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Generalized screen generic lightlike submanifolds

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Abstract. In this paper, we introduce generalized screen generic lightlike submanifolds of indefinite Kaehler manifolds which is new class and an umbrella of invariant (complex), screen real, CR, SCR, GCR and screen generic lightlike submanifolds. We give a non-trivial example for new class of submanifolds and find new conditions for the induced connection to be a metric connection. Then we obtain a characterization of such lightlike submanifolds in a complex space form. Moreover, we find some necessary and sufficient conditions for minimal generalized screen generic lightlike submanifolds and give an example of minimal generalized screen generic lightlike submanifold.

1. Introduction

In differential geometry, one of the most interesting topics is theory of submanifolds. According to the behaviour of the tangent bundle of a submanifold, we have three classes of submanifolds: holomorphic submanifolds, totally real submanifolds and CR-submanifolds.

The theory of lightlike submanifolds in geometry was firstly established and studied by Duggal and Bejancu [7]. They found that a non-degenerate screen distribution was employed to produce a non-intersecting lightlike transversal vector bundle of the tangent bundle. In this way, they defined the notion of CR-lightlike submanifolds of an indefinite Kaehler manifold as a generalization of lightlike real hypersurfaces of indefinite Kaehler manifolds and showed that CR-lightlike submanifolds do not contain invariant and totally real lightlike submanifolds in [3]. After that, Şahin and Güneş investigated geodesic property of CR-lightlike submanifolds and the integrability of distributions in CR-lightlike submanifolds [23]. Further, Duggal and Şahin published a nice book focusing on the study of lightlike submanifolds, pertaining to the field of differential geometry in [11]. This book provides a comprehensive examination of recent advancements in lightlike geometry. The investigation of the geometric properties of lightlike hypersurfaces and lightlike submanifolds has been the subject of research in several studies (see also: [4, 6, 9, 10, 13–22, 24]). Later on, Duggal and Şahin gave the notion of Screen Cauchy-Riemann (SCR)-lightlike submanifolds of an indefinite Kaehler manifold which contains complex and screen real subcases in [6]. However, there is no inclusion

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relation between screen Cauchy-Riemann and CR submanifolds. In [8], the authors, as a generalization of invariant submanifolds, screen real submanifolds, CR-lightlike and SCR-lightlike submanifolds, defined and studied GCR-lightlike submanifolds of Kaehler manifolds, respectively. On the other hand, as a generalization of invariant lightlike, screen real lightlike and generic lightlike submanifolds of indefinite Kaehler manifolds, Doğan et al. introduced screen generic lightlike submanifolds and investigate the geometry of submanifolds in [5] (see also:[12]). The growing importance of lightlike submanifolds in mathematical physics, in particular, their use in relativity and many more, motivated the authors to study lightlike submanifolds extensively. Motivated by above studies, we ask the following question:

Are there any lightlike submanifolds of indefinite Kaehler manifolds which contains invariant (complex), screen real, CR, SCR, GCR and screen generic lightlike submanifolds?

To answer this question, we introduce a new class called generalized screen generic lightlike submanifolds of indefinite Kaehler manifolds. The paper is organized as follows: In section 2, we summarize basic materials on lightlike submanifolds, indefinite Kaehler manifolds and indefinite complex space form which will be useful throughout this paper. In section 3, we introduce generalized screen generic lightlike submanifold of indefinite Kaehler manifolds as a generalization of invariant (complex), screen real, CR, SCR, GCR and screen generic lightlike submanifolds and give a non-trivial example of the new class of submanifolds. In the last section, we investigate some properties of minimal generalized screen generic lightlike submanifolds and give an example of minimal generalized screen generic lightlike submanifold.

2. Preliminaries

Let $(\bar{M}, \bar{g})$ be a real $(m+n)-$dimensional semi-Riemannian manifold of constant index q, such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an $m-$dimensional submanifold of $(\bar{M}, \bar{g})$, where g is the induced metric of $\bar{g}$ on M. If $\bar{g}$ is degenerate on the tangent bundle TM of M then M is named a lightlike submanifold of $(\bar{M}, \bar{g})$. For a degenerate metric g on M

$$TM^{\perp} = \cup \{u \in T_x\bar{M} : \bar{g}(u, v) = 0, \forall v \in T_x\bar{M}, x \in M\} \tag{1}$$

is a degenerate $n-$dimensional subspace of $T_x\bar{M}$. Hence, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. Thus, there exists a subspace $Rad(T_xM) = T_xM \cap T_xM^{\perp}$ which is known as radical (null) space. If the mapping $Rad(TM) : x \in M \longrightarrow Rad(T_xM)$, defines a smooth distribution, named radical distribution on M of rank $r > 0$ then the submanifold M of $(\bar{M}, \bar{g})$ is named an $r-$lightlike submanifold.

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM. This means that

$$TM = S(TM) \perp Rad(TM) \tag{2}$$

and $S(TM^{\perp})$ is a complementary vector subbundle to $Rad(TM)$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}_{|M}$ and $Rad(TM)$ in $S(TM^{\perp})^{\perp}$, respectively. Then, we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}), \tag{3}$$

$$T\bar{M}\,|_M = TM \oplus tr(TM) = \{Rad(TM) \oplus ltr(TM)\} \perp S(TM) \perp S(TM^{\perp}). \tag{4}$$

Theorem 2.1. *[3] Let $(M, g, S(TM), S(TM^{\perp}))$ be an $r-$lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Suppose U is a coordinate neighbourhood of M and ξ_j, $j \in \{1, .., r\}$ is a basis of $\Gamma(Rad(TM)_{|U})$. Then, there exist a complementary vector subbundle $ltr(TM)$ of $Rad(TM)$ in $S(TM^{\perp})^{\perp}_{|U}$ and a basis $\{N_i\}$, $i \in \{1, .., r\}$ of $\Gamma(ltr(TM)_{|U})$ such that*

$$\bar{g}\left(N_i, \xi_j\right) = \delta_{ij}, \quad \bar{g}\left(N_i, N_j\right) = 0 \tag{5}$$

for any $i, j \in \{1, .., r\}$.

We say that a submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $(\bar{M}, \bar{g})$ is
Case 1: $r-$lightlike if $r < min\{m, n\}$,
Case 2: Coisotropic if $r = n < m$, $S(TM^{\perp}) = \{0\}$,
Case 3: Isotropic if $r = m < n$, $S(TM) = \{0\}$,
Case 4: Totally lightlike if $r = m = n$, $S(TM) = \{0\} = S(TM^{\perp})$.
Let $\bar{\nabla}$ be the Levi-Civita connection on $(\bar{M}, \bar{g})$. Then, using (4), we have

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + h(X, Y), & (6)\\ \bar{\nabla}_X \bar{N} &= -A_{\bar{N}} X + \nabla^t_X \bar{N}, & (7) \end{aligned}$$

for any $X, Y \in \Gamma(TM)$ and $\bar{N} \in \Gamma(tr(TM))$, where $\{\nabla_X Y, A_{\bar{N}} X\}$ and $\left\{h(X, Y), \nabla^t_X \bar{N}\right\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. ∇ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$, respectively. Considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$ in (6), respectively, (6) and (7) become

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + h^l(X, Y) + h^s(X, Y), & (8)\\ \bar{\nabla}_X N &= -A_N X + \nabla^l_X N + D^s(X, N), & (9)\\ \bar{\nabla}_X W &= -A_W X + \nabla^s_X W + D^l(X, W), & (10) \end{aligned}$$

for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$, where $h^l(X, Y) = Lh(X, Y)$, $h^s(X, Y) = Sh(X, Y)$, $\nabla_X Y, A_N X, A_W X \in \Gamma(TM)$, $\nabla^s_X W, D^s(X, N) \in \Gamma(S(TM^{\perp}))$ and $\nabla^l_X N, D^l(X, W) \in \Gamma(ltr(TM))$. Hence, using (8)-(10) and letting into account that $\bar{\nabla}$ is a metric connection we derive

$$\begin{aligned} \bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^l(X, W)) &= g(A_W X, Y), & (11)\\ \bar{g}(D^s(X, N), W) &= \bar{g}(A_W X, N), & (12)\\ \bar{g}(h^l(X, Y), \xi) + \bar{g}(Y, h^l(X, \xi)) + g(Y, \nabla_X \xi) &= 0. & (13) \end{aligned}$$

Let Q be a projection of TM on $S(TM)$. Thus, using (2) we obtain

$$\begin{aligned} \nabla_X QY &= \nabla^*_X QY + h^*(X, QY), & (14)\\ \nabla_X \xi &= -A^*_{\xi} X + \nabla^{*t}_X \xi, & (15) \end{aligned}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where $\{\nabla^*_X QY, A^*_{\xi} X\}$ and $\left\{h^*(X, QY), \nabla^{*t}_X \xi\right\}$ belong to $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$, respectively.

