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Double inertial steps with single projection method for variational inequalities involving non-Lipschitz mappings

Duong Viet Thong[a,*], Vu Tien Dung[b], Hoang Thi Thanh Tam[c], Pham Thi Huong Huyen[c]

[a]*Fundamental Sciences Faculty, National Economics University, Hanoi City, Vietnam*
[b]*Department of Mathematics, University of Science, Vietnam National University, 334 Nguyen Trai, Thanh Xuan, Hanoi, Vietnam*
[c]*Faculty of Economics Mathematics, National Economics University, Hanoi City, Vietnam*

Abstract. In this work, we investigate a projection-type method with double inertial terms for solving pseudomonotone variational inequality in real Hilbert spaces. The proposed algorithm combines the techniques of double inertial steps and the projection and contraction method. A sufficient condition for weak convergence is established under pseudomonotonicity and uniform continuity assumptions. Finally, some numerical results are also provided to demonstrate the effectiveness of our method and compare it with recent related methods in the literature

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty, closed and convex subset of H. Let $F : H \to H$ be a single-valued continuous mapping. The classical variational inequality (VI) in the sense of Fichera [15] and Stampacchia [28] (see also Kinderlehrer and Stampacchia [20], Facchinei and Pang [14]) which is formulated as follows: Find a point $x^* \in C$ such that

$$\langle Fx^*, x - x^* \rangle \geq 0 \quad \forall x \in C. \tag{1}$$

We denote by $Sol(C, F)$ the solution set of the VI (1), which is assumed to be nonempty.

One of the projection methods for solving the problem (1) is extragradient method introduced Korpelevich [21] (also by Antipin [3] independently). The extragradient method has the following form:

$$\begin{cases} y_n = P_C(x_n - \tau F x_n), \\ x_{n+1} = P_C(x_n - \tau F y_n), \end{cases} \tag{2}$$

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* Corresponding author: Duong Viet Thong
Email addresses: thongduongviet@neu.edu.vn (Duong Viet Thong), duzngvt@gmail.com (Vu Tien Dung), tamht@neu.edu.vn (Hoang Thi Thanh Tam), huyenph@neu.edu.vn (Pham Thi Huong Huyen)
ORCID iDs: https://orcid.org/0000-0003-1753-7237 (Duong Viet Thong), https://orcid.org/0009-0003-5380-2170 (Hoang Thi Thanh Tam), https://orcid.org/0009-0007-6249-2396 (Pham Thi Huong Huyen)

where $\tau \in (0, 1/L)$ and $F : C \to H$ be monotone and L-Lipschitz continuous operator. Recently, the extragradient method method has given conclusive results assuming monotone and the Lipschitz continuous mappings (see, e.g., [1, 4, 7, 8, 12, 31]).

Observe that the extragradient method requires the evaluation of two orthogonal projections onto C per iteration. The first method which overcomes this obstacle is the projection and contraction method (PC) of He [17] and Sun [29]. For each iteration $n \in \mathbb{N}$ generates point y_n in the spirit of (2):

$$y_n = P_C(x_n - \tau_n F x_n),$$

and then the next iterate x_{n+1} is generated via the following

$$x_{n+1} = x_n - \gamma \Delta_n d(x_n, y_n), \tag{3}$$

where $\gamma \in (0, 2)$,

$$\Delta_n := \frac{\langle x_n - y_n, d(x_n, y_n)\rangle}{\|d(x_n, y_n)\|^2},$$

and

$$d(x_n, y_n) := x_n - y_n - \tau_n(Fx_n - Fy_n),$$

where $F : C \to H$ be monotone and L-Lipschitz continuous operator and $\tau_n \in (0, 1/L)$ or τ_n is updated by some adaptive rule as follows:

$$\tau_n \|Fx_n - Fy_n\| \le \mu \|x_n - y_n\|, \;\; \mu \in (0, 1).$$

The weak convergence analysis of (3) has been studied by several authors in the literature. See, for example, [5, 11]. We see that the method (3) only needs one projection to be computed per iteration. Therefore, it may be more efficient and cheaper than the extragradient method.

The inertial technique, which originates from a discrete version of a second order dissipative dynamical system [2, 26], is often used to accelerate the convergence rate of algorithms. Recently, inertial techniques have been applied to design algorithms for solving variational inequalities and it received great attention from many authors, see, e.g., [11, 13, 27, 30, 32, 34, 35].

The main purpose of this paper is to further improve the projection and contraction method with double inertial extrapolation steps for solving the problem (1) where F is pseudomonotone and non-Lipschitz continuous mappings in Hilbert spaces. We establish weak convergence of the sequence generated by our algorithm under the assumptions of uniformly continuous on bounded subsets of H instead of F is Lipschitz continuous as in [32].

The article is organized as follows: in Sect. 2, we recall some concepts and lemmas which will be used in the proof of main results and, in Sect. 3, a general inertial projection and contraction method with a line-search procedure is introduced and its weak convergence. In Sect. 4, a numerical example is reported to illustrate the performance of the proposed algorithm. Sect. 5, final conclusions are given.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. The weak convergence of $\{x_n\}$ to x is denoted by $x_n \rightharpoonup x$ as $n \to \infty$, while the strong convergence of $\{x_n\}$ to x is written as $x_n \to x$ as $n \to \infty$. For all $x, y \in H$ we have

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle.$$

Definition 2.1. ([6]) *Let $T : H \to H$ be an operator. Then*

1. *T is called L-Lipschitz continuous with constant* $L > 0$ *if*

$$\|Tx - Ty\| \leq L\|x - y\| \quad \forall x, y \in H,$$

if $L = 1$ *then the operator T is called nonexpansive and if* $L \in (0, 1)$*, T is called a contraction.*

2. *T is called monotone if*

$$\langle Tx - Ty, x - y\rangle \geq 0 \quad \forall x, y \in H;$$

3. *T is called pseudomonotone in the sense of Karamardian [19] if*

$$\langle Tx, y - x\rangle \geq 0 \Longrightarrow \langle Ty, y - x\rangle \geq 0 \quad \forall x, y \in H; \tag{4}$$

4. *T is called* α*-strongly monotone if there exists a constant* $\alpha > 0$ *such that*

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|x - y\|^2 \quad \forall x, y \in H;$$

5. *T is called* α*-strongly pseudomonotone if there exists a constant* $\alpha > 0$ *such that*

$$\langle Tx, y - x\rangle \geq 0 \Longrightarrow \langle Ty, y - x\rangle \geq \alpha\|x - y\|^2 \quad \forall x, y \in H;$$

6. *The operator T is called sequentially weakly continuous if for each sequence* $\{x_n\}$ *we have:* x_n *converges weakly to x implies* Tx_n *converges weakly to Tx.*

We note that (4) is only one of the definitions of pseudomonotonicity which can be found in the literature. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that $\|x - P_C x\| \leq \|x - y\| \, \forall y \in C$. P_C is called the *metric projection* of H onto C. It is known that P_C is nonexpansive. For properties of the metric projection, the interested reader could be referred to Section 3 in [16].

