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Approximation by a new sequence of operators involving Laguerre polynomials

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Abstract. This paper presents a new integral approach for operators using the modified Laguerre polynomials and Păltănea basis function to approximate functions over the interval $[0, \infty)$. Further, the universal Korovkin's theorem is established to investigate the approximation properties of the proposed operators. Convergence analysis is examined through various analytical methods, including the Lipschitz class, Peetre's K-functional, the second-order modulus of smoothness, and the modulus of continuity. The Voronovskaja-type asymptotic formula and approximation results in weighted spaces are also obtained. Finally, we employ Mathematica to present numerical examples that visually confirm the theoretical results.

1. Introduction

For the weight function $f^{(\alpha)}(x) = x^{\alpha} \exp(-x)$, $\alpha > -1$, the generalized Laguerre polynomials $L_{\kappa}^{(\alpha)}(x)$ are orthogonal over the interval $[0, \infty)$. These polynomials can be expressed using the Rodrigues formula (cf.[31] pp. 203-204]) as

$$L_{\kappa}^{(\alpha)}(x) = \frac{f^{(-\alpha)}(-x)}{\kappa!} D^{k}(f^{(k+\alpha)}(-x)) = \sum_{\iota=0}^{\kappa} \frac{(-1)^{\iota}}{\iota!} \binom{\kappa+\alpha}{\kappa-\iota} x^{\iota}, \text{where } D = \frac{d}{dx},$$

and $L_{\kappa}^{(\alpha)}(x)$ can be generated with the following generating function

$$(1-z)^{-\alpha-1} \exp\left(-\frac{xz}{1-z}\right) = \sum_{\kappa=0}^{\infty} L_{\kappa}^{(\alpha)}(x) z^{\kappa}.$$

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Recently, Gupta [14] introduced the operators based on modified Laguerre polynomials over the interval $[0, \infty)$. These operators are defined as follows:

$$P_n^{\alpha}(f;x) = \exp\left(\frac{-nx}{2}\right)2^{-\alpha-1}\sum_{\kappa=0}^{\infty}2^{-\kappa}L_{\kappa}^{(\alpha)}\left(\frac{-nx}{2}\right)f\left(\frac{\kappa}{n}\right), \ \ \alpha > -1.$$

Let $f : [0, \infty) \to \mathbb{R}$ be an integrable function. For $\rho > 0$ and $\beta > 0$, Păltănea [29] defined the modification of Szász-Mirakjan operators as follows:

$$B_{\rho}^{(\beta)}(f;x) = \exp(-\rho x)\left(f(0) + \sum_{\kappa=1}^{\infty}\frac{(\rho x)^{\kappa}}{\kappa!}\int_0^{\infty}\frac{\rho\beta\exp(-\rho\beta z)(\rho\beta z)^{\kappa\beta-1}}{\Gamma(\kappa\beta)}f(z)dz\right).$$

Cheney and Sharma [10] proposed operators based on Laguerre polynomials that are intended to explore the variation-diminishing properties and convergence. Öksüzer et al. [26] introduced Bézier variant of operators involving Laguerre polynomials and studied the rate of convergence for these operators. Özarslan [27] proposed the q-Laguerre type positive linear operators and examined the approximation properties of these operators. Deshwal et al. [11] constructed Bézier variant of an operator based on Laguerre polynomials. The author examined the degree of approximation and rate of convergence for these operators. Much recent work has been devoted to integral-type modification of operators applying the Păltănea basis function. Verma and Gupta [32] proposed the Jakimovski–Leviatan–Păltănea operators. They estimated approximation properties for the operators. Ansari et. al [3] constructed modified Păltănea operators with Gould–Hopper polynomials, and estimated the approximation properties by using the weighted modulus of continuity. Mishra and Deo [21] introduced Apostol-Genocchi-Păltănea operators and studied the asymptotic-type results, such as the Voronovskaya theorem and the quantitative Voronovskaya theorem. To see more work relevant to this area, one may refer ([1, 2, 4–9, 15, 16, 18, 20, 22–25, 28, 30]).
Motivated by the above work, we introduce a new class of operators, defined using modified Laguerre polynomials and the Păltănea basis function, as follows:

