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Birth and death processes associated with little q-Laguerre orthogonal polynomials

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Abstract. We study birth and death processes with rates $\lambda_n = q^n(1 - aq^{n+1})$ and $\mu_n = aq^n(1 - q^n)$, for $n \geq 0$ and $0 < a, q < 1$. Using the associated generating functions, we show that the corresponding orthogonal polynomials generalize the little q-Laguerre polynomials. We also provide the minimal solution of the three-term recurrence relation and derive explicit formulas for the convergents of the continued fractions associated with the little q-Laguerre orthogonal polynomials.

1. Introduction

A birth–death model is a continuous-time Markov process (denoted in the sequel by *BDP*) that is often used to study how the number of individuals in a population evolves over time. More precisely, a *BDP* is a special case of a continuous-time Markov process on the non-negative integers, that is, $\{X_t,\ t \in [0, \infty[\}$, with state space $\{0, 1, 2, \ldots\}$, in which only jumps to adjacent states are permitted. The transition probabilities are denoted by $P_{m,n}(t) = \Pr\{X_{t+s} = n \mid X_s = m\} = \Pr\{X_t = n \mid X_0 = m\}$, where $P_{m,n}(t)$ represents the probability that the process moves from state m to state n during a time period of length t.

These processes have been used as models in population growth, in queuing systems, and in many other fields of both theoretical and applied interest [4, 24].

To construct a general *BDP*, we need to define the rules according to which the number of particles evolves. To this aim, we specify the behaviour of the process during a very short time interval dt, when there are n particles in the system.

If dt is very small, the probability that a birth or a death occurs during the interval $(t, t + dt)$ with rate r is approximately $r\,dt$. Therefore, the transition probability $P_{m,n}(t)$, for small t, describing the probability

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that the process $\{X_t\}$ moves from state m to state n, is defined by

$$P_{m,n}(t) = \begin{cases} \lambda_m t + o(t) & \text{if} \quad n = m+1, \\ 1 - (\lambda_m + \mu_m)t + o(t) & \text{if} \quad n = m, \\ \mu_m t + o(t) & \text{if} \quad n = m-1, \\ o(t) & \text{if} \quad |n-m| > 1. \end{cases} \tag{1}$$

where λ_n and μ_n are the birth and death rates at the state n respectively. It is always assumed that $\lambda_n > 0$, $\mu_{n+1} > 0$ for $n \geq 0$ and $\mu_0 = 0$.

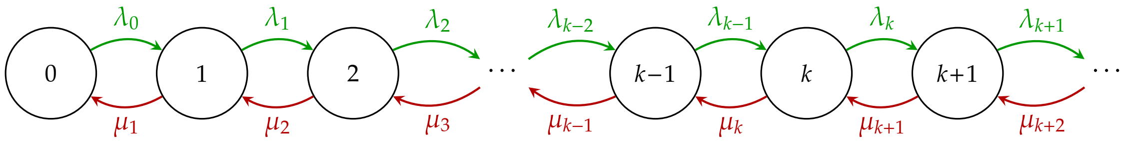

Figure 1: Transition diagram of a birth–death process.

The solution of (1) always satisfies the following conditions:

$$0 < \sum_{n\geq 0} P_{m,n}(t) \leq 1, \quad \text{and} \quad P_{m,n}(s+t) = \sum_{k\geq 0} P_{m,k}(s) P_{k,n}(t), \tag{2}$$

known as the Chapman–Kolmogorov equation, and $P_{m,n}(0) = \delta_{m,n}$, where $\delta_{m,n}$ is the Kronecker delta function.

In the sequel, we only consider $\sum_{n\geq 0} P_{m,n}(t) = 1$. The inequality $\sum_{n\geq 0} P_{m,n}(t) < 1$ expresses a model where $\mu_0 > 0$ (no honest), meaning the particle may disappear in the initial state either by going to infinity or by absorption at state -1.

