathcal{T})$, conversely, assume that $x_0 \neq \mathcal{T}x_0$. Then we have $d(\mathcal{T}x_0,x_0) > 0$ and using the inequality (13) with the hypothesis $x_0 \in Z_\varphi$, we find

$$\max\{d(x_0,\mathcal{T}x_0), \varphi(\mathcal{T}x_0)\} + \varphi(x_0) \leq k\max\{d(x_0,x_0), \varphi(x_0)\} + \varphi(x_0) = 0,$$

and hence

$$\max\{d(x_0,\mathcal{T}x_0), \varphi(\mathcal{T}x_0)\} = 0.$$

This implies $d(x_0,\mathcal{T}x_0) = 0$, which is a contradiction with our assumption. Therefore, it should be $\mathcal{T}x_0 = x_0$, that is, $x_0 \in Fix(\mathcal{T}) \cap Z_\varphi$.

Now we have two cases.

Case 1. If $\rho = 0$, then clearly $C_{x_0,\rho} = \{x_0\} \subset Fix(\mathcal{T}) \cap Z_\varphi$ and hence, the circle $C_{x_0,\rho}$ is a φ-fixed circle of T.

Case 2. Let $\rho > 0$. For any $x \in C_{x_0,\rho}$ with $\mathcal{T}x \neq x$, we have

$$\max\{d(x,\mathcal{T}x), \varphi(\mathcal{T}x)\} + \varphi(x) \leq k\max\{d(x,x_0), \varphi(x)\}.$$

If $\max\{d(x,x_0),\varphi(x)\} = d(x,x_0) = \rho$, then by the definition of the number ρ, we get

$$\max\{d(x,\mathcal{T}x),\varphi(\mathcal{T}x),\varphi(x)\} \leq k\rho \leq kd(x,\mathcal{T}x)$$

and so $d(x,\mathcal{T}x) \leq kd(x,\mathcal{T}x)$, a contradiction by the hypothesis $k \in (0,1)$.

If $\max\{d(x,x_0),\varphi(x)\} = \varphi(x)$, we obtain

$$\max\{d(x,\mathcal{T}x),\varphi(\mathcal{T}x)\} + \varphi(x) \leq k\varphi(x)$$

and hence $\varphi(x) \leq k\varphi(x)$, a contradiction.

Then, it should be $\mathcal{T}x = x$, that is, $x \in Fix(\mathcal{T})$. By the hypothesis $\varphi(x) \leq d(\mathcal{T}x,x)$, we have $\varphi(x) = 0$ for all $x \in C_{x_0,\rho}$. This implies $x \in Fix(\mathcal{T}) \cap Z_\varphi$ for all $x \in C_{x_0,\rho}$. Thus, we find $C_{x_0,\rho} \subset Fix(\mathcal{T}) \cap Z_\varphi$ and so, $C_{x_0,\rho}$ is a φ-fixed circle of $\mathcal{T}$. □

Theorem 2.14. *Let (X,d) be a metric space, the number ρ be defined as in (3), and $\mathcal{T} : X \to X$ be a type 2 φ_{x_0}-contraction with the point $x_0 \in X$ and the given function $\varphi : X \to [0,\infty)$. If $x_0 \in Z_\varphi$ and*

$$\varphi(x) \leq d(\mathcal{T}x,x)$$

for each $x \in D_{x_0,\rho}$, then $D_{x_0,\rho}$ is a φ-fixed disc of $\mathcal{T}$.

Proof. The proof is obtained in a similar way to that of Theorem 2.13. □

Definition 2.15. Let $\mathcal{T}$ be a self-mapping of a metric space (X,d), and $\varphi : X \to [0,\infty)$ be a given function. If there exists a point $x_0 \in X$ such that

$$d(\mathcal{T}x,x) > 0 \Rightarrow \max\{d(x,\mathcal{T}x),\varphi(\mathcal{T}x)\} + \varphi(x) \leq k\max\{M(x,x_0),\varphi(x)\} + \varphi(x_0), \tag{14}$$

for each $x \in X$ and some $k \in \left(0,\frac{1}{2}\right)$, then $\mathcal{T}$ is called a generalized type 2 φ_{x_0}-contraction.

