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Metallic contact-like manifolds

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Abstract. In this paper, we introduce a new class of manifolds, called metallic contact-like manifolds, inspired by the analogy between the contact structure on contact manifolds and the structure induced on the tangent bundle of a real hypersurface of an almost Metallic Riemannian manifold. Necessary and sufficient conditions for the integrability and normality of the constructed structure are derived, and an explicit example is provided to illustrate the theory.

1. Introduction

Manifold theory is one of the important research areas of modern differential geometry. Manifolds play a crucial role in solving problems in natural and engineering sciences and continue to inspire new applications in these fields.

Manifolds endowed with specific linear transformations between their tangent spaces exhibit rich geometric properties. Such manifolds have been extensively studied in differential geometry. For example, almost complex manifolds, almost contact manifolds, almost product manifolds, and the relationships between these manifolds have been studied extensively by many authors [2], [3], [6], [7], [9], [13]. The definition of Sasakian manifolds, which has an important place in contact manifolds, was given by the Japanese mathematician Sasaki [16] in the 1960s. Mathematicians such as Gray [11], Ogiue [13], and Boothby–Wang [3] have made significant contributions to differential geometry. Today, many mathematicians are working on this subject, especially Blair [2].

There are close relations between contact manifolds, symplectic manifolds and complex manifolds. The symplectic manifold is defined in even-dimensional spaces, and the contact manifold is defined in odd-dimensional spaces. They are spaces that complement each other in these aspects. On the horizontal distribution where the contact form is zero, the differential of the contact form gives a symplectic form. Tensor in contact Riemannian structure gives almost complex structure on the horizontal distribution [2]. Contact geometry has applications in many fields. In his articles, Geiges presented in detail how contact geometry is applied in the fields of physics, mechanics, optics, thermodynamics and control theory [8], [9].

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Now, we recall some basic notions and results which will be needed throughout the paper [5], [12], [15], [19].

On a $2n+1$ dimensional differentiable manifold M, if there is a 1- form η satisfying the condition $\eta\wedge(d\eta)^n \neq 0$, the manifold M is called a contact manifold, and the form η is called a contact form. Since the dimension of the contact manifold is $2n+1$ and $\eta \wedge (d\eta)^n \neq 0$, the form $\eta \wedge (d\eta)^n$ is a volume form over M and thus the contact manifold M is orientable. The sub-bundle formed by vector field $\mathcal{X}_1$ satisfying the condition $\eta(\mathcal{X}_1) = 0$ on the contact manifold (M, η) is called the horizontal sub-bundle or the contact sub-bundle and is denoted by H . On the contact manifold (M^{2n+1}, η) there is a single vector field $\tilde{\xi}$ for each vector field $\mathcal{X}_1$, with $d\eta(\tilde{\xi}, \mathcal{X}_1) = 0$ and $\eta(\tilde{\xi}) = 1$. The vector field $\tilde{\xi}$ here is called the characteristic vector field. Every point of the contact manifold (M, η) has local coordinates $(x_1, \cdots, x_n, y_1, \cdots, y_n, z)$ in its neighborhood such that $\eta = dz - \sum_{i=1}^{n} y_i dx_i$. If there is a metric g and (1,1) tensor φ, on M satisfying following conditions for each vector fields $\mathcal{X}_1, \mathcal{Y}_1$, then the structure $(M, \eta, \tilde{\xi}, g, \varphi)$ is called the contact metric structure [2].

$$*\varphi\tilde{\xi} = 0, \qquad \eta \circ \varphi = 0, \qquad \eta\tilde{\xi} = 1,$$

$$*\varphi^2 = -I + \eta \otimes \tilde{\xi},$$

$$*\eta(\mathcal{X}_1) = g(\mathcal{X}_1, \tilde{\xi}),$$

$$*g(\varphi\mathcal{X}_1, \varphi\mathcal{Y}_1) = g(\mathcal{X}_1, \mathcal{Y}_1) - \eta(\mathcal{X}_1)\eta(\mathcal{Y}_1),$$

where η is a 1-form , $\tilde{\xi}$ is a vector field and φ is a (1,1) tensor field.

Yano [18] defined the f-structure on the manifold as a generalization of complex and contact manifolds. Later, Goldberg and Yano [10] developed this structure and defined the concept of polynomial structure on a manifold. Crasmareanu and Hretcanu [4] initiated the theory of Golden manifolds by describing a polynomial structure called the golden structure on a differentiable manifold. If the tensor field Φ satisfies the equation $\Phi^2 = \Phi + I$, with a (1,1) tensor field Φ on any differentiable real manifold M, the tensor field Φ is called a golden structure on the manifold M. In other words, the golden structure Φ on a differentiable real manifold M is a second-order polynomial structure given by the polynomial structure $Q(\mathcal{X}_1) = \mathcal{X}_1^2 - \mathcal{X}_1 - I$. Golden Riemannian manifolds and their geometry have been investigated by many researchers. Golden manifolds show that polynomial structures are useful tools for generating new geometric structures on C^∞ differentiable manifolds. In [14] were defined a silver structure with the polynomial structure $Q(\mathcal{X}_1) = \mathcal{X}_1^2 - 2\mathcal{X}_1 - I$ and a bronze structure with the polynomial structure $Q(\mathcal{X}_1) = \mathcal{X}_1^2 - \mathcal{X}_1 - 2I$ with respectively. Also in 2013, Hretcanu and Crasmareanu [5] reported a type of (1,1) tensor field $\mathcal{J}$ with the polynomial structure $Q(\mathcal{X}_1) = \mathcal{X}_1^2 - p\mathcal{X}_1 - qI$, which is a generalization of the golden structure. They defined and studied a metallic structure with the help of the tensor field. In the following years, many researchers have worked on this subject, for example, Akyol [1] conducted a study on metallic transformations between metallic Riemannian manifolds.

As a generalization of the golden ratio in the metallic ratio, it can be used in more diverse situations in the same fields.

The positive solution of the equation $\mathcal{X}_1^2 - p\mathcal{X}_1 - qI = 0$ is called a member of the family of metallic ratios. The solution set is denoted by

$$\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}.$$

These members are also called (p, q)- metallic numbers. The relation $G(n+1) = pG(n) + qG(n-1), n \geq 1$ is given where $p, q, G(0) = a$ and $G(1) = b$ are real numbers. The ratio $G(n+1)/G(n)$ of two consecutive generalized secondary Fibonacci numbers according to the values of p, q,

- $p = q = 1$, determined by the ratio of two sequential classical Fibonacci numbers converge to the golden mean $\phi = \frac{1+\sqrt{5}}{2}$,
- $p = 2$ and $q = 1$, assigned by the ratio of two sequential Pell numbers converge to the silver mean $\sigma_{2,1} = 1 + \sqrt{2}$,

- $p = 3$ and $q = 1$ which plays an important role in the study of dynamical systems and quasi-crystals, converge to the bronze mean $\sigma_{3,1} = \frac{3+\sqrt{3}}{2}$
- $p = 1$ and $q = 2$ converge to the copper mean $\sigma_{1,2} = 2$,
- $p = 1$ and $q = 3$ converge to the nickel $\sigma_{1,3} = \frac{1+\sqrt{13}}{2}$ mean [5].