$$R_n^{(\alpha,\beta)}(f;x) = \exp\left(\frac{-nx}{2}\right)2^{-\alpha-1}\sum_{\kappa=0}^{\infty}2^{-\kappa}L_{\kappa}^{(\alpha)}\left(\frac{-nx}{2}\right)\int_0^{\infty}\mathcal{I}_{\kappa,n}^{\beta}(z)f(z)dz, \tag{1}$$

where $\mathcal{I}_{\kappa,n}^{\beta}(z) = \frac{n\beta\exp(-n\beta z)(n\beta z)^{\kappa\beta-1}}{\Gamma(\kappa\beta)}$ and $\beta > 0$.
We observe that for different values of parameters α and β, these operators (1) improve the approximation of the function f.
This is an overview of the profile presented in the paper. The main goal of this study is to introduce a new class of operators over the interval $[0, \infty)$. The moments associated with the proposed operators are addressed in Section 2. In Section 3, we used Peetre's K-functional, the known Voronovskaja-type asymptotic formula, the modulus of continuity, the Lipschitz class, and the second-order modulus of smoothness to examine the convergence traits of the proposed operators. Section 4 discussed the convergence of the operators in weighted spaces and estimated the approximation using the weighted modulus of continuity. We use visual and numerical illustrations in the last section to support the results.
We compute several lemmas that will be helpful in upcoming proofs of the main results.

Lemma 1.1. *[29]. For the test function $\varrho_{\jmath}(z) = z^{\jmath}$ and $\jmath = 0, 1, 2, \ldots$, we know*

$$\int_0^{\infty}\mathcal{I}_{\kappa,n}^{\beta}(z)\varrho_{\jmath}(z)dz = \frac{(\kappa\beta)_{\jmath}}{(n\beta)^{\jmath}},$$

where $(x)_{\jmath} = \prod_{s=1}^{\jmath}(x + s - 1), (x)_0 = 1.$

Lemma 1.2. *[14]. For the operators $P_n^{\alpha}(f;x)$ and $\jmath = 0,1,2,3,4$, we get*

$$
\begin{aligned}
P_n^{\alpha}(\varrho_0(z);x) &= 1,\\
P_n^{\alpha}(\varrho_1(z);x) &= x + \frac{1+\alpha}{n},\\
P_n^{\alpha}(\varrho_2(z);x) &= x^2 + \frac{x(2\alpha n + 5n) + \alpha^2 + 4\alpha + 3}{n^2},\\
P_n^{\alpha}(\varrho_3(z);x) &= x^3 + \frac{x^2(3\alpha n^2 + 12n^2) + x(3\alpha^2 n + 21\alpha n + 31n) + \alpha^3 + 9\alpha^2 + 21\alpha + 13}{n^3},\\
P_n^{\alpha}(\varrho_4(z);x) &= x^4 + \frac{x^3(4\alpha n^3 + 22n^3) + x^2(6\alpha^2 n^2 + 60\alpha n^2 + 133n^2)}{n^4}\\
&\quad + \frac{x(4\alpha^3 n + 54\alpha^2 n + 208\alpha n + 233n) + \alpha^4 + 16\alpha^3 + 78\alpha^2 + 138\alpha + 75}{n^4}.
\end{aligned}
$$

Lemma 1.3. *For $\varrho_\jmath(z) = z^\jmath$ and $\jmath = 0,1,2,3,4$, then the moments of the operators (1) are as follows:*

