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Numerical analysis of some deformable fractional problems based on the perturbation method

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Abstract. This paper presents solutions for several deformable fractional problems. Initially, by applying Banach's contraction principle, the existence and uniqueness of the fixed point are established. Then, using the reductive perturbation method, it is shown that a solution can be determined that closely approximates the exact result, without needing specific information about the solution required by the d'Alembert method. Our findings could assist in achieving a better alignment between theoretical and numerical outcomes, and in gaining insights into the characteristics of certain deformable fractional problems.

1. Introduction

In various scientific and engineering applications, it becomes necessary to compute fractional derivatives or integrals in order to solve a fractional partial or ordinary differential equation. The fractional calculus has been used to solve various problems in physics, engineering, and other scientific fields.

A recent investigation by Anwar et al. [8] formulated the mathematical problem for flow analysis through a set of partially coupled partial differential equations. The model was extended by incorporating fractional derivatives, and the problem was solved using the Laplace transform method to yield accurate analytical solutions. Alquran [6] presented solutions for certain types of fractional problems, where the proposed examples model a variety of physical applications and natural processes.

Srivastava et al. [18] introduced an innovative computational approach for solving physics problems that involve fractional-order differential equations, including the well-known Bagley-Torvik method. Albalawi [4] also utilized fractional calculus to solve and examine the time-fractional heat transfer equation, along

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with the nonlinear fractional porous media equation exhibiting cubic nonlinearity. Recently, Almatrafi [5] employed the tanh-function method to find specific traveling wave solutions to the space-time fractional symmetric regularized long wave equation, which is applied to study space-charge waves and shallow water waves.

Recent research has demonstrated the utility of deformable fractional derivatives for accurately representing physical phenomena [10, 14, 16]. The development of various types of fractional derivatives, notably deformable fractional derivatives, as proposed in [2], has broadened the scope of applied mathematics research. The efficient and reliable numerical methods in fractional calculus are in great demand, as seen from the references [2, 12, 13, 20] and the references therein.

Recent advances in controllability theory have extended to more general classes of systems involving fractional dynamics, stochastic perturbations, and nonsmooth effects. For instance, Ahmed and Ragusa [1] studied Sobolev-type conformable fractional stochastic evolution inclusions subject to nonlocal conditions and Clarke subdifferential operators. By combining tools from fractional calculus, stochastic analysis, and nonsmooth analysis, they established sufficient conditions ensuring nonlocal controllability and illustrated the applicability of their results through an example. This work highlights the growing interest in fractional and stochastic models with nonlocal features, which better capture memory, randomness, and nonsmooth behaviors observed in complex physical systems.

In the context of fractional inequalities, recent studies have emphasized the role of fractional operators with non-singular kernels in deriving new bounds for convex functions. For example, in [9], the authors established several new integral inequalities for quasi-convex and (h,m)-convex functions by employing the Caputo-Fabrizio fractional integral operator. Their approach relies on the algebraic properties of this operator, together with classical definitions of convexity and elementary analysis techniques, thereby extending the scope of fractional inequalities to broader classes of convex functions.

In the study of conformable fractional differential equations, new analytical and semi-analytical techniques have been developed to address singular perturbation problems. Danaei and Jahanshahi [11] proposed a conformable fractional variational iteration method tailored to such equations, showing that exact solutions can often be obtained within the first few iterations. Through several illustrative examples, they demonstrated the accuracy and efficiency of the method by comparing the obtained results with those of the corresponding classical differential equations, highlighting its potential for practical applications.

Recent efforts have also focused on the development of hybrid analytical-numerical schemes for solving fractional nonlinear evolution equations arising in physics. Nadeem and Wahash [15] investigated the fractional Kundu-Eckhaus equation and the fractional massive Thirring problem, which appear in quantum field theory, weakly nonlinear water waves, and nonlinear optics. By combining He's fractional complex transform, the homotopy perturbation method, and a two-scale approach, they derived rapidly convergent series solutions without requiring restrictive assumptions. Their results, supported by numerical simulations and error analysis, demonstrate that this hybrid scheme provides an efficient and accurate tool for handling nonlinear fractional models.

In this paper, we present an overview of two theoretical and numerical methods that can be employed to attain approximate solutions for some deformable fractional problems. To the best of our understanding, this type of study has not been addressed in existing literature. Therefore, in the second section of this paper, we aim to concentrate on exploring the solutions, as well as their uniqueness and stability, for certain deformable fractional differential equations.

Our paper is organized as follows. In Section 2, the problem is stated and theoretical results are demonstrated. Section 3 provides the numerical results. In this part, we consider four examples to demonstrate the efficacy of the method in resolving nonlinear singular perturbation problems. Finally, we summarize the numerical results in Section 4.