Let

$$\mathcal{J}^2 = p\mathcal{J} + qI, \tag{1}$$

with linear endomorphism of vector fields on a Lie algebra $\chi(M)$ in a manifold M , where p, q are positive integers and I is the unit operator. In this case, (1,1) tensor field $\mathcal{J}$ is called the metallic structure. If

$$g(\mathcal{J}\mathcal{X}_1, \mathcal{Y}_1) = g(\mathcal{X}_1, \mathcal{J}\mathcal{Y}_1),$$

for $\mathcal{X}_1, \mathcal{Y}_1 \in \chi(M)$, the Riemannian metric g is said to be compatible with the tensor $\mathcal{J}$. It can be easily seen that

$$g(\mathcal{J}\mathcal{X}_1, \mathcal{J}\mathcal{Y}_1) = pg(\mathcal{X}_1, \mathcal{J}\mathcal{Y}_1) + qg(\mathcal{X}_1, \mathcal{Y}_1).$$

Let (M, g) be a Riemannian manifold given with metallic structure $\mathcal{J}$. If g and $\mathcal{J}$ are compatible, then the triple $(M, g, \mathcal{J})$ is called a metallic Riemannian manifold and the tensor $\mathcal{J}$ is called a metallic Riemannian structure. Metallic structure $\mathcal{J}$ has the following features,

- For every $n \geq 1$ integer, $G(n)$ generalized with $n \geq 0$, $G(0) = 0$ and $G(1) = 1$ the secondary Fibonacci sequence: $\mathcal{J}^n = G(n)\mathcal{J} + qG(n-1)I$.
- $\mathcal{J}$ is the isomorphism in the tangent space T_xM for every $x \in M$. $\mathcal{J}$ is not a metallic structure, which is the inverse of the $\mathcal{J}$ structure defined by $\bar{\mathcal{J}} = \mathcal{J}^{-1} = \frac{1}{q}\mathcal{J} - \frac{p}{q}I$. But the polynomial quadratic equation is $q\bar{\mathcal{J}}^2 + p\bar{\mathcal{J}} - I = 0$.
- The eigenvalues of $\mathcal{J}$ are the metallic numbers $\sigma_{p,q}$ and $p - \sigma_{p,q}$ [5].

Let M be a manifold and $\mathcal{P}$ be an almost product structure on M . In this case, $\mathcal{P}$ on M produces two metallic structures given as $\mathcal{J}_1 = \frac{p}{2}I + (\frac{2\sigma_{p,q}-p}{2}\mathcal{P})$, $\mathcal{J}_2 = \frac{p}{2}I - (\frac{2\sigma_{p,q}-p}{2}\mathcal{P})$. Inversely, any metallic structure $\mathcal{J}$ on M produces two-product structures on M as $\mathcal{P} = \pm(\frac{2}{2\sigma_{p,q}-p}\mathcal{J} - \frac{p}{2\sigma_{p,q}-p}I)$. In particular, if the almost product structure $\mathcal{P}$ is Riemannian, then $\mathcal{J}_1$ and $\mathcal{J}_2$ are metallic Riemannian structures. In the metallic Riemannian manifold $(M, \mathcal{J})$, the complementary distributions D_l and D_m corresponding to the projection operators are given by $l = \frac{\sigma_{p,q}}{2\sigma_{p,q}-p}I - \frac{1}{2\sigma_{p,q}-p}\mathcal{J}$ and $m = \frac{\sigma_{p,q}-p}{2\sigma_{p,q}-p}I + \frac{1}{2\sigma_{p,q}-p}\mathcal{J}$. The l and m operators satisfy the following relations:

1− $\quad l + m = I, \quad l^2 = l, \quad m^2 = m,$
2− $\quad \mathcal{J} \cdot l = l \cdot \mathcal{J} = (p - \sigma_{p,q})l,$
3− $\quad \mathcal{J} \cdot m = m \cdot \mathcal{J} = \sigma_{p,q}m.$

Hence, complementary distributions D_l and D_m corresponding to projections l and m are defined. These distributions are orthogonal. The real hypersurface of a metallic Riemannian manifold will now be considered. For this purpose, some reminders about hypersurfaces will be made. The covariant differential operator defined by the g metric reduced on the hypersurface M is denoted by ∇ , and the covariant differential operator on a metallic Riemannian manifold $\bar{M}$ is denoted by $\bar{\nabla}$. The Weingarten operator on TM is denoted by A with respect to the local unit normal vector field N of the hypersurface M in the metallic Riemannian manifold.

Let M be a real hypersurface of $\bar{M}$ metallic Riemannian manifold. In this case, the structure $\sum = (\mathcal{P}, g, u, \xi, a)$ satisfies the following equations [5].

$$\mathcal{P}^2(\mathcal{X}_1) = p\mathcal{P}(\mathcal{X}_1) + q\mathcal{X}_1 - u(\mathcal{X}_1)\tilde{\xi}, \tag{2}$$
$$u(\mathcal{P}(\mathcal{X}_1)) = (p-a)u(\mathcal{X}_1), \tag{3}$$
$$u(\tilde{\xi}) = q + pa - a^2, \tag{4}$$
$$\mathcal{P}(\tilde{\xi}) = (p-a)\tilde{\xi}, \tag{5}$$
$$u(\mathcal{X}_1) = g(\mathcal{X}_1, \tilde{\xi}),$$
$$g(\mathcal{P}\mathcal{X}_1, \mathcal{Y}_1) = g(\mathcal{X}_1, \mathcal{P}\mathcal{Y}_1),$$
$$g(\mathcal{P}\mathcal{X}_1, \mathcal{P}\mathcal{Y}_1) = pg(\mathcal{X}_1, \mathcal{P}\mathcal{Y}_1) + qg(\mathcal{X}_1, \mathcal{Y}_1) - u(\mathcal{X}_1)u(\mathcal{Y}_1). \tag{6}$$

This structure is the structure reduced on the tangent bundle of a hypersurface of a metallic Riemannian manifold.

A differentiable manifold M that satisfies the above conditions will be called an almost metallic contact-like manifold.

Theorem 1.1. *Let (M^{2k}, Φ) be a symplectic manifold. In this case the following equation is hold*

$$g(\mathcal{X}_1, \mathcal{J}\mathcal{Y}_1) = \Phi(\mathcal{X}_1, \mathcal{Y}_1), \tag{7}$$

where g is a Riemannian metric and $\mathcal{J}$ is an almost complex structure[2].

In this paper, we construct a new type of manifold by taking advantage of the similarity between the contact structure on contact manifolds and the structure on the tangent bundle of a real hypersurface of an almost Metallic Riemannian manifold. In the construction process of the contact manifolds, firstly, a complex structure (linear transformation) is defined on the two dimensional manifold, which is the product of the contact manifold and the real numbers. By examining the integrability of this complex structure, it is seen that contact manifolds are investigated.