Using the equations given above, we derive

$$\begin{aligned} \bar{g}(h^l(X, QY), \xi) &= g(A^*_{\xi} X, QY), & (16)\\ \bar{g}(h^*(X, QY), N) &= g(A_N X, QY), & (17)\\ \bar{g}(h^l(X, \xi), \xi) &= 0, \quad A^*_{\xi} \xi = 0. & (18) \end{aligned}$$

Generally, ∇ on M is not metric connection. Since $\bar{\nabla}$ is a metric connection, from (8), we obtain

$$(\nabla_X g)(Y, Z) = \bar{g}(h^l(X, Y), Z) + \bar{g}(h^l(X, Z), Y).$$

However, it is important to note that $\nabla^{\star}$ is a metric connection on $S(TM)$. We denote curvature tensor of a lightlike submanifold by R, then the Gauss equation for lightlike submanifolds is given by

$$\begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z + A_{h^l(X,Z)} Y - A_{h^l(Y,Z)} + A_{h^s(X,Z)} Y\\ &- A_{h^s(Y,Z)} X + (\nabla_X h^l)(Y, Z) - (\nabla_Y h^l)(X, Z)\\ &+ D^l(X, h^s(Y, Z)) - D^l(Y, h^s(X, Z))\\ &+ (\nabla_X h^s)(Y, Z) - (\nabla_Y h^s)(X, Z)\\ &+ D^s(X, h^l(Y, Z)) - D^s(Y, h^l(X, Z)) \qquad (19) \end{aligned}$$

for any $X, Y, Z \in \Gamma(TM)$.

Theorem 2.2. *Let M be an $r-$lightlike submanifold of a semi-Riemannian manifold $\bar{M}$. Then ∇ is a metric connection iff $Rad(TM)$ is a parallel distribution with respect to ∇ [3].*

Definition 2.3. *A lightlike submanifold (M, g) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be totally umbilical in $\bar{M}$ if there is a smooth transversal vector field $H \in \Gamma(tr(TM))$ on M, called the transversal curvature vector field of M, such that*

$$h(X, Y) = H\bar{g}(X, Y) \tag{20}$$

for any $X, Y \in \Gamma(TM)$. In case $H = 0$, M is called totally geodesic [4].

Using (8) and (20) it is easy to see that M is totally umbilical iff on each coordinate neighborhood U there exists smooth vector fields $H^l \in \Gamma(ltr\,(TM))$ and $H^s \in \Gamma(S\,(TM^{\perp}))$ such that

$$h^l(X, Y) = \bar{g}(X, Y)H^l,\ h^s(X, Y) = \bar{g}(X, Y)H^s \text{ and } D^l(X, W) = 0 \tag{21}$$

for any $X, Y \in \Gamma(TM)$ and $W \in \Gamma(S\,(TM^{\perp}))$.

Now, we recall some basic definitions of indefinite Kaehler manifolds. Let $(\bar{M}, \bar{g})$ be a $2k$-dimensional semi-Riemannian manifold and the constant index of $\bar{g}$ be q, $0 < q < 2k$. An almost complex structure J on $\bar{M}$ is a tensor field of type (1,1) and an endomorphism of $T_p\bar{M}$, such that $J^2 = -I$, $\forall p \in \bar{M}$, where I denotes the identity transformation of $T_p\bar{M}$. Then, $(\bar{M}, J, \bar{g})$ is called an almost complex manifold. If semi-Riemannian metric $\bar{g}$ of an almost complex manifold $\bar{M}$ satisfies the following conditions, then $(\bar{M}, J, \bar{g})$ is an indefinite almost Hermitian manifold

$$\bar{g}(X, Y) = \bar{g}(JX, JY)\,, \quad \forall X, Y \in \Gamma(T\bar{M}). \tag{22}$$

Let $\bar{\nabla}$ be the Levi-Civita connection on an indefinite almost Hermitian manifold $(\bar{M}, J, \bar{g})$ and J be parallel with respect to $\bar{\nabla}$, i.e.,

$$(\bar{\nabla}_X J)Y = 0\,, \quad \forall X, Y \in \Gamma(T\bar{M}). \tag{23}$$

Then $\bar{M}$ is called an indefinite Kaehler manifold.
If holomorphic sectional curvature of an indefinite Kaehler manifold is constant c, then it is called an indefinite complex space form and denoted by $\bar{M}(c)$. The curvature tensor field of $\bar{M}(c)$ is given by

$$\begin{aligned} \bar{R}(X, Y)Z \;&=\; \frac{c}{4}\{\bar{g}(Y, Z)X - \bar{g}(X, Z)Y + \bar{g}(JY, Z)JX - \bar{g}(JX, Z)JY \\ &+\; 2\bar{g}(X, JY)JZ\}, \end{aligned} \tag{24}$$

for any $X, Y \in \Gamma(T\bar{M})$ [1].

3. Generalized Screen Generic Lightlike Submanifolds

Definition 3.1. *Let M be a real r-lightlike submanifold of an indefinite Kaehler manifold manifold $(\bar{M}, \bar{g})$. Then we say that M is a generalized screen generic lightlike submanifold if the condition (A) and (B) are holded:*
(A) There exist two subbundles D_1 and D_2 of $Rad(TM)$ such that

$$Rad(TM) = D_1 \oplus D_2,\ \ J(D_1) = D_1,\ J(D_2) \subset S(TM). \tag{25}$$

(B) There exist two subbundles D_0 and D' of $S\,(TM)$ such that

$$S(TM) = \{J(D_2) \oplus J(L_2)\} \perp D' \perp D_0,\ J(D_0) = D_0, \tag{26}$$

where D_0 is a non-degenerate distribution on M, $ltr\,(TM) = L_1 \oplus L_2$, respectively.

From definition of a generalized screen generic lightlike submanifold, TM of M is decomposed as

$$TM = D \oplus D^{\mu} \tag{27}$$

where

$$D = D_0 \oplus J(D_2) \oplus RadTM, \tag{28}$$

$$D^{\mu} = J(L_2) \perp D' \tag{29}$$

and

$$J(D^{'}) \nsubseteq S(TM) \ \ and \ \ J(D^{'}) \nsubseteq S(TM^{\perp}).$$

It is clear that D is invariant. Besides, we have

$$ltr(TM) = L_1 \oplus L_2, J(L_1) = L_1, J(L_2) \subset S(TM). \tag{30}$$

Now, we denote the projections from $\Gamma(TM)$ to $\Gamma(J(L_2))$ and $\Gamma(D')$ by P_1 and P_2; respectively. Thus, we write

$$X = QX + P_1X + P_2X \tag{31}$$
$$= QX + PX \tag{32}$$

and

$$JX = \Phi X + \omega P_1 X + w P_2 X \tag{33}$$
$$= \Phi X + \omega X \tag{34}$$

for any $X \in \Gamma(TM)$; where $QX \in \Gamma(D)$ and $PX \in \Gamma(D^{\mu})$ and ΦX and ωX are tangential and transversal parts of JX, respectively. Besides, it is clear that $J(D^{'}) \neq D^{'}$.
On the other hand, for a vector field $Y \in \Gamma(D^{'})$, we have

$$JY = \Phi Y + \omega Y \tag{35}$$

such that, $\Phi Y \in \Gamma(D^{'})$ and $\omega Y \in \Gamma(S(TM^{\perp}))$.
Similarly, for $W \in \Gamma(tr(TM))$, we get following decomposition

$$JW = BW + CW \tag{36}$$

where BW is tangential part and CW is transversal part of JW, respectively.

If $D_1 \neq \{0\}$, $D_2 \neq \{0\}$, $D_0 \neq \{0\}$ and $D' \neq \{0\}$, then M is called a proper generalized screen generic lightlike submanifold of an indefinite Kaehler manifold $(\bar{M}, \bar{g})$. For proper generalized screen generic lightlike submanifold we note that the following features:

1. The condition (A) implies that $dim(Rad(TM) \geq 3$.
2. The condition (B) implies $dim(D) = 2s \geq 6$, $dim(D') = 2p \geq 2$ and $dim(D_2) = dim(L_2)$. Thus $dim(M) \geq 9$ and $dim(\bar{M}) \geq 14$.
3. Any proper 9-dimensional generalized screen generic lightlike submanifold must be 3-lightlike.
4. (A) imply that index$(\bar{M}) \geq 3$.