$$
\begin{aligned}
R_n^{(\alpha,\beta)}(\varrho_0(z);x) &= 1,\\
R_n^{(\alpha,\beta)}(\varrho_1(z);x) &= x + \frac{1+\alpha}{n},\\
R_n^{(\alpha,\beta)}(\varrho_2(z);x) &= x^2 + \frac{nx(2\alpha\beta + 5\beta + 1) + \beta(\alpha^2 + 4\alpha + 3) + \alpha + 1}{\beta n^2},\\
R_n^{(\alpha,\beta)}(\varrho_3(z);x) &= x^3 + \frac{x^2(3\alpha\beta^2 + 12\beta^2 + 3\beta)}{\beta^2 n}\\
&\quad + \frac{x(3\alpha^2\beta^2 + 21\alpha\beta^2 + 31\beta^2 + 6\alpha\beta + 15\beta + 2)}{\beta^2 n^2}\\
&\quad + \frac{\alpha^3\beta^2 + 9\alpha^2\beta^2 + 21\alpha\beta^2 + 13\beta^2 + 3\alpha^2\beta + 12\alpha\beta + 9\beta + 2\alpha + 2}{\beta^2 n^3},\\
R_n^{(\alpha,\beta)}(\varrho_4(z);x) &= x^4 + \frac{x^3(4\alpha\beta^3 + 6\beta^2 + 22\beta^3)}{\beta^3 n}\\
&\quad + \frac{x^2(11\beta + 72\beta^2 + 18\alpha\beta^2 + 133\beta^3 + 60\alpha\beta^3 + 6\alpha^2\beta^3}{\beta^3 n^2}\\
&\quad + \frac{x(4\alpha^3\beta^3 + 54\alpha^2\beta^3 + 208\alpha\beta^3 + 233\beta^3 + 18\alpha^2\beta^2 + 126\alpha\beta^2)}{\beta^3 n^3}\\
&\quad + \frac{x(186\beta^2 + 22\alpha\beta + 55\beta + 6)}{\beta^3 n^3} + \frac{\alpha^4\beta^3 + 16\alpha^3\beta^3 + 78\alpha^2\beta^3}{\beta^3 n^4}\\
&\quad + \frac{138\alpha\beta^3 + 75\beta^3 + 6\alpha^3\beta^2 + 54\alpha^2\beta^2 + 126\alpha\beta^2 + 78\beta^2 + 11\alpha^2\beta}{\beta^3 n^4}\\
&\quad + \frac{44\alpha\beta + 33\beta + 6\alpha + 6}{\beta^3 n^4}.
\end{aligned}
$$

Proof. By Lemma 1.1 and Lemma 1.2, we can easily calculate the following moments for the proposed operators. □

Lemma 1.4. *Let $\mu_{n,\jmath}(x) := R_n^{(\alpha,\beta)}((\varrho_1(z) - x\varrho_0(z))^\jmath;x)$ and $\jmath = 1,2,4$, we calculate easily the central moments by*

Lemma 1.3, we get

$$\mu_{n,1}(x) = \frac{1+\alpha}{n},$$
$$\mu_{n,2}(x) = \frac{x(3\beta+1)}{\beta n} + \frac{\alpha^2\beta + 4\alpha\beta + 3\beta + \alpha + 1}{\beta n^2},$$
$$\mu_{n,4}(x) = \frac{3x^2(9\beta^2+6\beta+1)}{\beta^2 n^2} + \frac{x(124\alpha\beta^3 + 181\beta^3 + 6\alpha^2\beta^2 + 78\alpha\beta^2 + 150\beta^2 + 14\alpha\beta + 47\beta + 6)}{\beta^3 n^3} + \frac{\alpha^4\beta^3 + 16\alpha^3\beta^3 + 78\alpha^2\beta^3 + 138\alpha\beta^3 + 75\beta^3 + 6\alpha^3\beta^2 + 54\alpha^2\beta^2}{\beta^3 n^4} + \frac{126\alpha\beta^2 + 78\beta^2 + 11\alpha^2\beta + 44\alpha\beta + 33\beta + 6\alpha + 6}{\beta^3 n^4}.$$

Remark 1.5. *With a normed space* $C^B_{[0,\infty)} = \{f \in C[0,\infty) : f$ *is bounded over* $[0,\infty)\}$ *that has the norm*

$$\|f\| = \sup_{x\in[0,\infty)} |f(x)|. \tag{2}$$

Lemma 1.6. *For each* $f \in C^B_{[0,\infty)}$ *and* $x \in [0,\infty)$, *then*

$$|R_n^{(\alpha,\beta)}(f;x)| \le \|f\|. \tag{3}$$

2. Direct Result and Asymptotic Formula

Theorem 2.1. *Let* $f \in C^B_{[0,\infty)}$. *Then,* $\lim_{n\to\infty} R_n^{(\alpha,\beta)}(f;x) = f(x)$ *holds uniformly on compact subsets of* $[0,\infty)$.