Sasaki and Hatakeyema [17] defined an almost complex structure $\mathcal{J}$ on $M \times \mathbb{R}$ by

$$\mathcal{J}(\mathcal{X}_1, f\frac{d}{dt}) = (\varphi\mathcal{X}_1 - f\tilde{\xi}, \eta(\mathcal{X}_1)\frac{d}{dt}),$$

where f is a C^∞ real-valued function on $M \times \mathbb{R}$. Considering the Nijenhuis torsion $N_{\mathcal{J}}$ of $\mathcal{J}$, they computed $N_{\mathcal{J}}((\mathcal{X}_1, 0), (\mathcal{Y}_1, 0))$ and $N_{\mathcal{J}}((\mathcal{X}_1, 0), (0, \frac{d}{dt}))$ which gave rise to four tensors N^1, N^2, N^3, N^4 given as

$$\begin{aligned} N^1(\mathcal{X}_1, \mathcal{Y}_1) &= N_\varphi(\mathcal{X}_1, \mathcal{Y}_1) + 2d\eta(\mathcal{X}_1, \mathcal{Y}_1)\tilde{\xi}, \\ N^2(\mathcal{X}_1, \mathcal{Y}_1) &= (\mathcal{L}_{\varphi\mathcal{X}_1}\eta)(\mathcal{Y}_1) - (\mathcal{L}_{\varphi\mathcal{Y}_1}\eta)(\mathcal{X}_1), \\ N^3(\mathcal{X}_1) &= (\mathcal{L}_{\tilde{\xi}}\varphi)(\mathcal{X}_1), \\ N^4(\mathcal{X}_1) &= (\mathcal{L}_{\tilde{\xi}}\eta)(\mathcal{X}_1), \end{aligned}$$

where $\mathcal{L}$ denotes Lie derivative. It is clear that the almost contact structure $(\varphi, \tilde{\xi}, \eta)$ is normal if and only if these four tensors vanish .

Theorem 1.2. *For an almost contact structure $(\varphi, \tilde{\xi}, \eta)$ the vanishing of N^1 implies the vanishing N^2, N^3 and N^4[14].*

Then, the normality condition is

$$N_\varphi(\mathcal{X}_1, \mathcal{Y}_1) + 2d\eta(\mathcal{X}_1, \mathcal{Y}_1)\tilde{\xi} = 0.$$

Namely φ is integrable if and only if $N^1 = 0$ implies $N^2 = N^3 = N^4 = 0$. An almost contact structure is said to be normal if $N^1 = 0$, that is, if the almost complex structure on $M \times \mathbb{R}$ is integrable.

Theorem 1.3. *Let $(\varphi, \tilde{\xi}, \eta, g)$ be a contact metric structure. Then the tensors N^2 and N^4 vanish. Moreover N^3 vanishes iff the characteristic vector field $\tilde{\xi}$ is Killing vector field with respect to g [17].*

Lemma 1.4. *For an almost contact metric structure $(\varphi, \tilde{\xi}, \eta, g)$, the covariant derivative of φ is given by [17]*

$$\begin{aligned} 2g((\nabla_{\mathcal{X}_1}\varphi)\mathcal{Y}_1, \mathcal{Z}_1) &= 3d\Phi(\mathcal{X}_1, \varphi\mathcal{Y}_1, \varphi\mathcal{Z}_1) - 3d\Phi(\mathcal{X}_1, \mathcal{Y}_1, \mathcal{Z}_1) + g(N^1(\mathcal{Y}_1, \mathcal{Z}_1), \varphi\mathcal{X}_1) \\ &+ N^2(\mathcal{Y}_1, \mathcal{Z}_1)\eta(\mathcal{X}_1) + 2d\eta(\varphi\mathcal{Y}_1, \mathcal{X}_1)\eta(\mathcal{Z}_1) \\ &- 2d\eta(\varphi\mathcal{Z}_1, \mathcal{X}_1)\eta(\mathcal{Y}_1). \end{aligned}$$

2. Metallic Contact-Like Manifolds

Let $(M, \mathcal{P}, g, u, \tilde{\xi}, a)$ be a metallic contact-like manifold. Consider a product manifold $M \times \mathbb{R}$. A vector field on $M \times \mathbb{R}$ is given by $(\mathcal{X}_1, f\frac{d}{dt})$, where $\mathcal{X}_1$ is a tangent vector field on M and f, β is a C^∞ real-valued function on $M \times \mathbb{R}$. Let us define a linear map $\mathcal{J}$ in the tangent space of M.

$$\begin{aligned} \mathcal{J} : \chi(M \times \mathbb{R}) &\rightarrow \chi(M \times \mathbb{R}) \\ (\mathcal{X}_1, f\frac{d}{dt}) &\rightarrow \mathcal{J}(\mathcal{X}_1, f\frac{d}{dt}) = (\mathcal{P}\mathcal{X}_1 + f\tilde{\xi}, [\beta u(\mathcal{X}_1)]\frac{d}{dt}). \end{aligned} \tag{8}$$

Then, we applied $\mathcal{J}$ to both sides in (8), we obtain

$$\mathcal{J}^2(\mathcal{X}_1, f\frac{d}{dt}) = \mathcal{J}(\mathcal{P}\mathcal{X}_1 + f\tilde{\xi}, [\beta u(\mathcal{X}_1)]\frac{d}{dt}).$$

By using (2), we have

$$\mathcal{J}^2(\mathcal{X}_1, f\frac{d}{dt}) = (\mathcal{P}(\mathcal{P}\mathcal{X}_1 + f\tilde{\xi}) + \beta u(\mathcal{X}_1)\tilde{\xi}, [\beta u(\mathcal{P}\mathcal{X}_1 + f\tilde{\xi})]\frac{d}{dt}).$$

Then, using (2), (3), (4) and (5), we get

$$\mathcal{J}^2(\mathcal{X}_1, f\frac{d}{dt}) = (p\mathcal{P}\mathcal{X}_1 + q\mathcal{X}_1 - u(\mathcal{X}_1)\tilde{\xi} + f(p-a)\tilde{\xi} + \beta u(\mathcal{X}_1)\tilde{\xi}, [\beta(p-a)u(\mathcal{X}_1) + f\beta u(\tilde{\xi})]\frac{d}{dt}). \tag{9}$$

On the other hand, with the help of (1), we have

$$\begin{aligned} \mathcal{J}^2(\mathcal{X}_1, f\frac{d}{dt}) &= p\mathcal{J}(\mathcal{X}_1, f\frac{d}{dt}) + q(\mathcal{X}_1, f\frac{d}{dt}), \\ \mathcal{J}^2(\mathcal{X}_1, f\frac{d}{dt}) &= (p\mathcal{P}\mathcal{X}_1 + q\mathcal{X}_1 + pf\tilde{\xi}, (p\beta u(\mathcal{X}_1) + qf)\frac{d}{dt}). \end{aligned} \tag{10}$$

Equality of (9) and (10) we have,

$$\beta^2 u(\tilde{\xi}) - (2q + pa)\beta + q = 0. \tag{11}$$

The following equation is obtained from the solution of (11).

$$\beta_{1,2} = \frac{2q + pa \mp \sqrt{(2q + pa)^2 - 4u(\tilde{\xi})q}}{2u(\tilde{\xi})}.$$

Similarly, equality of (9) and (10) we have,

$$f = \frac{(\beta - 1)u(\tilde{\xi})}{a}.$$

In here if we choose $f = 0$, then $\beta = 1$ and in this case $a = p - \sigma$. As a result, the following equation provides equation (1).