2. Theoretical Study of the Problem

2.1. Basic properties of deformable derivative

In this section, we introduce the key concepts related to deformable fractional derivatives [2, 12, 13, 20]. Let $C = C([a,\ b], \mathbb{R})$ represent the Banach space of continuous functions from $[a, b]$ to $\mathbb{R}$ equipped with the norm

$$\|u\|_C = \sup_{x\in[a,\,b]} |u(x)|. \tag{2.1}$$

Definition 2.1. ([2]) *Let $u : [a, b] \to \mathbb{R}$ be a continuous function and τ, α positive numbers with $0 \le \tau \le 1$ and $\tau + \alpha = 1$. The deformable derivative of u of order τ at $x \in I = [a, b]$ is defined by*

$$(D^\tau u)(x) = \lim_{\varepsilon\to 0} \frac{(1+\varepsilon\alpha)\, u(x+\varepsilon\tau) - u(x)}{\varepsilon}. \tag{2.2}$$

If the limit exists, u is $\tau-$differentiable at x. If $\tau = 1$, then $\alpha = 0$, we recover the usual derivative. Therefore, the deformable derivative is more general than the usual derivative.

Definition 2.2. ([2]) *For $\tau \in (0, 1]$, the $\tau-$integral of the function $u \in L^1([a, b], \mathbb{R}_+)$ is defined by*

$$(I_a^\tau u)(x) = \frac{1}{\tau} e^{-\frac{\alpha}{\tau}x} \int_a^x e^{\frac{\alpha}{\tau}t} u(t)\, dt, \quad x \in [a, b], \tag{2.3}$$

where $\tau + \alpha = 1$. When $a = 0$, we write $(I^\tau u)$ instead of writing $\left(I_0^\tau u\right)$.

The next theorem compiles the key properties of the operators D^τ and I_a^τ, which are essential for the paper.

Theorem 2.3. ([2]) *Let $\tau \in (0, 1]$ be such that $\tau + \alpha = 1$. Then*

1. *The operators D^τ and I_a^τ are linear.*
2. *The operators D^τ and I_a^τ are commutative.*
3. *$D^\tau(\sigma) = \alpha\sigma$ for all constant $\sigma \in \mathbb{R}$.*
4. *$D^\tau(uv) = (D^\tau u)\, v + \tau u D v$.*
5. *Let u be a continuous function on $[a, b]$. Then $I_a^\tau u$ is $\tau-$differentiable in (a, b) and we have*

$$D^\tau (I_a^\tau u)(x) = u(x) \tag{2.4}$$

$$I_a^\tau (D^\tau u)(x) = u(x) - e^{\frac{\alpha}{\tau}(a-x)} u(a). \tag{2.5}$$

2.2. Existence and uniqueness of solution for a Ulam-Hyers stable problem

In this paper, we seek to find solutions of the following deformable fractional differential equation of the form

$$\begin{cases} D^\tau D^\tau y(x) + P(x, y) D^\tau y(x) + q(x) y(x) = 0 \\ y(0) = a,\ D^\tau y(0) = \lambda,\ a, \lambda \in \mathbb{R},\ x \in [0\ T],\ T > 0 \end{cases} \tag{2.6}$$

where $P(x, y), q(x)$ are continuous functions and $\tau + \delta = 1,\ 0 < \tau < 1$.

It was proven in reference [3] (Theorem 4) using D'Alambert's approach that if y_1 is a solution of equation (2.6), then the second solution is given by

$$y_2 = \tau y_1 \int \Big[\frac{\Lambda}{y_1^2} \exp\Big(\frac{-2\delta}{\tau} x - \int P\, d_\tau x\Big)\Big] d_\tau x \tag{2.7}$$

where Λ is an arbitrary constant and $d_\tau x = \dfrac{1}{\tau} d\, x$. It is clear from Eq. (2.7) that y_2 can't be found if we do not have y_1. As we can see, this method is not effective if we do not know a particular solution y_1 which will be used to calculate the general solution y_2.

2.2.1. Existence and uniqueness

The question that arises is does Problem (2.6) have solutions and under which conditions? A technique for solving the given problem consists in reformulating it and seeking its solution as being the fixed point of a certain operator to be defined. Banach's contraction principle is a very powerful tool to prove the existence and uniqueness of the fixed point. Under some assumptions on $P(x,y), q(x)$ and using Banach's contraction principle, we can prove that Problem (2.6) has a unique solution. Indeed, we can rewrite the given problem as

$$\begin{cases} D^{\tau}D^{\tau}y(x) = f\left(x, y(x), D^{\tau}y(x)\right) \\ y\left(0\right) = a,\ D^{\tau}y\left(0\right) = \lambda,\ a, \lambda \in \mathbb{R},\ x \in [0\ T],\ T > 0. \end{cases} \tag{2.8}$$

where $\tau + \delta = 1,\ 0 < \tau < 1$, and $f\left(x, y(x), D^{\tau}y(x)\right) = -P(x,y)D^{\tau}y(x) - q(x)y(x)$ is a function satisfying the following assumptions:

- (A_1) $f:\ [0\ T]\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ is continuous.
- (A_2) There exists a positive real number σ such that:

$$\left|f\left(x, y_1, z_1\right) - f\left(x, y_2, z_2\right)\right| \leq \sigma\left[\left|y_1 - y_2\right| + \left|z_1 - z_2\right|\right],\quad \forall y_1, y_2, z_1, z_2 \in \mathbb{R}, x \in [0, T].$$

- (A_3) $\exists\eta > 0,$ such that $\sup\limits_{x\in[0\ T]}\left|f\left(x, 0, 0\right)\right| \leq \eta.$

Let Ω be the Banach space defined by $\Omega = \left\{y \in C\ \ :\ D^{\tau}y \in C\right\}$ equipped with the norm

$$\|y\|_{\Omega} = \|y\|_{C} + \|D^{\tau}y\|_{C}. \tag{2.9}$$

The solution of Problem (2.8) when it exists is of the form given by the following theorem.