We need to recall the following Lemmas, which are useful for the later convergence analysis.

Lemma 2.2. ([2]) *Let* $\{\varphi_n\}$*,* $\{\delta_n\}$ *and* $\{\alpha_n\}$ *be sequences in* $[0, +\infty)$ *such that*

$$\varphi_{n+1} \leq \varphi_n + \alpha_n(\varphi_n - \varphi_{n-1}) + \delta_n \; \forall n \geq 1, \sum_{n=1}^{+\infty} \delta_n < +\infty,$$

and there exists a real number α *with* $0 \leq \alpha_n \leq \alpha < 1$ *for all* $n \in \mathbb{N}$*. Then the following hold:*
i) $\sum_{n=1}^{+\infty}[\varphi_n - \varphi_{n-1}]_+ < +\infty$*, where* $[t]_+ := \max\{t, 0\}$*;*
ii) there exists $\varphi^* \in [0, +\infty)$ *such that* $\lim_{n\to+\infty} \varphi_n = \varphi^*$.

Lemma 2.3. ([25]) *Let C be a nonempty set of H and* $\{x_n\}$ *be a sequence in H such that the following two conditions hold:*
i) for every $x \in C$*,* $\lim_{n\to\infty} \|x_n - x\|$ *exists;*
ii) every sequential weak cluster point of $\{x_n\}$ *is in C.*
Then $\{x_n\}$ *converges weakly to a point in C.*

Lemma 2.4. [10] *For* $x \in H$ *and* $\alpha \geq \beta > 0$ *the following inequalities hold.*

$$\frac{\|x - P_C(x - \alpha Fx)\|}{\alpha} \leq \frac{\|x - P_C(x - \beta Fx)\|}{\beta},$$

$$\|x - P_C(x - \beta Fx)\| \leq \|x - P_C(x - \alpha Fx)\|,$$

where $F : C \to H$ *is a nonlinear mapping.*

Lemma 2.5. [18] *Let* H_1 *and* H_2 *be two real Hilbert spaces. Suppose* $F : H_1 \to H_2$ *is uniformly continuous on bounded subsets of* H_1 *and M is a bounded subset of* H_1*. Then* $F(M)$ *(the image of M under F) is bounded.*

Lemma 2.6. [9] *Consider the problem* $Sol(C, F)$ *with C being a nonempty, closed, convex subset of a real Hilbert space H and* $F : C \to H$ *being pseudo-monotone and continuous. Then,* x^* *is a solution of* $Sol(C, F)$ *if and only if*

$$\langle Fx, x - x^*\rangle \geq 0 \quad \forall x \in C.$$

3. Weak Convergence Analysis

In this section, we propose modified projection method for solving VIs. We assume that the following conditions hold:

Condition 3.1. *The solution set* $\mathrm{Sol}(C,F)$ *is nonempty.*

Condition 3.2. *The mapping* $F : H \to H$ *is pseudomonotone on* H*, that is,*

$$\langle Fx, y-x\rangle \geq 0 \Longrightarrow \langle Fy, y-x\rangle \geq 0 \quad \forall x,y \in H.$$

In addition, the mapping $F : H \to H$ satisfies the condition

$$\{z_n\} \subset C,\ z_n \rightharpoonup z \Longrightarrow \|Fz\| \leq \liminf_{n\to\infty} \|Fz_n\|. \tag{5}$$

Condition 3.3. $F : H \to H$ *is uniformly continuous on bounded subsets of* H*.*

The proposed algorithm is of the form:

Algorithm 3.4.

Initialization: *Given* $\rho, l \in (0,1), \mu \in (0,1), \gamma \in (0,2), \theta \in [0,1], \alpha, \lambda \in (0,1)$*. Let* $s_0, s_1 \in H$ *be arbitrary.*
Iterative Steps: *Given the current iterate* s_n*, calculate* s_{n+1} *as follows:*

Step 1. *Compute*

$$\begin{cases} u_n = s_n + \alpha(s_n - s_{n-1}) \\ w_n = s_n + \theta(s_n - s_{n-1}) \end{cases}$$

and

$$y_n = P_C(w_n - \tau_n F w_n),$$

where τ_n *is chosen to be the largest* $\tau \in \{\rho, \rho l, \rho l^2, ...\}$ *satisfying*

$$\tau\|Fy_n - Fw_n\| \leq \mu\|y_n - w_n\|. \tag{6}$$

If $w_n = y_n$ *or* $Fw_n = 0$ *then stop and* w_n *is a solution of (1). Otherwise*

Step 2. *Compute*

$$z_n = w_n - \gamma\Delta_n\Phi_n,$$

where

$$\Delta_n := \begin{cases} \dfrac{\langle w_n - y_n, \Phi_n\rangle}{\|\Phi_n\|^2} & \text{if } \Phi_n \neq 0, \\ 0 & \text{if } \Phi_n = 0, \end{cases}$$

and

$$\Phi_n := w_n - y_n - \tau_n(Fw_n - Fy_n).$$

Step 3. *Compute*

$$s_{n+1} = (1-\lambda)u_n + \lambda z_n.$$

Set $n := n+1$ *and go to* ***Step 1****.*

Lemma 3.5. *Assume that the mapping* $F : H \to H$ *is uniformly continuous on bounded subsets of* H*. The Armijo-line search rule (6) is well defined. In addition, we have* $\tau_n \leq \rho$*.*