Theorem 2.16. *Let (X,d) be a metric space, the number μ be defined as in (4), and $\mathcal{T} : X \to X$ be a generalized type 2 φ_{x_0}-contraction with the point $x_0 \in X$ and the given function $\varphi : X \to [0,\infty)$. If $x_0 \in Z_\varphi$ and the inequalities*

$$d(\mathcal{T}x,x_0) \leq \mu,$$

$$\varphi(x) \leq d(\mathcal{T}x,x)$$

hold for each $x \in C_{x_0,\mu}$, then $C_{x_0,\mu}$ is a φ-fixed circle of $\mathcal{T}$.

Proof. Suppose that $d(\mathcal{T}x_0,x_0) > 0$. Using the inequality (14) and the hypothesis $x_0 \in Z_\varphi$, we find

$$\begin{aligned}\max\{d(x_0,\mathcal{T}x_0),\varphi(\mathcal{T}x_0)\} + \varphi(x_0) &\leq k\max\{M(x_0,x_0),\varphi(x_0)\} + \varphi(x_0)\\ &\leq kM(x_0,x_0) = kd(x_0,\mathcal{T}x_0)\end{aligned}$$

and hence

$$\max\{d(x_0,\mathcal{T}x_0),\varphi(\mathcal{T}x_0)\} \leq kd(x_0,\mathcal{T}x_0).$$

This implies $d(x_0,\mathcal{T}x_0) \leq kd(x_0,\mathcal{T}x_0)$, which is a contradiction since $k \in \left(0,\frac{1}{2}\right)$. It should be $\mathcal{T}x_0 = x_0$, that is, we get $x_0 \in Fix(\mathcal{T})$. Therefore, we have $x_0 \in Fix(T) \cap Z_\varphi$.

If $\mu = 0$, then clearly $C_{x_0,\mu} = \{x_0\} \subset Fix(\mathcal{T}) \cap Z_\varphi$ and hence, the circle $C_{x_0,\mu}$ is a φ-fixed circle of $\mathcal{T}$.

Now, let $\mu > 0$. For any $x \in C_{x_0,\mu}$ with $\mathcal{T}x \neq x$, we have

$$\max\{d(x,\mathcal{T}x),\varphi(\mathcal{T}x)\} + \varphi(x) \leq k\max\{M(x,x_0),\varphi(x)\}.$$

If $\max\{M(x,x_0),\varphi(x)\} = \varphi(x)$, we obtain

$$\max\{d(x,\mathcal{T}x),\varphi(\mathcal{T}x)\} + \varphi(x) \leq k\varphi(x)$$

and hence $\varphi(x) \leq k\varphi(x)$, a contradiction by the hypothesis $k \in \left(0, \frac{1}{2}\right)$.

Let $\max\{M(x,x_0), \varphi(x)\} = M(x,x_0)$. We have

$$\begin{aligned} M(x,x_0) &= \max\left\{d(x,x_0), d(x,\mathcal{T}x), d(x_0,\mathcal{T}x_0), \left[\frac{d(x,\mathcal{T}x_0)+d(x_0,\mathcal{T}x)}{1+d(x,\mathcal{T}x)+d(x_0,\mathcal{T}x_0)}\right]d(x,x_0)\right\} \\ &\leq \max\left\{\mu, d(x,\mathcal{T}x), 0, \left[\frac{\mu+\mu}{1+d(x,Tx)}\right]\mu\right\} \\ &= \max\left\{\mu, d(x,\mathcal{T}x), 0, \frac{2\mu^2}{1+d(x,\mathcal{T}x)}\right\}. \end{aligned}$$

If $M(x,x_0) = d(x,\mathcal{T}x)$ then we get

$$\max\{d(x,\mathcal{T}x), \varphi(\mathcal{T}x)\} + \varphi(x) \leq kd(x,\mathcal{T}x)$$

and so $d(x,\mathcal{T}x) \leq kd(x,\mathcal{T}x)$, a contradiction by the hypothesis $k \in \left(0, \frac{1}{2}\right)$. If $M(x,x_0) = \frac{2\mu^2}{1+d(x,\mathcal{T}x)}$ then by the definition of the number μ we have

