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Another generalized angle related to norm derivatives in Banach spaces

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Abstract. Motivated by recent work on generalized angles in Banach spaces, we introduce a new angle $A_\rho(x, y)$ based on norm derivatives for two nonzero vectors x and y. We first study its elementary properties and then use it to characterize strict convexity in Banach spaces. Furthermore, we define two new geometric constants, $D_\rho^B(X)$ and $D_g^B(X)$, induced from norm derivatives. We explore the relationship between $D_\rho^B(X)$ and existing geometric constants, as well as properties like non-squareness and uniform convexity. A characterization of the Radon plane with an affine-regular hexagonal unit sphere is given in terms of $D_\rho^B(X)$. Finally, we establish some estimates for $D_g^B(X)$ and connect it to non-squareness.

1. Introduction

Throughout the paper, assume that $(X, \|\cdot\|)$ is a real Banach space of dimension at least two. Let B_X and S_X be the closed unit ball and the unit sphere of X, respectively.

Suppose that the norm of X derives from an inner product $\langle\cdot,\cdot\rangle$. Then there exists a natural orthogonality relationship defined by

$$x \perp y \iff \langle x, y\rangle = 0,$$

for $x, y \in X$. In general Banach spaces, there are several notions of orthogonality, among which one of the most distinguished is Birkhoff orthogonality introduced by Birkhoff [4]:

$$x \perp_B y \iff \|x + \lambda y\| \geq \|x\|, \quad \forall \lambda \in \mathbb{R}.$$

Also, for given $x, y \in X$, the isosceles orthogonality in X (see [10]) is defined by

$$x \perp_I y \iff \|x + y\| = \|x - y\|.$$

Another possible concept of orthogonality is related to norm derivatives. Amir [2] defined these derivatives as follows:

$$\rho_\pm(x, y) = \lim_{t\to 0^\pm} \frac{\|x + ty\|^2 - \|x\|^2}{2t} = \|x\| \lim_{t\to 0^\pm} \frac{\|x + ty\| - \|x\|}{t}.$$

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The convexity of the norm ensures that the above definitions are meaningful. Recently, some scholars have begun to study orthogonality relations based on norm derivatives. Miličić [17] proposed a mapping $\langle \cdot, \cdot \rangle_g : X \times X \to \mathbb{R}$ defined as follows:

$$\rho(x, y) = \langle y, x \rangle_g = \frac{\rho_-(x, y) + \rho_+(x, y)}{2}.$$

These functionals are generalized forms of inner products in Hilbert spaces.

In a Hilbert space X, the equality

$$\|x + y\|^4 - \|x - y\|^4 = 8(\|x\|^2 \langle x, y \rangle + \|y\|^2 \langle y, x \rangle) \quad (x, y \in X) \tag{1}$$

holds, which is equivalent to the parallelogram equality

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2) \quad (x, y \in X).$$

It is evident that the equality

$$\|x + y\|^4 - \|x - y\|^4 = 8(\|x\|^2 \rho(x, y) + \|y\|^2 \rho(y, x)) \quad (x, y \in X) \tag{2}$$

is a generalization of the equality (1) in Banach spaces. The author in [18] refers to a space satisfying equality (2) as a quasi-inner-product space and shows that such a space is smooth. Moreover, $\rho(x, y)$ may differ from $\rho(y, x)$ for some elements x and y in the Banach space X, as demonstrated by the following example:

Example 1.1. *Consider the Banach space $X = \mathbb{R}^3$ endowed with the norm:*

$$\|(x_1, x_2, x_3)\| = |x_1| + |x_2| + |x_3|.$$

If $x = (1, 0, 0)$ and $y = (1, 1, 0)$, then $\rho(x, y) \neq \rho(y, x)$.

Proof. It is easy for us to obtain

$$\rho_-(x, y) = \|x\| \lim_{t \to 0^-} \frac{\|x + ty\| - \|x\|}{t} = \lim_{t \to 0^-} \frac{1 - 1}{t} = 0,$$

$$\rho_+(x, y) = \|x\| \lim_{t \to 0^+} \frac{\|x + ty\| - \|x\|}{t} = \lim_{t \to 0^-} \frac{1 + 2t - 1}{t} = 2.$$

Hence we derive

$$\rho(x, y) = \frac{\rho_-(x, y) + \rho_+(x, y)}{2} = 1.$$

Similarly, it is not hard to compute

$$\rho_-(y, x) = 2, \quad \rho_+(y, x) = 2,$$

then $\rho(y, x) = 2$. Thus we have $\rho(x, y) = 1 \neq 2 = \rho(y, x)$. □

It is well known that in Hilbert spaces $(X, \langle \cdot, \cdot \rangle)$, the angle between two nonzero vectors x and y is defined as $A(x, y) = \arccos \frac{\langle x, y \rangle}{\|x\| \cdot \|y\|}$, where $\|x\| = \langle x, x \rangle^{\frac{1}{2}}$ denotes the induced norm on X. Note that $A(x, y) = \frac{\pi}{2}$ if and only if $x \perp y$. Furthermore, for a real Banach space X, Miličić [19] considered the following two types of angles between two nonzero elements x and y:

$$\underset{g}{\angle}(x, y) = \arccos \frac{\rho(x, y) + \rho(y, x)}{2\|x\|\|y\|},$$

$$\angle_g(x, y) = \arccos \frac{\rho(x, y)}{\|x\|\|y\|}.$$

In 2007, Miličić [20] introduced the concept of the ρ-angle between two nonzero vectors x and y in a smooth Banach space X as follows

$$\angle_\rho(x,y) = \arccos \frac{\|x\|^2\rho(x,y) + \|y\|^2\rho(y,x)}{\|x\|\|y\|(\|x\|^2 + \|y\|^2)}.$$

In 2019, another notion of g-angle between two nonzero vectors x and y in a Banach space X was considered in [23]. Denoted by $A_g(x,y)$, it is defined by the formula

$$A_g(x,y) = \arccos \frac{\rho(y,x)}{\|x\| \cdot \|y\|}.$$

Example 1.1 shows that $A_g(x,y)$ may differ from $\angle g(x,y)$. Note that $A_g(x,y) = \frac{\pi}{2}$ if and only if $y \perp_\rho x$. In fact, these angle concepts in Banach spaces coincide with the usual angle in Hilbert spaces.