Proof. If $w_n \in Sol(C, F)$ then $w_n = P_C(w_n - \rho F w_n)$ and $m_n = 0$. We consider the situation $w_n \notin Sol(C, F)$ and assume the contrary that for all m we have

$$\rho l^m \|FP_C(w_n - \rho l^m F w_n) - F w_n\| > \mu \|P_C(w_n - \rho l^m F w_n) - w_n\|. \tag{7}$$

This implies that

$$\|F w_n - FP_C(w_n - \rho l^m F w_n)\| > \mu \frac{\|w_n - P_C(w_n - \rho l^m F w_n)\|}{\rho l^m}. \tag{8}$$

We consider the follows cases.
Case 1: $w_n \in C$. Since P_C is continuous, we have $\lim_{m\to\infty} \|w_n - P_C(w_n - \rho l^m F w_n)\| = 0$. From the uniform continuity of the operator F on bounded subsets of C it implies that

$$\lim_{m\to\infty} \|F w_n - FP_C(w_n - \rho l^m F w_n)\| = 0. \tag{9}$$

Combining (8) and (9) we get

$$\lim_{m\to\infty} \frac{\|w_n - P_C(w_n - \rho l^m F w_n)\|}{\rho l^m} = 0. \tag{10}$$

Assume that $z_m = P_C(w_n - \rho l^m F w_n)$ we have

$$\langle z_m - w_n + \rho l^m F w_n, x - z_m\rangle \geq 0 \ \ \forall x \in C.$$

This implies that

$$\langle \frac{z_m - w_n}{\gamma l^m}, x - z_m\rangle + \langle F w_n, x - z_m\rangle \geq 0 \ \ \forall x \in C. \tag{11}$$

Taking the limit $m \to \infty$ in (11) and using (10) we obtain

$$\langle F w_n, x - w_n\rangle \geq 0 \ \ \forall x \in C,$$

which implies that $w_n \in Sol(C, F)$ this is a contraction.
Case 2: $w_n \notin C$. Since F is uniformly continuous on bounded subsets of H. Hence, there exists $M > 0, \delta > 0$ such that

$$\|Fz - F(P_C w_n)\| \leq M \ \ \forall z \in V,$$

where $V := \{u \in H : \|u - P_C w_n\| \leq \delta\}$. By $l \in (0, 1)$, it implies that there exists $N \in \mathbb{N}$ such that

$$\rho l^m \|F w_n\| < \delta \ \ \forall m > N.$$

Thus, we get

$$\|P_C(w_n - \rho l^m F w_n) - P_C w_n\| \leq \rho l^m \|F w_n\| < \rho \ \ \forall m > N.$$

This implies that $P_C(w_n - \rho l^m F w_n) \in V \ \ \forall m > N$. Therefore, we deduce

$$\|F(P_C(w_n - \rho l^m F w_n)) - F(P_C w_n)\| \leq M \ \ \forall m > N,$$

which implies that the sequence $\{F(P_C(w_n - \rho l^m F w_n))\}_m$ is bounded. Hence

$$\lim_{m\to\infty} \rho l^m \|F(P_C(w_n - \rho l^m F w_n)) - F w_n\| = 0,$$

which contradits (7). □

Lemma 3.6. *Assume that Conditions 3.1–3.3 hold. Let $\{w_n\}$ be any sequence generated by Algorithm 3.4. If there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $\{w_{n_k}\}$ converges weakly to $z \in C$ and $\lim_{k\to\infty} \|w_{n_k} - y_{n_k}\| = 0$ then $z \in Sol(C, F)$.*

Proof. We have $y_{n_k} = P_C(w_{n_k} - \tau_{n_k} F w_{n_k})$ thus,

$$\langle w_{n_k} - \tau_{n_k} F w_{n_k} - y_{n_k}, x - y_{n_k}\rangle \leq 0 \ \ \forall x \in C.$$

or equivalently

$$\frac{1}{\tau_{n_k}}\langle w_{n_k} - y_{n_k}, x - y_{n_k}\rangle \leq \langle F w_{n_k}, x - y_{n_k}\rangle \ \ \forall x \in C.$$

This implies that

$$\frac{1}{\tau_{n_k}}\langle w_{n_k} - y_{n_k}, x - y_{n_k}\rangle + \langle F w_{n_k}, y_{n_k} - w_{n_k}\rangle \leq \langle F w_{n_k}, x - w_{n_k}\rangle \ \ \forall x \in C. \tag{12}$$

Now, we show that

$$\liminf_{k\to\infty}\langle F w_{n_k}, x - w_{n_k}\rangle \geq 0. \tag{13}$$

For showing this, we consider two possible cases. Suppose first that $\liminf_{k\to\infty} \tau_{n_k} > 0$. We have $\{w_{n_k}\}$ is a bounded sequence, F is uniformly continuous on bounded subsets of C. By Lemma 2.5, we get that $\{F w_{n_k}\}$ is bounded. Taking $k \to \infty$ in (12) since $\|w_{n_k} - y_{n_k}\| \to 0$, we get

$$\liminf_{k\to\infty}\langle F w_{n_k}, x - w_{n_k}\rangle \geq 0.$$

Now, we assume that $\liminf_{k\to\infty} \tau_{n_k} = 0$. Assume $z_{n_k} = P_C(w_{n_k} - \tau_{n_k}.l^{-1} F w_{n_k})$, we have $\tau_{n_k} l^{-1} > \tau_{n_k}$. Applying Lemma 2.4, we obtain

$$\|w_{n_k} - z_{n_k}\| \leq \frac{1}{l}\|w_{n_k} - y_{n_k}\| \to 0 \text{ as } k \to \infty.$$

Consequently, $z_{n_k} \rightharpoonup z \in C$, this implies that $\{z_{n_k}\}$ is bounded, which the uniformly continuity of the operator F on bounded subsets of C it follows that

$$\|F w_{n_k} - F z_{n_k}\| \to 0 \text{ as } k \to \infty. \tag{14}$$

By the Armijo line-search rule (6) we have

$$\tau_{n_k} l^{-1}\|F w_{n_k} - F P_C(w_{n_k} - \tau_{n_k} l^{-1} F w_{n_k})\| > \mu\|w_{n_k} - P_C(w_{n_k} - \tau_{n_k} l^{-1} F w_{n_k})\|.$$

That is,

$$\frac{1}{\mu}\|F w_{n_k} - F z_{n_k}\| > \frac{\|w_{n_k} - z_{n_k}\|}{\tau_{n_k} l^{-1}}. \tag{15}$$