$$\begin{aligned} \max\{d(x,\mathcal{T}x), \varphi(\mathcal{T}x)\} + \varphi(x) &\leq k\frac{2\mu^2}{1+d(x,\mathcal{T}x)} \\ &\leq \frac{2k\left(\sqrt{d(x,\mathcal{T}x)}\right)^2}{1+d(x,\mathcal{T}x)} \\ &< 2k(x,\mathcal{T}x) \end{aligned}$$

and hence $d(x,\mathcal{T}x) < 2k(x,\mathcal{T}x)$, a contradiction by the hypothesis $k \in \left(0, \frac{1}{2}\right)$. Then, it should be $\mathcal{T}x = x$, that is, $x \in Fix(\mathcal{T})$ in all of the above cases. By the hypothesis $\varphi(x) \leq d(\mathcal{T}x, x)$, we have $\varphi(x) = 0$ for all $x \in C_{x_0,\mu}$. This implies $x \in Fix(\mathcal{T}) \cap Z_\varphi$ for all $x \in C_{x_0,\mu}$. Therefore, we find $C_{x_0,\mu} \subset Fix(\mathcal{T}) \cap Z_\varphi$ and hence, $C_{x_0,\mu}$ is a φ-fixed circle of $\mathcal{T}$. □

Theorem 2.17. *Let (X,d) be a metric space, the number μ be defined as in (4), and $\mathcal{T} : X \to X$ be a generalized type 2 φ_{x_0}-contraction with the point $x_0 \in X$ and the given function $\varphi : X \to [0,\infty)$. If $x_0 \in Z_\varphi$ and the inequalities*

$$d(\mathcal{T}x, x_0) \leq \mu,$$

$$\varphi(x) \leq d(\mathcal{T}x, x)$$

hold for each $x \in D_{x_0,\mu}$, then $D_{x_0,\mu}$ is a φ-fixed disc of $\mathcal{T}$.

Proof. The proof is obtained in a similar way to that of Theorem 2.16. □

2.3. *φ-Fixed Circle (resp. φ-Fixed Disc) Results via Type 3 φ_{x_0}-Contractions*

We define another type of a φ_{x_0}-contraction.

Definition 2.18. Let $\mathcal{T}$ be a self-mapping of a metric space (X,d), and $\varphi : X \to [0,\infty)$ be a given function. If there exists a point $x_0 \in X$ such that

$$d(\mathcal{T}x, x) > 0 \Rightarrow d(x,\mathcal{T}x) + \varphi(\mathcal{T}x) + \varphi(x) \leq k\left[d(x,x_0) + \varphi(x) + \varphi(x_0)\right], \tag{15}$$

for each $x \in X$ and some $k \in (0,1)$, then $\mathcal{T}$ is called a type 3 φ_{x_0}-contraction.

Theorem 2.19. *Let (X,d) be a metric space, the number ρ be defined as in (3), and $\mathcal{T} : X \to X$ be a type 3 φ_{x_0}-contraction with the point $x_0 \in X$ and the given function $\varphi : X \to [0,\infty)$. If $x_0 \in Z_\varphi$ and*

$$\varphi(x) \leq d(\mathcal{T}x, x)$$

for each $x \in C_{x_0,\rho}$, then $C_{x_0,\rho}$ is a φ-fixed circle of $\mathcal{T}$.

Proof. If $d(Tx_0, x_0) > 0$, using the inequality (15) and the hypothesis $x_0 \in Z_\varphi$, we find

$$d(x_0, \mathcal{T}x_0) + \varphi(\mathcal{T}x_0) + \varphi(x_0) \leq k[d(x_0, x_0) + \varphi(x_0) + \varphi(x_0)] = 0,$$

and hence

$$d(x_0, \mathcal{T}x_0) + \varphi(\mathcal{T}x_0) = 0.$$

This implies $d(x_0, \mathcal{T}x_0) = 0$, which is a contradiction with our assumption. Therefore, it should be $\mathcal{T}x_0 = x_0$, that is, $x_0 \in Fix(\mathcal{T}) \cap Z_\varphi$.

Now we have two cases.

Case 1. If $\rho = 0$, then clearly $C_{x_0,\rho} = \{x_0\} \subset Fix(\mathcal{T}) \cap Z_\varphi$ and hence, the circle $C_{x_0,\rho}$ is a φ-fixed circle of T.