It is a classical fact that $P_{m,n}(t)$ satisfies the following system of differential equations, called the forward Chapman–Kolmogorov equation:

$$\frac{d}{dt} P_{m,n}(t) = \mu_n P_{m,n-1}(t) - (\lambda_n + \mu_n) P_{m,n}(t) + \lambda_n P_{m,n+1}(t). \tag{3}$$

The correspondence between *BDP*, continued fractions, and orthogonal polynomial systems was investigated early in the literature by S. Karlin and J. L. McGregor in [14]. Karlin and McGregor developed a formal theory of general *BDP* that expresses their transition probabilities in terms of a sequence of orthogonal polynomials and a spectral measure. More precisely, they show in [14] that the solution of (1) can be derived from the family of polynomials $(Q_n(x))_{n\geq 0}$, which satisfy the recursion relation:

$$-xQ_n(x) = \mu_n Q_{n-1}(x) - (\lambda_n + \mu_n) Q_n(x) + \lambda_n Q_{n+1}(x), \tag{4}$$

with the initial conditions

$$Q_0(x) = 1, \qquad Q_1(x) = \frac{\lambda_0 + \mu_0 - x}{\lambda_0}. \tag{5}$$

The transition probability is then given by

$$P_{m,n}(t) = \pi_m \int_0^\infty Q_m(x) Q_n(x) e^{-tx} \, d\mu(x), \tag{6}$$

where $\pi_0 = 1$, $\pi_n = \dfrac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}$ for $n > 0$, and $d\mu$ is a non-negative measure with respect to which the polynomials $(Q_n)_{n\geq 0}$ are orthogonal.

We mention that G. E. H. Reuter showed in [23] that if

$$\sum_{n\geq 0}\left(\pi_n + \frac{1}{\lambda_n \pi_n}\right) = \infty, \tag{7}$$

then μ is unique (the problem is then called determinate). For more information on determinate and indeterminate measures associated with a moment sequence, we refer, for example, to [1] for the scalar case and to [6] for the operator case.

For the relation with continued fractions, we start with the initial state $m = 0$ and write

$$f(s) = \mathcal{L}(P_{0,0})(s) = \int_{\mathbb{R}} P_{0,0}(t)e^{-st}\,dt$$

for the Laplace transform of $P_{0,0}$. Then, the expansion of $f(s)$ as a continued fraction is given by

$$f(s) = \cfrac{1}{s+\lambda_0 - \cfrac{\lambda_0\mu_1}{s+\lambda_1+\mu_1 - \cfrac{\lambda_1\mu_2}{s+\lambda_2+\mu_2 - \cdots}}}. \tag{8}$$

To simplify the previous notation, let us denote $\frac{a}{b+*} = \frac{a}{b+}*$ and we set $a_1 = 1$, $a_n = -\lambda_{n-2}\mu_{n-1}$, $b_1 = s+\lambda_0$, $b_n = s+\lambda_{n-1}+\mu_{n-1}$ for $n \geq 2$. Then (8) becomes $f(s) = \frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+}\cdots$. We denote the kth convergent of the Laplace transform $f(s)$ by $f^{(k)}(s) := \frac{a_1}{b_1+}\frac{a_2}{b_2+}\cdots\frac{a_k}{b_k}$, which, thanks to the definitions of a_i and b_i for $1 \leq i \leq k$, is a rational function of the variable s. In this context, we introduce $Q_k^*(s)$ and $Q_k(s)$ as the numerator and the denominator, respectively, so that $f^{(k)}(s) = \frac{Q_k^*(s)}{Q_k(s)}$.

In [5], F. W. Crawford and M. A. Suchard obtained expressions for the Laplace transforms of these transition probabilities and made explicit an important derivation connecting transition probabilities and continued fractions.

The computational method in [5] is based on the work of J. Murphy and M. O'Donohoe [20] for numerically computing the transition probabilities of a general *BDP* with arbitrary birth and death rates.

Murphy and O'Donohoe showed that it is possible to find exact expressions for the Laplace transforms of the transition probabilities of a general *BDP* using continued fractions. However, this approach does not, in general, lead to exact expressions for the transition probabilities themselves. Efficient approximations of these transition probabilities are obtained by taking the inverse Laplace transform of a truncation of the associated continued fraction.

Despite the simplicity of the representation (6) for the transition probabilities, it is usually difficult to find the polynomials $(Q_n)_{n\geq 0}$ even for simple models. Indeed, analytic expressions for the transition probabilities of a *BDP* are known only in some simple cases, where the rates arise from specific applicable phenomena. Among these models, we have:

- The general linear case $\lambda_n = an + b$, $\mu_n = cn + d$. When $b = d = 0$, it was treated by Kendall [17] and exhibits Charlier polynomials. The case $d = 0$ was solved in [15], leading to Meixner polynomials for $a = c$ and Laguerre polynomials for $a \neq c$. The case $d \neq 0$ is analyzed in [2, 10].