Based on equality (2) and the concept of ρ-angle, we define the A_ρ-angle between two nonzero elements x and y in Banach spaces as follows:

$$A_\rho(x,y) = \arccos \frac{\|x+y\|^4 - \|x-y\|^4}{8\|x\|\|y\|(\|x\|^2 + \|y\|^2)}.$$

Note that $A_\rho(x,y) = \frac{\pi}{2}$ if and only if $x \perp_I y$. This angle generalizes the usual angle in Hilbert spaces and coincides with $\angle_\rho(x,y)$ in quasi-inner-product spaces.

Recently, quantitative studies of the difference between Birkhoff orthogonality and isosceles orthogonality have been carried out (see [1, 12, 24]):

$$D(X) = \inf\left\{\inf_{\lambda\in\mathbb{R}} \|x + \lambda y\| : x, y \in S_X, x \perp_I y\right\},$$

$$D'(X) = \sup\left\{\|x+y\| - \|x-y\| : x, y \in S_X, x \perp_B y\right\}.$$

On account of the notion of $A_\rho(x,y)$, we introduce the following geometric constant:

$$\begin{aligned} D_\rho^B(X) &= \sup\left\{\cos A_\rho(x,y) : x, y \in S_X, x \perp_B y\right\} \\ &= \sup\left\{\frac{\|x+y\|^4 - \|x-y\|^4}{16} : x, y \in S_X, x \perp_B y\right\}. \end{aligned}$$

The constant $D_\rho^B(X)$ represents the supremum of $\cos \angle_\rho(x,y)$ over right triangles. Furthermore, since $\angle_\rho(x,y) = \frac{\pi}{2}$ when $x \perp_I y$, this constant measures the discrepancy between Birkhoff orthogonality and isosceles orthogonality.

Using the angle $\angle_g(x,y)$, we define another constant to quantify the difference between Birkhoff orthogonality and ρ-orthogonality. This constant also corresponds to the supremum of $\cos \angle_g(x,y)$ over right triangles:

$$\begin{aligned} D_{g,B}(X) &= \sup\left\{\cos \angle_g(x,y) : x, y \in S_X, x \perp_B y\right\} \\ &= \sup\left\{\rho(x,y) : x, y \in S_X, x \perp_B y\right\}. \end{aligned}$$

In [6], Ghosh et al. introduced the parameter $\Gamma(X)$ and investigated several of its key geometric properties:

$$\Gamma(X) = \sup\left\{|\rho(x,y)| : x, y \in S_X, x \perp_B y\right\}.$$

Directly from the definition, we have $D_{g,B}(X) = \Gamma(X)$.

However, Example 1.1 shows that the angle $A_g(x,y)$ may be different from $\angle_g(x,y)$. Based on this, we introduce the following geometric constant:

$$\begin{aligned} D_g^B(X) &= \sup\left\{\cos A_g(x,y) : x, y \in S_X, x \perp_B y\right\} \\ &= \sup\left\{\rho(y,x) : x, y \in S_X, x \perp_B y\right\}. \end{aligned}$$

This parameter represents the supremum of $\cos A_g(x,y)$ over right triangles.

This paper is organized as follows:

In Section 2, we discuss elementary properties of the angle $A_\rho(x,y)$ and apply it to characterize strict convexity in Banach spaces.

In Section 3, we study the geometric constant $D_\rho^B(X)$ related to $A_\rho(x,y)$. First, we examine the relationship between $D_\rho^B(X)$ and existing geometric constants. We then investigate the connection between $D_\rho^B(X)$ and non-squareness. Furthermore, we relate $D_\rho^B(X)$ to $D'(X)$ in Minkowski spaces and study $D_\rho^B(X)$ in Radon planes. We use the upper bound of $D_\rho^B(X)$ to characterize Radon planes with affine-regular hexagonal unit spheres.

In Section 4, we introduce the constant $D_g^B(X)$ related to $A_g(x,y)$. We discuss its relation to existing constants, show that $D_g^B(X)$ takes a fixed value on smooth Radon planes, and compute its values in specific spaces. Finally, we establish the connection between $D_g^B(X)$ and non-squareness.

In Section 5, we summarize the results derived in this paper.

2. The angle $A_\rho(x,y)$

In this section, we present some basic properties of the angle $A_\rho(x,y)$ and one of its applications.

2.1. Some basic properties of the angle $A_\rho(x,y)$

The following proposition describes the properties of the angle $A_\rho(x,y)$ based on the idea and techniques from [8, Fact 2.6].

Proposition 2.1. *The ρ-angle $A_\rho(\cdot,\cdot)$ satisfies the following properties:*

(1) If x and y are of the same direction, then $A_\rho(x,y)=0$; if x and y are of the opposite direction, then $A_\rho(x,y)=\pi$ (part of parallelism property);

(2) $A_\rho(x,y)=A_\rho(y,x)$ for each $x,y\in X\setminus\{0\}$ (symmetry property);

(3) $A_\rho(kx,ky)=A_\rho(x,y)$ and $A_\rho(kx,-ky)=\pi-A_\rho(x,y)$ for each $x,y\in X\setminus\{0\}$ and $k\in\mathbb{R}\setminus\{0\}$ (part of homogeneity property);

(4) If $x_n\to x$ and $y_n\to y$ (in norm), then $A_\rho(x_n,y_n)\to A_\rho(x,y)$ (continuity property).

Proof. (1) Assume that x and y are two non-zero vectors with $y=kx$. Then we derive

$$\begin{aligned}
A_\rho(x,y) &= \arccos\frac{\|x+y\|^4-\|x-y\|^4}{8(\|x\|^2+\|y\|^2)\|x\|\|y\|}\\
&= \arccos\frac{\|x+kx\|^4-\|x-kx\|^4}{8(\|x\|^2+\|kx\|^2)\|x\|\|kx\|}\\
&= \arccos\frac{((1+k)^4-(1-k)^4)\|x\|^4}{8(1+k^2)|k|\|x\|^4}\\
&= \arccos\frac{8(1+k^2)k}{8(1+k^2)|k|}.
\end{aligned}$$

Hence if x and y are of the same direction (i.e., $k>0$), then $A_\rho(x,y)=\arccos(1)=0$; if x and y are of the opposite direction (i.e., $k<0$), then $A_\rho(x,y)=\arccos(-1)=\pi$.