Proposition 3.2. *A generalized screen generic lightlike submanifold M of an indefinite Kaehler manifold $(\bar{M}, \bar{g})$ is screen generic lightlike submanifold iff $D_2 = \{0\}$.*

We easily observe that
(a) If $D_2 = D' = \{0\}$, then M is an invariant lightlike submanifold.
(b) If D' is anti-invariant and $D_1 = \{0\}$, then M is CR-lightlike submanifold.
(c) If D' is anti-invariant and $D_2 = D_0 = \{0\}$, then M is screen real lightlike submanifold.
(d) If D' is anti-invariant and $D_2 = \{0\}$, then M is SCR-lightlike submanifold.
(e) If D' is anti-invariant, then M is GCR-lightlike submanifold.
(f) If $D_2 = \{0\}$, then M is screen generic lightlike submanifold.

Proposition 3.3. *There exist no coisotropic, isotropic or totally lightlike proper generalized screen generic lightlike submanifolds M of an indefinite Kaehler manifold. Any isotropic or totally lightlike generalized screen generic lightlike submanifold M is an invariant submanifold. Furthermore, a coisotropic generalized screen generic lightlike submanifold is invariant lightlike submanifold iff $D_2 = \{0\}$.*

Proof. Assume that M is a proper generalized screen generic lightlike submanifold. From definition of proper generalized screen generic lightlike submanifold, we know that $D_1 \neq \{0\}$, $D_2 \neq \{0\}$, $D_0 \neq \{0\}$ and $D' \neq \{0\}$, that is both $S(TM)$ and $S(TM^\perp)$ are non-zero. Thus, M can not be a coisotropic, isotropic or totally lightlike submanifold. On the other hand, if M is isotropic, then $S(TM) = 0$ which implies that $D_0 = 0$, $D^{'} = 0$ and $J(D_2) = 0$. Therefore we derive $TM = Rad(TM) = J(Rad(TM))$, which is invariant respect to J.
If M is totally lightlike, then $S(TM) = 0$ and $S(TM^\perp) = 0$. Hence, $TM = Rad(TM)$, which implies M is invariant.
Finally, let M be coisotropic. Then $S(TM^\perp) = 0$ implies $\mu = 0$ and the real parts of $\omega(D^{'}) = 0$. Thus, if M is invariant, then $D_2 = \{0\}$. Conversely, suppose that $D_2 = \{0\}$. Then $TM = D_0 \oplus \Phi(D^{'}) \oplus Rad(TM) = D_0 \oplus \Phi(D^{'}) \oplus D_1$ i.e. M is invariant. Thus the proof is completed. □

Example 3.4. *Let $(\bar{M} = \mathbb{R}^{16}_4, \bar{g})$ be a semi-Euclidean space, where $\bar{g}$ is of signature $(-,-,-,-,+,+,+,+,+,+,+,+,+,+,+,+)$ with respect to canonical basis $(\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7, \partial x_8, \partial x_9, \partial x_{10}, \partial x_{11}, \partial x_{12}, \partial x_{13}, \partial x_{14}, \partial x_{15}, \partial x_{16})$. Consider the complex structure J defined by*

$$J(x_1, x_2, ..., x_{15}, x_{16}) = (x_2, -x_1, x_4, -x_3, x_6, -x_5, x_8, -x_7, x_{10}, - x_9, x_{12}, -x_{11}, x_{14}, -x_{13}, x_{16}, -x_{15}).$$

Suppose that M is a submanifold of $\mathbb{R}^{16}_4$ defined by

$$\begin{aligned}
x_1 &= u_3 \sin\alpha - u_4 \cos\alpha + \frac{u_5}{2}\cos\alpha\ ,\ x_2 = u_3\cos\alpha + u_4 \sin\alpha - \frac{u_5}{2}\sin\alpha,\\
x_3 &= u_1, x_4 = u_2, x_5 = u_3, x_6 = u_4 + \frac{u_5}{2}, x_7 = u_1, x_8 = u_2, x_9 = -u_9\cos\alpha,\\
x_{10} &= u_8\cos\alpha, x_{11} = -\cos u_6 \sinh u_7, x_{12} = \sin u_6 \cosh u_7, x_{13} = u_8,\\
x_{14} &= 0, x_{15} = u_8 \sin\alpha, x_{16} = u_9 \sin\alpha.
\end{aligned}$$

A local frame of TM is given by

$$\begin{aligned}
Z_1 &= \partial x_3 + \partial x_7, Z_2 = \partial x_4 + \partial x_8, Z_3 = \sin\alpha \partial x_1 + \cos\alpha \partial x_2 + \partial x_5,\\
Z_4 &= -\cos\alpha\partial x_1 + \sin\alpha\partial x_2 + \partial x_6, Z_5 = \frac{1}{2}(\cos\alpha\partial x_1 - \sin\alpha\partial x_2 + \partial x_6),\\
Z_6 &= \sin u_6 \sinh u_7 \partial x_{11} + \cos u_6 \cosh u_7 \partial x_{12},\\
Z_7 &= -\cos u_6 \cosh u_7 \partial x_{11} + \sin u_6 \sinh u_7 \partial x_{12},\\
Z_8 &= \cos\alpha\partial x_{10} + \partial x_{13} + \sin\alpha\partial x_{15}, Z_9 = -\cos\alpha\partial x_9 + \sin\alpha\partial x_{16}.
\end{aligned}$$

Thus M is a $3-$lightlike submanifold of $\mathbb{R}^{16}_4$ with $Rad(TM) = Sp\{Z_1, Z_2, Z_3\}$, $D_1 = Sp\{Z_1, Z_2\}$, $D_2 = Sp\{Z_3\}$. It is easy to see $JZ_1 = Z_2 \in \Gamma(D_1)$ and $JZ_3 = Z_4$, hence $D_1 = Sp\{Z_1, Z_2\}$ and $D_2 = Sp\{Z_3\}$. On the other hand, since $JZ_6 = Z_7 \in \Gamma(S(TM))$, we derive $D_0 = Sp\{Z_6, Z_7\}$ and $D' = Sp\{Z_8, Z_9\}$. By direct calculations, we derive the lightlike transversal bundle spanned by

$$N_1 = \frac{1}{2}(-\partial x_3 + \partial x_7), N_2 = \frac{1}{2}(-\partial x_4 + \partial x_8), N_3 = \frac{1}{2}(-\sin\alpha\partial x_1 - \cos\alpha\partial x_2 + \partial x_5).$$

Then we see that $L_1 = Sp\{N_1, N_2\}$, $L_2 = Sp\{N_3\}$ *and the screen transversal bundle spanned by*

$$\begin{aligned} W_1 &= \partial x_{14}, W_2 = -\cos\alpha\partial x_{10} + \partial x_{13} - \sin\alpha\partial x_{15}, \\ W_3 &= \sin\alpha\partial x_9 + \cos\alpha\partial x_{16}, W_4 = \sin\alpha\partial x_{10} - \cos\alpha\partial x_{15}, \end{aligned}$$

where $\mu = Sp\{W_3, W_4\}$, $JW_3 = W_4$. *Since*

$$\begin{aligned} JZ_8 &= -\cos\alpha\partial x_9 + \partial x_{14} + \sin\alpha\partial x_{16} = Z_9 + W_1, \\ JZ_9 &= -\cos\alpha\partial x_{10} - \sin\alpha\partial x_{15} = \frac{-Z_8 + W_2}{2} \end{aligned}$$

then, M is a proper generalized screen generic lightlike submanifold of $\mathbb{R}^{16}_4$.