$$\mathcal{J}(\mathcal{X}_1, f\frac{d}{dt}) = (\mathcal{P}\mathcal{X}_1 + \frac{\beta - 1}{a}u(\mathcal{X}_1)\tilde{\xi}, \beta u(\mathcal{X}_1)\frac{d}{dt}).$$

The linear map $\mathcal{J}$ defined in the tangent space of M is a metallic structure.
Consider the Nijenhuis tensor $N_{\mathcal{J}}$ of $\mathcal{J}$, if we compute $N_{\mathcal{J}}((\mathcal{X}_1,0),(\mathcal{Y}_1,0))$ and $N_{\mathcal{J}}((\mathcal{X}_1,0),(0,\frac{d}{dt}))$, we get

$$\begin{aligned} N_{\mathcal{J}}((\mathcal{X}_1,0\frac{d}{dt}),(\mathcal{Y}_1,0\frac{d}{dt})) &= \mathcal{J}^2[(\mathcal{X}_1,0\frac{d}{dt}),(\mathcal{Y}_1,0\frac{d}{dt})]+[\mathcal{J}(\mathcal{X}_1,0\frac{d}{dt}),\mathcal{J}(\mathcal{Y}_1,0\frac{d}{dt})] \\ &- \mathcal{J}[\mathcal{J}(\mathcal{X}_1,0\frac{d}{dt}),(\mathcal{Y}_1,0\frac{d}{dt})]-\mathcal{J}[(\mathcal{X}_1,0\frac{d}{dt}),\mathcal{J}(\mathcal{Y}_1,0\frac{d}{dt})]. \end{aligned} \tag{12}$$

On the other hand, we know,

$$[(\mathcal{X}_1,f\frac{d}{dt}),(\mathcal{Y}_1,g\frac{d}{dt})]=([\mathcal{X}_1,\mathcal{Y}_1],(\mathcal{X}_1(g)-\mathcal{Y}_1(f))\frac{d}{dt}). \tag{13}$$

Then, by using (8) and (13) in (12), we obtain

$$\begin{aligned} N_{\mathcal{J}}((\mathcal{X}_1,0\frac{d}{dt}),(\mathcal{Y}_1,0\frac{d}{dt})) &= (\{p\mathcal{P}[\mathcal{X}_1,\mathcal{Y}_1]+q[\mathcal{X}_1,\mathcal{Y}_1]+[\mathcal{P}\mathcal{X}_1,\mathcal{P}\mathcal{Y}_1] \\ &\quad -\mathcal{P}[\mathcal{P}\mathcal{X}_1,\mathcal{Y}_1]-\mathcal{P}[\mathcal{X}_1,\mathcal{P}\mathcal{Y}_1]+\mathcal{Y}_1(u(\mathcal{X}_1))\tilde{\xi}-\mathcal{X}_1(u(\mathcal{Y}_1))\tilde{\xi}\}, \\ &\quad \{\beta pu([\mathcal{X}_1,\mathcal{Y}_1])+\beta\mathcal{P}\mathcal{X}_1(u(\mathcal{Y}_1))-\beta\mathcal{P}\mathcal{Y}_1(u(\mathcal{X}_1))-\beta u([\mathcal{P}\mathcal{X}_1,\mathcal{Y}_1]) \\ &\quad -\beta u([\mathcal{X}_1,\mathcal{P}\mathcal{Y}_1])\frac{d}{dt}\}). \end{aligned} \tag{14}$$

On the other hand, we have the following equations.

$$\begin{aligned} N_{\mathcal{P}}(\mathcal{X}_1,\mathcal{Y}_1) &= \mathcal{P}^2[\mathcal{X}_1,\mathcal{Y}_1]+[\mathcal{P}\mathcal{X}_1,\mathcal{P}\mathcal{Y}_1]-\mathcal{P}[\mathcal{P}\mathcal{X}_1,\mathcal{Y}_1]-\mathcal{P}[\mathcal{X}_1,\mathcal{P}\mathcal{Y}_1], && (15)\\ (\mathcal{L}_{\mathcal{P}\mathcal{X}_1}u)\mathcal{Y}_1 &= \mathcal{P}\mathcal{X}_1(u(\mathcal{Y}_1))-u([\mathcal{P}\mathcal{X}_1,\mathcal{Y}_1]), && (16)\\ (\mathcal{L}_{\tilde{\xi}}u)\mathcal{X}_1 &= \tilde{\xi}(u(\mathcal{X}_1))-u([\tilde{\xi},\mathcal{X}_1]), && (17)\\ 2du(\mathcal{X}_1,\mathcal{Y}_1) &= \mathcal{X}_1(u(\mathcal{Y}_1))-\mathcal{Y}_1(u(\mathcal{X}_1))-u([\mathcal{X}_1,\mathcal{Y}_1]), && (18)\\ (\mathcal{L}_{\tilde{\xi}}\mathcal{P})\mathcal{X}_1 &= [\tilde{\xi},\mathcal{P}\mathcal{X}_1]-\mathcal{P}[\tilde{\xi},\mathcal{X}_1]. && (19) \end{aligned}$$

Then, by using (15), (16), (17), (18) and (19) into (14) we have,

$$\begin{aligned} [\mathcal{J},\mathcal{J}]((\mathcal{X}_1,0),(\mathcal{Y}_1,0)) &= (N_{\mathcal{P}}(\mathcal{X}_1,\mathcal{Y}_1)-2du(\mathcal{X}_1,\mathcal{Y}_1)\tilde{\xi},[\beta((L_{\mathcal{P}\mathcal{X}_1}u)\mathcal{Y}_1 \\ &- (\mathcal{L}_{\mathcal{P}\mathcal{Y}_1}u)\mathcal{X}_1)+\beta pu([\mathcal{X}_1,\mathcal{Y}_1])]\frac{d}{dt}), \end{aligned}$$

where

$$\begin{aligned} N^{(1)}(\mathcal{X}_1,\mathcal{Y}_1) &= N_{\mathcal{P}}(\mathcal{X}_1,\mathcal{Y}_1)-2du(\mathcal{X}_1,\mathcal{Y}_1)\tilde{\xi}, \\ N^{(2)}(\mathcal{X}_1,\mathcal{Y}_1) &= \beta((L_{\mathcal{P}\mathcal{X}_1}u)\mathcal{Y}_1-(\mathcal{L}_{\mathcal{P}\mathcal{Y}_1}u)\mathcal{X}_1)+\beta pu([\mathcal{X}_1,\mathcal{Y}_1])]\frac{d}{dt}). \end{aligned}$$

On the other hand, we have the following equation.

$$\begin{aligned} [\mathcal{J},\mathcal{J}]((\mathcal{X}_1,0),(0,\frac{d}{dt})) &= \mathcal{J}^2[(\mathcal{X}_1,0),(0,\frac{d}{dt})]+[\mathcal{J}(\mathcal{X}_1,0),\mathcal{J}(0,\frac{d}{dt})] \\ &- \mathcal{J}[\mathcal{J}(\mathcal{X}_1,0),(0,\frac{d}{dt})]-\mathcal{J}[(\mathcal{X}_1,0),\mathcal{J}(0,\frac{d}{dt})]. \end{aligned} \tag{20}$$

By using (13) in (20), we find

$$[\mathcal{J},\mathcal{J}]((\mathcal{X}_1,0),(0,\frac{d}{dt}))=([\mathcal{P}\mathcal{X}_1,\tilde{\xi}]-\mathcal{P}[\mathcal{X}_1,\tilde{\xi}],(-\beta\tilde{\xi}u(\mathcal{X}_1)-\beta u([\mathcal{X}_1,\tilde{\xi}]))\frac{d}{dt}). \tag{21}$$

In (21) if we get

$$\begin{aligned} N^{(3)} &= [\mathcal{P}\mathcal{X}_1, \tilde{\xi}] - \mathcal{P}[\mathcal{X}_1, \tilde{\xi}] = -(L_{\tilde{\xi}}\mathcal{P})\mathcal{X}_1, \\ N^{(4)} &= (-\beta\tilde{\xi}u(\mathcal{X}_1) - \beta u([\mathcal{X}_1, \tilde{\xi}])) = -\beta(L_{\tilde{\xi}}u)\mathcal{X}_1. \end{aligned} \tag{22}$$

Therefore we obtain the following theorem.