(2) Suppose that x and y are two non-zero elements. Then we have

$$\begin{aligned}
A_\rho(x,y) &= \arccos\frac{\|x+y\|^4-\|x-y\|^4}{8(\|x\|^2+\|y\|^2)\|x\|\|y\|}\\
&= \arccos\frac{\|y+x\|^4-\|y-x\|^4}{8(\|y\|^2+\|x\|^2)\|y\|\|x\|}\\
&= A_\rho(y,x).
\end{aligned}$$

(3) Assume that x and y are two non-zero vectors. Then we derive

$$\begin{aligned}A_\rho(kx,ky) &= \arccos \frac{\|kx+ky\|^4-\|kx-ky\|^4}{8(\|kx\|^2+\|ky\|^2)\|kx\|\|ky\|}\\ &= \arccos \frac{k^4(\|x+y\|^4-\|x-y\|^4)}{8k^4(\|x\|^2+\|y\|^2)\|x\|\|y\|}\\ &= \arccos \frac{\|x+y\|^4-\|x-y\|^4}{8(\|x\|^2+\|y\|^2)\|x\|\|y\|}\\ &= A_\rho(x,y)\end{aligned}$$

and

$$\begin{aligned}A_\rho(kx,-ky) &= \arccos \frac{\|kx-ky\|^4-\|kx+ky\|^4}{8(\|kx\|^2+\|-ky\|^2)\|kx\|\|-ky\|}\\ &= \arccos \frac{-k^4(\|x+y\|^4-\|x-y\|^4)}{8k^4(\|x\|^2+\|y\|^2)\|x\|\|y\|}\\ &= \arccos(-\frac{\|x+y\|^4-\|x-y\|^4}{8(\|x\|^2+\|y\|^2)\|x\|\|y\|})\\ &= \pi - A_\rho(x,y).\end{aligned}$$

(4) The continuity of $A_\rho(x,y)$ follows from the continuity of the norm and arccos. □

In Euclidean spaces, the base angles of an isosceles triangle are equal, and each angle of an equilateral triangle is $\frac{\pi}{3}$. We investigate whether these properties extend to the angle $A_\rho(x,y)$ in Banach spaces. The following two propositions, based on techniques from [20, Theorems 6 and 7], address this question.

Proposition 2.2. *Let X be a Banach space and x,y be two non-zero linearly independent elements in X with $\left(y-\frac{1}{2}x\right)\perp_I \frac{3}{2}x$. If $\|y-x\|=\|y\|$, then $A_\rho(x,y)=A_\rho(-x,y-x)$.*

Proof. Assume that $\left(y-\frac{1}{2}x\right)\perp_I \frac{3}{2}x$. Then we have

$$\|y+x\| = \left\|\left(y-\frac{1}{2}x\right)+\frac{3}{2}x\right\| = \left\|\left(y-\frac{1}{2}x\right)-\frac{3}{2}x\right\| = \|y-2x\|.$$

Thus it follows that

$$\begin{aligned}\cos A_\rho(x,y) &= \frac{\|x+y\|^4-\|x-y\|^4}{8(\|x\|^2+\|y\|^2)\|x\|\|y\|}\\ &= \frac{\|y-2x\|^4-\|y\|^4}{8(\|x\|^2+\|y-x\|^2)\|x\|\|y-x\|}\\ &= \frac{\|-x+(y-x)\|^4-\|-x-(y-x)\|^4}{8(\|-x\|^2+\|y-x\|^2)\|(-x)\|\|y-x\|}\\ &= \cos A_\rho(-x,y-x),\end{aligned}$$

which implies that the desired conclusion. □

Corollary 2.3. *Let X be a Hilbert space and x,y be two non-zero linearly independent vectors in X with $\|y-x\|=\|y\|$. Then $A(x,y)=A(-x,y-x)$.*

Proof. Suppose that x,y are two non-zero linearly independent elements with $\|y-x\|=\|y\|$ in the Hilbert space. Then we have

$$\|y\|^2-2\langle x,\ y\rangle+\|x\|^2=\|y-x\|^2=\|y\|^2,$$

which implies that $\|x\|^2 = 2\langle x,\ y\rangle$. On the other hand, it is easily seen that

$$\begin{aligned}\left\|\left(y-\frac{1}{2}x\right)-\frac{3}{2}x\right\|^2 &= \|y-2x\|^2 = \|(y-x)-x\|^2\\ &= \|y-x\|^2 - 2\langle y-x,\ x\rangle + \|x\|^2\\ &= \|y\|^2 - 2\langle y,\ x\rangle + 2\|x\|^2 + \|x\|^2\\ &= \|y\|^2 + 2\|x\|^2 = \|y\|^2 + 2\langle x,\ y\rangle + \|x\|^2\\ &= \|y+x\|^2 = \left\|\left(y-\frac{1}{2}x\right)+\frac{3}{2}x\right\|^2,\end{aligned}$$

which means $\left(y-\frac{1}{2}x\right) \perp_I \frac{3}{2}x$. Thus it follows from the above proposition that the desired conclusion holds. □

Remark 2.4. *It is natural to wonder whether the condition $\left(y-\frac{1}{2}x\right) \perp_I \frac{3}{2}x$ is unnecessary for Proposition 2.2. As a matter of fact, the answer is negative. For instance, consider $X = (\mathbb{R}^2, \|\cdot\|_1)$. Taking $x = (1,1)$ and $y = (0,2)$. It is evident that $\|y-2x\| = 2 \neq 4 = \|y+x\|$, which implies that $\left(y-\frac{1}{2}x\right) \perp_I \frac{3}{2}x$ does not hold. However, it is easily seen that $A_\rho(x,y) = \arccos\frac{15}{16} \neq \frac{\pi}{2} = \arccos 0 = A_\rho(-x, y-x)$.*