Theorem 3.5. *Let M be a generalized screen generic lightlike submanifold of an indefinite Kaehler manifold* $(\bar{M}, \bar{g})$. *Then* ∇ *is a metric connection iff the following conditions are satisfied:*

$$\begin{aligned} A^*_{JY}X - \nabla^{*t}_X JY \in \Gamma(J(D_2) \oplus D_1), Y \in \Gamma(D_1) \\ -\nabla^*_X JY - h^*(X, JY) \in \Gamma(J(D_2 \oplus D_1), Y \in \Gamma(D_2) \\ Bh(X, JY) = 0, Y \in \Gamma(RadTM). \end{aligned}$$

Proof. Let ∇ be a metric connection. From (23) we obtain

$$\bar{\nabla}_X Y = -J\bar{\nabla}_X JY \tag{37}$$

for any $X \in \Gamma(TM)$ and $Y \in \Gamma(Rad(TM))$. Thus from (6), (34) and (36) we obtain

$$\nabla_X Y + h(X, Y) = -J(\nabla_X JY + h(X, JY)), \tag{38}$$

for any $X \in \Gamma(TM)$ and $Y \in \Gamma(Rad(TM))$. Using (15) in (37) and taking the tangential parts from the above equation give

$$\nabla_X Y = \Phi A^*_{JY}X - \Phi\nabla^{*t}_X JY - Bh(X, JY), \tag{39}$$

for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D_1)$. Similarly, if we take $Y \in \Gamma(D_2)$ in (38) and using (14), we derive

$$\nabla_X Y = -\Phi\nabla^*_X JY - \Phi h^*(X, JY) - Bh(X, JY). \tag{40}$$

Therefore considering Theorem 2.2, the proof comes from (39) and (40). □

Theorem 3.6. *Let M be a generalized screen generic lightlike submanifold of an indefinite Kaehler manifold* $(\bar{M}, \bar{g})$. *Then* D_0 *is integrable iff the following conditions are satisfied:*

$$g(\nabla^*_X JY - \nabla^*_Y JX, \Phi Z) = \bar{g}(B(h^s(X, JY) - h^s(Y, JX)), Z), \tag{41}$$

$$\bar{g}(h^l(X, JY), \xi) = \bar{g}(h^l(Y, JX), \xi), \tag{42}$$

$$\bar{g}(h^*(X, JY), N_1) = \bar{g}(h^*(Y, JX), N_1), \tag{43}$$

$$g(\nabla^*_X JY - \nabla^*_Y JX, BN) = \bar{g}(h^*(Y, JX) - h^*(X, JY), CN), \tag{44}$$

for any $X, Y \in \Gamma(D_0)$, $Z \in \Gamma(D')$, $\xi \in \Gamma(D_2)$, $N \in \Gamma(ltr(TM))$ *and* $N_1 \in \Gamma(L_2)$.

Proof. From definition of generalized screen generic lightlike submanifold, D_0 is integrable iff for any $X, Y \in \Gamma(D_0)$, $[X, Y] \in \Gamma(D_0)$, that is,

$$g([X, Y], Z) = g([X, Y], J\xi) = g([X, Y], JN_1) = \bar{g}([X, Y], N) = 0$$

for any $Z \in \Gamma(D^{'})$, $\xi \in \Gamma(D_2)$, $N \in \Gamma(ltr(TM))$ and $N_1 \in \Gamma(L_2)$.
Using that $\bar{\nabla}$ is a metric connection and (8), (14), (22), (23), (34) and (36), we derive

$$g([X,Y],Z) = g(\nabla^*_X JY - \nabla^*_Y JX, \Phi Z) - g(B(h^s(X,JY) - h^s(Y,JX)), Z), \tag{45}$$

$$g([X,Y], J\xi) = \bar{g}(h^l(Y,JX) - h^l(X,JY), \xi), \tag{46}$$

$$g([X,Y], JN_1) = \bar{g}(h^*(Y,JX) - h^*(X,JY), N_1), \tag{47}$$

$$\bar{g}([X,Y],N) = \bar{g}(\nabla^*_X JY - \nabla^*_Y JX + h^*(X,JY) - h^*(Y,JX), JN), \tag{48}$$

which holds (41)-(44). This completes the proof. □

Theorem 3.7. *Let M be a generalized screen generic lightlike submanifold of an indefinite Kaehler manifold $(\bar{M}, \bar{g})$. Then D is integrable if and only if the following conditions are satisfied:*

$$g(\nabla^*_X JY - \nabla^*_Y JX, \Phi Z) = g(B(h^s(X,JY) - h^s(Y,JX)), Z), \tag{49}$$

$$\bar{g}(h^l(X,JY), \xi) = \bar{g}(h^l(Y,JX), \xi) \tag{50}$$

for any $X, Y \in \Gamma(D)$, $Z \in \Gamma(D^{'})$ and $\xi \in \Gamma(D_2)$.

Proof. From definition of generalized screen generic lightlike submanifold, D is integrable iff for any $X, Y \in \Gamma(D)$, $[X,Y] \in \Gamma(D)$, that is,

$$g([X,Y],Z) = \bar{g}([X,Y], J\xi) = 0$$

for any $Z \in \Gamma(D^{'})$ and $\xi \in \Gamma(D_2)$.
Since $\bar{\nabla}$ is a metric connection and using (8), (14), (22), (23), (34) and (36), we get

$$g([X,Y],Z) = g(\nabla^*_X JY - \nabla^*_Y JX, \Phi Z) - g(B(h^s(X,JY) - h^s(Y,JX)), Z), \tag{51}$$

$$g([X,Y], J\xi) = \bar{g}(h^l(Y,JX) - h^l(X,JY), \xi), \tag{52}$$

which satisfy (49) and (50). Hence, the proof is completed. □

Theorem 3.8. *Let M be a generalized screen generic-lightlike submanifold of an indefinite Kaehler manifold $(\bar{M}, \bar{g})$. Then, $D^{'}$ is integrable iff the following conditions are satisfied:*

$$\nabla_Z \Phi W - \nabla_W \Phi Z - A_{\omega W} Z + A_{\omega Z} W \in \Gamma(D^{'}), \tag{53}$$

$$\bar{g}(h^l(Z,JW), \xi) = \bar{g}(h^l(W,JZ), \xi) \tag{54}$$

for any $Z, W \in \Gamma(D^{'})$ and $\xi \in \Gamma(D_2)$.

Proof. From definition of generalized screen generic lightlike submanifold, $D^{'}$ is integrable iff, for any $Z, W \in \Gamma(D^{'})$, $X \in \Gamma(D_0)$, $\xi \in \Gamma(D_2)$, $N' \in \Gamma(L_2)$ and $N \in \Gamma(ltr(TM))$,

$$g([Z,W], X) = g([Z,W], J\xi) = g([Z,W], JN') = \bar{g}([Z,W], N) = 0.$$

Using (8), (10), (22), (23) and (34), we derive

$$g([Z,W],X) = g(\nabla_Z \Phi W - \nabla_W \Phi Z - A_{\omega W} Z + A_{\omega Z} W, JX), \tag{55}$$

$$g([Z,W], J\xi), = \bar{g}(h^l(W,JZ) - h^l(Z,JW), \xi), \tag{56}$$

$$\bar{g}([Z,W], JN') = -\bar{g}(\nabla_Z \Phi W - \nabla_W \Phi Z + A_{wZ} W - A_{wW} Z, N'), \tag{57}$$

$$\bar{g}([Z,W], N) = \bar{g}(\nabla_Z \Phi W - \nabla_W \Phi Z - A_{\omega W} Z + A_{\omega Z} W, JN). \tag{58}$$

From (55)-(58), we get (53) and (54). □

Theorem 3.9. *Let M be a generalized screen generic lightlike submanifold of an indefinite Kaehler manifold $(\bar{M}, \bar{g})$. Then, D is parallel iff the following conditions are satisfied:*

$$\nabla_X \Phi Z - A_{\omega Z} X \text{ has no components on } \Gamma(D_0 \perp J(L_2)), \tag{59}$$

$$h^l(X, \Phi Z) = -D^l(X, \omega Z) \tag{60}$$

and $Bh^l(X, JY) = 0$, for any $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D')$.

Proof. D is parallel iff

$$g(\nabla_X Y, Z) = g(\nabla_X Y, J\xi) = 0$$

for any $X, Y \in \Gamma(D)$, $Z \in \Gamma(D')$ and $\xi \in \Gamma(D_2)$. Considering (8), (10), (22), (23) and (34) we get

$$g(\nabla_X Y, Z) = -\bar{g}(\nabla_X \Phi Z + h^l(X, \Phi Z) - A_{\omega Z} X + D^l(X, \omega Z), JY), \tag{61}$$

$$g(\nabla_X Y, J\xi) = -\bar{g}(h^l(X, JY), \xi), \tag{62}$$

which completes proof. □

Theorem 3.10. *Let M be a generalized screen generic lightlike submanifold of an indefinite Kaehler manifold $(\bar{M}, \bar{g})$. Then, D' is parallel iff the following conditions are satisfied:*

$$\nabla_Z \Phi W - A_{\omega W} Z \in \Gamma(D'), \tag{63}$$

$$h^l(Z, \Phi W) = -D^l(Z, \omega W) \tag{64}$$

for any $X \in \Gamma(D)$ and $Z, W \in \Gamma(D')$.