Theorem 2.1. *For an almost metallic contact structure $(\mathcal{P}, \tilde{\xi}, \eta)$, with condition*

$$2du(\mathcal{P}\mathcal{X}_1, \mathcal{P}\mathcal{Y}_1) = qu([\mathcal{X}_1, \mathcal{Y}_1]),$$

the vanishing of $N^{(1)}$ implies the vanishing of $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$.

Proof. We have the following equation.

$$N^{(1)}(\mathcal{X}_1, \mathcal{Y}_1) = N_{\mathcal{P}}(\mathcal{X}_1, \mathcal{Y}_1) - 2du(\mathcal{X}_1, \mathcal{Y}_1)\tilde{\xi} = 0. \tag{23}$$

Then, we get $\mathcal{Y}_1 = \tilde{\xi}$ and we use (15), (18) into (23), we have

$$\mathcal{P}^2[\mathcal{X}_1, \tilde{\xi}] + [\mathcal{P}\mathcal{X}_1, \mathcal{P}\tilde{\xi}] - \mathcal{P}[\mathcal{X}_1, \mathcal{P}\tilde{\xi}] - \mathcal{P}[\mathcal{P}\mathcal{X}_1, \tilde{\xi}] - (\mathcal{X}_1(u(\tilde{\xi})) - \tilde{\xi}(u(\mathcal{X}_1)) - u([\mathcal{X}_1, \tilde{\xi}]))\tilde{\xi} = 0. \tag{24}$$

Then using (2), (4), (5) and (17) into (24), we get

$$q[\mathcal{X}_1, \tilde{\xi}] + (p - a)[\mathcal{P}\mathcal{X}_1, \tilde{\xi}] - \mathcal{P}[\mathcal{P}\mathcal{X}_1, \tilde{\xi}] + a\mathcal{P}[\mathcal{X}_1, \tilde{\xi}] + \tilde{\xi}(u(\mathcal{X}_1))\tilde{\xi} = 0. \tag{25}$$

Then, we applied u to both sides in (25) and using (3), (4) we obtain

$$(L_{\tilde{\xi}}u)\mathcal{X}_1 = 0. \tag{26}$$

Similarly, following the same method, we obtain the following equation.

$$(L_{\tilde{\xi}}u)\mathcal{Y}_1 = 0. \tag{27}$$

Then, by using (26) and (27) into (22) we find $N^{(4)}(\mathcal{X}_1, \mathcal{Y}_1) = 0$. On the other hand, we take $\mathcal{X}_1 = \mathcal{P}\mathcal{X}_1$ into (26), we get

$$(L_{\tilde{\xi}}u)\mathcal{P}\mathcal{X}_1 = \tilde{\xi}(u(\mathcal{P}\mathcal{X}_1)) - u([\tilde{\xi}, \mathcal{P}\mathcal{X}_1]) = 0. \tag{28}$$

By using (3) into (28), we obtain

$$(p - a)u([\tilde{\xi}, \mathcal{X}_1]) - u([\tilde{\xi}, \mathcal{P}\mathcal{X}_1]) = 0. \tag{29}$$

Then, we applied $\mathcal{P}$ to both sides in (25) and using (2), (5) and (26), we get

$$(q + ap)\mathcal{P}[\mathcal{X}_1, \tilde{\xi}] + a\mathcal{P}[\tilde{\xi}, \mathcal{P}\mathcal{X}_1] - au([\mathcal{X}_1, \tilde{\xi}])\tilde{\xi} - q[\mathcal{P}\mathcal{X}_1, \tilde{\xi}] + q[\tilde{\xi}, \mathcal{P}\mathcal{X}_1] - aq[\tilde{\xi}, \mathcal{X}_1] = 0.$$

With the help of (19), (26) and (29), we have

$$g((L_{\tilde{\xi}}\mathcal{P})\mathcal{X}_1, \tilde{\xi}) = 0. \tag{30}$$

From (30), we obtain $(L_{\tilde{\xi}}\mathcal{P})\mathcal{X}_1 = 0$, where $(L_{\tilde{\xi}}\mathcal{P})\mathcal{X}_1$ is not perpendicular $\tilde{\xi}$.
Then we find $N^{(3)}(\mathcal{X}_1, \mathcal{Y}_1) = 0$. If we take $\mathcal{X}_1 = \mathcal{P}\mathcal{X}_1$ in (23), we get

$$\begin{aligned} &\mathcal{P}^2[\mathcal{P}\mathcal{X}_1, \mathcal{Y}_1] + [\mathcal{P}^2\mathcal{X}_1, \mathcal{P}\mathcal{Y}_1] - \mathcal{P}[\mathcal{P}\mathcal{X}_1, \mathcal{P}\mathcal{Y}_1] - \mathcal{P}[\mathcal{P}^2\mathcal{X}_1, \mathcal{Y}_1] \\ &-(\mathcal{P}\mathcal{X}_1(u(\mathcal{Y}_1)) - \mathcal{Y}_1(u(\mathcal{P}\mathcal{X}_1)) - u([\mathcal{P}\mathcal{X}_1, \mathcal{Y}_1]))\tilde{\xi} = 0. \end{aligned} \tag{31}$$

At the same time, by taking the necessary calculations and using (2), (16) into (31) we obtain,

$$\begin{aligned} &q[\mathcal{P}\mathcal{X}_1, \mathcal{Y}_1] - u([\mathcal{P}\mathcal{X}_1, \mathcal{Y}_1])\tilde{\xi} + p[\mathcal{P}\mathcal{X}_1, \mathcal{P}\mathcal{Y}_1] + q[\mathcal{X}_1, \mathcal{P}\mathcal{Y}_1] - q\mathcal{P}[\mathcal{X}_1, \mathcal{Y}_1] - \mathcal{P}[\mathcal{P}\mathcal{X}_1, \mathcal{P}\mathcal{Y}_1] \\ &-u(x)[\tilde{\xi}, \mathcal{P}\mathcal{Y}_1] + \mathcal{P}\mathcal{Y}_1(u(\mathcal{X}_1))\tilde{\xi} + u(\mathcal{X}_1)\mathcal{P}[\tilde{\xi}, \mathcal{Y}_1] - \mathcal{P}\mathcal{X}_1(u(\mathcal{Y}_1))\tilde{\xi} = 0. \end{aligned} \tag{32}$$