Proposition 2.5. *Let X be a Banach space with $\|x+y\|^2 = \|x\|^2 + \|y\|^2 + \|x-y\|^2$, where x and y are two non-zero linearly independent elements in X. If $\|x\| = \|y\| = \|x-y\|$, i.e., let the triangle $(0,x,y)$ be equilateral. Then $A_\rho(x,y) = \frac{\pi}{3}$.*

Proof. For any two non-zero linearly independent vectors x and y in X, we can derive

$$\begin{aligned}\cos\angle_\rho(x,y) &= \frac{\|x+y\|^4 - \|x-y\|^4}{8(\|x\|^2+\|y\|^2)\|x\|\|y\|}\\ &= \frac{9\|x\|^4 - \|x\|^4}{16\|x\|^4}\\ &= \frac{1}{2},\end{aligned}$$

which implies the desired result. □

Corollary 2.6. *Let X be a Hilbert space and x, y be two non-zero linearly independent vectors in X. If $\|x\| = \|y\| = \|x-y\|$, then $A(x,y) = \frac{\pi}{3}$.*

Proof. Suppose that x, y are two non-zero linearly independent elements such that $\|x\| = \|y\| = \|x-y\|$, then it follows from the parallelogram law in Hilbert space that we have

$$\|x+y\|^2 = 2(\|x\|^2 + \|y\|^2) - \|x-y\|^2 = 3\|x\|^2 = \|x\|^2 + \|y\|^2 + \|x-y\|^2.$$

Hence we get the desired conclusion by the above proposition. □

Remark 2.7. *It is natural to ask whether the term $\|x+y\|^2 = \|x\|^2 + \|y\|^2 + \|x-y\|^2$ is not needed for Proposition 2.5. In fact, the answer is negative. For example, consider $X = (\mathbb{R}^2, \|\cdot\|_\infty)$. Let $x = (1,1)$ and $y = (1,0)$. Then we have $\|x\| = \|y\| = \|x-y\| = 1$. And it is obvious that $\|x+y\|^2 = 4 \neq 3 = \|x\|^2 + \|y\|^2 + \|x-y\|^2$. Also we can derive that $A_\rho(x,y) = \arccos\frac{15}{16} \neq \frac{\pi}{3}$.*

2.1.1. An application of the angle $A_\rho(x,y)$

It is well known that the strict convexity of a Banach space X is essential for the uniqueness of the best approximation element in all bounded closed convex and nonempty set $A \subset X$ for every $x \in X \setminus A$. Namely, if X is a strictly convex Banach space, A is a bounded closed convex and nonempty set in X and $x \in X \setminus A$ and if $y \in A$ is satisfied with

$$\|x-y\| = d(x,A) = \inf\{\|x-z\| : z \in A\},$$

then for any $z \in A$, $z \neq y$, we have $d(x,A) < \|x-z\|$. Assume that X is not strictly convex, then there exists a bounded closed convex and nonempty set $A \subset X$ and $x \in X \setminus A$ such that there exists a continuum of points in A that attains the distance $d(x,A)$.

By use of the ρ-angle, we can check the strict convexity of a Banach space $(X, \|\cdot\|)$. First, we recall the definition of strict convexity as follows.

Definition 2.8. *([7]) Let X be a Banach space. Then X is strictly convex if whenever $\|x\| + \|y\| = \|x+y\|$, where x and y are two nonzero elements, we have $y = \lambda x$ for some real number $\lambda > 0$.*

Now, we give the relationship between the ρ-angle and strict convexity based on the idea of [22, Theorem 3.3].

Theorem 2.9. *Let X be a Banach space. Then the following statements are equivalent:*
(1) X is strictly convex.
(2) If $\cos A_\rho(x,y) = 1$, where $x, y \neq 0$, then $y = \lambda x$ for some real number $\lambda > 0$.

Proof. Assume that (1) holds and that $\cos A_\rho(x,y) = 1$. Then

$$\|x+y\|^4 - \|x-y\|^4 = 8(\|x\|^2 + \|y\|^2)\|x\|\|y\| = (\|x\| + \|y\|)^4 - (\|x\| - \|y\|)^4,$$

which implies that $\|x+y\| = \|x\| + \|y\|$. Thus we have $y = \lambda x$ for some real $\lambda > 0$ by Definition 2.8.

On the other hand, suppose that (2) holds but (1) does not hold. Since X is not strictly convex, then there exist $x, y \in S_X$ such that $x \neq y$, for any $\lambda \in (0,1)$, we have $\|\lambda x + (1-\lambda)y\| = 1$. In particular, we can derive $\|x+y\| = 2$ and

$$\|x+2y\| = 3\left\|\frac{1}{3}x + \frac{2}{3}y\right\| = 3.$$

It is easy for us to obtain $\|x+2y\| = \|x+y\| + \|y\|$. And we also have

$$\begin{aligned}\cos A_\rho(x+y,y) &= \frac{\|(x+y)+y\|^4 - \|(x+y)-y\|^4}{8(\|x+y\|^2 + \|y\|^2)\|x+y\|\|y\|} \\ &= \frac{\|x+2y\|^4 - \|x\|^4}{80} \\ &= \frac{3^4 - 1^4}{80} = 1.\end{aligned}$$

Since $\cos A_\rho(x+y,y) = 1$, condition (2) yields $y = \lambda(x+y)$ for some $\lambda > 0$, which is impossible (as x and y are not collinear). This completes the proof. □

In a quasi-inner-product space, we have $\angle_\rho(x,y) = A_\rho(x,y)$ for any $x, y \in X$. Moreover, such spaces are strictly convex by [21, Corollary 1]. Therefore, the preceding theorem yields the following conclusion:

Corollary 2.10. *Let X be a quasi-inner-product space. If $\cos \angle_\rho(x,y) = 1$, where $x, y \neq 0$, then $y = \lambda x$ for some real number $\lambda > 0$.*

3. The geometric constant related to the angle $A_\rho(x, y)$

In this section, we investigate the following geometric constant, which is connected to the angle $A_\rho(x, y)$:

$$D_\rho^B(X) = \sup\left\{\cos A_\rho(x, y) : x, y \in S_X, x \perp_B y\right\}$$
$$= \sup\left\{\frac{\|x+y\|^4 - \|x-y\|^4}{16} : x, y \in S_X, x \perp_B y\right\}.$$

3.1. Relations with other constants

First, we discuss the relationship between $D_\rho^B(X)$ and $D'(X)$.