- In the quadratic case, the coefficients $\lambda_n = (N-n)(n+a)$, $\mu_n = n(n+b)$ and $\lambda_n = n^2+an+b$, $\mu_n = n^2+cn+d$ first appeared in applications related to genetic models [11, 16, 25] and are linked to continuous dual Hahn polynomials.

- The quartic case $\lambda_n = (4n+1)(4n+2)^2(4n+3)$, $\mu_n = (4n-1)(4n)^2(4n+1)$ and the cubic case $\lambda_n = (3n+3c+1)^2(3n+3c+2)$, $\mu_n = (3n+3c)^2(3n+3c+1)$ were treated by G. Valent in [8, 29].

Because of the difficulty in identifying the orthogonal polynomials associated with a *BDP*, numerous areas where birth–death processes are applicable are not covered by the previous cases (nor by any possible polynomial model). For example, in queuing theory, one considers a birth–death process where the states of the system represent the number of customers in a queue with a single server and an infinite waiting room.

A number of authors have studied the birth–death process for such queuing models [21, 30]. E. H. Ismail, in [9], explicitly found the orthogonal polynomials (Al-Salam–Carlitz q-polynomials) corresponding to the arrival rate $\lambda_n = \lambda q^n$ and the departure rate $\mu_n = \mu q(1 - q^n)$, for $n = 0, 1, \dots$, with $0 < q < 1$.

The purpose of this paper is to give the spectral representation of the transition probabilities of the birth–death process with the following geometric parameters:

$$\lambda_n = q^n(1 - aq^{n+1}), \qquad \mu_n = aq^n(1 - q^n), \quad n \geq 0, \quad 0 < a, q < 1. \tag{9}$$

Section 2 contains some basic relations between birth–death processes and orthogonal polynomial theory, as well as the preliminary material needed in the remainder of the paper.

In Section 3, we study the orthogonal polynomials associated with the birth–death process (9). We compute the generating functions and give the expressions of the polynomials of the first kind $(Q_n(x))_{n\geq 0}$. We outline an approach previously used by several authors [3, 19].

In Section 4, we derive an explicit formula for the numerator polynomials $(Q_n^*(x))_{n\geq 0}$ in the continued fractions whose denominators are $(Q_n(x))_{n\geq 0}$.

In Section 5, we study the convergence of the continued fractions (8). Finally, in Section 6, we use a theorem of Pincherle [22] concerning minimal solutions of three-term recurrence relations to evaluate explicitly the continued fractions (8) for the little q-Laguerre polynomials.

2. Basic Tools

2.1. Askey scheme and classical orthogonal polynomials associated with BDP

A system of polynomials $(p_n(x))_{n\geq 0}$, where $p_n(x)$ is of degree n for all $n \in \{0, 1, 2, \dots\}$, is called orthogonal on an interval $[a, b]$ with respect to a non-negative measure $d\mu(x)$ if

$$\int_a^b p_m(x)p_n(x)\,d\mu(x) = 0, \quad m \neq n, \quad m, n \in \{0, 1, 2, \dots\},$$

and

$$\int_a^b p_n^2(x)\,d\mu(x) = \alpha_n \neq 0, \quad n \in \{0, 1, 2, \dots\}.$$

It is clear that $(p_k)_{k\leq n}$ forms a basis of $\mathbb{R}_n[X]$, the space of polynomials of degree less than or equal to n. It follows that p_{n+1} is orthogonal to $\mathbb{R}_n[X]$ for every $n \geq 0$. Now, since $xp_n(x) \in \mathbb{R}_{n+1}[X]$, we have

$$xp_n(x) = \sum_{k=0}^{n+1} a_{n,k}p_k(x), \quad a_{n,k} = \frac{1}{\alpha_k}\int_a^b xp_n(x)p_k(x)\,d\mu(x).$$