Theorem 2.4. *For $y \in \Omega$, y is a solution of* (2.8) *if and only if*

$$y(x) = ae^{-\frac{\delta}{\tau}x} + \frac{\lambda}{\tau}xe^{-\frac{\delta}{\tau}x} + \frac{1}{\tau^2}e^{-\frac{\delta}{\tau}x}\int_0^x\int_0^s e^{\frac{\delta}{\tau}t}f\left(t, y(t), D^{\tau}y(t)\right)\,dt\,ds.$$

Proof. If $y \in \Omega$ is a solution of Problem (2.8), then we have

$$I^{\tau}\left(I^{\tau}\left(D^{\tau}D^{\tau}y(x)\right)\right) = I^{\tau}\left(I^{\tau}\left(f\left(x, y(x), D^{\tau}y(x)\right)\right)\right), \tag{2.10}$$

Applying (2.5) together with the property of linearity, we obtain

$$\begin{aligned} I^{\tau}\left(I^{\tau}\left(D^{\tau}D^{\tau}y(x)\right)\right) &= I^{\tau}\left(D^{\tau}y(x) - I^{\tau}\left(e^{-\frac{\delta}{\tau}x}D^{\tau}y(0)\right)\right) \\ &= y(x) - y(0)e^{-\frac{\delta}{\tau}x} - \lambda I^{\tau}\left(e^{-\frac{\delta}{\tau}x}\right), \end{aligned}$$

that is,

$$I^{\tau}\left(I^{\tau}\left(D^{\tau}D^{\tau}y(x)\right)\right) = y(x) - ae^{-\frac{\delta}{\tau}x} - \frac{\lambda}{\tau}xe^{-\frac{\delta}{\tau}x}, \tag{2.11}$$

On the other hand

$$I^{\tau}\left(I^{\tau}\left(f\left(x, y(x), D^{\tau}y(x)\right)\right)\right) = \frac{1}{\tau^2}e^{-\frac{\delta}{\tau}x}\int_0^x\int_0^s e^{\frac{\delta}{\tau}t}f\left(t, y(t), D^{\tau}y(t)\right)\,dt\,ds. \tag{2.12}$$

Hence, from (2.10), (2.11),(2.12), we have

$$y(x) = ae^{-\frac{\delta}{\tau}x} + \frac{\lambda}{\tau}xe^{-\frac{\delta}{\tau}x} + \frac{1}{\tau^2}e^{-\frac{\delta}{\tau}x}\int_0^x\int_0^s e^{\frac{\delta}{\tau}t}f\left(t, y(t), D^\tau y(t)\right)\,dt\,ds. \tag{2.13}$$

Conversely, if

$$y(x) = ae^{-\frac{\delta}{\tau}x} + \frac{\lambda}{\tau}xe^{-\frac{\delta}{\tau}x} + I^\tau\left(I^\tau\left(f\left(x, y(x), D^\tau y(x)\right)\right)\right),$$

then applying (2.4) and using linearity property, we have

$$\begin{aligned} D^\tau D^\tau y(x) &= D^\tau\left(D^\tau\left(ae^{-\frac{\delta}{\tau}x}\right) + \frac{\lambda}{\tau}D^\tau\left(xe^{-\frac{\delta}{\tau}x}\right) + I^\tau\Big(f\Big(x, y(x), D^\tau y(x)\Big)\Big)\right) \\ &= D^\tau\left(\lambda e^{-\frac{\delta}{\tau}x} + I^\tau\Big(f\Big(x, y(x), D^\tau y(x)\Big)\Big)\right) \\ &= D^\tau\left(\lambda e^{-\frac{\delta}{\tau}x}\right) + D^\tau\left(I^\tau\Big(f\Big(x, y(x), D^\tau y(x)\Big)\Big)\right) \\ &= f\Big(x, y(x), D^\tau y(x)\Big). \end{aligned} \tag{2.14}$$

Moreover, we can easily verify that $y(0) = a$ and $D^\tau y(0) = \lambda$. □

The following theorem ensures the existence and uniqueness of the solution to Problem (2.8).

Theorem 2.5. *Assume that $(A_1) - (A_3)$ are satisfied. If $\sigma\left(\frac{1}{\delta^2} + \frac{1}{\delta}\right) < 1$, then Problem (2.8) has a unique solution.*

Proof. We will use a fixed point approach to prove Theorem 2.5. Let $\Omega = \{y \in C : \ D^\tau y \in C\}$, we consider the operator $\psi : \ \Omega \longrightarrow \Omega$ such that

$$\psi y(x) = ae^{-\frac{\delta}{\tau}x} + \frac{\lambda}{\tau}xe^{-\frac{\delta}{\tau}x} + \frac{1}{\tau^2}e^{-\frac{\delta}{\tau}x}\int_0^x\int_0^s e^{\frac{\delta}{\tau}t}f\left(t, y(t), D^\tau y(t)\right)\,dt\,ds,$$

which we will prove that it has a fixed point y^* satisfying $y^* = \psi y^*$ and consequently y^* becomes a solution of Problem (2.8). Our main tool is Banach's contraction principle.