Proposition 3.1. *Let X be a Banach space. Then* $\frac{1}{4}D'(X) \leq D_\rho^B(X) \leq 2D'(X)$.

Proof. For any $x, y \in S_X$ with $x \perp_B y$, the inequality $\|x \pm y\| \leq 2$ holds. Thus we can obtain

$$\begin{aligned}\frac{\|x+y\|^4 - \|x-y\|^4}{16} &= \frac{(\|x+y\|^2 + \|x-y\|^2)(\|x+y\|^2 - \|x-y\|^2)}{16}\\ &\leq \frac{1}{2}(\|x+y\|^2 - \|x-y\|^2)\\ &= \frac{1}{2}(\|x+y\| + \|x-y\|)(\|x+y\| - \|x-y\|)\\ &\leq 2(\|x+y\| - \|x-y\|),\end{aligned}$$

which implies the right inequality.

On the other hand, for each pair of elements $x, y \in S_X$ with $x \perp_B y$, then one can get $\|x + y\| \geq 1$. Hence it follows that

$$\begin{aligned}\frac{\|x+y\|^4 - \|x-y\|^4}{16} &= \frac{(\|x+y\|^2 + \|x-y\|^2)(\|x+y\|^2 - \|x-y\|^2)}{16}\\ &\geq \frac{1}{8}(\|x+y\|^2 - \|x-y\|^2)\\ &= \frac{1}{8}(\|x+y\| + \|x-y\|)(\|x+y\| - \|x-y\|)\\ &\geq \frac{1}{4}(\|x+y\| - \|x-y\|),\end{aligned}$$

which means the left inequality. This completes the proof. □

By the above proposition and [1, Theorem 5.17], we derive the lower bound of $D_\rho^B(X)$.

Corollary 3.2. *Let X be a Banach space. Then* $D_\rho^B(X) \geq 0$.

The above proposition, combined with [1, Theorem 5.17], leads to the following conclusion. This result implies that the lower bound of $D_\rho^B(X)$ characterizes Hilbert space.

Corollary 3.3. *Let X be a Banach space. Then the following statements are equivalent:*
(1) $D_\rho^B(X) = 0$;
(2) $D'(X) = 0$;
(3) X is a Hilbert space.

Du and Li [5] proposed the following coefficient, known as the modulus of convexity related to Birkhoff orthogonality:

$$\delta_X(B) = \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in S_X, x \perp_B y\right\}.$$

Below, we study the relation between $D_\rho^B(X)$ and $\delta_X(B)$.

Proposition 3.4. *Let X be a Banach space. Then* $D_\rho^B(X) \leq \frac{16(1-\delta_X(B))^4-1}{16}$.

Proof. By the definition of $\delta_X(B)$, we have

$$\delta_X(B) \leq 1 - \frac{\|x+y\|}{2},$$

for any $x, y \in S_X$ such that $x \perp_B y$. Then we derive $\|x+y\| \leq 2(1-\delta_X(B))$. Since for all $x, y \in S_X$ with $x \perp_B y$, the inequality $\|x-y\| \geq 1$ holds. Then it follows that

$$\frac{\|x+y\|^4 - \|x-y\|^4}{16} \leq \frac{16(1-\delta_X(B))^4 - 1}{16},$$

which implies that the desired result. □

By the above proposition and [5, Proposition 2.3], the following conclusion holds:

Corollary 3.5. *Let X be a Banach space. Then* $D_\rho^B(X) \leq \frac{15}{16}$.

According to Corollary 3.3, the lower bound of $D_\rho^B(X)$ can be attained. Next, we provide an example to illustrate that the upper bound of $D_\rho^B(X)$ is sharp.

Example 3.6. *Let* $X = (\mathbb{R}^2, \|\cdot\|_\infty)$. *Then* $D_\rho^B(X) = \frac{15}{16}$.

Proof. Suppose that $x = (1,1)$ and $y = (1,0)$. It is clear that $x, y \in S_X$ such that $x \perp_B y$, and it is easily seen that $\|x+y\| = 2$ and $\|x-y\| = 1$. Then it follows from Corollary 3.5 that deduce

$$\frac{15}{16} = \frac{\|x+y\|^4 - \|x-y\|^4}{16} \leq D_\rho^B(X) \leq \frac{15}{16},$$

which means that the desired conclusion. □

3.2. *Some applications of the coefficient* $D_\rho^B(X)$

We begin by studying the relationship between non-squareness and the parameter $D_\rho^B(X)$. Now recall that the space X is said to be non-square [13], if for every $x, y \in S_X$, $\min\{\|x+y\|, \|x-y\|\} < 2$.

Theorem 3.7. *Let X be a Banach space with* $D_\rho^B(X) < \frac{15}{16}$. *Then X is non-square.*

Proof. Suppose conversely that X is not non-square. Then there exist $x, y \in S_X$ such that $\|x+y\| = \|x-y\| = 2$. Hence it is easily seen that $\|tx + (1-t)y\| = 1$ and $\|tx + (1-t)(-y)\| = 1$ for any $t \in [0,1]$. Now let $u = x$ and $v = \frac{x+y}{2}$, it is clear that $u, v \in S_X$. In order to prove $u \perp_B v$, one only need to consider the following three cases:

Case 1: $\lambda \geq 0$. Then we have

$$\|u + \lambda v\| = \left\|x + \lambda\frac{x+y}{2}\right\| = (1+\lambda)\left\|\frac{2+\lambda}{2(1+\lambda)}x + \frac{\lambda}{2(1+\lambda)}y\right\| = 1 + \lambda \geq 1 = \|u\|.$$