From $xp_k(x) \in \mathbb{R}_{k+1}[X]$, we derive that $a_{n,k} = 0, \quad$ for $0 \leq k < n - 1$. Thus, orthogonal polynomials satisfy a three-term recurrence relation of the form

$$xp_n(x) = a_{n,n+1}p_{n+1}(x) + a_{n,n}p_n(x) + a_{n,n-1}p_{n-1}(x), \quad n \geq 0, \tag{10}$$

with $p_{-1}(x) = 0$, $p_0(x) = 1$, and $a_{0,-1} = 0$.
In fact, equation (10) characterizes the family of orthogonal polynomials with respect to a non-negative measure μ if $\dfrac{a_{n,n-1}}{a_{n-1,n}} > 0, \quad n > 0$.

Classical orthogonal polynomials are often expressed in terms of generalized hypergeometric series using classical expansion methods. The explicit identification of such series is, however, in general difficult; see [18, page 5].

The generalized hypergeometric series ${}_rF_s$ is defined by

$$ {}_rF_s\Big(a_1,\dots,a_r;b_1,\dots,b_s;x\Big) := \sum_{n=0}^{\infty} \frac{(a_1,\dots,a_r)_n}{(b_1,\dots,b_s)_n}\frac{x^n}{n!}, \tag{11} $$

where $(a)_n$ denotes the Pochhammer symbol (or rising factorial) defined as

$$ (a)_0 = 1, \qquad (a)_n = a(a+1)\cdots(a+n-1) = \prod_{i=0}^{n-1}(a+i), \quad n \geq 1, $$

and, for multiple parameters,

$$ (a_1,\dots,a_r)_n = (a_1)_n\cdots(a_r)_n. $$

Similarly, the q-shifted factorials (or q-Pochhammer symbols) are defined by

$$ (a;q)_0 = 1, \qquad (a;q)_n = \prod_{i=0}^{n-1}(1-aq^i), \qquad (a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i), $$

and for multiple parameters,

$$ (a_1,\dots,a_k;q)_n = (a_1;q)_n\cdots(a_k;q)_n. $$

Askey-scheme of hypergeometric lists several interesting finite systems of orthogonal polynomials. Next are some of these polynomials known in the literature.

- **Laguerre polynomials:** $a_n = -(n+1),\ b_n = -(a_n+c_n),\ c_n = -(n+\alpha),\ \alpha > -1,$

  $$ L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\ {}_1F_1(-n;\alpha+1;x). $$

- **Dual Hahn polynomials:** $a_n = (n+\alpha+1)(n-N),\ b_n = -(a_n+c_n),\ c_n = n(n-\beta-N-1),\ -1<\alpha,\ \beta < -N,\ n \leq N,$

  $$ R_n(\lambda(x);\alpha,\beta,N) = {}_3F_2(-n,-x,x+\alpha+\beta+1;\alpha+1,-N;1), $$

  where $\lambda(x) = x(x+\alpha+\beta+1)$.

- **Meixner polynomials:** $a_n = \gamma(\beta+n)(1-\gamma)^{-1},\ b_n = -(a_n+c_n),\ c_n = n(1-\gamma)^{-1},\ \beta>0,\ 0<\gamma<1,$

  $$ M_n(x;\beta,\gamma) = {}_2F_1(-n,-x;\beta;1-1/\gamma). $$

- **Charlier polynomials:** $a_n = -a,\ b_n = -(a_n+c_n),\ c_n = n,\ a>0,$

  $$ C_n(x;a) = {}_0F_1(-;-x;-1/a). $$

- **Hermite polynomials:** $a_n = \dfrac{1}{2},\ b_n = 0,\ c_n = n,$

  $$ H_n(x) = (2x)^n\ {}_2F_0(-n/2,-(n-1)/2;-;-1/x^2). $$

- **Al-Salam–Carlitz q-polynomials:** $a_n = 1,\ b_n = (a+1)q^n,\ c_n = -aq^{n-1}(1-q^n),\ 0<q<1,$

  $$ U_n^{(a)}(x;q) = (-a)^n q^{\binom{n}{2}}\ {}_2\phi_1(q^{-n},1/x;0;q,qx/a). $$

2.2. Continued Fractions

Let $\frac{Q_n^*(s)}{Q_n(s)}$ be the nth convergent of the continued fraction

$$\frac{1}{x+\lambda_0-}\frac{\lambda_0\mu_0}{x+\lambda_1+\mu_1-}\cdots\frac{\lambda_{n-1}\mu_n}{x+\lambda_{n-1}+\mu_{n-1}}\cdots, \tag{12}$$