Step-1: Let ρ be a positive number such that

$$\rho > \frac{|a| + |\lambda|\left(\frac{T}{\tau} + 1\right) + \eta\left(\frac{1}{\delta^2} + \frac{1}{\delta}\right)}{1 - \sigma\left(\frac{1}{\delta^2} + \frac{1}{\delta}\right)}$$

and $B_\rho = \{y \in \Omega, \ \ \|y\|_\Omega \le \rho\}$.

We claim that $\psi B_\rho \subset B_\rho$. Indeed, let $y \in \Omega$, we have

$$\Big|\psi y(x)\Big| \le |a|e^{-\frac{\delta}{\tau}x} + |\frac{\lambda}{\tau}||x|e^{-\frac{\delta}{\tau}x} + \frac{1}{\tau^2}e^{-\frac{\delta}{\tau}x}\int_0^x\int_0^s e^{\frac{\delta}{\tau}t}\Big|f\left(t, y(t), D^\tau y(t)\right)\Big|\,dt\,ds,$$

$$\begin{aligned} \Big|\psi y(x)\Big| \le |a| + |\frac{\lambda}{\tau}|T + \frac{1}{\tau^2}e^{-\frac{\delta}{\tau}x}\int_0^x\int_0^s e^{\frac{\delta}{\tau}t}\Big|f\left(t, y(t), D^\tau y(t)\right) - f\left(t, 0, 0\right)\Big|\,dt\,ds + \\ \frac{1}{\tau^2}e^{-\frac{\delta}{\tau}x}\int_0^x\int_0^s e^{\frac{\delta}{\tau}t}\Big|f\left(t, 0, 0\right)\Big|\,dt\,ds, \end{aligned}$$

$$\begin{aligned}\left|\psi y(x)\right| &\leq |a|+|\frac{\lambda}{\tau}|T+\frac{1}{\delta^2}\left(\sigma\rho+\eta\right)\left[\left(1-e^{-\frac{\delta}{\tau}x}\right)-\frac{1}{\delta\tau^2}e^{-\frac{\delta}{\tau}x}\right]\\ &\leq |a|+|\frac{\lambda}{\tau}|T+\frac{1}{\delta^2}\left(\sigma\rho+\eta\right),\\ &\leq |a|+\frac{T}{\tau}|\lambda|+\frac{\sigma}{\delta^2}\rho+\frac{1}{\delta^2}\eta.\end{aligned}$$

Furthermore, in a same way, we obtain

$$\begin{aligned}\left|D^\tau\psi y(x)\right| &\leq |\lambda|+\frac{1}{\delta}\left(\sigma\rho+\eta\right)\left(1-e^{-\frac{\delta}{\tau}x}\right)\\ &\leq |\lambda|+\frac{1}{\delta}\left(\sigma\rho+\eta\right),\\ &\leq |\lambda|+\frac{\sigma}{\delta}\rho+\frac{1}{\delta}\eta.\end{aligned}$$

Hence

$$\begin{aligned}\|\psi y\|_\Omega &= \|\psi y\|_C+\|D^\tau\psi y\|_C,\\ &\leq |a|+\frac{T}{\tau}|\lambda|+\frac{\sigma}{\delta^2}\rho+\frac{1}{\delta^2}\eta+|\lambda|+\frac{\sigma}{\delta}\rho+\frac{1}{\delta}\eta,\\ &\leq |a|+|\lambda|\left(\frac{T}{\tau}+1\right)+\eta\left(\frac{1}{\delta^2}+\frac{1}{\delta}\right)+\sigma\left(\frac{1}{\delta^2}+\frac{1}{\delta}\right)\rho,\\ &\leq 1-\sigma\left(\frac{1}{\delta^2}+\frac{1}{\delta}\right)+\sigma\left(\frac{1}{\delta^2}+\frac{1}{\delta}\right)\rho,\\ &\leq \rho.\end{aligned}$$

Step-2: We claim that ψ is a contraction. Indeed, let $y,z\in\Omega$. For all $x\in[0\ T]$, we have

$$\begin{aligned}\left|\psi y(x)-\psi z(x)\right| &= \frac{1}{\tau^2}e^{-\frac{\delta}{\tau}x}\int_0^x\int_0^s e^{\frac{\delta}{\tau}t}\left|f\left(t,y(t),D^\tau y(t)\right)-f\left(t,z(t),D^\tau z(t)\right)\right|dt\,ds,\\ &\leq \frac{\sigma}{\delta^2}\|y-z\|_\Omega\left(1-e^{-\frac{\delta}{\tau}x}\right),\\ &\leq \frac{\sigma}{\delta^2}\|y-z\|_\Omega.\end{aligned}$$

We also have

$$\left|D^\tau\psi y(x)-D^\tau\psi z(x)\right| \leq \frac{\sigma}{\delta}\|y-z\|_\Omega.$$

Then,

$$\begin{aligned}\|\psi y-\psi z\| &\leq \frac{1}{\delta^2}\|y-z\|_\Omega+\frac{1}{\delta}\|y-z\|_\Omega,\\ &\leq \sigma\left(\frac{1}{\delta^2}+\frac{1}{\delta}\right)\|y-z\|_\Omega,\\ &\leq \mu\|y-z\|_\Omega,\end{aligned}$$

with $\mu=\sigma\left(\frac{1}{\delta^2}+\frac{1}{\delta}\right)<1$.