Case 2: $-2 \leq \lambda \leq 0$. Then we have

$$\|u + \lambda v\| = \left\|x + \lambda\frac{x+y}{2}\right\| = \left\|\left(1 - \left(-\frac{\lambda}{2}\right)\right)x + \left(-\frac{\lambda}{2}\right)(-y)\right\| = 1 = \|u\|.$$

Case 3: $\lambda \leq -2$. Then we have

$$\|u + \lambda v\| = \left\|x + \lambda\frac{x+y}{2}\right\| = (-\lambda - 1)\left\|\frac{2+\lambda}{2(1+\lambda)}x + \frac{\lambda}{2(1+\lambda)}y\right\| = -\lambda - 1 \geq 1 = \|u\|.$$

Moreover, it is easy to see that

$$\|u+v\| = \left\|x + \frac{x+y}{2}\right\| = 2\left\|\frac{3}{4}x + \frac{1}{4}y\right\| = 2$$

and

$$\|u-v\| = \left\|x - \frac{x+y}{2}\right\| = \frac{1}{2}\|x-y\| = 1.$$

Then it follows from Corollary 3.5 that we deduce

$$\frac{15}{16} = \frac{\|u+v\|^4 - \|u-v\|^4}{16} \leq D_\rho^B(X) \leq \frac{15}{16},$$

which contradicts that $D_\rho^B(X) < \frac{15}{16}$. This completes the proof. □

It is natural to ask whether the converse of Theorem 3.7 holds. The following example, however, shows that it does not.

Example 3.8. *Assume that $X = \mathbb{R}^2$ is equipped with the norm endowed by*

$$\|x\|_{\ell_\infty - \ell_1} = \|(x_1, x_2)\|_{\ell_\infty - \ell_1} = \begin{cases} \|(x_1,x_2)\|_\infty, & x_1x_2 \geq 0, \\ \|(x_1,x_2)\|_1, & x_1x_2 \leq 0. \end{cases}$$

Then $D_\rho^B(X) = \frac{15}{16}$.

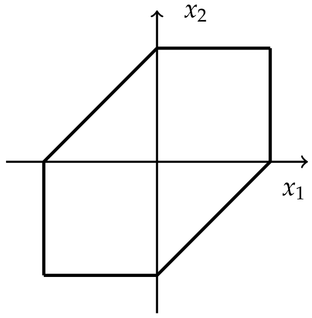

Figure 1. The unit sphere of $(\mathbb{R}^2, \ell_\infty - \ell_1)$

Proof. It is obvious that the space X is not non-square. For $x = (1,0)$ and $y = (1,1)$ in S_X, we have $x \perp_B y$ by the proof of [5, Example 2.3]. It follows that

$$\|x+y\| = \|(2,1)\| = 2, \ \ \|x-y\| = \|(0,-1)\| = 1.$$

Thus, by Corollary 3.5, we have

$$\frac{15}{16} = \frac{\|x+y\|^4 - \|x-y\|^4}{16} \leq D_\rho^B(X) \leq \frac{15}{16},$$

which implies that $D_\rho^B(X) = \frac{15}{16}$. □

Next, it is natural to wonder what kind of spaces can imply $D_\rho^B(X) < \frac{15}{16}$. Now we give an answer.

Theorem 3.9. *Let X be a uniformly convex Banach space. Then $D_\rho^B(X) < \frac{15}{16}$.*

Proof. Assume that X is uniformly convex. Then $\delta_X(B) > 0$ by [5, Corollary 2.2]. Thus the desired conclusion holds by Proposition 3.4. □

It is well known that Minkowski geometry—the geometry of real finite-dimensional Banach spaces—is closely related to Euclidean geometry and was addressed in the fourth Hilbert problem (see [9]). This discipline connects deeply with other mathematical fields such as distance geometry and convex geometry. Problems in Minkowski geometry are also studied in discrete and computational geometry and in operations research (see [15, 16]). A Minkowski space is a real finite-dimensional Banach space, and a two-dimensional Minkowski space is called a Minkowski plane.

In the following, we study the relation between the constant $D_\rho^B(X)$ and $D'(X)$ in Minkowski spaces. First, we present the following result, which is based on the idea in [1, Theorem 5.18].

Theorem 3.10. *Let X be a Banach space and $x, y \in S_X$. Then the following statements are equivalent:*
(1) $x \perp_B y$ and $\|x+y\| - \|x-y\| = 1$;
(2) $x \perp_B y$ and $\frac{\|x+y\|^4 - \|x-y\|^4}{16} = \frac{15}{16}$;
(3) The segments $[x, y]$ and $[x, x-y]$ are included in S_X.

Proof. On the one hand, the statement (1) and statement (3) are equivalent by [1, Theorem 5.18]. Now, we only need to prove that (1) and (2) are equivalent.

Suppose (1) is true. For any $x, y \in S_X$ satisfying $x \perp_B y$, it follows that $\|x + \lambda y\| \geq 1$ for all $\lambda \in \mathbb{R}$. Consequently, $\|x+y\| \leq 2$ and $\|x-y\| \geq 1$. Now, if $\|x+y\| - \|x-y\| = 1$, then

$$1 = \|x+y\| - \|x-y\| \leq 2 - \|x-y\|,$$

which implies that $\|x-y\| \leq 1$. Thus we have $\|x-y\| = 1$ and $\|x+y\| = 2$. By a direct calculation, the statement (2) holds. Similarly, if the statement (2) is true, we can also obtain that (1) holds. This completes the proof. □

Corollary 3.11. *Let X be a Minkowski space. Then $D_\rho^B(X) = \frac{15}{16}$ if and only if $D'(X) = 1$.*

It is clear that the usual orthogonality in Hilbert spaces is always symmetric. However, the Birkhoff orthogonality is not symmetric in general Banach spaces. James [11] derived the following conclusion and also illustrated that the assumption on the dimension of the space X in this result cannot be omitted.

Theorem 3.12. *([11]) A Banach space X whose dimension is at least three is a Hilbert space if and only if Birkhoff orthogonality is symmetric.*

Later, the authors give the definition of Radon planes in [15].