Proof. D' is parallel iff

$$g(\nabla_Z W, X) = g(\nabla_Z W, J\xi) = g(\nabla_Z W, JN') = \bar{g}(\nabla_Z W, N) = 0$$

for any $Z, W \in \Gamma(D')$, $X \in \Gamma(D_0)$, $\xi \in \Gamma(D_2)$ and $N \in \Gamma((ltr(TM))$. Using (8), (10), (22), (23) and (34) we obtain

$$g(\nabla_Z W, X) = g(\nabla_Z \Phi W - A_{\omega W} Z, JX), \tag{65}$$

$$g(\nabla_Z W, JN') = \bar{g}(\nabla_Z \Phi W - A_{\omega W} Z, N'), \tag{66}$$

$$\bar{g}(\nabla_Z W, N) = \bar{g}(\nabla_Z \Phi W - A_{\omega W} Z, JN), \tag{67}$$

$$g(\nabla_Z W, J\xi) = -\bar{g}(h^l(Z, \Phi W) + D^l(Z, \omega W), \xi). \tag{68}$$

Therefore from (65)-(68) we see that $\nabla_X \Phi Z - A_{wZ} X \in \Gamma(D')$ and considering (68) we derive (64). □

Theorem 3.11. *Let M be a lightlike submanifold of an indefinite complex space form $\bar{M}(c)$ with $c \neq 0$ Then, M is a generalized screen generic lightlike submanifold of $\bar{M}(c)$ iff:*

(i) The maximal invariant subspaces of TpM, $p \in M$, define a distribution

$$D = D_0 \oplus J(D_2) \oplus Rad(TM),$$

where $Rad(TM) = D_1 \oplus D_2$ and D_0 is a non-degenerate invariant distribution.

(ii) There exists a lightlike transversal vector bundle $ltr(TM)$ such that

$$\bar{g}(\bar{R}(X, Y)N, N') = 0,$$

for any $X, Y \in \Gamma(D_0)$, $N, N' \in \Gamma(ltr(TM))$.

(iii) There exists a non-degenerate distribution D' on M such that

$$\bar{g}(\bar{R}(X, Y)Z, V) \neq 0\,,$$

for any $X, Y \in \Gamma(D)$, $Z, V \in \Gamma(D')$.

(iv) There exists a non-degenerate distribution μ on $S(TM^\perp)$ such that

$$\bar{g}(\bar{R}(X, Y)Z, W) \neq 0,$$

where for any $X, Y \in \Gamma(D)$, $Z \in \Gamma(D')$, $W \in \Gamma(TM^\perp)$, but $W \notin \Gamma(\mu)$.

Proof. Let M be a generalized screen generic lightlike submanifold of $\bar{M}(c)$, $c \neq 0$. From (i), $D = D_0 \oplus J(D_2) \oplus Rad(TM)$ is maximal invariant subspaces. Next from (24), we obtain

$$\bar{g}(\bar{R}(X,Y)N,N') = \frac{c}{2} g(X,JY)\bar{g}(JN,N') = 0$$

for any $X,Y \in \Gamma(D_0)$, N, $N' \in \Gamma(ltr(TM))$. Since $g(X,JY) \neq 0$ and $\bar{g}(JN,N') = 0$, we get $\bar{g}(\bar{R}(X,Y)N,N') = 0$. Thus (ii) holds. If we use (24) for any $X,Y \in \Gamma(D)$ and $Z,V \in \Gamma(D^{'})$, we derive

$$\bar{g}(\bar{R}(X,Y)Z,V) = \frac{c}{2}\bar{g}(X,JY)\bar{g}(JZ,V) \neq 0$$

then (iii) holds.
Similarly, for any $X,Y \in \Gamma(D)$, $Z \in \Gamma(D^{'})$, $W \in \Gamma(TM^{\perp})$, we have

$$\bar{g}(\bar{R}(X,Y)Z,W) = \frac{c}{2}\bar{g}(X,JY)\bar{g}(JZ,W) \neq 0$$

then (iv) holds.

$\Longleftarrow$) : Conversely, we suppose that (i), (ii), (iii) and (iv) are satisfied. From (i), $D = D_0 \oplus J(D_2) \oplus Rad(TM)$ is maximal invariant subspaces and $Rad(TM) = D_1 \oplus D_2$. Then, we see that a part, D_2, of $Rad(TM)$ is a distribution on M such that $J(D_2) \cap Rad(TM) = \{0\}$. It also shows that the other part of $Rad(TM)$ is invariant. Thus (A) of the definition of generalized screen generic lightlike submanifold is satisfied. Therefore, we can choose a screen distribution containing $J(D_2)$ and D_0 (since D_0 is non-degenerate). $ltr(TM)$ orthogonal to $S(TM)$ implies that $\bar{g}(JN,\xi) = -\bar{g}(N,J\xi) = 0$ for $\xi \in \Gamma(D_2)$. Hence we conclude that some part of $J(ltr(TM))$ defines a distribution on M, say $J(L_2)$. On the other hand, from (ii) we derive $\frac{c}{2}g(X,JY)\bar{g}(JN,N') = 0$ for any $X,Y \in \Gamma(D_0)$, N, $N' \in \Gamma(ltr(TM))$. Since $c \neq 0$ and D_0 is non-degenerate we conclude that $\bar{g}(JN,N') = 0$ that is $J(ltr(TM)) \cap Rad(TM) = \{0\}$. Moreover, if $\bar{g}(\xi,N) = 1$ for $\xi \in \Gamma(D_2)$, $N \in \Gamma(L_2)$ then we have $\bar{g}(J\xi,JN) = 1$. This shows that $J(L_2)$ is not orthogonal to $J(D_2)$ and hence, it is not orthogonal to D. From (iii), we see that there exists a non-degenerate anti-invariant distribution $D^{'}$ on $S(TM)$ such that since $\bar{g}(JZ,V) \neq 0$, for any $Z,V \in \Gamma(D^{'})$, then

$$\Gamma(J(D^{'})) \subset \Gamma(D^{'}) \subset \Gamma(S(TM)). \tag{69}$$

On the other hand, from (iv), we have $\bar{g}(JZ,W) = -\bar{g}(Z,JW)$, for any $X,Y \in \Gamma(D_0)$, $Z \in \Gamma(D^{'})$, $W \in \Gamma(TM^{\perp})$, but $W \notin \Gamma(\mu)$. So, it is clear that

$$\Gamma(J(D^{'})) \subset \Gamma(S(TM^{\perp})). \tag{70}$$

Thus, from (69) and (70), we obtain neither $\Gamma(J(D^{'}))$ is in $\Gamma(D^{'})$ totally, nor $\Gamma(J(D^{'}))$ is in $\Gamma(S(TM^{\perp}))$ totally, which satisfies (B) of the definition of generalized screen generic lightlike submanifold. This completes the proof. □

Definition 3.12. *We say that M is a D-geodesic generalized screen generic lightlike submanifold if its second fundamental form h satisfies*

$$h(X,Y) = 0\,, \quad \forall X,Y \in \Gamma(D). \tag{71}$$

It is easy to see that M is a D-geodesic generalized screen generic lightlike submanifold if

$$h^{l}(X,Y) = h^{s}(X,Y) = 0\,, \quad \forall X,Y \in \Gamma(D). \tag{72}$$

On the other hand, if h satisfies

$$h(X,Y) = 0\,, \quad \forall X \in \Gamma(D)\,, \ Y \in \Gamma(D^{\mu}) \tag{73}$$

then M is called a mixed geodesic generalized screen generic lightlike submanifold.