By Banach's fixed point theorem, we deduce that the operator ψ admits a unique fixed point and it follows that Problem (2.8) has a unique solution. Now, we can confirm that if $P(x,y)$ and $q(x)$ are chosen so that

$$f\left(x,y(x),D^\tau y(x)\right)=-P(x,y)D^\tau y(x)-q(x)y(x)$$

satisfies assumptions $(A_1)-(A_3)$ and δ is as in Theorem 2.5, then Problem (2.6) admits a unique solution. □

Remark 2.6. *Theorem 2.5 ensures existence and uniqueness of solution for Problem* (2.6)*, however it is not always possible to calculate analytically this solution. Generally, in differents fields of science, researchers are content to find approximate solutions which best converge towards the exact solution. Among the techniques which can allow us to confirm such convergence we distinguish the study of Ulam-Hyers stability.*

2.2.2. Ulam-Hyers stability

Definition 2.7. ([7]) *Problem* (2.8) *is said to be Ulam-Hyers stable if it exists a positive real number* $\nu > 0$ *such that for all* $\epsilon > 0$ *and for all* $w \in \Omega$*, satisfying*

$$\left|D^{\tau} D^{\tau} w(x) - f(x, w(x), D^{\tau} w(x))\right| \leq \epsilon, \text{ with } w(0) = a, D^{\tau} w(0) = \lambda \ , \ x \in [0, T]$$

there exists $y^* \in \Omega$ *is a solution of Problem* (2.8) *such that*

$$|w(x) - y^*(x)| \leq \nu\epsilon, \ x \in [0, T].$$

Let us assume that $(A_1) - (A_3)$ are satisfied, the following theorem is very important.

Theorem 2.8. *If* $\sigma\left(\frac{1}{\delta^2} + \frac{1}{\delta}\right) < 1$, *then Problem* (2.8) *is Ulam-Hyers stable.*

Proof. Let $\varepsilon > 0$ and $w \in \Omega$ be such that

$$\left|D^{\tau} D^{\tau} w(x) - f(x, w(x), D^{\tau} w(x))\right| \leq \epsilon, \text{ with } w(0) = a, D^{\tau} w(0) = \lambda, \ x \in [0, T]. \tag{2.15}$$

Using (2.15), we can easily show that there exists a real number $\kappa > 0$ such that

$$\|w - \psi w\|_{\Omega} \leq \kappa\varepsilon.$$

Let y^* be the unique solution of Problem (2.8) which exists according to Theorem 2.5. It is immediately seen that

$$\begin{aligned} \|w - y^*\|_{\Omega} &\leq \|w - \psi w\|_{\Omega} + \|\psi w - y^*\|_{\Omega}, \\ &\leq \|w - \psi w\|_{\Omega} + \|\psi w - \psi y^*\|_{\Omega}, \\ &\leq \kappa\varepsilon + \mu\|w - y^*\|_{\Omega}, \end{aligned}$$

hence

$$(1 - \mu)\,\|w - y^*\|_{\Omega} \leq \kappa\varepsilon,$$

since $(1 - \mu) > 0$, we have

$$\|w - y^*\|_{\Omega} \leq \nu\varepsilon,$$

where

$$\nu = \frac{\kappa}{1 - \mu} > 0.$$

That is, Problem (2.8) is Ulam-Hyers stable. Therefore, for each approximate solution there exists a an exact solution and they are connected by the

$$\|w - y^*\|_{\Omega} \leq \nu\varepsilon,$$

where w designates the approximate solution and y^* the exact solution of the given Problem. □

3. Numerical Analysis

3.1. Singular perturbation method

In numerous physical situations, obtaining an exact solution to the entire problem is unfeasible, necessitating the use of various approximation and perturbation techniques to achieve the desired outcome.

A boundary-layer problem serves as an example of this type of situation. We examine a one-dimensional differential equation boundary value problem on the unit interval, where the highest derivative of the equation is multiplied by a small parameter ε^2. This paper will analyze the singularly-perturbed boundary value problem. The problem requires considering parameters that change over different scales, influencing the solution's behavior. The objective of this study is to investigate how variations in these parameters affect the solution's characteristics.

$$\varepsilon^2 y''(x) + H(x,y)\, y'(x) + G(x,y)\, y(x) = 0 \tag{3.1}$$

with the boundary conditions

$$y(0) = a, \quad y(1) = b. \tag{3.2}$$

It was proven in [19] (section 3) that Problem (3.2) admit a solution of the form

$$y(x) = T(x) + Z(t), \quad \text{with } t = \frac{x}{\varepsilon},$$

where T and Z are functions satisfying some additional conditions. Driven by the extensive body of research on deformable fractional derivatives and their applications, along with the work in [19], our goal is to make a contribution by numerically solving a problem investigated by Ahuja et al. in [3].