Definition 3.13. *([15]) A Minkowski plane in which the Birkhoff orthogonality is symmetric is called Radon planes.*

It is well known that Radon planes have a lot of fascinating, almost-Euclidean properties. For instance, the radial projection $R : X \to X$, defined by:

$$R(x) = \begin{cases} x, & \|x\| \leq 1, \\ \frac{x}{\|x\|}, & \|x\| > 1, \end{cases}$$

satisfies $\|R(x) - R(y)\| \leq \|x - y\|$ for any $x, y \in X$, that is, the radial projection on the Radon plane X is non-expansive.

Note that the space X considered in Example 3.8 is a Radon plane (see [11]). By an affine-regular hexagon we mean any non-degenerate affine image of the regular hexagon (see [14]). Hence, by Example 3.8 and Figure 1, the upper bound of $D_\rho^B(X)$ is attained when the unit sphere S_X is an affine-regular hexagon. It is natural to ask whether the upper bound of $D_\rho^B(X)$ is attained if and only if the unit sphere is an affine-regular hexagon, under the assumption that X is a Radon plane. We provide a positive answer to this question.

Theorem 3.14. *Let X be a Radon plane. Then $D_\rho^B(X) = \frac{15}{16}$ if and only if its unit sphere is an affine-regular hexagon.*

Proof. Assume that $D_\rho^B(X) = \frac{15}{16}$, then it follows from Proposition 3.4 and [5, Corollary 2.2] that we have $\delta_X(B) = 0$. Thus we can obtain S_X is an affine-regular hexagon by [5, Theorem 2.18].

On the other hand, if S_X is an affine-regular hexagon, then there are $u, v \in S_X$ such that $\pm u$, $\pm v$, $\pm(u+v)$ are the vertices of S_X (see Figure 2).

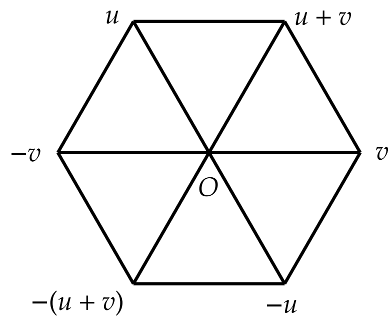

Figure 2. Affine-regular hexagonal unit sphere.

Let $x = u + v$ and $y = u$, it is obvious that $x, y \in S_X$. By the proof of [5, Theorem 2.18], we derive $x \perp_B y$. Moreover, we also have

$$\|x - y\| = \|v\| = 1$$

and

$$\|x + y\| = \|2u + v\| = 2\left\|u + \frac{1}{2}v\right\| = 2\left\|\frac{1}{2}(u+v) + \frac{1}{2}u\right\| = 2.$$

Hence it follows from the above equalities and Corollary 3.5 that we deduce

$$\frac{15}{16} = \frac{\|x+y\|^4 - \|x-y\|^4}{16} \le D_\rho^B(X) \le \frac{15}{16},$$

which means that $D_\rho^B(X) = \frac{15}{16}$. This completes the proof. □

From the above theorem and Corollary 3.11, the following conclusion holds:

Corollary 3.15. *Let X be a Radon plane. Then $D'(X) = 1$ if and only if its unit sphere is an affine-regular hexagon.*

4. The constant related to $A_g(x, y)$

Firstly, we discuss the relationship between $D_g^B(X)$ and $\delta_X(B)$.

Proposition 4.1. *Let X be a Banach space. Then $D_g^B(X) \le 1 - 2\delta_X(B)$.*

Proof. For any $x, y \in S_X$ with $x \perp_B y$, it follows from [19, Lemma 1] that

$$\rho(y, x) \le \|x + y\| - 1.$$

Thus we can obtain

$$\begin{aligned} D_g^B(X) &= \sup\{\rho(y,x) : x, y \in S_X, x \perp_B y\} \\ &\le \sup\{\|x+y\| : x, y \in S_X, x \perp_B y\} - 1 \\ &= 2(1 - \delta_X(B)) - 1 = 1 - 2\delta_X(B). \end{aligned}$$

This completes the proof. □

By the preceding proposition and [5, Proposition 2.3], we derive the upper bound of $D_g^B(X)$.

Corollary 4.2. *Let X be a Banach space. Then* $D_g^B(X) \leq 1$.

The above proposition and [5, Corollary 2.2] yield the following corollary:

Corollary 4.3. *Let X be a uniformly convex Banach space. Then* $D_g^B(X) < 1$.

Below, we provide an example to illustrate that the upper bound of $D_g^B(X)$ is sharp.

Example 4.4. *Let* $X = (\mathbb{R}^2, \|\cdot\|_\infty)$. *Then* $D_g^B(X) = 1$.

Proof. Assume that $x = (1, 1)$ and $y = (1, 0)$. It is clear that $x, y \in S_X$ such that $x \perp_B y$. And we can derive

$$\rho(y, x) = \rho_-(y, x) = \rho_+(y, x) = 1.$$

Thus it follows from Corollary 4.2 that

$$1 = \rho(y, x) \leq D_g^B(X) \leq 1,$$

which means that $D_g^B(X) = 1$. □

Du and Li [5] introduced the following coefficient, called the modulus of smoothness related to Birkhoff orthogonality:

$$\rho_X(B) = \sup\left\{1 - \frac{\|x + y\|}{2} : x, y \in S_X, x \perp_B y\right\}.$$

Now, we establish the relationship between $D_g^B(X)$ and $\rho_X(B)$.

Proposition 4.5. *Let X be a Banach space. Then* $D_g^B(X) \geq 2\rho_X(B) - 1$.

Proof. For any $x, y \in X$, from [19, Lemma 1], we deduce

$$\rho(y, x) \geq 1 - \|y - x\| = 1 - \|x - y\|.$$

Thus it follows from the homogeneity of Birkhoff orthogonality and the above inequality that we have

$$\begin{aligned} D_g^B(X) &= \sup\{\rho(y, x) : x, y \in S_X, x \perp_B y\} \\ &\geq \sup\{1 - \|x - y\| : x, y \in S_X, x \perp_B y\} \\ &= 1 - \inf\{\|x + y\| : x, y \in S_X, x \perp_B y\} \\ &= 1 - 2(1 - \rho_X(B)) \\ &= 2\rho_X(B) - 1, \end{aligned}$$

which completes the proof. □

In fact, the above proposition and [5, Corollary 3.2] yield the following corollary, which may give the best possible lower bound for $D_g^B(X)$. However, we do not know if there exists a Banach space that attains this bound.