Proof. By Lemma 1.3, we can easily see that $\lim_{n\to\infty} R_n^{(\alpha,\beta)}(\varrho_\jmath(z);x)$ *for* $\jmath = 0,1,2,3,4$ *and hence the well known Bohman-Korovkin's theorem due to [19], operators* $R_n^{(\alpha,\beta)}$ *converge uniformly on every compact subset of* $[0,\infty)$ *to* $f(x)$. □

In this section, we now explore some important results.
For $f \in C^B_{[0,\infty)}$, the Peetre's K-functional is given by

$$K_2(f,\delta) = \inf_{\psi\in W_{C^2([0,\infty))}} \{\|f-\psi\| + \delta\|\psi''\|\}, \text{ where } \delta > 0, \text{ and}$$

$$W_{C^2([0,\infty))} = \{\psi \in C^B_{[0,\infty)} : \psi', \psi'' \in C^B_{[0,\infty)}\}.$$

$$K_2(f,\delta) \le M\omega_2(f,\sqrt{\delta}), \tag{4}$$

$$\text{where } \omega_2(f,\sqrt{\delta}) = \sup_{0<\epsilon\le\sqrt{\delta}} \left(\sup_{x,x+\epsilon,x+2\epsilon\in[0,\infty)} |f(x+2\epsilon) - 2f(x+\epsilon) + f(x)| \right)$$

is called the second order of the modulus of continuity of f.
The expression

$$\omega(f,\delta) = \sup_{|z-x|\le\delta} |f(z) - f(x)| \tag{5}$$

is the modulus of continuity of f, where $x, z \in [0,\infty)$.

Theorem 2.2. *For $f \in C^B_{[0,\infty)}$ and $x \in [0,\infty)$, then*

$$|R_n^{(\alpha,\beta)}(f;x) - f(x)| \leq M\omega_2(f, \sqrt{\delta/2}) + \omega\left(f, \frac{1+\alpha}{n}\right),$$

where $M > 0$ is constant and $\delta = R_n^{(\alpha,\beta)}((z-x)^2;x) + \left(\frac{1+\alpha}{n}\right)^2$.

Proof. Firstly, we introduce

$$\mathcal{H}_n^{(\alpha,\beta)}(f;x) = R_n^{(\alpha,\beta)}(f;x) - f\left(x + \frac{1+\alpha}{n}\right) + f(x). \tag{6}$$

Let $\psi \in W_{C^2([0,\infty))}$, then by Taylor's theorem

$$\psi(z) = \psi(x) + (z-x)\psi'(x) + \frac{1}{2}\int_x^z (z-u)\psi''(u)du. \tag{7}$$

Applying $\mathcal{H}_n^{(\alpha,\beta)}$ on (7),

$$\mathcal{H}_n^{(\alpha,\beta)}(\psi;x) = \psi(x) + \psi'(x)\mathcal{H}_n^{(\alpha,\beta)}(z-x;x) + \frac{1}{2}\mathcal{H}_n^{(\alpha,\beta)}\left(\int_x^z (z-u)\psi''(u)du;x\right).$$

By Lemma 1.4 and (6),

$$\mathcal{H}_n^{(\alpha,\beta)}(z-x;x) = 0.$$

$$\text{Therefore,}\quad |\mathcal{H}_n^{(\alpha,\beta)}(\psi;x) - \psi(x)| \leq \mathcal{H}_n^{(\alpha,\beta)}\left(\int_x^z (z-u)|\psi''(u)|du;x\right) \leq \|\psi''\|\left|\mathcal{H}_n^{(\alpha,\beta)}\left(\int_x^z (z-u)du;x\right)\right|.$$

By (6),

$$\begin{aligned}|\mathcal{H}_n^{(\alpha,\beta)}(\psi;x) - \psi(x)| &\leq \|\psi''\|\left(\left|(R_n^{(\alpha,\beta)}\left(\int_x^z (z-u)du;x\right)\right| + \left|\int_x^{x+\frac{1+\alpha}{n}}\left(x + \frac{1+\alpha}{n} - u\right)du\right|\right)\\ &\leq \|\psi''\|\left(R_n^{(\alpha,\beta)}((z-x)^2;x) + \left(\frac{1+\alpha}{n}\right)^2\right) = \delta\|\psi''\|.\end{aligned} \tag{8}$$

It follows that

$$\begin{aligned}R_n^{(\alpha,\beta)}(f;x) - f(x) &= \mathcal{H}_n^{(\alpha,\beta)}(f-\psi;x) - (f-\psi)(x)\\ &\quad + \mathcal{H}_n^{(\alpha,\beta)}(\psi;x) - \psi(x) + f\left(x + \frac{1+\alpha}{n}\right) - f(x).\end{aligned}$$

$$\begin{aligned}|R_n^{(\alpha,\beta)}(f;x)-f(x)| &\le |\mathcal{H}_n^{(\alpha,\beta)}(f-\psi;x)-(f-\psi)(x)|+|\mathcal{H}_n^{(\alpha,\beta)}(\psi;x)-\psi(x)|\\ &\quad+\left|f\left(x+\frac{1+\alpha}{n}\right)-f(x)\right|\\ &\le 2\|f-\psi\|+\delta\|\psi''\|+\omega\left(f,\frac{1+\alpha}{n}\right).\end{aligned}$$