Combining (14) and (15) we obtain

$$\lim_{k\to\infty}\frac{\|w_{n_k} - z_{n_k}\|}{\tau_{n_k} l^{-1}} = 0.$$

Furthermore, we have

$$\langle w_{n_k} - \tau_{n_k} l^{-1} F w_{n_k} - z_{n_k}, x - z_{n_k}\rangle \leq 0 \ \ \forall x \in C.$$

This implies that

$$\frac{1}{\tau_{n_k} l^{-1}} \langle w_{n_k} - z_{n_k}, x - z_{n_k} \rangle + \langle F w_{n_k}, z_{n_k} - w_{n_k} \rangle \le \langle F w_{n_k}, x - w_{n_k} \rangle \ \ \forall x \in C. \tag{16}$$

Taking the limit $k \to \infty$ in (16) we get

$$\liminf_{k\to\infty} \langle F w_{n_k}, x - w_{n_k} \rangle \ge 0.$$

Therefore, the inequality (13) is proved. Next, we show that $z \in Sol(C, F)$.

Now we choose a sequence $\{\epsilon_k\}$ of positive numbers decreasing and tending to 0. For each k, we denote by N_k the smallest positive integer such that

$$\langle F w_{n_j}, x - w_{n_j} \rangle + \epsilon_k \ge 0 \ \ \forall j \ge N_k, \tag{17}$$

where the existence of N_k follows from (13). Since $\{\epsilon_k\}$ is decreasing, it is easy to see that the sequence $\{N_k\}$ is increasing. Furthermore, for each k, since $\{w_{N_k}\} \subset C$ we have $F w_{N_k} \neq 0$ and, setting

$$u_{N_k} = \frac{F w_{N_k}}{\|F w_{N_k}\|^2},$$

we have $\langle F w_{N_k}, u_{N_k} \rangle = 1$ for each k. Now, we can deduce from (17) that for each k

$$\langle F w_{N_k}, x + \epsilon_k u_{N_k} - w_{N_k} \rangle \ge 0.$$

Since the fact that F is pseudo-monotone, we get

$$\langle F(x + \epsilon_k u_{N_k}), x + \epsilon_k u_{N_k} - w_{N_k} \rangle \ge 0.$$

This implies that

$$\langle F x, x - w_{N_k} \rangle \ge \langle F x - F(x + \epsilon_k u_{N_k}), x + \epsilon_k u_{N_k} - w_{N_k} \rangle - \epsilon_k \langle F x, u_{N_k} \rangle. \tag{18}$$

Now, we show that $\lim_{k\to\infty} \epsilon_k u_{N_k} = 0$. Indeed, we have $w_{n_k} \rightharpoonup z$ as $k \to \infty$. Since F satisfies the condition (5). We have

$$0 < \|Fz\| \le \liminf_{k\to\infty} \|F w_{n_k}\| \ \ \text{(note that } Fz \neq 0 \text{ otherwise, } z \text{ is a solution).}$$

Since $\{w_{N_k}\} \subset \{w_{n_k}\}$ and $\epsilon_k \to 0$ as $k \to \infty$, we obtain

$$0 \le \limsup_{k\to\infty} \|\epsilon_k u_{N_k}\| = \limsup_{k\to\infty} \left(\frac{\epsilon_k}{\|F w_{n_k}\|} \right) \le \frac{\limsup_{k\to\infty} \epsilon_k}{\liminf_{k\to\infty} \|F w_{n_k}\|} = 0,$$

which implies that $\lim_{k\to\infty} \epsilon_k u_{N_k} = 0$.

Now, letting $k \to \infty$, then the right hand side of (18) tends to zero by F is uniformly continuous, $\{w_{N_k}\}, \{u_{N_k}\}$ are bounded and $\lim_{k\to\infty} \epsilon_k u_{N_k} = 0$. Thus, we get

$$\liminf_{k\to\infty} \langle F x, x - w_{N_k} \rangle \ge 0.$$

Hence, for all $x \in C$ we have

$$\langle F x, x - z \rangle = \lim_{k\to\infty} \langle F x, x - w_{N_k} \rangle = \liminf_{k\to\infty} \langle F x, x - w_{N_k} \rangle \ge 0.$$

By Lemma 2.6 we obtain $z \in Sol(C, F)$ and the proof is complete. □

Lemma 3.7. *Assume that Conditions 3.1–3.2 hold. Let* $\{z_n\}, \{w_n\}, \{y_n\}$ *be three sequences generated by Algorithm 3.4. Then the following statements hold:*

$$\|z_n - x^*\|^2 \leq \|w_n - x^*\|^2 - \frac{2-\gamma}{\gamma}\|z_n - w_n\|^2 \quad \forall x^* \in Sol(C, F)$$

and

$$\|w_n - y_n\|^2 \leq \frac{(1+\mu)^2}{[(1-\mu)\gamma]^2}\|w_n - z_n\|^2. \tag{19}$$

Proof. We prove the first confirmation. Indeed, from $y_n = P_C(w_n - \tau_n F w_n)$ we get

$$\langle w_n - y_n - \tau_n F w_n, y_n - x^* \rangle \geq 0, \tag{20}$$

By the pseudomonotonicity of F and $x^* \in Sol(C, F)$ we have

$$\langle F y_n, y_n - x^* \rangle \geq \langle F x^*, y_n - x^* \rangle \geq 0. \tag{21}$$

Adding (20) and (21) we get

$$\langle y_n - x^*, w_n - y_n - \tau_n(F w_n - F y_n) \rangle \geq 0.$$

Thus

$$\langle y_n - x^*, \Phi_n \rangle = \langle y_n - x^*, w_n - y_n - \tau_n(F w_n - F y_n) \rangle \geq 0. \tag{22}$$

Using (22) we have

$$\begin{aligned}\langle w_n - x^*, \Phi_n \rangle &= \langle w_n - y_n, \Phi_n \rangle + \langle y_n - x^*, \Phi_n \rangle \\ &\geq \langle w_n - y_n, \Phi_n \rangle.\end{aligned} \tag{23}$$