Proposition 3.13. *The distribution D of a generalized screen generic lightlike submanifold M of $\bar{M}$ is a totally geodesic foliation in $\bar{M}$ iff M is D-geodesic and D is parallel respect to ∇ on M.*

Proof. Suppose that D defines a totally geodesic foliation in $\bar{M}$, that is, $\bar{\nabla}_X Y \in \Gamma(D)$, for any $X, Y \in \Gamma(D)$. Then, we get

$$\bar{g}(\bar{\nabla}_X Y, Z) = \bar{g}(\bar{\nabla}_X Y, J\xi') = \bar{g}(\bar{\nabla}_X Y, \xi) = \bar{g}(\bar{\nabla}_X Y, W) = 0$$

for any $Z \in \Gamma(D')$, $\xi' \in \Gamma(D_2)$, $\xi \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. Considering (8), we obtain

$$\begin{aligned} \bar{g}(\bar{\nabla}_X Y, \xi) &= \bar{g}(h^l(X, Y), \xi), \\ \bar{g}(\bar{\nabla}_X Y, W) &= \bar{g}(h^s(X, Y), W) \end{aligned}$$

then, it is clear that, for any $X, Y \in \Gamma(D)$, $h^l(X, Y) = h^s(X, Y) = 0$. In other words, M is D-geodesic and D is parallel respect to ∇ on M.
Conversely, we assume that M is D-geodesic and D is parallel respect to ∇ on M. Since $h^l(X, Y) = h^s(X, Y) = 0$, for any $X, Y \in \Gamma(D)$, then $\bar{\nabla}_X Y \in \Gamma(TM)$. On the other hand, since D is parallel on M, using (8), we have $\bar{\nabla}_X Y \in \Gamma(D)$. This completes the proof. □

Theorem 3.14. *Let M be a generalized screen generic lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then, M is mixed geodesic iff the followings are holded:*
(i) $g(A_{\omega Z}X - \nabla_X \Phi Z, J\xi) = \bar{g}(h^l(X, \Phi Z + \nabla^t_X \omega Z, J\xi)$,
(ii) $g(A_{\omega Z}X - \nabla_X \Phi Z, BW) = \bar{g}(h^s(X, \Phi Z) + \nabla^s_X \omega Z, CW)$,
for any $X \in \Gamma(D)$, $Z \in \Gamma(D^{\mu})$.

Proof. Suppose that M is mixed geodesic, then from (72), $\bar{g}(h^l(X, Z), \xi) = 0$ and $\bar{g}(h^s(X, Z), W) = 0$ for any $X \in \Gamma(D)$, $Z \in \Gamma(D^{\mu})$, $\xi \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. Thus from (7), (8), (22), (23), (34) and (36), we derive

$$\begin{aligned} &\bar{g}(h^l(X, Z), \xi) = g(\nabla_X \Phi Z + h^l(X, \Phi Z) - A_{\omega Z}X + \nabla^t_X \omega Z, J\xi), \\ &\bar{g}(h^s(X, Z), W) = g(\nabla_X \Phi Z - A_{\omega Z}X, BW) + \bar{g}(h^s(X, \Phi Z) + \nabla^t_X \omega Z, CW) \end{aligned}$$

which proves our assertion. □

For any $Y \in \Gamma(TM)$, differentiating (34) and using (8), (9), (10), (22), (23), (33), (34) and (36), we derive

$$\begin{aligned} &\nabla_X \Phi Y + h^l(X, \Phi Y) + h^s(X, \Phi Y) - A_{\omega P_1 Y}X + \nabla^l_X \omega P_1 Y \\ &+ D^s(X, \omega P_1 Y) - A_{\omega P_2} Y + \nabla^s_X \omega P_2 Y + D^l(X, \omega P_2 Y) = \Phi \nabla_X Y \\ &+ \omega P_1 \nabla_X Y + \omega P_2 \nabla_X Y + Bh^l(X, Y) + Ch^l(X, Y) + Bh^s(X, Y) + Ch^s(X, Y). \end{aligned}$$

Taking tangential, lightlike transversal and screen transversal parts of this equation, we obtain

$$(\nabla_X \Phi)Y = A_{\omega P_1 Y}X + A_{\omega P_2 Y}X + Bh^l(X, Y) + Bh^s(X, Y), \tag{74}$$

$$h^l(X, \Phi Y) + \nabla^l_X \omega P_1 Y + D^l(X, \omega P_2 Y) = \omega P_1 \nabla_X Y + Ch^l(X, Y), \tag{75}$$

$$h^s(X, \Phi Y) + D^s(X, \omega P_1 Y) + \nabla^s_X \omega P_2 Y = \omega P_2 \nabla_X Y + Ch^s(X, Y). \tag{76}$$

Lemma 3.15. *Let M be a generalized screen generic lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then M is mixed geodesic iff*

$$\omega P_1(-\nabla_X \Phi Z + A_{\omega P_1 Z}X + A_{\omega P_2 Z}X) = C(h^l(X, \Phi Z) + \nabla^l_X \omega P_1 Z + D^l(X, \omega P_2 Z)), \tag{77}$$

$$\omega P_2(-\nabla_X \Phi Z + A_{\omega P_1 Z}X + A_{\omega P_2 Z}X) = C(h^s(X, \Phi Z) + D^s(X, \omega P_1 Z) + \nabla^s_X \omega P_2 Z) \tag{78}$$

for any $X \in \Gamma(D)$, $Z \in \Gamma(D^{\mu})$.

Proof. Using (8), (9), (10), (22), (23), (33) and (36), we obtain

$$\begin{aligned} h(X,Z) &= -J(\bar{\nabla}_X(\Phi Z + \omega P_1 Z + \omega P_2 Z)) - \nabla_X Z \\ &= -J(\nabla_X \Phi Z + h^l(X,\Phi Z) + h^s(X,\Phi Z) \\ &\quad -A_{\omega P_1 Z}X + \nabla^l_X \omega P_1 Z + D^s(X,\omega P_1 Z) \\ &\quad -A_{\omega P_2 Z}X + \nabla^s_X \omega P_2 Z + D^l(X,\omega P_2 Z) - \nabla_X Z \end{aligned}$$

for any $X \in \Gamma(D)$, $Z \in \Gamma(D^\mu)$. Considering (34) and (36) and taking transversal part of this equation, we have

$$\begin{aligned} h(X,Z) &= -(\omega(\nabla_X \Phi Z) + Ch^l(X,\Phi Z) + Ch^s(X,\Phi Z) \\ &\quad -\omega A_{\omega P_1 Z}X + C\nabla^l_X \omega P_1 Z + CD^s(X,\omega P_1 Z) \\ &\quad -\omega A_{\omega P_2 Z}X + C\nabla^s_X \omega P_2 Z + CD^l(X,\omega P_2 Z)). \end{aligned}$$

Hence, $h(X,Z) = 0$ iff (77) and (78) are satisfied. □

Lemma 3.16. *Let M be a generalized screen generic lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then, we have*

$$\nabla_X Z = \Phi A_{\omega Z}X - \Phi\nabla_X \Phi Z - Bh(X,\Phi Z) - B\nabla^l_X \omega P_1 Z - BD^s(X,\omega P_1 Z) - B\nabla^s_X \omega P_2 Z - CD^l(X,\omega P_2 Z)$$

for any $X \in \Gamma(D_0)$, $Z \in \Gamma(D^\mu)$.

Proof. Considering (8), (9), (10), (22), (23), (33) and (36), we get

$$\begin{aligned} \nabla_X Z &= -\Phi\nabla_X \Phi Z - Bh^l(X,\Phi Z) - Bh^s(X,\Phi Z) + \Phi A_{\omega P_1 Z}X \\ &\quad -B\nabla^l_X \omega P_1 Z - BD^s(X,\omega P_1 Z) + \Phi A_{\omega P_2 Z}X \\ &\quad -B\nabla^s_X \omega P_2 Z - CD^l(X,\omega P_2 Z), \end{aligned}$$

for any $X \in \Gamma(D_0)$, $Z \in \Gamma(D^\mu)$. If we take tangential parts of last equation, then we obtain

$$\nabla_X Z = \Phi A_{\omega Z}X - \Phi\nabla_X \Phi Z - Bh(X,\Phi Z) - B\nabla^l_X \omega P_1 Z - BD^s(X,\omega P_1 Z) - B\nabla^s_X \omega P_2 Z - CD^l(X,\omega P_2 Z)$$

which proves our assertion. □

Theorem 3.17. *There exist no totally umbilical proper generalized screen generic lightlike submanifold of an indefinite complex space form $\bar{M}(c)$, $c \neq 0$.*

Proof. Suppose that M is a totally umbilical proper generalized screen generic lightlike submanifold of $\bar{M}(c)$, $c \neq 0$. Then, from (19) and (24) we get

$$\bar{R}(X,JX)Z = -\frac{c}{2}g(X,X)JZ \tag{79}$$

and

$$\bar{R}(X,JX)Z = (\nabla_X h^s)(JX,Z) - (\nabla_{JX} h^s)(X,Z), \tag{80}$$

for any $X \in \Gamma(D_0)$, $Z \in \Gamma(D^\mu)$. Since M is totally umbilical, from (21) we obtain

$$(\nabla_X h^s)(JX,Z) = -g(\nabla_X JX,Z)H^s - g(JX,\nabla_X Z)H^s. \tag{81}$$

We know that

$$g(JX,Z) = 0.$$

From the last equation, we have

$$(\bar{\nabla}_X g)(JX, Z) = g(\nabla_X JX, Z) + g(JX, \nabla_X Z) = 0. \tag{82}$$