Then, we applied to u both sides in (32) and using (3), (4), (29) we derive

$$-q(L_{\mathcal{P}\mathcal{X}_1}u)\mathcal{Y}_1 + q(L_{\mathcal{P}\mathcal{Y}_1}u)\mathcal{X}_1 - u(\mathcal{X}_1)[u((L_{\tilde{\xi}}\mathcal{P})\mathcal{Y}_1)] - 2adu(\mathcal{P}\mathcal{X}_1, \mathcal{P}\mathcal{Y}_1) - qu(\mathcal{P}[\mathcal{X}_1, \mathcal{Y}_1]) = 0. \tag{33}$$

Similarly, we take $\mathcal{Y}_1 = \mathcal{P}\mathcal{Y}_1$ into (23), we obtain the following equation.

$$-q(L_{\mathcal{P}\mathcal{X}_1}u)\mathcal{Y}_1 + q(L_{\mathcal{P}\mathcal{Y}_1}u)\mathcal{X}_1 + u(\mathcal{Y}_1)[u((L_{\tilde{\xi}}\mathcal{P})\mathcal{X}_1)] - 2adu(\mathcal{P}\mathcal{X}_1, \mathcal{P}\mathcal{Y}_1) - qu(\mathcal{P}[\mathcal{X}_1, \mathcal{Y}_1]) = 0. \tag{34}$$

By using (33) and (34), we get

$$\frac{q}{\beta}N^{(2)}(\mathcal{X}_1, \mathcal{Y}_1) + 2adu(\mathcal{P}\mathcal{X}_1, \mathcal{P}\mathcal{Y}_1) - aqu([\mathcal{X}_1, \mathcal{Y}_1] = 0. \tag{35}$$

In this case, we obtain

$$N^{(2)}(\mathcal{X}_1, \mathcal{Y}_1) = 0$$

which completes the proof. □

The following theorem is clear from the previous theorem.

Theorem 2.2. *Let $(\mathcal{P}, \tilde{\xi}, u, g)$ be a metallic contact metric structure. Then the tensors $N^{(2)}$ and $N^{(4)}$ vanish. Moreover $N^{(3)}$ vanishes if and only if the characteristic vector field $\tilde{\xi}$ is Killing vector field with respect to g.*

Lemma 2.3. *For an almost metallic contact metric structure $(\mathcal{P}, \tilde{\xi}, u, g)$, the covariant derivative of $\mathcal{P}$ is given by*

$$\begin{aligned}
2g((\nabla_{\mathcal{X}_1}\mathcal{P})\mathcal{Y}_1, \mathcal{Z}_1) &= \frac{1}{q}3d\Phi(\mathcal{X}_1, \mathcal{P}\mathcal{Y}_1, \mathcal{P}\mathcal{Z}_1) - \frac{p}{q}3d\Phi(\mathcal{X}_1, \mathcal{Y}_1, \mathcal{P}\mathcal{Z}_1) - 3d\Phi(\mathcal{X}_1, \mathcal{Y}_1, \mathcal{Z}_1) \\
&+ \frac{1}{q}g(N^{(1)}(\mathcal{Y}_1, \mathcal{Z}_1), \mathcal{X}_1) + \frac{1}{q\beta}N^{(2)}(\mathcal{Y}_1, \mathcal{Z}_1)u(\mathcal{X}_1) + \frac{1}{q}2du(\mathcal{P}\mathcal{Y}_1, \mathcal{X}_1)u(\mathcal{Z}_1) \\
&+ \frac{1}{q}2du(\mathcal{P}\mathcal{Z}_1, \mathcal{X}_1)u(\mathcal{Y}_1) + 2\frac{p-a}{q}\mathcal{X}_1(u(\mathcal{Z}_1))u(\mathcal{Y}_1) + 2\frac{p-a}{q}u(\mathcal{Z}_1)\mathcal{X}_1(u(\mathcal{Y}_1)) \\
&- \frac{p}{q}\mathcal{P}\mathcal{Y}_1\Phi(\mathcal{X}_1, \mathcal{Z}_1) + \frac{2}{q}\Phi([\mathcal{X}_1, \mathcal{P}\mathcal{Y}_1], \mathcal{P}\mathcal{Z}_1) - \frac{p}{q}\Phi([\mathcal{X}_1, \mathcal{P}\mathcal{Y}_1], \mathcal{Z}_1) \\
&+ \frac{p}{q}\Phi([\mathcal{P}\mathcal{Y}_1, \mathcal{Z}_1], \mathcal{X}_1) - \frac{2}{q}u(\mathcal{X}_1)\mathcal{P}\mathcal{Z}_1(u(\mathcal{Y}_1)) - 2\Phi([\mathcal{X}_1, \mathcal{Y}_1], \mathcal{Z}_1) \\
&+ \frac{2}{q}\Phi([\mathcal{Y}_1, \mathcal{P}\mathcal{Z}_1], \mathcal{P}\mathcal{X}_1) + \frac{2}{q}u([\mathcal{Y}_1, \mathcal{P}\mathcal{Z}_1])u(\mathcal{X}_1) - 2\Phi([\mathcal{Y}_1, \mathcal{Z}_1], \mathcal{X}_1) \\
&- \frac{p}{q}\Phi([\mathcal{Y}_1, \mathcal{Z}_1], \mathcal{P}\mathcal{X}_1) + \frac{p-pq-a}{q}u([\mathcal{Y}_1, \mathcal{Z}_1])u(\mathcal{X}_1) + \frac{p-a}{q}2du(\mathcal{Y}_1, \mathcal{Z}_1)u(\mathcal{X}_1) \\
&+ \frac{p}{q}\mathcal{Y}_1\Phi(\mathcal{P}\mathcal{Z}_1, \mathcal{X}_1) - \frac{p}{q}\Phi([\mathcal{X}_1, \mathcal{Y}_1], \mathcal{P}\mathcal{Z}_1) - \frac{2p}{q}\Phi([\mathcal{Y}_1, \mathcal{P}\mathcal{Z}_1], \mathcal{X}_1).
\end{aligned}$$

Proof. Recall that the Levi-Civita connection ∇ of g is given by

$$\begin{aligned}
2g(\nabla_{\mathcal{X}_1}\mathcal{Y}_1, \mathcal{Z}_1) &= \mathcal{X}_1 g(\mathcal{Y}_1, \mathcal{Z}_1) + \mathcal{Y}_1 g(\mathcal{X}_1, \mathcal{Z}_1) - \mathcal{Z}_1 g(\mathcal{X}_1, \mathcal{Y}_1) \\
&+ g([\mathcal{X}_1, \mathcal{Y}_1], \mathcal{Z}_1) + g([\mathcal{Z}_1, \mathcal{X}_1], \mathcal{Y}_1) - g([\mathcal{Y}_1, \mathcal{Z}_1], \mathcal{X}_1)
\end{aligned}$$

and that coboundary formula for d on a 2-form Φ is

$$\begin{aligned}
d\Phi(\mathcal{X}_1, \mathcal{Y}_1, \mathcal{Z}_1) &= \frac{1}{3}\{\mathcal{X}_1\Phi(\mathcal{Y}_1, \mathcal{Z}_1) + \mathcal{Y}_1\Phi(\mathcal{X}_1, \mathcal{Z}_1) + \mathcal{Z}_1\Phi(\mathcal{X}_1, \mathcal{Y}_1) \\
&- \Phi([\mathcal{X}_1, \mathcal{Y}_1], \mathcal{Z}_1) - \Phi([\mathcal{Z}_1, \mathcal{X}_1], \mathcal{Y}_1) - \Phi([\mathcal{Y}_1, \mathcal{Z}_1], \mathcal{X}_1)\}.
\end{aligned}$$