Case 2. Let $\rho > 0$. For any $x \in C_{x_0,\rho}$ with $\mathcal{T}x \neq x$, we have

$$\begin{aligned} d(x, \mathcal{T}x) + \varphi(\mathcal{T}x) + \varphi(x) &\leq k[d(x, x_0) + \varphi(x)] \\ &< d(x, x_0) + \varphi(x) \end{aligned}$$

and so,

$$d(x, \mathcal{T}x) + \varphi(\mathcal{T}x) < d(x, x_0) = \rho.$$

This last inequality implies $d(x, \mathcal{T}x) < \rho$, which is a contradiction by the definition of ρ. Then, it should be $\mathcal{T}x = x$, that is, $x \in Fix(\mathcal{T})$. Since we have $\varphi(x) \leq d(\mathcal{T}x, x)$ for all $x \in C_{x_0,\rho}$, we obtain $\varphi(x) = 0$ and hence, $x \in Fix(\mathcal{T}) \cap Z_\varphi$ for all $x \in C_{x_0,\rho}$. Thus, we get $C_{x_0,\rho} \subset Fix(\mathcal{T}) \cap Z_\varphi$, that is, $C_{x_0,\rho}$ is a φ-fixed circle of $\mathcal{T}$. □

Theorem 2.20. *Let (X, d) be a metric space, the number ρ be defined as in (3), and $\mathcal{T} : X \to X$ be a type 3 φ_{x_0}-contraction with the point $x_0 \in X$ and the given function $\varphi : X \to [0, \infty)$. If $x_0 \in Z_\varphi$ and*

$$\varphi(x) \leq d(\mathcal{T}x, x)$$

for each $x \in D_{x_0,\rho}$, then $D_{x_0,\rho}$ is a φ-fixed disc of $\mathcal{T}$.

Proof. The proof is obtained in a similar way to that of Theorem 2.16. □

Definition 2.21. Let $\mathcal{T}$ be a self-mapping of a metric space (X, d), and $\varphi : X \to [0, \infty)$ a given function. If there exists a point $x_0 \in X$ such that

$$d(\mathcal{T}x, x) > 0 \Rightarrow d(x, \mathcal{T}x) + \varphi(\mathcal{T}x) + \varphi(x) \leq k[M(x, x_0) + \varphi(x) + \varphi(x_0)], \tag{16}$$

for each $x \in X$ and some $k \in \left(0, \frac{1}{2}\right)$, then $\mathcal{T}$ is called a generalized type 3 φ_{x_0}-contraction.

Theorem 2.22. *Let (X, d) be a metric space, the number μ be defined as in (4), and $\mathcal{T} : X \to X$ be a generalized type 3 φ_{x_0}-contraction with the point $x_0 \in X$ and the given function $\varphi : X \to [0, \infty)$. If $x_0 \in Z_\varphi$ and the inequalities*

$$d(\mathcal{T}x, x_0) \leq \mu,$$

$$\varphi(x) \leq d(\mathcal{T}x, x)$$

hold for each $x \in C_{x_0,\mu}$, then $C_{x_0,\mu}$ is a φ-fixed circle of $\mathcal{T}$.

Proof. If $d(\mathcal{T}x_0, x_0) > 0$, using the inequality (16) and the hypothesis $x_0 \in Z_\varphi$, we find

$$\begin{aligned} d(x_0, \mathcal{T}x_0) + \varphi(\mathcal{T}x_0) + \varphi(x_0) &\leq k[M(x_0, x_0) + \varphi(x_0) + \varphi(x_0)] \\ &\leq kM(x_0, x_0) = kd(x_0, \mathcal{T}x_0) \end{aligned}$$

and hence

$$d(x_0, \mathcal{T}x_0) + \varphi(\mathcal{T}x_0) \leq kd(x_0, \mathcal{T}x_0).$$

This implies $d(x_0, \mathcal{T}x_0) \leq kd(x_0, \mathcal{T}x_0)$, which is a contradiction since $k \in \left(0, \frac{1}{2}\right)$. Then, it should be $\mathcal{T}x_0 = x_0$, that is, we get $x_0 \in Fix(\mathcal{T})$. Therefore, we have $x_0 \in Fix(\mathcal{T}) \cap Z_\varphi$.