with $\lambda_n\mu_{n+1}\neq 0$ for $n\geq 0$. Then the polynomials $Q_n^*(x)$ and $Q_n(x)$ are solutions of the recurrence relation

$$y_{n+1}(x)=[x-(\lambda_n+\mu_n)]y_n(x)-\lambda_{n-1}\mu_n y_{n-1}(x),\quad n>0, \tag{13}$$

with the initial values

$$Q_0^*(x)=0,\quad Q_1^*(x)=1,\quad Q_0(x)=1,\quad Q_1(x)=x+1-aq.$$

A solution $(Q_n^{**}(x))_{n\geq 1}$ of (13) is called a minimal solution if

$$\lim_{n\to+\infty}\frac{Q_n^{**}(x)}{y_n(x)}=0$$

for any other solution $y_n(x)$ of orthogonal polynomials.

It is shown in [7] that the existence of a minimal solution of a three-term recurrence relation is closely related to the determinacy of the associated moment problem. In addition, S. Pincherle in [22, Capitulo 3] showed that the continued fraction (8) converges if and only if the recurrence relation (13) has a minimal solution.

3. Little q-Laguerre birth and death process

3.1. Little q-Laguerre polynomials

We introduce the q-Laguerre rates as follows:

$$\lambda_n=q^n(1-aq^{n+1}),\quad \mu_n=aq^n(1-q^n),\quad n=0,1,\ldots,\text{ where } 0<a,q<1. \tag{14}$$

From direct computations, we obtain

$$\pi_n=\frac{(aq;q)_n}{(aq)^n(q;q)_n},$$

from which it follows that

$$\sum_{n\geq 0}\frac{1}{\lambda_n\pi_n}=\infty.$$

We deduce that the associated series (7) is divergent, and therefore the transition rates (14) induce a unique birth-death process.

Theorem 3.1. *The orthogonal polynomials $(Q_n(x))_{n\geq 1}$ associated with the BDP (9) are precisely the little q-Laguerre polynomials of first kind given by the expression*

$$Q_n(x)=\sum_{k=0}^{n}\frac{q^k(q^{-n};q)_k}{(aq;q)_k(q;q)_k}x^k.$$

Proof. Let $(Q_n(x))_{n\geq 1}$ be the orthogonal polynomials associated with the birth and death process (14). The polynomials $(Q_n(x))_{n\geq 1}$ are generated recursively by the three-term recurrence relation

$$-xQ_n(x) = q^n(1-aq^{n+1})Q_{n+1}(x) - q^n(1-a(q^{n+1}+q^n-1))Q_n(x) + aq^n(1-q^n)Q_{n-1}(x),$$

with $Q_0(x) = 1$ and $Q_1(x) = \dfrac{1-aq-x}{1-aq}$.

The family of polynomials

$$P_n(x) = \frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n} Q_n(x)$$

satisfies the modified three-term recurrence relation

$$xP_n(x) = (1-(a+1)q^{n+1} + aq^{2n+2})P_{n+1}(x) + (q^n(1+a) - aq^{2n}(1+q))P_n(x) + aq^{2n-1}P_{n-1}(x), \tag{15}$$

with

$$P_0(x) = 1, \quad P_1(x) = \frac{1-aq-x}{(1-aq)(1-q)}. \tag{16}$$

We define the generating function associated with the polynomials $(P_n(x))_{n\geq 1}$ by

$$P(x,t) := \sum_{n\geq 0} P_n(x)t^n.$$

Since the series converges, $P(x,t)$ is analytic in t for $|t| < 1$.