Also from (7) and (2), (6) we find

$$\Phi(\mathcal{P}\mathcal{X}_1,\mathcal{P}\mathcal{Y}_1)=p\Phi(\mathcal{P}\mathcal{X}_1,\mathcal{Y}_1)+q\Phi(\mathcal{X}_1,\mathcal{Y}_1)-(p-a)u(\mathcal{X}_1)u(\mathcal{Y}_1), \tag{36}$$

$$\Phi(\mathcal{X}_1,\mathcal{P}\mathcal{Y}_1)=\Phi(\mathcal{P}\mathcal{X}_1,\mathcal{Y}_1). \tag{37}$$

Therefore we have

$$2g((\nabla_{\mathcal{X}_1}\mathcal{P})\mathcal{Y}_1,\mathcal{Z}_1) = 2g(\nabla_{\mathcal{X}_1}\mathcal{P}\mathcal{Y}_1,\mathcal{Z}_1)-2g(\nabla_{\mathcal{X}_1}\mathcal{Y}_1,\mathcal{P}\mathcal{Z}_1).$$

By using (36) and (37) in the above equation, we obtain

$$\begin{aligned}
2g((\nabla_{\mathcal{X}_1}\mathcal{P})\mathcal{Y}_1,\mathcal{Z}_1) &= \mathcal{X}_1\Phi(\mathcal{Y}_1,\mathcal{Z}_1)+\frac{1}{q}\Phi(\mathcal{P}\mathcal{X}_1,\mathcal{Z}_1)\mathcal{P}\mathcal{Y}_1-\frac{p}{q}\Phi(\mathcal{X}_1,\mathcal{Z}_1)\mathcal{P}\mathcal{Y}_1)\\
&+ \frac{1}{q}u(\mathcal{X}_1)u(\mathcal{Z}_1)\mathcal{P}\mathcal{Y}_1-\mathcal{Z}_1\Phi(\mathcal{X}_1,\mathcal{Y}_1)+\frac{1}{q}\Phi(\mathcal{P}[\mathcal{X}_1,\mathcal{P}\mathcal{Y}_1],\mathcal{Z}_1)\\
&- \frac{p}{q}\Phi([\mathcal{X}_1,\mathcal{P}\mathcal{Y}_1],\mathcal{Z}_1)+\frac{1}{q}u([\mathcal{X}_1,\mathcal{P}\mathcal{Y}_1])u(\mathcal{Z}_1)+\Phi([\mathcal{Z}_1,\mathcal{X}_1],\mathcal{Y}_1)\\
&- \frac{1}{q}\Phi(\mathcal{P}[\mathcal{P}\mathcal{Y}_1,\mathcal{Z}_1],\mathcal{X}_1)+\frac{p}{q}\Phi([\mathcal{P}\mathcal{Y}_1,\mathcal{Z}_1],\mathcal{X}_1)-\frac{1}{q}u([\mathcal{P}\mathcal{Y}_1,\mathcal{Z}_1])u(\mathcal{X}_1)\\
&+ \frac{1}{q}\mathcal{P}\mathcal{Z}_1\Phi(\mathcal{P}\mathcal{X}_1,\mathcal{Y}_1)-\frac{p}{q}\mathcal{P}\mathcal{Z}_1\Phi(\mathcal{X}_1,\mathcal{Y}_1)+\frac{1}{q}\mathcal{P}\mathcal{Z}_1u(\mathcal{X}_1)u(\mathcal{Y}_1)\\
&- \frac{1}{q}\Phi(\mathcal{P}[\mathcal{P}\mathcal{Z}_1,\mathcal{X}_1],\mathcal{Y}_1)+\frac{p}{q}\Phi([\mathcal{P}\mathcal{Z}_1,\mathcal{X}_1],\mathcal{Y}_1)-\frac{1}{q}u([\mathcal{P}\mathcal{Z}_1,\mathcal{X}_1])u(\mathcal{Y}_1)\\
&+ \frac{1}{q}\Phi(\mathcal{P}[\mathcal{Y}_1,\mathcal{P}\mathcal{Z}_1],\mathcal{X}_1)-\frac{p}{q}\Phi([\mathcal{Y}_1,\mathcal{P}\mathcal{Z}_1],\mathcal{X}_1)+\frac{1}{q}u([\mathcal{Y}_1,\mathcal{P}\mathcal{Z}_1])u(\mathcal{X}_1)\\
&- \mathcal{X}_1\Phi(\mathcal{Y}_1,\mathcal{Z}_1)-\mathcal{Y}_1\Phi(\mathcal{X}_1,\mathcal{Z}_1)-\Phi([\mathcal{X}_1,\mathcal{Y}_1],\mathcal{Z}_1).
\end{aligned}$$

By straightforward direct computations, we find desired equation and the proof is completed. □

Example. Let $\mathcal{J}:E^{k+l}\to E^{k+l}$ be a $(1,1)$ tensor field described for any point $(x^1,\cdots,x^k,y^1,\cdots,y^l)\in E^{k+l}$ by:

$$\mathcal{J}((x^1,\cdots,x^k,y^1,\cdots,y^l)=(\sigma_{p,q}x^1,\cdots,\sigma_{p,q}x^k,(p-\sigma_{p,q})y^1,\cdots,(p-\sigma_{p,q})y^l)$$

It follows that $\mathcal{J}^2=p\mathcal{J}+qI$ and the scalar product $<,>$ on E^{k+l} is $\mathcal{J}-$compatible. Thus $(E^{k+l},<,>,\mathcal{J})$ is a metallic Riemannian manifold.
In E^{k+l} we can get the hypersphere

$$S^{k+l-1}(r)=\{(x^1,\cdots,x^k,y^1,\cdots,y^l):\sum_{i=1}^{k}(x^i)^2+\sum_{j=1}^{l}(y^j)^2=r_1^2+r_2^2=r^2\},$$

which is a submanifold of codimension 1 in E^{k+l}.