If $\mu = 0$, then clearly $C_{x_0,\mu} = \{x_0\} \subset Fix(\mathcal{T}) \cap Z_\varphi$ and hence, the circle $C_{x_0,\mu}$ is a φ-fixed circle of $\mathcal{T}$.

Now, let $\mu > 0$. For any $x \in C_{x_0,\mu}$ with $\mathcal{T}x \neq x$, we have

$$\begin{aligned} d(x, \mathcal{T}x) + \varphi(\mathcal{T}x) + \varphi(x) &\leq k[M(x, x_0) + \varphi(x)] \\ &< M(x, x_0) + \varphi(x) \end{aligned}$$

and so,

$$d(x, \mathcal{T}x) + \varphi(\mathcal{T}x) < M(x, x_0).$$

By the inequality $d(\mathcal{T}x, x_0) \leq \mu$, we have

$$M(x, x_0) \leq \max\left\{\mu, d(x, \mathcal{T}x), 0, \frac{2\mu^2}{1 + d(x, \mathcal{T}x)}\right\}.$$

If $M(x, x_0) = d(x, \mathcal{T}x)$, then we get

$$d(x, \mathcal{T}x) + \varphi(\mathcal{T}x) \leq kd(x, \mathcal{T}x)$$

and so $d(x, \mathcal{T}x) \leq kd(x, \mathcal{T}x)$, a contradiction by the hypothesis $k \in \left(0, \frac{1}{2}\right)$. If $M(x, x_0) = \frac{2\mu^2}{1+d(x,Tx)}$ then by the definition of the number μ we have

$$\begin{aligned} \max\{d(x, \mathcal{T}x), \varphi(\mathcal{T}x)\} + \varphi(x) &\leq k\frac{2\mu^2}{1 + d(x, \mathcal{T}x)} \\ &\leq \frac{2k\left(\sqrt{d(x, \mathcal{T}x)}\right)^2}{1 + d(x, \mathcal{T}x)} \\ &< 2k(x, \mathcal{T}x) \end{aligned}$$

and hence $d(x, \mathcal{T}x) < 2k(x, \mathcal{T}x)$, a contradiction by the hypothesis $k \in \left(0, \frac{1}{2}\right)$. Then, it should be $Tx = x$, that is, $x \in Fix(\mathcal{T})$ in all of the above cases. By the hypothesis $\varphi(x) \leq d(\mathcal{T}x, x)$, we have $\varphi(x) = 0$ for all $x \in C_{x_0,\mu}$. This implies $x \in Fix(\mathcal{T}) \cap Z_\varphi$ for all $x \in C_{x_0,\mu}$. Hence, we obtain $C_{x_0,\mu} \subset Fix(\mathcal{T}) \cap Z_\varphi$, and so $C_{x_0,\mu}$ is a φ-fixed circle of $\mathcal{T}$. □

Theorem 2.23. *Let (X, d) be a metric space, the μ be defined as in (4), and $\mathcal{T} : X \to X$ be a generalized type 3 φ_{x_0}-contraction with the point $x_0 \in X$ and the given function $\varphi : X \to [0, \infty)$. If $x_0 \in Z_\varphi$ and the inequalities*

$$d(\mathcal{T}x, x_0) \leq \mu,$$

$$\varphi(x) \leq d(\mathcal{T}x, x)$$

hold for each $x \in D_{x_0,\mu}$, then $D_{x_0,\mu}$ is a φ-fixed disc of $\mathcal{T}$.

Proof. The proof is obtained in a similar way to that of Theorem 2.22. □

Finally, we provide an example of a self-mapping $\mathcal{T}$ satisfying all conditions of Theorem 2.2, Theorem 2.7, Theorem 2.13, Theorem 2.16, Theorem 2.19 and Theorem 2.22.