In this contribution, we seek to solve Eq. (2.6). Our solution is an approximative solution given by a perturbative method. The approximative solution is compared to the exact one in order to see the effectiveness of our proposed solution. To do this let's firstly rewrite equation (2.6) in the following form,

$$y''(x) + \frac{1}{\tau}\left(2\delta + P(x,y)\right)\, y'(x) + \frac{1}{\tau^2}\left(\delta^2 + \delta P(x,y) + q(x)\right)\, y(x) = 0 \tag{3.3}$$

that we can write as fellow,

$$y''(x) + H(x,y)\, y'(x) + G(x,y)\, y(x) = 0, \tag{3.4}$$

where

$$H(x,y) = \frac{1}{\tau}\left(2\delta + P(x,y)\right), \tag{3.5}$$

$$G(x,y) = \frac{1}{\tau^2}\left(\delta^2 + \delta P(x,y) + q(x)\right). \tag{3.6}$$

3.2. Some applications

In this part, we give four examples that are solved to illustrate the precision and application of the singular perturbation method.

3.2.1. Example 1

First, we consider the following equation

$$D^{\frac{1}{2}}D^{\frac{1}{2}}y + D^{\frac{1}{2}}y + \frac{1}{4}y = 0 \tag{3.7}$$

where $\tau = 1/2$ and $\delta = 1/2$. To get a solution of such a problem one can use the D'Alembert method [17] where the first solution is given by $y_1 = x$. In order to get y_2 we must calculate the following integral

$$y_2 = x\left[\int_{0.5}^{1} \frac{e^{-2x+\int_{0.5}^{1} 2\,dx}}{x^2}dx\right] \tag{3.8}$$

on interval $x = [0.5, 1.5]$. For $k = 1$, the numerical results are given by

x	y_2
0.1	0.07052765735
0.2	0.14105531470
0.3	0.21158297200
0.4	0.28211062940
0.5	0.35263828680
0.6	0.42316594410
0.7	0.49369360140
0.8	0.56422125880
0.9	0.63474891620
1	0.70527657350
1.1	0.77580423080
1.2	0.84633188820
1.3	0.91685954560
1.4	0.98738720290
1.5	1.05791486000

Table 1: Numerical solution using d'Alembert method.

The main difficulties with d'Alambert method is that to get the appropriate solution y_2, the solution y_1 must be must be known previously. In order to overcome this problem, we have to solve Eq. (3.7) using the singular perturbation method. The primary benefit of this approach is that it does not rely on any particular solution to obtain the exact result. Equation (3.7) can then be modified and expressed as,

$$\varepsilon^2\frac{d^2y}{dx^2} + \frac{1}{\tau}(2\delta+1)\frac{dy}{dx} + \frac{1}{\tau^2}\left(\delta^2+\delta+\frac{1}{4}y\right)y(x) = 0, \quad y(0) = 1, \quad y(1) = 1 \tag{3.9}$$

By application of the singular perturbation method, Eq. (3.9) admits an exact solution given by

$$y(x) = \frac{\left(7e^{-\frac{1}{8}} - 7\right)e^{-\frac{4x}{\varepsilon}} - 6\,e^{-\frac{1}{8}} + 7e^{-\frac{x}{8}}}{e^{-\frac{1}{8}}} \tag{3.10}$$

for $\varepsilon = 10^{-3}$, we get the following table of results:

x	$\varepsilon = 10^{-1}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$
0.1	1.81643490	1.83350580	1.83350580
0.2	1.73588380	1.73619650	1.73619650
0.3	1.64009010	1.64009580	1.64009580
0.4	1.54518890	1.54518900	1.54518900
0.5	1.45146120	1.45146120	1.45146120
0.6	1.35889770	1.35889770	1.35889770
0.7	1.26748400	1.26748400	1.26748400
0.8	1.17720580	1.17720580	1.17720580
0.9	1.08804910	1.08804910	1.08804910
1	1.00000000	1.00000000	1.00000000
1.1	0.91304457	0.91304457	0.91304457
1.2	0.82716948	0.82716948	0.82716948
1.3	0.74236091	0.74236091	0.74236091
1.4	0.65860594	0.65860594	0.65860594
1.5	0.57589143	0.57589143	0.57589143

Table 2: Approximate numerical solutions using singular perturbation method.

Figure 1 displays the validity domain of the numerical solutions achieved by the utilized methods. The two solutions demonstrate two disparate behaviors. As can be seen in Figure 1, the solutions converge in the domain $x \in [0.1, 1.2]$ and diverge in the domain $x \in [1.2, 1.5]$.