$$A=\frac{\sigma_{p,q}r_1^2+(p-\sigma_{p,q})r_2^2}{r^2}$$

$$\tilde{\xi}=\frac{2\sigma_{p,q}-p}{r^3}(r_2^2x^i,-r_1^2y^j)$$

$$u(\mathcal{X}_1)=\frac{2\sigma_{p,q}-p}{r}\mu$$

$$\mathcal{P}(\mathcal{X}_1)=(\sigma_{p,q}\mathcal{X}^i-\frac{2\sigma_{p,q}-p}{r^2}\mu x^i,(p-\sigma_{p,q})\mathcal{Y}^j-\frac{2\sigma_{p,q}-p}{r^2}\mu y^j)$$

where $\mathcal{X}_1 = (\mathcal{X}^i, \mathcal{Y}^j) = (\mathcal{X}^1, \cdots \mathcal{X}^k, \mathcal{Y}^1, \cdots \mathcal{Y}^l)$ is a tangent vector field on $S^{k+l-1}(r)$ and $\mu = \sum_{i=1}^k x^i \mathcal{X}^i = -\sum_{j=1}^l y^j \mathcal{Y}^j$, we get

$$\begin{aligned}
\mathcal{J}((\mathcal{X}^i, \mathcal{Y}^j), f\frac{d}{dt}) &= (\mathcal{P}(\mathcal{X}^i, \mathcal{Y}^j) + \frac{(\beta-1)u(\tilde{\xi})}{a}\tilde{\xi}, \beta u(\mathcal{X}^i, \mathcal{Y}^j)\frac{d}{dt}) \\
&= ((\sigma_{p,q}\mathcal{X}^i - \frac{2\sigma_{p,q}-p}{r^2}\mu x^i, (p-\sigma_{p,q})\mathcal{Y}^j - \frac{2\sigma_{p,q}-p}{r^2}\mu y^j) \\
&+ \frac{(\beta-1)u(\tilde{\xi})}{a}.\frac{2\sigma_{p,q}-p}{r^3}(r_2^2 x^i, -r_1^2 y^j), \beta\frac{2\sigma_{p,q}-p}{r}\mu\frac{d}{dt}) \\
&= ((\sigma_{p,q}\mathcal{X}^i - \frac{2\sigma_{p,q}-p}{r^2}(\mu - \frac{(\beta-1)u(\tilde{\xi})}{ar}r_2^2)x^i, \\
&\quad (p-\sigma_{p,q})\mathcal{Y}^j - \frac{2\sigma_{p,q}-p}{r^2}(\mu + \frac{(\beta-1)u(\tilde{\xi})}{ar}r_1^2)y^j), \\
&\quad \beta\frac{2\sigma_{p,q}-p}{r}\mu\frac{d}{dt}).
\end{aligned}$$

If we calculate $\mathcal{J}^2$, then we have the following equation.

$$\begin{aligned}
\mathcal{J}^2((\mathcal{X}^i, \mathcal{Y}^j), f\frac{d}{dt}) &= \mathcal{J}(((\sigma_{p,q}\mathcal{X}^i - \frac{2\sigma_{p,q}-p}{r^2}(\mu - \frac{(\beta-1)u(\tilde{\xi})}{ar}r_2^2)x^i, \\
&\quad (p-\sigma_{p,q})\mathcal{Y}^j - \frac{2\sigma_{p,q}-p}{r^2}(\mu + \frac{(\beta-1)u(\tilde{\xi})}{ar}r_1^2)y^j), \\
&\quad \beta\frac{2\sigma_{p,q}-p}{r}\mu\frac{d}{dt})) \\
&= (((\sigma_{p,q})^2\mathcal{X}^i + (\frac{2\sigma_{p,q}-p}{r^2})^2\mu(\mu - (\frac{(\beta-1)u(\tilde{\xi})}{ar} - \beta)r_2^2)x^i, \\
&\quad (p-\sigma_{p,q})^2\mathcal{Y}^j + (\frac{2\sigma_{p,q}-p}{r^2})^2\mu(\mu + (\frac{(\beta-1)u(\tilde{\xi})}{ar} - \beta)r_1^2)y^j), \\
&\quad \beta\frac{2\sigma_{p,q}-p}{r}\mu\frac{d}{dt}).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\mathcal{J}^2((\mathcal{X}^i, \mathcal{Y}^j), f\frac{d}{dt}) &= p\mathcal{J}((\mathcal{X}^i, \mathcal{Y}^j), f\frac{d}{dt}) + q((\mathcal{X}^i, \mathcal{Y}^j), f\frac{d}{dt}) \\
&= p((\sigma_{p,q}\mathcal{X}^i - \frac{2\sigma_{p,q}-p}{r^2}(\mu - \frac{(\beta-1)u(\tilde{\xi})}{ar}r_2^2)x^i, \\
&\quad (p-\sigma_{p,q})\mathcal{Y}^j - \frac{2\sigma_{p,q}-p}{r^2}(\mu + \frac{(\beta-1)u(\tilde{\xi})}{ar}r_1^2)y^j), \\
&\quad \beta\frac{2\sigma_{p,q}-p}{r}\mu\frac{d}{dt}) + q((\mathcal{X}^i, \mathcal{Y}^j), f\frac{d}{dt}) \\
&= (((p\sigma_{p,q}+q)\mathcal{X}^i - p\frac{2\sigma_{p,q}-p}{r^2}(\mu - \frac{(\beta-1)u(\tilde{\xi})}{ar}r_2^2)x^i, \\
&\quad (p(p-\sigma_{p,q})+q)\mathcal{Y}^j - p\frac{2\sigma_{p,q}-p}{r^2}(\mu + \frac{(\beta-1)u(\tilde{\xi})}{ar}r_1^2)y^j), \\
&\quad (\beta p\frac{2\sigma_{p,q}-p}{r}\mu + q\frac{(\beta-1)u(\tilde{\xi})}{a})\frac{d}{dt}).
\end{aligned}$$

Thus, we find

$$u(\tilde{\xi}) = \frac{(2\sigma_{p,q} - p)\mu^2 + pr^2\mu + (2\sigma - p)r_2^2)ar}{(pr_2^2r^2 + (2\sigma_{p,q} - p)r_2^2)(\beta - 1)}$$

or

$$u(\tilde{\xi}) = \frac{(-(2\sigma_{p,q} - p)\mu^2 - pr^2\mu + \beta\mu(2\sigma - p)r_1^2)ar}{(pr_1^2r^2 + \mu(2\sigma_{p,q} - p)r_1^2)(\beta - 1)}.$$

Thus, the transition map defined from metallic Riemannian manifold E^{k+l} to its tangent bundle of hypersurface $S^{k+l-1}(r)$ is obtained.

3. Conclusions

In this paper, we have introduced a new geometric structure called a metallic contact-like manifold, inspired by the analogy between the contact structure on classical contact manifolds and the structure induced on the tangent bundle of a real hypersurface of an almost Metallic Riemannian manifold [2], [5], [17]. The integrability and normality conditions of this new structure were explicitly derived, and necessary and sufficient conditions ensuring its well-definedness were obtained. Furthermore, the theoretical framework was illustrated through a concrete example constructed within a metallic Riemannian manifold [5].

The proposed structure provides a natural generalization that connects metallic Riemannian geometry [5], [12], [15] with classical contact and symplectic geometry [2], [9]. In particular, it extends ideas originally developed by Sasaki and Hatakeyama [17] for almost contact manifolds to the metallic context, thereby enriching the theory of polynomial-type geometries such as golden and silver structures [4], [14].

While the results establish a consistent theoretical framework for metallic contact-like manifolds, further investigations may focus on curvature properties, invariant submanifolds, and possible applications in mathematical physics and mechanics, following the geometric approaches discussed in [8], [9]. Such studies could deepen the understanding of how metallic structures interact with contact and symplectic frameworks in differential geometry.

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