Example 2.24. Let $X = \{-6, -4, -2, 0, 1, 2, 4, 5\} \cup [6, \infty)$ with the usual metric $d(x, y) = |x - y|$ and consider the self-mapping $\mathcal{T} : X \to X$ defined by

$$\mathcal{T}x = \begin{cases} x & , \quad x \neq 5 \\ 1 & ; \quad x = 5 \end{cases}$$

and the function $\varphi : X \to [0,\infty)$ defined by

$$\varphi(x) = \begin{cases} x^5 - 20x^3 + 64x & ; \quad x \in X - \{1,5\} \\ 0 & ; \quad x \in \{-6,1,5\} \end{cases}.$$

We have

$$\begin{aligned} \rho &= \inf\{d(\mathcal{T}x, x) : x \in X, x \neq \mathcal{T}x\} \\ &= |5-1| = 4 \end{aligned}$$

and

$$\begin{aligned} \mu &= \inf\left\{\sqrt{d(\mathcal{T}x,x)} : x \in X, x \neq \mathcal{T}x\right\} \\ &= \sqrt{4} = 2. \end{aligned}$$

Observe that $Fix(\mathcal{T}) = X - \{5\}$, $Z_\varphi = \{-6,-4,-2,0,1,2,4,5\}$ and $Fix(\mathcal{T}) \cap Z_\varphi = \{-6,-4,-2,0,1,2,4\}$.

Now, we show that $\mathcal{T}$ is a type 1 φ_{x_0}-contraction with the point $x_0 = 0$ and $k = \frac{9}{10}$. Indeed, we have

$$\max\{|5-1|, 0, 0\} = 4 \leq \frac{9}{10} \max\{|5-0|, 0, 0\} = \frac{45}{10}$$

for $x = 5$. Clearly, the conditions of Theorem 2.2 are satisfied by $\mathcal{T}$. We get

$$C_{0,4} = \{-4, 4\} \subset Fix(\mathcal{T}) \cap Z_\varphi.$$

Hence, the circle $C_{0,4}$ is a φ-fixed circle of $\mathcal{T}$.

$\mathcal{T}$ is a generalized type 1 φ_{x_0}-contraction with the point $x_0 = -4$ and $k = \frac{1}{4}$. Indeed, we have

$$\begin{aligned} M(5,4) &= \max\left\{d(5,-4), d(5,1), d(-4,-4), \left[\frac{d(5,-4) + d(-4,1)}{1 + d(5,1) + d(-4,-4)}\right] d(5,-4)\right\} \\ &= \max\left\{9, 4, 0, \frac{126}{5}\right\} = \frac{126}{5} \end{aligned}$$

and so,

$$\max\{|5-1|, 0, 0\} = 4 \leq \frac{1}{4} \max\left\{\frac{126}{5}, 0, 0\right\} = \frac{63}{10}.$$

$\mathcal{T}$ also satisfies all conditions of Theorem 2.7 and $C_{-4,2} = \{-6,-2\}$ is another φ-fixed circle of $\mathcal{T}$.

Similarly, it is easy to verify that $\mathcal{T}$ is a type 2 φ_{x_0}-contraction with the point $x_0 = 0$ and $k = \frac{9}{10}$; a generalized type 2 φ_{x_0}-contraction with the point $x_0 = -4$ and $k = \frac{1}{4}$; a type 3 φ_{x_0}-contraction with the point $x_0 = 0$ and $k = \frac{9}{10}$; a generalized type 2 φ_{x_0}-contraction with the point $x_0 = -4$ and $k = \frac{1}{4}$. The discs $D_{0,4} = \{-4,-2,0,1,2,4\}$ and $D_{-4,2} = \{-6,-4,-2\}$ are φ-fixed discs of $\mathcal{T}$.

3. Conclusions and future work

We have provided some solutions to a recent open problem related to the geometric properties of φ-fixed points of self-mappings of a metric space in the non-unique fixed point cases. We have seen that any zero of a given function will produce a fixed circle (resp. fixed disc) contained in the fixed point set of a self-mapping on a metric space. Considering the examples given in the paper, we have deduce that a φ-fixed circle (resp. φ-fixed disc) need not to be unique. Hence, the investigation of the uniqueness condition(s) of a φ-fixed circle (resp. φ-fixed disc) can be considered as a future scope of this paper.