On the other hand, we have

$$\|z_n - x^*\|^2 = \|w_n - \gamma\Delta_n\Phi_n - x^*\|^2 = \|w_n - x^*\|^2 - 2\gamma\Delta_n\langle w_n - x^*, \Phi_n \rangle + \gamma^2\eta_n^2\|\Phi_n\|^2. \tag{24}$$

It implies from (23) and (24) that

$$\|z_n - x^*\|^2 \leq \|w_n - x^*\|^2 - 2\gamma\Delta_n\langle w_n - y_n, \Phi_n \rangle + \gamma^2\eta_n^2\|\Phi_n\|^2. \tag{25}$$

Since $\Delta_n = \dfrac{\langle w_n - y_n, \Phi_n \rangle}{\|\Phi_n\|^2}$, we get

$$\Delta_n\|\Phi_n\|^2 = \langle w_n - y_n, \Phi_n \rangle. \tag{26}$$

Substituting (26) into (25), we get

$$\begin{aligned}\|z_n - x^*\|^2 &\leq \|w_n - x^*\|^2 - 2\gamma\Delta_n\langle w_n - y_n, \Phi_n \rangle + \gamma^2\Delta_n\langle w_n - y_n, \Phi_n \rangle \\ &= \|w_n - x^*\|^2 - (2-\gamma)\gamma\Delta_n\langle w_n - y_n, \Phi_n \rangle \\ &= \|w_n - x^*\|^2 - \gamma(2-\gamma)\|\Delta_n\Phi_n\|^2 \\ &= \|w_n - x^*\|^2 - \frac{2-\gamma}{\gamma}\|\gamma\Delta_n\Phi_n\|^2 \\ &= \|w_n - x^*\|^2 - \frac{2-\gamma}{\gamma}\|w_n - z_n\|^2.\end{aligned}$$

Next, we show the second statement. Using (6) we have

$$\begin{aligned}\|\Phi_n\| &= \|w_n - y_n - \tau_n(Fw_n - Fy_n)\| \\ &\geq \|w_n - y_n\| - \tau_n\|Fw_n - Fy_n\| \\ &\geq \|w_n - y_n\| - \mu\|w_n - y_n\| \\ &= (1-\mu)\|w_n - y_n\|,\end{aligned}$$

and it is easy to see that $\|\Phi_n\| \leq (1+\mu)\|w_n - y_n\|$. Now, we will shall the lemma. We have

$$\begin{aligned}\langle w_n - y_n, \Phi_n\rangle &= \langle w_n - y_n, w_n - y_n - \tau_n(Fw_n - Fy_n)\rangle \\ &= \|w_n - y_n\|^2 - \tau_n\langle w_n - y_n, Fw_n - Fy_n\rangle \\ &\geq \|w_n - y_n\|^2 - \tau_n\|w_n - y_n\|\|Fw_n - Fy_n\| \\ &\geq \|w_n - y_n\|^2 - \mu\|w_n - y_n\|^2 \\ &= (1-\mu)\|w_n - y_n\|^2. \end{aligned} \tag{27}$$

This implies that

$$\begin{aligned}\|w_n - y_n\|^2 \leq \frac{1}{1-\mu}\langle w_n - y_n, \Phi_n\rangle = \frac{1}{1-\mu}\Delta_n\|\Phi_n\|^2 &= \frac{1}{\Delta_n(1-\mu)}\|\Delta_n\Phi_n\|^2 \\ &= \frac{1}{\Delta_n(1-\mu)\gamma^2}\|w_n - z_n\|^2. \end{aligned} \tag{28}$$

Using the definition of $\{\Delta_n\}$, (27) and $\|\Phi_n\| \leq (1+\mu)\|w_n - y_n\|$ we get

$$\frac{1}{\Delta_n} = \frac{\|\Phi_n\|^2}{\langle w_n - y_n, \Phi_n\rangle} \leq \frac{1}{1-\mu}\frac{\|\Phi_n\|^2}{\|w_n - y_n\|^2} \leq \frac{\left(1+\mu\right)^2}{(1-\mu)}. \tag{29}$$

Substituting (29) into (28) we obtain

$$\|w_n - y_n\|^2 \leq \frac{(1+\mu)^2}{[(1-\mu)\gamma]^2}\|w_n - z_n\|^2. \tag{30}$$

The proof of lemma is completed. □

Now, we give our main results.

Theorem 3.8. *Assume that Conditions 3.1–3.3 hold. If the factor λ of Algorithm 3.4 is chosen such that $0 < \lambda < \frac{1}{1+\delta}$ and $0 \leq \alpha < \frac{\delta - \sqrt{2\delta}}{\delta}$ for some $\delta > 2$ then the sequence $\{s_n\}$ is generated by Algorithm 3.4 converges weakly to an element $x^* \in Sol(C, F)$.*

Proof. **Claim 1.**

$$\Lambda_{n+1} - \Lambda_n \leq -\Big[\frac{(1-\lambda)}{\lambda}(1-\alpha)^2 - (1-\lambda)(1+\alpha)\alpha - \lambda(1+\theta)\theta\Big]\|s_{n+1} - s_n\|^2,$$

where

$$\Lambda_n := \|s_n - x^*\|^2 - [(1-\lambda)\alpha + \lambda\theta]\|s_{n-1} - x^*\|^2 + \frac{(1-\lambda)}{\lambda}(1-\alpha)\|s_n - s_{n-1}\|^2.$$

Indeed, by Lemma 3.7, we deduce

$$\|z_n - x^*\|^2 \leq \|w_n - x^*\|^2. \tag{31}$$

On the other hand, we have

$$\begin{aligned}\|s_{n+1}-x^*\|^2 &= \|(1-\lambda)u_n+\lambda z_n-x^*\|^2\\ &= \|(1-\lambda)(u_n-x^*)+\lambda(z_n-x^*)\|^2\\ &= (1-\lambda)\|u_n-x^*\|^2+\lambda\|z_n-x^*\|^2-\lambda(1-\lambda)\|u_n-z_n\|^2. \end{aligned} \tag{32}$$

Substituting the inequality (31) into (32), we deduce

$$\|s_{n+1}-x^*\|^2 \le (1-\lambda)\|u_n-x^*\|^2+\lambda\|w_n-x^*\|^2-\lambda(1-\lambda)\|u_n-z_n\|^2. \tag{33}$$