Thus, from (81) and (82) we get

$$(\nabla_X h^s)(JX, Z) = 0. \tag{83}$$

Similarly, we have

$$(\nabla_{JX} h^s)(X, Z) = 0. \tag{84}$$

Hence, using (79), (80), (83) and (84) we obtain

$$\frac{c}{2} g(X, X)JZ = 0,$$

that is, $c = 0$ is derived. This is a contradiction which completes the proof. □

Lemma 3.18. *Let M be a totally umbilical proper generalized screen generic lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then $H^s \notin \Gamma(\mu)$.*

Proof. Let M be a totally umbilical proper generalized screen generic lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then, considering (76) we get

$$g(X, JY)H^s = g(X, Y)CH^s + \omega P_2 \nabla_X Y, \quad \forall X, Y \in \Gamma(D_0).$$

If we get $X = JY$, then we obtain

$$g(Y, Y)CH^s = \omega P_2 \nabla_{JY} Y.$$

Thus, we get $H^s \notin \Gamma(\mu)$, which completes the proof. □

Theorem 3.19. *Let M be a totally umbilical generalized screen generic lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. If distribution D_0 is integrable then the induced connection ∇ is a metric connection.*

Proof. Suppose that M is totally umbilical generalized screen generic lightlike submanifold. Then, considering (75) we derive

$$g(X, JY)H^l = \omega P_1 \nabla_X Y + g(X, Y)CH^l$$

for any $X, Y \in \Gamma(D_0)$. From the last equation,

$$g(X, JY)H^l - g(Y, JX)H^l = \omega P_1 [X, Y]$$

is derived. Since D_0 is integrable, for any $X, Y \in \Gamma(D_0)$, then $[X, Y] \in \Gamma(D_0)$. If we take $X = JY$ in last equation then we get

$$g(Y, Y)H^l = 0.$$

Since D_0 is non-degenerate, then $H^l = 0$ that is, from (20) $h^l = 0$. Hence the proof is completed. □

4. Minimal generalized screen generic lightlike submanifolds

Definition 4.1. *We say that a lightlike submanifold M of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is minimal if:*
(i) $h^s = 0$ on $Rad(TM)$ and
(ii) $trh = 0$, where trace is written with respect to g restricted to $S(TM)$.

It has been proved in [2] that the above definition is independent of $S(TM)$ and $S(TM^{\perp})$, but it depends on $tr(TM)$.

Example 4.2. *Let $\bar{M} = (\mathbb{R}^{16}_4, \bar{g})$ be a semi-Euclidean space with signature $(-,-,-,-,+,+,+,+,+,+,+,+,+,+,+,+)$ and $(x_1, x_2, x_3, x_4, ..., x_{15}, x_{16})$ be the standard coordinate system of $\mathbb{R}^{16}_4$. Consider the complex structure J defined by*

$$\begin{aligned} J(x_1, x_2, ..., x_{15}, x_{16}) &= (-x_2, x_1, -x_4, x_3, -x_6, x_5, -x_8, x_7, -x_{11}\cos\alpha - x_{10}\sin\alpha, \\ &\quad -x_{12}\cos\alpha + x_9\sin\alpha, x_9\cos\alpha + x_{12}\sin\alpha, \\ &\quad x_{10}\cos\alpha - x_{11}\sin\alpha, -x_{14}, x_{13}, -x_{16}, x_{15}). \end{aligned}$$

Let M be a submanifold of $\mathbb{R}^{16}_4$ given by

$$\begin{aligned} x_1 &= u_1\sin\alpha + u_2,\ x_2 = u_1 - u_2\sin\alpha,\ x_3 = u_3,\ x_4 = u_4 - \frac{u_5}{2}, \\ x_5 &= u_1\cos\alpha, x_6 = -u_2\cos\alpha, x_7 = u_3,\ x_8 = u_4 + \frac{u_5}{2}, \\ x_9 &= \sin u_6 \sinh u_7, x_{10} = \sin u_6\cosh u_7, x_{11} = -\cos u_6\cosh u_7, \\ x_{12} &= -\cos u_6\sinh u_7,\ x_{13} = u_8 - u_9,\ x_{14} = u_8 + u_9,\ x_{15} = 0, x_{16} = 0. \end{aligned}$$

Then, TM is spanned by

$$\begin{aligned} &Z_1 = \sin\alpha\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \cos\alpha\frac{\partial}{\partial x_5}, Z_2 = \frac{\partial}{\partial x_1} - \sin\alpha\frac{\partial}{\partial x_2} - \cos\alpha\frac{\partial}{\partial x_6}, \\ &Z_3 = \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_7}, Z_4 = \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_8}, Z_5 = \frac{1}{2}(-\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_8}), \\ &Z_6 = \cos u_6\sinh u_7\frac{\partial}{\partial x_9} + \cos u_6\cosh u_7\frac{\partial}{\partial x_{10}} + \sin u_6\cosh u_7\frac{\partial}{\partial x_{11}} + \sin u_6\sinh u_7\frac{\partial}{\partial x_{12}}, \\ &Z_7 = \sin u_6\cosh u_7\frac{\partial}{\partial x_9} + \sin u_6\sinh u_7\frac{\partial}{\partial x_{10}} - \cos u_6\sinh u_7\frac{\partial}{\partial x_{11}} - \cos u_6\cosh u_7\frac{\partial}{\partial x_{12}}, \\ &Z_8 = \frac{\partial}{\partial x_{13}} + \frac{\partial}{\partial x_{14}}, Z_9 = -\frac{\partial}{\partial x_{13}} + \frac{\partial}{\partial x_{14}}, \end{aligned}$$

where $Rad(TM) = \{Z_1, Z_2, Z_3\}$ and $D_0 = \{Z_8, Z_9\}$. By direct calculation, we obtain that $ltr(TM)$ is spanned by

$$\begin{aligned} N_1 &= \frac{1}{2}(-\sin\alpha\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} + \cos\alpha\frac{\partial}{\partial x_5}), N_2 = \frac{1}{2}(-\frac{\partial}{\partial x_1} + \sin\alpha\frac{\partial}{\partial x_2} - \cos\alpha\frac{\partial}{\partial x_6}), \\ N_3 &= \frac{1}{2}(-\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_7}). \end{aligned}$$

Furthermore, the screen transversal bundle is spanned by

$$\begin{aligned} &W_1 = -\cos u_6\cosh u_7\frac{\partial}{\partial x_9} + \cos u_6\sinh u_7\frac{\partial}{\partial x_{10}} + \sin u_6\sinh u_7\frac{\partial}{\partial x_{11}} - \sin u_6\cosh u_7\frac{\partial}{\partial x_{12}} \\ &W_2 = -\sin u_6\sinh u_7\frac{\partial}{\partial x_9} + \sin u_6\cosh u_7\frac{\partial}{\partial x_{10}} - \cos u\cosh u_7\frac{\partial}{\partial x_{11}} + \cos u_6\sinh u_7\frac{\partial}{\partial x_{12}} \\ &W_3 = \frac{\partial}{\partial x_{15}} + \frac{\partial}{\partial x_{16}}, W_4 = -\frac{\partial}{\partial x_{15}} + \frac{\partial}{\partial x_{16}}. \end{aligned}$$

Since $JW_1 \neq W_2$, then it is easy to see that $\mu = \{W_3, W_4\}$ and

$$JZ_6 = \cos\alpha Z_7 - \sin\alpha W_1,$$
$$JZ_7 = -\cos\alpha Z_6 - \sin\alpha W_2.$$

Then, $D' = Sp\{Z_6, Z_7\}$ and M is a generalized screen generic lightlike submanifold of $\mathbb{R}^{16}_4$.

On the other hand, by direct computations and using Gauss and Weingarten formulas, we get

$$\bar{\nabla}_{Z_i} Z_T = 0, \quad i = 1,2,3,4,5,8,9, \quad 1 \leq T \leq 9$$

and

$$h^l(Z_6, Z_6) = 0, \; h^l(Z_7, Z_7) = 0,$$
$$h^s(Z_6, Z_6) = -\frac{1}{\sinh^2 u_7 + \cosh^2 u_7} W_2,$$
$$h^s(Z_7, Z_7) = \frac{1}{\sinh^2 u_7 + \cosh^2 u_7} W_2.$$

Hence, we get

$$h^s(X, Z_1) = h^s(X, Z_2) = h^s(X, Z_3) = 0$$

for any $X \in \Gamma(TM)$, that is, $h^s = 0$ on $Rad(TM)$ and

$$trace\,|_{S(TM)}\, h = 0.$$

Then, it is clear that M is not totally geodesic, but it is a minimal generalized screen generic lightlike submanifold of $\bar{M} = \mathbb{R}^{16}_4$.