Taking the infimum of ψ on $W_{C^2([0,\infty))}$ of the right hand side of the inequality,

$$|R_n^{(\alpha,\beta)}(f;x)-f(x)|\le K_2\left(f,\frac{\delta}{2}\right)+\omega\left(f,\frac{1+\alpha}{n}\right).$$

By (4),

$$|R_n^{(\alpha,\beta)}(f;x)-f(x)|\le M\omega_2(f,\sqrt{\delta/2})+\omega\left(f,\frac{1+\alpha}{n}\right).$$

□

Here, we study ways to apply the Ditzian-Toitik moduli of smoothness [12] to find the degree of approximation:

$$\text{Let}\quad \omega^2_{\phi^\lambda}(f,\delta)=\sup_{0<\epsilon\le\delta}\left(\sup_{x,x+\epsilon\phi^\lambda,x-2\epsilon\phi^\lambda\in[0,\infty)}\left|f(x+\epsilon\phi^\lambda)-2f(x)+f(x-\epsilon\phi^\lambda)\right|\right)$$

and
$\mathfrak{D}^2_\lambda=\{\psi\in C^B_{[0,\infty)}:\psi'\in A.C._{loc}([0,\infty)),\ \|\phi^{2\lambda}\psi''\|<\infty\}$, where weight function $\phi(x)=\sqrt{x}$, $0\le\lambda\le1$. The corresponding K-functional is given by

$$K_{2,\phi^\lambda}(f,\delta^2)=\inf_{\psi'\in A.C._{loc}([0,\infty))}\{\|f-\psi\|+\delta^2\|\phi^{2\lambda}\psi''\|\}.$$

In view of [12], it follows that

$$K_{2,\phi^\lambda}(f,\delta^2)\sim\omega^2_{\phi^\lambda}(f,\delta).$$

Theorem 2.3. *For $f\in C^B_{[0,\infty)}$ and $x\in[0,\infty)$, then*

$$|R_n^{(\alpha,\beta)}(f;x)-f(x)|\le 4\omega^2_{\phi^\lambda}\left(f,\frac{\delta_n^{(1-\lambda)}(x)}{\sqrt{2n}}\right)+\omega\left(f,\frac{1+\alpha}{n}\right).$$

Proof. Let

$$\mathcal{H}_n^{(\alpha,\beta)}(f;x)=R_n^{(\alpha,\beta)}(f;x)-f\left(x+\frac{1+\alpha}{n}\right)+f(x). \tag{9}$$

For $\psi\in\mathfrak{D}^2_\lambda$, by Taylor's theorem,

$$\psi(z)=\psi(x)+(z-x)\psi'(x)+\frac{1}{2}\int_x^z(z-u)\psi''(u)du. \tag{10}$$

Applying $\mathcal{H}_n^{(\alpha,\beta)}$ on (10),

$$\mathcal{H}_n^{(\alpha,\beta)}(\psi;x)=\psi(x)+\psi'(x)\mathcal{H}_n^{(\alpha,\beta)}(z-x;x)+\frac{1}{2}\mathcal{H}_n^{(\alpha,\beta)}\left(\int_x^z(z-u)\psi''(u)du;x\right). \tag{11}$$

From (9), (1.5), and Lemma 1.4, we get

$$\mathcal{H}_n^{(\alpha,\beta)}(z-x;x)=0,$$

$$|\mathcal{H}_n^{(\alpha,\beta)}(f;x)|\leq 3\|f\|, \tag{12}$$

and

$$\mu_{n,2}(x)\leq \frac{1}{n}\delta_n^2(x),$$

where $\delta_n^2(x)=\frac{\phi^2(x)(3\beta+1)+\alpha^2\beta+4\alpha\beta+3\beta+\alpha+1}{\beta}$.
From ([12], p. 141), for $z<u<x$, we have

$$\frac{|z-u|}{\phi^{2\lambda}(u)}\leq\frac{|z-x|}{\phi^{2\lambda}(x)} \text{ and } \frac{|z-u|}{\delta_n^{2\lambda}(u)}\leq\frac{|z-x|}{\delta_n^{2\lambda}(x)}. \tag{13}$$