Note that

$$s_{n+1}=(1-\lambda)u_n+\lambda z_n$$

and this implies that

$$z_n-u_n=\frac{1}{\lambda}(s_{n+1}-u_n). \tag{34}$$

Substituting (34) into (33) we get

$$\|s_{n+1}-x^*\|^2 \le (1-\lambda)\|u_n-x^*\|^2+\lambda\|w_n-x^*\|^2-\frac{(1-\lambda)}{\lambda}\|s_{n+1}-u_n\|^2. \tag{35}$$

From the definition of w_n, we get

$$\begin{aligned}\|w_n-x^*\|^2 &= \|s_n+\theta(s_n-s_{n-1})-x^*\|^2\\ &= \|(1+\theta)(s_n-x^*)-\theta(s_{n-1}-x^*)\|^2\\ &= (1+\theta)\|s_n-x^*\|^2-\theta\|s_{n-1}-x^*\|^2+(1+\theta)\theta\|s_n-s_{n-1}\|^2\end{aligned} \tag{36}$$

and

$$\begin{aligned}\|u_n-x^*\|^2 &= \|s_n+\alpha(s_n-s_{n-1})-x^*\|^2\\ &= \|(1+\alpha)(s_n-x^*)-\alpha(s_{n-1}-x^*)\|^2\\ &= (1+\alpha)\|s_n-x^*\|^2-\alpha\|s_{n-1}-x^*\|^2+(1+\alpha)\alpha\|s_n-s_{n-1}\|^2.\end{aligned} \tag{37}$$

Substituting (36) and (37) into (35), we have

$$\begin{aligned}\|s_{n+1}-x^*\|^2 &\le (1-\lambda)((1+\alpha)\|s_n-x^*\|^2-\alpha\|s_{n-1}-x^*\|^2\\ &\quad+(1+\alpha)\alpha\|s_n-s_{n-1}\|^2)+\lambda((1+\theta)\|s_n-x^*\|^2-\theta\|s_{n-1}-x^*\|^2\\ &\quad+(1+\theta)\theta\|s_n-s_{n-1}\|^2)-\frac{(1-\lambda)}{\lambda}\|s_{n+1}-u_n\|^2\\ &= [(1-\lambda)(1+\alpha)+\lambda(1+\theta)]\|s_n-x^*\|^2-[(1-\lambda)\alpha+\lambda\theta]\|s_{n-1}-x^*\|^2\\ &\quad+((1-\lambda)(1+\alpha)\alpha+\lambda(1+\theta)\theta\|s_n-s_{n-1}\|^2-\frac{(1-\lambda)}{\lambda}\|s_{n+1}-u_n\|^2.\end{aligned} \tag{38}$$

Moreover, we get

$$\begin{aligned}\|s_{n+1}-u_n\|^2 &= \|s_{n+1}-s_n-\alpha(s_n-s_{n-1})\|^2\\ &= \|s_{n+1}-s_n\|^2+\alpha^2\|s_n-s_{n-1}\|^2-2\alpha\langle s_{n+1}-s_n, s_n-s_{n-1}\rangle\\ &\ge \|s_{n+1}-s_n\|^2+\alpha^2\|s_n-s_{n-1}\|^2-2\alpha\|s_{n+1}-s_n\|\|s_n-s_{n-1}\|\\ &\ge (1-\alpha)\|s_{n+1}-s_n\|^2+\left(\alpha^2-\alpha\right)\|s_n-s_{n-1}\|^2.\end{aligned} \tag{39}$$

Substituting (39) into (38) we deduce

$$\begin{aligned}||s_{n+1}-x^*||^2 &\le [(1-\lambda)(1+\alpha)+\lambda(1+\theta)]||s_n-x^*||^2-[(1-\lambda)\alpha+\lambda\theta]||s_{n-1}-x^*||^2\\ &\quad+[(1-\lambda)(1+\alpha)\alpha+\lambda(1+\theta)\theta-(\alpha^2-\alpha)\frac{(1-\lambda)}{\lambda}]||s_n-s_{n-1}||^2\\ &\quad-\frac{(1-\lambda)}{\lambda}(1-\alpha)||s_{n+1}-s_n||^2 \qquad (40)\\ &\le ||s_n-x^*||^2+[(1-\lambda)\alpha+\lambda\theta][||s_n-x^*||^2-||s_{n-1}-x^*||^2]\\ &\quad+[(1-\lambda)(1+\alpha)\alpha+\lambda(1+\theta)\theta-(\alpha^2-\alpha)\frac{(1-\lambda)}{\lambda}]||s_n-s_{n-1}||^2. \qquad (41)\end{aligned}$$

Let

$$\Lambda_n := ||s_n-x^*||^2-[(1-\lambda)\alpha+\lambda\theta]||s_{n-1}-x^*||^2+\frac{(1-\lambda)}{\lambda}(1-\alpha)||s_n-s_{n-1}||^2.$$

Using (40), we get

$$\Lambda_{n+1}-\Lambda_n \le -\Big[\frac{(1-\lambda)}{\lambda}(1-\alpha)^2-(1-\lambda)(1+\alpha)\alpha-\lambda(1+\theta)\theta\Big]||s_{n+1}-s_n||^2.$$

Claim 2.

$$\lim_{n\to\infty}||s_n-x^*||^2 \text{ exists.}$$

Indeed, we use the assumption $\lambda<\dfrac{1}{1+\delta}, \delta>2$ we get $\dfrac{1-\lambda}{\lambda}>\delta$ and we also obtain

$$\begin{aligned}\frac{(1-\lambda)}{\lambda}(1-\alpha)^2-(1-\lambda)(1+\alpha)\alpha-\lambda(1+\theta)\theta &> \delta(1-\alpha)^2-2(1-\lambda)-2\lambda\\ &= \delta\alpha^2-2\delta\alpha+\delta-2. \qquad (42)\end{aligned}$$

From $0\le\alpha<\dfrac{\delta-\sqrt{2\delta}}{\delta}$, it follows that $\delta\alpha^2-2\delta\alpha+\delta-2>0$ Using (42), we get

$$\Lambda_{n+1}-\Lambda_n < -(\delta\alpha^2-2\delta\alpha+\delta-2)||s_{n+1}-s_n||^2<0. \qquad (43)$$