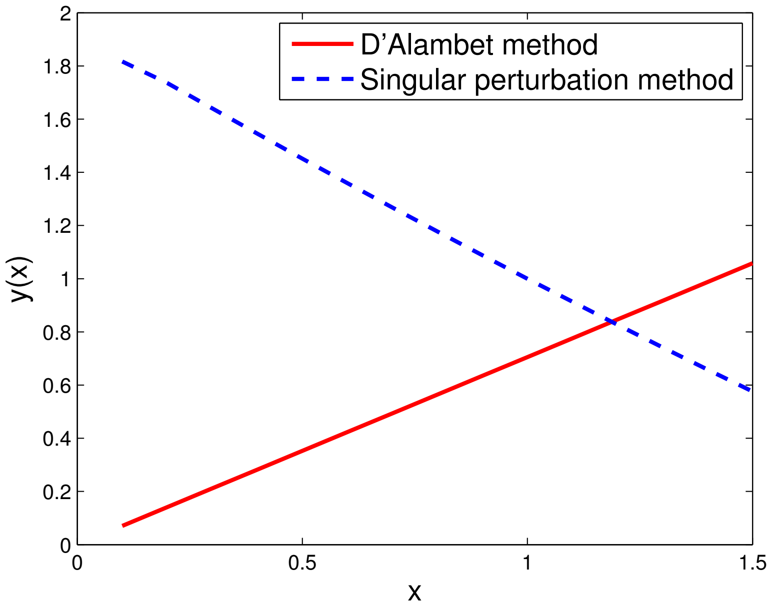

Figure 1: Comparison of the d'Alembert method and the singular perturbation method $\epsilon = 0.1$.

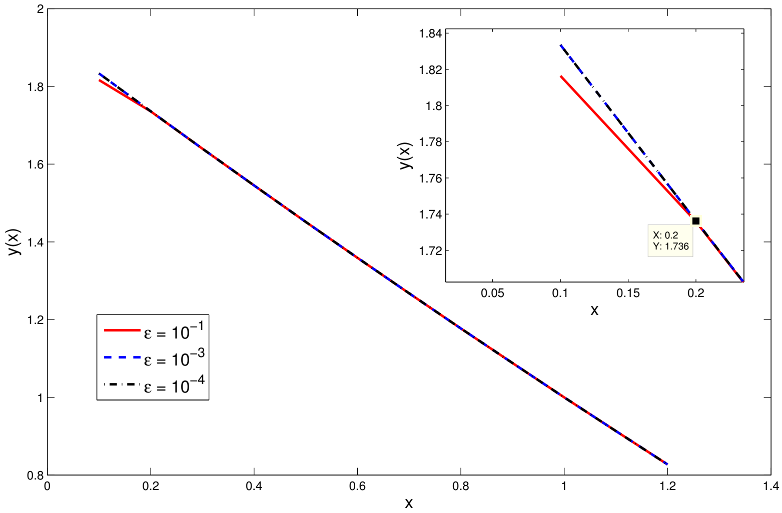

Figure 2: Comparison of numerical results of the singular perturbation method for different values of ε.

3.2.2. Example 2

Let us consider the following equation

$$D^{\frac{1}{2}}D^{\frac{1}{2}}y + D^{\frac{1}{2}}y = 0. \tag{3.11}$$

For $P(x, y) = 1$ and $q(x) = 0$, we obtain the following example:

$$\varepsilon^2 \frac{d^2y}{dx^2} + \frac{1}{\tau}(2\delta + 1)\frac{dy}{dx} + \frac{1}{\alpha^2}\left(\delta^2 + \delta\right) y(x) = 0, \quad y(0) = 1, \quad y(1) = 1. \tag{3.12}$$

For $\tau = 1/2$ and $\delta = 1/2$, the solution is given in the following form:

$$y(x) = \frac{x}{4} + \frac{3}{4} + \frac{1}{4}e^{\frac{-4x}{\varepsilon}}. \tag{3.13}$$

For $\varepsilon = 10^{-3}$, we have the following table:

x	Present ($\tau = 1/2$ and $\delta = 1/2$)
0	1.0000000
0.5 ε	0.78395882
1 ε	0.75482891
0.1	0.77500000
0.2	0.80000000
0.3	0.82500000
0.4	0.85000000
0.5	0.87500000
0.6	0.90000000
0.7	0.92500000
0.8	0.95000000
0.9	0.97500000
1	1.

Table 3: Approximate solution for Example 2.

3.2.3. Example 3

Consider the equation given below

$$D^{\frac{2}{5}} D^{\frac{2}{5}} y + (1+x) D^{\frac{2}{5}} y + x^2 = 0. \tag{3.14}$$

For $P(x,y) = 1 + x$ and $q(x) = x^2$, the following example is obtained

$$\varepsilon^2 \frac{d^2 y}{dx^2} + \frac{1}{\alpha} (2\beta + 1 + x) \frac{dy}{dx} + \frac{1}{\tau^2} \left(\delta^2 + \delta(1+x) + x^2\right) y(x) = 0. \tag{3.15}$$

$$y(0) = -2, \quad y(1) = 1. \tag{3.16}$$

For $\tau = 2/5$ and $\delta = 3/5$ the solution is given by

$$y(x) = \frac{1}{100} (-195\, Ln(11) - 2544\, Ln(2))\, e^{\frac{-11x}{2\varepsilon}} - \frac{3x^2}{4} + \frac{9x}{5} + \frac{636\, Ln(2)}{25} - \frac{159\, Ln(5x+11)}{25} - \frac{1}{20}. \tag{3.17}$$

For $\varepsilon = 10^{-3}$, we have the next table as follows:

x	Present ($\tau = 2/5$ and $\delta = 3/5$)
0	−2.0000000
0.5 ε	2.055502
1 ε	2.314252
0.1	2.222837
0.2	2.109658
0.3	1.992530
0.4	1.870586
0.5	1.743058
0.6	1.609260
0.7	1.468579
0.8	1.320465
0.9	1.164422
1	1.0000000

Table 4: Approximate solution for Example 3 at $\varepsilon = 10^{-3}$.