On the other hand, theoretical fixed point results and examples of self-mappings are important tools in the study of neural networks. The typical form of a partitioned real valued activation function is the following:

$$\mathfrak{f}(x)=\begin{cases}\mathfrak{f}_0(x) & ; \quad x<0\\ \mathfrak{f}_1(x) & ; \quad x\geq 0\end{cases},$$

where $\mathfrak{f}_0(x)$ and $\mathfrak{f}_2(x)$ are local functions for positive and negative regions (see [12] for more details). Many activation functions such as LReLU and SELU are in this form. The typical form of more complicated cases is the following:

$$\mathfrak{f}(x)=\begin{cases}\mathfrak{f}_0(x) & ; \quad x<x_0\\ \mathfrak{f}_1(x) & ; \quad x_0<x\leq x_1\\ \cdots & \\ \mathfrak{f}_{n-1}(x) & ; \quad x_{n-2}<x\leq x_{n-1}\\ \mathfrak{f}_n(x) & ; \quad x_{n-1}<x\end{cases}. \tag{17}$$

Now, we propose two examples of real and complex valued activation functions of which fixed point sets contain a φ-fixed circle.

Example 3.1. We define an example of a real valued activation function using the general form (17) by

$$\mathcal{T}x=\begin{cases}|x|+1 & ; \quad x<0\\ x & ; \quad 0<x\leq 4\\ x+1 & ; \quad 4<x\leq 8\\ x+2 & ; \quad x>8\end{cases},$$

for all $x\in\mathbb{R}$. Notice that we have

$$Fix(\mathcal{T})=(0,4] \text{ and } \rho=\inf\{d(\mathcal{T}x,x):x\in\mathbb{R},x\neq\mathcal{T}x\}=1.$$

If we define the function $\varphi:\mathbb{R}\to[0,\infty)$ by

$$\varphi(x)=\begin{cases}|x| & ; \quad x<0\\ 0 & ; \quad 0<x\leq 3\\ 8-x & ; \quad 3<x\leq 8\\ x & ; \quad x>8\end{cases},$$

we get

$$Z_\varphi=(0,3]\cup\{8\} \text{ and } Fix(\mathcal{T})\cap Z_\varphi=(0,3].$$

The fixed points of $\mathcal{T}$ belonging in the interval $(0,3]$ are special because of the reason that they are also zeros of the function φ. The circle $C_{2,1}=\{1,3\}$ (resp. the disc $D_{2,1}=[1,3]$) contained in the set $Fix(\mathcal{T})\cap Z_\varphi$ is a φ-fixed circle (resp. φ-fixed disc) of $\mathcal{T}$.

This approach can also be used to define complex activation functions as we have seen in the following example.

Example 3.2. Define the self-mapping $\mathcal{S}$ by

$$\mathcal{S}z=\begin{cases}-\overline{z} & ; \quad x<-4\\ z+1 & ; \quad -4<x\leq -2\\ z & ; \quad -2<x\leq 1\\ z-1 & ; \quad x>1\end{cases},$$

for all complex numbers $z = x + iy$. We have

$$Fix\,(\mathcal{S}) = \{z = x + iy \in \mathbb{C} : -2 < x \leq 1\}$$

and

$$\rho = \inf\{d\,(\mathcal{S}z, z) : z \in \mathbb{C}, z \neq \mathcal{S}z\} = 1.$$

Define the function $\varphi : \mathbb{C} \to [0, \infty)$ by

$$\varphi\,(z) = \begin{cases} |z| - 1 & ; \quad |z| \geq 1 \\ |z| & ; \quad |z| < 1 \end{cases}.$$

We find

$$Z_\varphi = \{z \in \mathbb{C} : |z| = 1\} \cup \{0\} \text{ and } Fix\,(T) \cap Z_\varphi = \{z \in \mathbb{C} : |z| = 1\} \cup \{0\}.$$

Hence, $C_{0,1} = \{z \in \mathbb{C} : |z| = 1\}$ (resp. $D_{0,1} = \{z \in \mathbb{C} : |z| \leq 1\}$) is a φ-fixed circle (resp. φ-fixed disc) of $\mathcal{S}$.

Thus, these kind general activation functions can be used in the study of neural networks to obtain more properties with a geometric approach.

Finally, we note that we were inspired by the definitions of the functions $F : [0, \infty)^3 \to [0, \infty)$ defined by

$$F\,(a, b, c) = \max\{a, b\} + c$$

and

$$F\,(a, b, c) = a + b + c.$$

New solutions to the φ-fixed circle (resp. φ-fixed disc) problem can be obtained by similar techniques as a future work using similar examples of functions F belonging to the set of functions $\mathcal{F}$ defined in [7] (see [7] for more details).

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