Multiplying both sides of (15) by t^{n+1} and summing over $n \geq 1$, using (16), we get the functional equation

$$(1-xt)P(x,t) - (a+1)(1-t)P(x,qt) + a(1-t)(1-tq)P(x,tq^2) = 0. \tag{17}$$

Setting

$$H(x,t) = \frac{(xt;q)_\infty}{(t;q)_\infty} P(x,t), \tag{18}$$

and using

$$(t;q)_n = \frac{(t;q)_\infty}{(tq^n;q)_\infty}, \tag{19}$$

we rewrite (17) as

$$H(x,t) - (a+1)H(x,qt) + a(1-qxt)H(x,q^2t) = 0. \tag{20}$$

Expanding

$$H(x,t) = \sum_{n\geq 0} h_n(x)t^n \tag{21}$$

and substituting into (20) gives

$$(1-aq^n)(1-q^n)h_n(x) - q^{2n-1}axh_{n-1}(x) = 0.$$

By induction, we find

$$h_n(x) = \frac{q^{n^2}a^n x^n}{(aq;q)_n(q;q)_n}, \quad n \geq 1, \quad h_0(x) = 1. \tag{22}$$

Inserting (22) into (18) gives

$$P(x,t)=\frac{(t;q)_\infty}{(xt;q)_\infty}\sum_{n\geq 0}\frac{q^{n^2}a^nx^n}{(aq;q)_n(q;q)_n}t^n.$$

Applying Heine's transformation [27, page 91] and Euler's formula [27, page 93], we finally obtain

$$Q_n(x)=\sum_{k=0}^{n}\frac{(qx)^k(q^{-n};q)_k}{(aq;q)_k(q;q)_k},$$

which are precisely the little q-Laguerre polynomials of first kind. □

3.2. Denominator Polynomials Associated with q-Laguerre BDP

We introduce a second solution of the recurrence relation (13) given by the family of polynomials $(Q_n^*(x))_{n\geq 1}$ of degree $(n-1)$, associated with the initial conditions

$$Q_0^*(x)=0,\quad Q_1^*(x)=1.$$

It is straightforward to see that the polynomials $(Q_n^*(x))_{n\geq 1}$ coincide with the denominators of the associated continued fraction (8). Moreover, by Markov's Theorem [28, page 57], we have

$$\lim_{n\to+\infty}\frac{Q_n^*(x)}{Q_n(x)}=\int_0^{+\infty}\frac{d\mu(y)}{x-y},\quad \text{with } x\notin\left\{q^k\mid k\in\mathbb{N}\right\}.$$

It is also well known that the polynomials $(Q_n^*(x))_{n\geq 1}$ admit the following integral representation:

$$Q_n^*(x)=\int_0^{+\infty}\frac{Q_n(x)-Q_n(y)}{x-y}\,d\mu(y),\quad n\geq 0.$$

See N. I. Akheizer [1, page 8] for further details.

Proposition 3.2. *The little q-Laguerre polynomials of the second kind $(Q_n^*(x))_{n\geq 1}$ associated with the BDP (9) are given by*

$$Q_n^*(x)=\sum_{k=0}^{n-1}\left[\sum_{l=k+1}^{n}a_l^{(n)}s_{l-k-1}\right]x^k.$$

Proof. Writing

$$Q_n(x)=\sum_{k=0}^{n}a_k^{(n)}x^k,$$

we obtain

$$a_k^{(n)}=\frac{(q^{-n};q)_kq^k}{(aq;q)_k(q;q)_k}.$$

For any fixed $y\in(0,+\infty)$ and $n\in\mathbb{N}^*$, we have

$$\frac{Q_n(x)-Q_n(y)}{x-y}=\sum_{k=1}^{n}\sum_{l=0}^{k-1}a_k^{(n)}y^{k-l-1}x^l=\sum_{k=0}^{n-1}\left[\sum_{l=k+1}^{n}a_l^{(n)}y^{l-k-1}\right]x^k.$$

By integrating the above expression with respect to $d\mu$, we deduce

$$Q_n^*(x) = \sum_{k=0}^{n-1}\left[\sum_{l=k+1}^{n} a_l^{(n)} s_{l-k-1}\right]x^k.$$

Moreover, the discrete measure of orthogonality of little q-Laguerre polynomials is given by

$$\mu = \sum_{k\geq 0} \frac{(aq)^k}{(q;q)_k}\,\delta_{q^k}, \quad \text{see [18, page 519]},$$

and the sequence of moments $(s_n)_{n\in\mathbb{N}}$ is

$$s_n = \int_0^{+\infty} y^n\, d\mu(y) = \sum_{k=0}^{+\infty} \frac{(aq^{n+1})^k}{(q;q)_k} = \frac{1}{(aq^{n+1};q)_\infty}.$$

It follows that

$$Q_n^*(x) = \sum_{k=0}^{n-1}\left[\sum_{l=k+1}^{n} \frac{(q^{-n};q)_l\, q^l}{(aq;q)_l(q;q)_l(aq^{l-k};q)_\infty}\right]x^k.$$

□

In the following result, we prove that the associated continued fraction is convergent.