3.2.4. Example 4

Let us consider the following equation

$$D^{\frac{1}{2}}D^{\frac{1}{2}}y + yD^{\frac{1}{2}}y = 0. \tag{3.18}$$

For $P(x, y) = y$, we obtain the following example

$$\varepsilon^2 \frac{d^2y}{dx^2} + \frac{1}{\alpha}(2\beta + y)\frac{dy}{dx} + \frac{1}{\alpha^2}\left(\beta^2 + \beta y\right) y(x) = 0, \quad y(0) = -1, \quad y(1) = 3.9995. \tag{3.19}$$

For $\tau = 1/2$ and $\delta = 1/2$ the solution is given by,

$$y(x) = \frac{3}{2} W_1 e^{\frac{2x}{3}} + \frac{1}{2} - \frac{3\, e^{\frac{-3x(1+W_1)}{\varepsilon}}}{e^{\frac{-3x(1+W_1)}{\varepsilon}} + 1} \tag{3.20}$$

where

$$W_1 = LambertW\left(\frac{2333\, e^{\frac{4999}{3000}}}{1000}\right). \tag{3.21}$$

For $\varepsilon = 10^{-3}$, we get the following table of results

x	Present (τ = 1/2 and δ = 1/2)	Present (τ = 1 and δ = 0)	Results in [16] for $q(x)$ = 0
0	−1.0000000	−1.0000000	−1.0000000
0.5 ε	3.2092800	2.5269382	2.5269382
1 ε	3.3213014	3.7624247	3.7624247
0.1	3.3877260	3.9995000	3.9995000
0.2	3.4537943	3.9995000	3.9995000
0.3	3.5203642	3.9995000	3.9995000
0.4	3.5874244	3.9995000	3.9995000
0.5	3.6549642	3.9995000	3.9995000
0.6	3.7229735	3.9995000	3.9995000
0.7	3.7914416	3.9995000	3.9995000
0.8	3.8603585	3.9995000	3.9995000
0.9	3.9297146	3.9995000	3.9995000
1	3.9995000	3.9995000	3.9995000

Table 5: Approximate solution for Example 4.

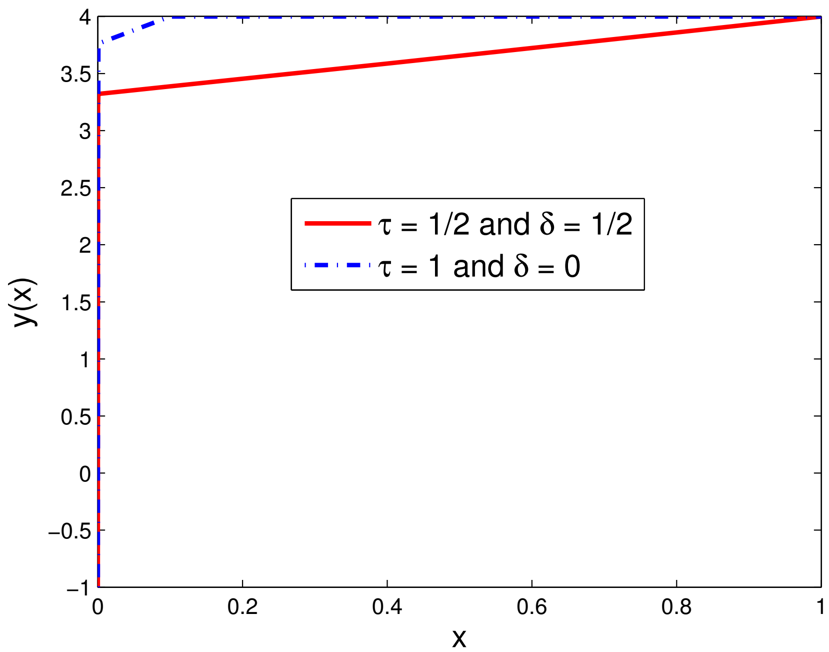

Figure 3: Approximate solution for Example 4 for different values of τ and δ.

4. Conclusion

In this work, we employ the perturbation technique to solve four deformable fractional examples, using various values of ϵ. We present the computational outcomes and compare them with the solutions provided by others works. Our findings indicate that our proposed method closely approximates the exact solution.

This perturbation approach enables us to find a solution that approaches the exact result, without requiring any information about the specific solution needed by the d'Alembert method. It offers a way to address the challenges posed by the d'Alembert method. It is a way to overcome the obstacles of the d'Alembert method. We disrupt the problem by multiplying y'' by ε. An interesting perspective for future research is to extend the present study to the case where $P(x, y)$ and $q(x)$ are discontinuous. Such problems naturally arise in physical models with abrupt parameter variations, leading to sharp transitions or internal layers. Adapting our perturbative approach with suitable matching conditions at the discontinuity points would open promising directions for further analysis.

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