Therefore, we have

$$\begin{aligned}\Lambda_n &= ||s_n-x^*||^2-[(1-\lambda)\alpha+\lambda\theta]||s_{n-1}-x^*||^2+\frac{(1-\lambda)}{\lambda}(1-\alpha)||s_n-s_{n-1}||^2\\ &\ge ||s_n-x^*||^2-\epsilon||s_{n-1}-x^*||^2,\end{aligned}$$

where $\epsilon:=(1-\lambda)\alpha+\lambda\theta$, (it is easy to see that $\epsilon:=(1-\lambda)\alpha+\lambda\theta\le\max\{\alpha,\theta\}<1$). This follows that

$$\begin{aligned}||s_n-x^*||^2 &\le \epsilon||s_{n-1}-x^*||^2+\Lambda_n\\ &\le \epsilon||s_{n-1}-x^*||^2+\Lambda_{n_0}\\ &\le \cdots\\ &\le \epsilon^{n-n_0}||s_{n_0}-x^*||^2+\Lambda_{n_0}(\epsilon^{n-n_0-1}+\cdots+1)\\ &\le \epsilon^{n-n_0}||s_{n_0}-x^*||^2+\frac{\Lambda_{n_0}}{1-\epsilon}. \qquad (44)\end{aligned}$$

For all $n\ge n_0$, we also have

$$\begin{aligned}\Lambda_{n+1} &= ||s_{n+1}-x^*||^2-\epsilon||s_n-x^*||^2\\ &\ge -\epsilon||s_n-x^*||^2. \qquad (45)\end{aligned}$$

From (44) and (45), we obtain

$$-\Lambda_{n+1} \le \epsilon \|s_n - x^*\|^2 \le \epsilon^{n-n_0+1} \|s_{n_0} - x^*\|^2 + \frac{\epsilon \Lambda_{n_0}}{1-\epsilon}.$$

It follows from (43) that

$$\begin{aligned}(\delta\alpha^2 - 2\delta\alpha + \delta - 2) \sum_{n=n_0}^{k} \|s_{n+1} - s_n\|^2 &\le \Lambda_{n_0} - \Lambda_{k+1} \\ &\le \lambda^{k-n_0+1} \|s_{n_0} - x^*\|^2 + \frac{\Lambda_{n_0}}{1-\epsilon} \\ &\le \|s_{n_0} - x^*\|^2 + \frac{\Lambda_{n_0}}{1-\epsilon}, \quad \forall k > n_0.\end{aligned}$$

This implies

$$\sum_{n=1}^{\infty} \|s_{n+1} - s_n\|^2 < +\infty. \tag{46}$$

This follows that

$$\lim_{n\to\infty} \|s_{n+1} - s_n\| = 0 \tag{47}$$

Moreover, from (41), (46) and Lemma 2.2, we have

$$\lim_{n\to\infty} \|s_n - x^*\|^2 = l.$$

Claim 3.

$$\lim_{n\to\infty} \|w_n - y_n\| = 0.$$

Indeed, we have

$$\begin{aligned}\|s_{n+1} - u_n\|^2 &= \|s_{n+1} - s_n - \alpha(s_n - s_{n-1})\|^2 \\ &= \|s_{n+1} - s_n\|^2 + \alpha^2 \|s_n - s_{n-1}\|^2 - 2\alpha \langle s_{n+1} - s_n, s_n - s_{n-1} \rangle.\end{aligned} \tag{48}$$

Combining (47) and (48), it follows that $\lim_{n\to\infty} \|s_{n+1} - u_n\|^2 = 0$. We have

$$\lim_{n\to\infty} \|u_n - z_n\| = \frac{1}{\lambda} \lim_{n\to\infty} \|s_{n+1} - u_n\| = 0 \tag{49}$$

and

$$\lim_{n\to\infty} \|w_n - s_n\| = \theta \lim_{n\to\infty} \|s_n - s_{n-1}\| = 0. \tag{50}$$

$$\lim_{n\to\infty} \|u_n - s_n\| = \alpha \lim_{n\to\infty} \|s_n - s_{n-1}\| = 0. \tag{51}$$

Combining (50) and (51), we deduce

$$\lim_{n\to\infty} \|u_n - w_n\| = 0. \tag{52}$$

It follows from (49) and (52) that

$$\lim_{n\to\infty} \|w_n - z_n\| = 0. \tag{53}$$

From (30) and (53) we get

$$\lim_{n\to\infty} \|w_n - y_n\| = 0. \tag{54}$$

Claim 4. The sequence $\{s_n\}$ converges weakly to an element of Sol(C,F). Indeed, using Claim 2, we get $\lim_{n\to\infty} \|s_n - x^*\|$ exists, hence the sequence $\{s_n\}$ is bounded. Now, we choose a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ such that $s_{n_k} \rightharpoonup z^*$. By (50), we have $w_{n_k} \rightharpoonup z^*$. From (54), we deduce $\lim_{n\to\infty} \|w_n - y_n\| = 0$, by Lemma 3.6, we get $z^* \in \mathrm{Sol}(C, F)$.
So, we show that:

1. For all $x^* \in \mathrm{Sol}(C, F)$, $\lim_{n\to\infty} \|s_n - x^*\|$ exists.
2. Each sequential weak cluster point of the sequence $\{s_n\}$ is in $\mathrm{Sol}(C, F)$.

Now, using Lemma 2.3, we conclude that the sequence $\{s_n\}$ converges weakly to an element of Sol(C,F). □

4. Numerical Illustrations

In this section, we present an numerical experiments in solving variational inequality problems. We compare the proposed algorithm with two well-known algorithms including Algorithm 3.1 of Cai et al. in [4], Algorithm 3 of Xie et al. [33]. All the numerical experiments are performed on an HP laptop with Intel(R) Core(TM)i5-6200U CPU 2.3GHz with 4 GB RAM. The programs are written in Matlab2018a.

In this example, we examine a variational model that addresses a fundamental problem in traffic networks—namely, the determination of network equilibrium flow—which is conventionally represented by the minimum-cost flow problem in operations research. The model is formulated through an appropriate variational inequality (see [22, 23]). A concise overview of the problem is provided below, and further details can be found in [22, 23].