Proposition 3.3. *The continued fraction associated with the BDP (9) is convergent.*

Proof. According to (8), for any $x \notin \operatorname{supp}(\mu) := \{q^k \mid k \in \mathbb{N}\}$, we formally have

$$f(x) = \lim_{n\to+\infty} \frac{Q_n^*(x)}{Q_n(x)} = \int_0^{+\infty} \frac{1}{x-y}\, d\mu(y).$$

Using the expression of μ, it follows that

$$\int_0^{+\infty} \frac{1}{x-y}\, d\mu(y) = \sum_{n=0}^{+\infty} \frac{(aq)^n}{(x-q^n)(q;q)_n} = \sum_{n=0}^{+\infty} a_n(x),$$

where

$$a_n(x) = \frac{(aq)^n}{|x-q^n|(q;q)_n}.$$

Now, since

$$\lim_{n\to+\infty} \frac{a_{n+1}(x)}{a_n(x)} = aq < 1,$$

we deduce that $f(x)$ is convergent for every $x \notin \operatorname{supp}(\mu)$. □

4. Minimal Solution

In this last section, we derive a closed form of the minimal solution of the recurrence relation (13), by using the Rodrigues formula from [12].

Proposition 4.1. *The minimal solution $(Q_n^{**}(x))_{n\geq 1}$ of the BDP (9) exists and admits the following expansion formula:*

$$Q_n^{**}(x) = (-1)^n q^{n\alpha+\binom{n+1}{2}}(1-q)\frac{(\frac{q^{n+1}}{x},q;q)_\infty (q;q)_n}{(q^{\alpha+1};q)_n(\frac{1}{x},qx;q)_\infty x^{n+1+\alpha}}\ {}_2\phi_1\Big(\frac{1}{x},0;\frac{q^{n+1}}{x};q,q^{\alpha+n+1}\Big). \tag{23}$$

Moreover, the associated continued fraction is given by

$$f(x) = \frac{1}{(x-q)\displaystyle\sum_{k=0}^{\alpha} q^{k-\alpha-1}}\frac{{}_2\phi_1(\frac{1}{x},0,\frac{q^2}{x};q,q^{\alpha+2})}{{}_2\phi_1(\frac{1}{x},0;\frac{q}{x};q,q^{\alpha+1})},\quad x\notin supp(\mu).$$

Proof. According to Pincherle's result [22, Capitulo 3], the minimal solution $(Q_n^{**}(x))_{n\geq 1}$ of (13) exists because the continued fraction (8) converges, as shown in the previous section. Moreover, it satisfies

$$w(x)Q_n^{**}(x) = Q_n(x)w(x)Q_0^{**}(x) - Q_n^{*}(x),$$

where $d\mu(y) = w(y)dx$ is absolutely continuous with $w(y)\geq 0$.

By [26, Proposition 5.21], $Q_n^{**}(x)$ admits the integral representation

$$Q_n^{**}(x) = \frac{1}{w(x)}\int_0^\infty \frac{Q_n(y)}{x-y}\,d\mu(y),\quad n>0,\quad x\notin \mathrm{supp}(w) = \mathrm{supp}(\mu). \tag{24}$$

The little q-Laguerre polynomials admit the basic hypergeometric representation [18, page 518]:

$$Q_n(x) = {}_2\phi_1(q^{-n},0;aq;q,qx),$$

with the orthogonality relation [18, page 519]:

$$\sum_{k=0}^{\infty}\frac{(aq)^k}{(q;q)_k}Q_m(q^k)Q_n(q^k) = \frac{1}{\pi_n(aq;q)_\infty}\delta_{mn},\quad 0<a,q<1.$$