Theorem 4.3. *Let M be a proper generalized screen generic lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then M is minimal iff*

$$traceA^*_{\xi_q|S(TM)} = 0, traceA_{W_\beta|S(TM)} = 0 \tag{85}$$

$\bar{g}(Y, D^l(X, W)) = 0$ for any $X, Y, \xi_q \in \Gamma(Rad(TM))$ and $W, W_\beta \in \Gamma(S(TM^\perp))$, where $q \in \{1, 2, ..., r\}$ and $\beta \in \{1, 2, ..., n-r\}$.

Proof. We know that $h^l = 0$ on $Rad(TM)$ [2]. Definition of a generalized screen generic lightlike submanifold, M is minimal iff

$$\sum_{i=1}^{2s} \epsilon_i h(X_i, X_i) + \sum_{j=1}^{p} h(J\xi_j, J\xi_j) + \sum_{j=1}^{p} h(JN_j, JN_j) + \sum_{\alpha=1}^{k} \epsilon_\alpha h(Z_\alpha, Z_\alpha) = 0$$

and $h^s = 0$ on $Rad(TM)$, where $2s = dim(D_0)$, $p = dim(D_2)$ and $k = dim(D')$. From (11), we have $h^s = 0$ on $Rad(TM)$ iff $\bar{g}(Y, D^l(X, W)) = 0$, for any $X, Y \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM^\perp))$. Moreover, we get

$$\begin{aligned} traceh \;|\;_{S(TM)} = {} & \frac{1}{r}\sum_{q=1}^{r}\sum_{j=1}^{p} \bar{g}(h^l(J\xi_j, J\xi_j), \xi_q)N_q + \bar{g}(h^l(JN_j, JN_j), \xi_q)N_q \\ & + \frac{1}{n-r}\sum_{j=1}^{p}\sum_{\beta=1}^{n-r} \epsilon_\beta\{\bar{g}(h^s(J\xi_j, J\xi_j), W_\beta)W_\beta + \bar{g}(h^s(JN_j, JN_j), W_\beta)W_\beta\} \\ & + \sum_{\beta=1}^{n-r} \epsilon_\beta \frac{1}{n-r}\{\sum_{i=1}^{2s} \bar{g}(h^s(X_i, X_i), W_\beta)W_\beta + \sum_{\alpha=1}^{k} \bar{g}(h^s(Z_\alpha, Z_\alpha), W_\beta)W_\beta\} \\ & + \sum_{q=1}^{r} \frac{1}{r}\{\sum_{i=1}^{2s} \bar{g}(h^l(X_i, X_i), \xi_q)N_q + \sum_{\alpha=1}^{k} \bar{g}(h^l(Z_\alpha, Z_\alpha), \xi_q)N_q\}. \end{aligned} \tag{86}$$

Using (11) and (16) in (86), we obtain

$$
\begin{aligned}
traceh \mid_{S(TM)} &= \frac{1}{r}\sum_{q=1}^{r}\sum_{j=1}^{p} g(A^*_{\xi_q}J\xi_j, J\xi_j)N_q + g(A^*_{\xi_q}JN_j, JN_j)N_q \\
&+\frac{1}{n-r}\sum_{j=1}^{p}\sum_{\beta=1}^{n-r}\epsilon_\beta\{g(A_{W_\beta}J\xi_j, J\xi_j)W_\beta + g(A_{W_\beta}JN_j, JN_j)W_\beta\} \\
&+\sum_{\beta=1}^{n-r}\epsilon_\beta\frac{1}{n-r}\{\sum_{i=1}^{2s} g(A_{W_\beta}X_i, X_i)W_\beta + \sum_{\alpha=1}^{k} g(A_{W_\beta}Z_\alpha, Z_\alpha)W_\beta\} \\
&+\sum_{q=1}^{r}\frac{1}{r}\{\sum_{i=1}^{2s} g(A^*_{\xi_q}X_i, X_i)N_q + \sum_{\alpha=1}^{k} g(A^*_{\xi_q}Z_\alpha, Z_\alpha)N_q\},
\end{aligned}
\tag{87}
$$

which completes the proof. □

Theorem 4.4. *Let M be a totally umbilical generalized screen generic lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then M is minimal iff*

$$traceA^*_{\xi_q}\mid_{D_0\oplus D'} = traceA_{W_T}\mid_{D_0\oplus D'} = 0 \tag{88}$$

for any $\xi_q \in \Gamma(Rad(TM))$ and $W_T \in \Gamma(S(TM^\perp))$, where $q \in \{1, 2, ..., r\}$ and $T \in \{1, 2, ..., n-r\}$.

Proof. M is minimal iff $h^s = 0$ on $Rad(TM)$ and $traceh = 0$ on $S(TM)$, i.e.

$$
\begin{aligned}
traceh \mid_{S(TM)} &= traceh\mid_{D_0} + traceh\mid_{J(D_2)} + traceh\mid_{J(L_2)} + traceh\mid_{D'} \\
&= \sum_{i=1}^{2a}\epsilon_i h(e_i, e_i) + \sum_{j=1}^{b} h(J\xi_j, J\xi_j) + \sum_{j=1}^{b} h(JN_j, JN_j) + \sum_{k=1}^{c}\epsilon_k h(e_k, e_k).
\end{aligned}
\tag{89}
$$

where $2a = dim(D_0)$, $b = dim(D_2)$ and $c = dim(D')$. Considering (20) in (89) we obtain

$$
\begin{aligned}
traceh \mid_{S(TM)} &= traceh\mid_{D_0} + traceh\mid_{D'} \\
&= \sum_{i=1}^{2a} h(e_i, e_i) + \sum_{k=1}^{c} h(e_k, e_k).
\end{aligned}
$$

If we choose an orthonormal basis of $S(TM)$ as $\{e_i\}_{i=1}^{m-r}$, then we derive

$$
\begin{aligned}
traceh\mid_{S(TM)} &= \sum_{i=1}^{2a}\varepsilon_i\left[h^l(e_i, e_i) + h^s(e_i, e_i)\right] + \sum_{k=1}^{c}\epsilon_k\left[h^l(e_k, e_k) + h^s(e_k, e_k)\right] \\
&= \sum_{i=1}^{2a}\varepsilon_i[\frac{1}{r}\sum_{q=1}^{r}\bar{g}(h^l(e_i, e_i), \xi_q)N_q + \frac{1}{n-r}\sum_{T=1}^{n-r}\bar{g}(h^s(e_i, e_i), W_T)W_T] \\
&+\sum_{k=1}^{c}\varepsilon_k[\frac{1}{r}\sum_{q=1}^{r}\bar{g}(h^l(e_k, e_k), \xi_q)N_q + \frac{1}{n-r}\sum_{T=1}^{n-r}\bar{g}(h^s(e_k, e_k), W_T)W_T].
\end{aligned}
\tag{90}
$$

Using (11) and (16) in (90), we obtain

$$
\begin{aligned}
traceh\mid_{S(TM)} &= \sum_{i=1}^{2a}\varepsilon_i[\frac{1}{r}\sum_{q=1}^{r}\bar{g}(A^*_{\xi_q}e_i, e_i)N_q + \frac{1}{n-r}\sum_{T=1}^{n-r} g(A_{W_T}e_i, e_i)W_T] \\
&+\sum_{k=1}^{c}\varepsilon_k[\frac{1}{r}\sum_{q=1}^{r}\bar{g}(A^*_{\xi_q}e_k, e_k)N_q + \frac{1}{n-r}\sum_{T=1}^{n-r} g(A_{W_T}e_k, e_k)W_T].
\end{aligned}
$$

Therefore we derive

$$traceh\mid_{S(TM)}= traceA^{*}_{\xi_k}\mid_{D_0\oplus D'} +traceA_{W_T}\mid_{D_0\oplus D'} .$$

Hence, we get

$$traceA^{*}_{\xi_k}\mid_{D_0\oplus D'}=0 \quad and \quad traceA_{W_T}\mid_{D_0\oplus D'}=0.$$

This completes the proof. □

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