Notation

The following notations are employed:

- f_i denotes the flow on arc $A_i := (r, s)$, and $f := (f_1, \dots, f_n)^\top$ represents the vector of flows across all arcs;
- Each arc A_i is associated with an upper capacity bound d_i, and $\boldsymbol{d} := (d_1, \dots, d_n)^\top$;
- $c_i(\boldsymbol{f})$ denotes the cost on arc A_i as a function of the flow, for all $i = 1, \dots, n$, and $\boldsymbol{c}(\boldsymbol{f}) := (c_1(\boldsymbol{f}), \dots, c_n(\boldsymbol{f}))^\top$; it is assumed that $\boldsymbol{c}(\boldsymbol{f}) \geq 0$;
- q_j denotes the net flow balance at node j for $j = 1, \dots, p$, and $\boldsymbol{q} := (q_1, \dots, q_p)^\top$;
- $\Gamma = (\gamma_{ij}) \in \mathbb{R}^{p\times n}$ is the node–arc incidence matrix, whose entries are defined as

$$\gamma_{ij} = \begin{cases} -1, & \text{if node } i \text{ is the starting node of arc } A_j, \\ +1, & \text{if node } i \text{ is the ending node of arc } A_j, \\ 0, & \text{otherwise.} \end{cases}$$

Variational Inequality Formulation

A flow vector $\boldsymbol{f}$ is said to be a *variational equilibrium flow* for the capacitated network model if and only if it satisfies the following variational inequality (see [22]):

$$\text{Find } \boldsymbol{f}^* \in K_f \text{ such that } \langle \boldsymbol{c}(\boldsymbol{f}^*), \boldsymbol{f} - \boldsymbol{f}^* \rangle \geq 0, \quad \forall \boldsymbol{f} \in K_f,$$

where the feasible set K_f is defined as

$$K_f = \{\boldsymbol{f} \in \mathbb{R}^n \mid \Gamma \boldsymbol{f} = \boldsymbol{q},\ 0 \leq \boldsymbol{f} \leq \boldsymbol{d}\}.$$

This problem reduces to the classical *minimum-cost network flow problem* when the cost function $\boldsymbol{c}(\boldsymbol{f})$ is independent of the flow, i.e., $\boldsymbol{c}(\boldsymbol{f}) := (c_{ij}, (i,j) \in A)$.

Numerical Example

In the numerical experiment (see [22], Example 2.1), we consider the following data:

$$\Gamma = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \boldsymbol{q} = (-2,0,0,0,0,2)^{\mathsf{T}}, \quad \boldsymbol{d} = (2,1,1,1,1,1,2,2)^{\mathsf{T}}.$$

The cost function is defined as $\boldsymbol{c}(\boldsymbol{f}) := C\boldsymbol{f}$, where $C = \mathrm{diag}(\boldsymbol{D})$ and

$$\boldsymbol{D} = (5.5, 1, 2, 3, 4, 50, 3.5, 1.5).$$

The corresponding solution to this network flow problem is obtained as

$$\boldsymbol{f}^* = (1.000, 1.000, 0.1575, 0.8425, 0.885, 0.115, 1.0425, 0.9575)^{\mathsf{T}}.$$

The initial values are chosen as follows: $x_0 = (1,1,\cdots,1) \in R^8, x_1 = 0.9x_0$ for all Algorithms. We choose $\rho = 0.01, l = 0.6, \mu = 0.5, \gamma = 1.2, \theta = 0.5, \alpha = 0.01, \lambda = 0.3$ for our Algorithm 1, $\gamma = 1.0, l = 0.6, \mu = 0.4, \beta_n = \frac{0.8}{(n+1)}, f(x) = 0.5x$ for Algorithm 3.1 and $\gamma = 1.0, \beta = 0.8, l = 0.5, \mu = 0.5, \alpha = 0.5, \kappa = 0.5, \eta_n = \frac{1}{(n+1)}, f(x) = 0.5x$ and

$$\alpha_n = \begin{cases} \min\left\{\alpha, \dfrac{\xi_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \alpha, & \text{if } x_n = x_{n-1}, \end{cases}$$

with $\xi_n = \dfrac{1}{(n+1)^{1.1}}$ for Algorithm 3. To terminate algorithms, we use the condition $D_n := \|x_n - f^\star\| \leq 10^{-4}$ or iterations ≥ 500. The numerical result is described in Figure 1.

Figure 1 gives the errors of our proposed algorithm, Algorithm 3.1 of Cai et al.[4] , Algorithm 3 of Xie et al. [33] as well as their execution times and the number of iteration. They show that the proposed algorithm behaves better than other algorithms.

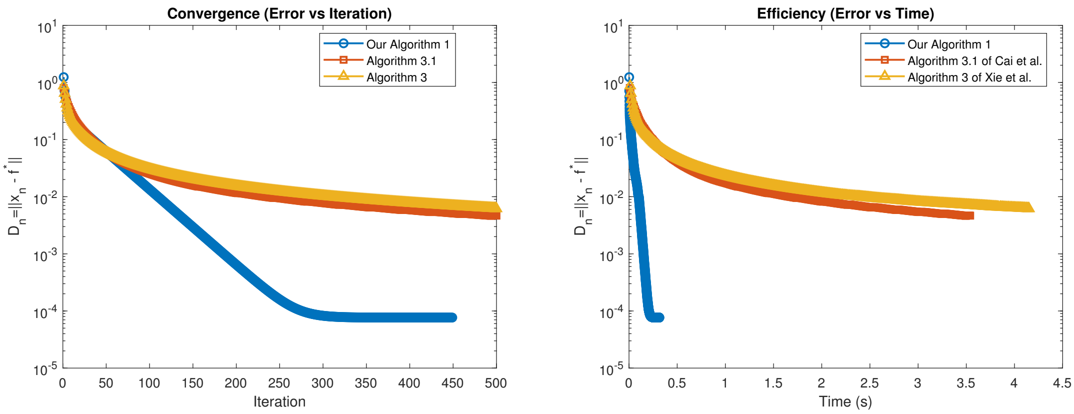

Figure 1: Comparison of all Algorithms in Example.

5. Conclusions

In this paper, based on the projection method we introduce an algorithm with double inertial steps for solving a variational inequality problem involving pseudomonotone mappings in Hilbert spaces. Weak convergence is given under pseudomonotoni city and uniform continuity assumptions. Numerical experiments are presented to illustrate the performance of the proposed algorithm.

Ethical Approval

Not Applicable

Availability of supporting data

The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

Competing interests

The authors declare that they have no conflict of interest.

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