The associated Rodrigues-type formula is

$$w(x;\alpha;q)Q_n(x) = \frac{q^{n\alpha+\binom{n}{2}}(1-q)^n}{(q^{\alpha+1};q)_n}D^n_{q^{-1}}[w(x;\alpha+n;q)],\quad \alpha>-1, \tag{25}$$

where $w(x;\alpha;q) = (qx;q)_\infty x^\alpha$, and the q-difference operator D_q is defined by

$$D_q f(x) = \frac{f(x)-f(qx)}{x-qx}.$$

Replacing (25) into (24) yields

$$w(x;\alpha;q)Q_n^{**}(x) = \frac{q^{n\alpha+\binom{n}{2}}(1-q)^n}{(q^{\alpha+1};q)_n}\int_0^\infty \frac{1}{x-y}D^n_{q^{-1}}[w(y;\alpha+n;q)]d_q y,$$

where the Jackson q-integral is defined for an arbitrary function f over $[0,\infty[$ by

$$\int_0^\infty f(x)\,d_q x = (1-q)\sum_{n=-\infty}^{+\infty} q^n f(q^n).$$

Using the q-integration by parts formula, we obtain

$$w(x;\alpha;q)Q_n^{**}(x) = \frac{(-1)^n q^{n\alpha+\binom{n+1}{2}}(1-q)^n}{(q^{\alpha+1};q)_n}\int_0^\infty w(y;\alpha+n;q)D_{q^{-1}}^n \frac{1}{x-y}\, d_q y.$$

Finally, applying the standard q-difference identities and the Jackson integral, we arrive at

$$Q_n^{**}(x) = (-1)^n q^{n\alpha+\binom{n+1}{2}}(1-q)\frac{(\frac{q^{n+1}}{x},q;q)_\infty (q;q)_n}{(q^{\alpha+1};q)_n(\frac{1}{x},qx;q)_\infty x^{n+1+\alpha}}\,{}_2\phi_1\Big(\frac{1}{x},0;\frac{q^{n+1}}{x};q,q^{\alpha+n+1}\Big).$$

For the second assertion, by [22, Capitulo 3], when the minimal solution exists, the continued fraction (8) converges to

$$f(x) = -\frac{Q_1^{**}(x)}{Q_0^{**}(x)} = \frac{1}{(x-q)\sum_{k=0}^{\alpha} q^{k-\alpha-1}}\frac{{}_2\phi_1(\frac{1}{x},0,\frac{q^2}{x};q,q^{\alpha+2})}{{}_2\phi_1(\frac{1}{x},0;\frac{q}{x};q,q^{\alpha+1})}, \quad x \notin \mathrm{supp}(w).$$

□

Remark 4.2. *Let us conclude with some comments on the ergodicity of the BDP associated with the rates (9). Following [13], a BDP is said to be ergodic if*

$$\lim_{t\to+\infty} P_{m,n}(t) = p_n,$$

where the stationary distribution p_n is given by

$$p_n = \frac{\pi_n}{\sum_{l\geq 0}\pi_l}.$$

For the BDP associated with rates (9), we have

$$\lim_{n\to+\infty}\frac{\lambda_n}{\mu_n} = \frac{1}{a} > 1,$$

and hence

$$\sum_{n\geq 0}\pi_n = +\infty.$$

In particular, it follows that $p_n = 0$ for every n, which contradicts the normalization condition $\sum_{n\geq 0} p_n = 1$. Thus, the BDP with rates (9) is not ergodic.

Conclusion 4.3. *For the birth–death process defined by*

$$\lambda_n = q^n(1-aq^{n+1}), \quad \mu_n = aq^n(1-q^n), \quad n \geq 0, \quad 0 < a, q < 1,$$

the parameters a and q have a clear interpretation in terms of the process dynamics:

- *q: an exponential decay factor. The smaller q is, the faster the rates λ_n and μ_n decrease with increasing n, modeling a slowdown in transitions at higher states (e.g., saturation effects or limited resources).*
- *a: a parameter controlling the relative "death" or downward transition rate. A small a significantly reduces the downward rate (μ_n), while a value close to 1 balances upward (λ_n) and downward (μ_n) transitions, affecting the stationary distribution and ergodicity.*

Thus, q regulates how quickly transitions decay with the state n, while a adjusts the ratio between upward and downward rates, making this BDP useful for modeling systems with saturation, limited queues, or populations constrained by finite resources.

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