Corollary 4.6. *Let X be a Banach space. Then* $D_g^B(X) \geq 1 - \sqrt{2}$.

In what follows, we compute the exact value of $D_g^B(X)$ for a specific Banach space.

Example 4.7. *Suppose that* $X = \mathbb{R}^2$ *is endowed with the norm*

$$\|x\|_{\ell_\infty - \ell_1} = \|(x_1, x_2)\|_{\ell_\infty - \ell_1} = \begin{cases} \|(x_1, x_2)\|_\infty, & x_1 x_2 \geq 0, \\ \|(x_1, x_2)\|_1, & x_1 x_2 \leq 0. \end{cases}$$

Then $D_g^B(X) = \frac{1}{2}$.

Proof. After a lengthy but straightforward computation, we find $D_g^B(X) \le \frac{1}{2}$. Now taking $x = (-1, 0)$ and $y = (0, 1)$, it is obvious that $x, y \in S_X$ such that $x \perp_B y$. Moreover, we also have

$$\rho_-(y, x) = 0, \ \ \rho_+(y, x) = 1,$$

then we can obtain

$$\rho(y, x) = \frac{1}{2}(\rho_-(y, x) + \rho_+(y, x)) = \frac{1}{2}.$$

Hence the desired conclusion holds. □

Notice that the space $\mathbb{R}^2$ endowed with the norm $\|\cdot\|_{\ell^\infty - \ell^1}$ is a non-smooth Radon plane. In contrast, we now state the value of $D_g^B(X)$ for any smooth Radon plane.

Proposition 4.8. *Let X be a smooth Radon plane. Then* $D_g^B(X) = 0$.

Proof. Assume that X is a Radon plane. Then for any $x, y \in S_X$ with $x \perp_B y$, we have $y \perp_B x$. Moreover, since X is smooth by [3, Proposition 2.2.4], we derive $\rho(y, x) = 0$. This implies that $D_g^B(X) = 0$. □

Next, we also establish the connection between $D_g^B(X)$ and $D_\rho^B(X)$ in quasi-inner-product spaces.

Proposition 4.9. *Let X be a quasi-inner-product space. Then* $D_g^B(X) = 2D_\rho^B(X)$.

Proof. Assume that X is a quasi-inner-product space. Then it follows from the equality (2) that we deduce

$$\|x + y\|^4 - \|x - y\|^4 = 8(\rho(x, y) + \rho(y, x)).$$

for any $x, y \in S_X$. Moreover, since X is smooth by [18, Theorem 1], it follows from $x \perp_B y$ and [3, Proposition 2.2.4] that $\rho(x, y) = 0$. Therefore, the above equality yields

$$\begin{aligned} D_g^B(X) &= \sup\{\rho(y, x) : x, y \in S_X, x \perp_B y\} \\ &= \sup\left\{\frac{\|x + y\|^4 - \|x - y\|^4}{8} : x, y \in S_X, x \perp_B y\right\} \\ &= 2D_\rho^B(X). \end{aligned}$$

This completes the proof. □

From Corollary 3.3 and the above proposition, the following result holds:

Corollary 4.10. *Suppose that X is a quasi-inner-product space. Then X is a Hilbert space if and only if* $D_g^B(X) = 0$.

In the end, we study the relation between $D_g^B(X)$ and non-squareness.

Theorem 4.11. *Let X be a Banach space. If* $D_g^B(X) < 1$, *then X is non-square.*

Proof. Assume conversely that X is not non-square. Then there exist $x, y \in S_X$ such that $\|x + y\| = \|x - y\| = 2$. Let $u = x$ and $v = \frac{x+y}{2}$, it is easily seen that $x, y \in S_X$ such that $u \perp_B v$ by the proof of Theorem 3.7. Moreover, we can obtain

$$\|v + tu\| = \left\|\frac{x + y}{2} + tx\right\| = (1 + t)\left\|\frac{1 + 2t}{2(1 + t)}x + \frac{1}{2(1 + t)}y\right\| = 1 + t,$$

whenever $t \to 0$. Thus, we also derive

$$\begin{aligned} \rho_-(v, u) &= \|v\| \lim_{t \to 0^-} \frac{\|v + tu\| - \|v\|}{t} = \lim_{t \to 0^-} \frac{1 + t - 1}{t} = 1, \\ \rho_+(v, u) &= \|v\| \lim_{t \to 0^+} \frac{\|v + tu\| - \|v\|}{t} = \lim_{t \to 0^-} \frac{1 + t - 1}{t} = 1, \end{aligned}$$

Hence we have $\rho(v, u) = 1$. Therefore, from Corollary 4.2, it follows that

$$1 = \rho(v, u) \le D_g^B(X) \le 1,$$

which contradicts that $D_g^B(X) < 1$. This completes the proof. □

5. Conclusion

This paper defines a new angle $A_\rho(x, y)$ based on norm derivatives and studies its elementary properties. As an application, we provide an equivalent characterization of strict convexity in Banach spaces using this angle. We also introduce two new geometric constants related to norm derivatives and investigate their relations with existing constants. In particular, we examine the connections between the parameter $D_\rho^B(X)$ and geometric properties such as non-squareness and uniform convexity. Additionally, we study the relation between $D_\rho^B(X)$ and $D'(X)$ in Minkowski spaces and characterize the Radon plane with affine-regular hexagonal unit sphere in terms of $D_\rho^B(X)$. The relationship between the constant $D_g^B(X)$ and non-squareness is also explored. Several open problems remain for future research: Is the angle $A_\rho(x, y)$ related to other geometric properties? What is the best lower bound for $D_g^B(X)$? How are the constants $D_\rho^B(X)$ and $D_g^B(X)$ connected with other geometric properties? Further results on the new angle and these constants will be presented in future work for readers interested in the geometry of Banach spaces.

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