By (9), (11), and (13),

$$\begin{aligned}
|\mathcal{H}_n^{(\alpha,\beta)}(\psi;x)-\psi(x)| &\leq \left|\mathcal{H}_n^{(\alpha,\beta)}\left(\int_x^z (z-u)\psi''(u)du;x\right)\right|\\
&\leq \left|R_n^{(\alpha,\beta)}\left(\int_x^z (z-u)\psi''(u)du;x\right)\right|\\
&\quad+\left|\int_x^{x+\frac{1+\alpha}{n}}\left(x+\frac{1+\alpha}{n}-u\right)\psi''(u)du\right|,\\
&\leq \|\delta_n^{2\lambda}\psi''\|R_n^{(\alpha,\beta)}\left(\frac{(z-x)^2}{\delta_n^{2\lambda}(x)};x\right)+\frac{\|\delta_n^{2\lambda}\psi''\|}{\delta_n^{2\lambda}(x)}\left(\frac{1+\alpha}{n}\right)^2\\
&\leq \delta_n^{-2\lambda}(x)\|\delta_n^{2\lambda}\psi''\|\left(\mu_{n,2}(x)\right)+\delta_n^{-2\lambda}(x)\|\delta_n^{2\lambda}\psi''\|\left(R_n^{(\alpha,\beta)}((z-x);x)\right)^2\\
&\leq \delta_n^{-2\lambda}(x)\|\delta_n^{2\lambda}\psi''\|\frac{\delta_n^2(x)}{n}+\delta_n^{-2\lambda}(x)\|\delta_n^{2\lambda}\psi''\|\frac{\delta_n^2(x)}{n}.
\end{aligned}$$

Hence,

$$|\mathcal{H}_n^{(\alpha,\beta)}(\psi;x)-\psi(x)|\leq\frac{2\delta_n^{2(1-\lambda)}(x)}{n}\|\delta_n^{2\lambda}\psi''\|. \tag{14}$$

Using (1.5), (12), and (14), we get

$$\begin{aligned}
|\mathcal{H}_n^{(\alpha,\beta)}(f;x)-f(x)| &\leq |\mathcal{H}_n^{(\alpha,\beta)}(f-\psi;x)|+|\mathcal{H}_n^{(\alpha,\beta)}(\psi;x)-\psi(x)|+|f(x)-\psi(x)|\\
&\leq 4\|f-\psi\|+|\mathcal{H}_n^{(\alpha,\beta)}(\psi;x)-\psi(x)|\\
&\leq 4\|f-\psi\|+\frac{2\delta_n^{2(1-\lambda)}(x)}{n}\|\delta_n^{2\lambda}\psi''\|.
\end{aligned}$$

Hence,

$$\begin{aligned}
|R_n^{(\alpha,\beta)}(f;x)-f(x)| &\leq |\mathcal{H}_n^{(\alpha,\beta)}(f;x)-f(x)|+\left|f\left(x+\frac{1+\alpha}{n}\right)-f(x)\right|\\
&\leq 4\omega_{\phi^\lambda}^2\left(f,\frac{\delta_n^{(1-\lambda)}(x)}{\sqrt{2n}}\right)+\omega\left(f,\frac{1+\alpha}{n}\right).
\end{aligned}$$

□

Now, we discuss the Voronovskaja asymptotic formula [33] for operators $R_n^{(\alpha,\beta)}$.

Theorem 2.4. *For functions $f, f', f'' \in C^B_{[0,\infty)}$ and $x \in [0,\infty)$, then*

$$\lim_{n\to\infty} n[R_n^{(\alpha,\beta)}(f;x) - f(x)] = (1+\alpha)f'(x) + \frac{x(3\beta+1)}{2\beta}f''(x).$$

Proof. By Taylor's theorem

$$f(z) = f(x) + (z-x)f'(x) + \frac{(z-x)^2}{2}f''(x) + \epsilon_{\mathfrak{B}}(x;z)(z-x)^2, \tag{15}$$

where $\epsilon_{\mathfrak{B}}(x;z)$ is the Peano form of the remainder, $\epsilon_{\mathfrak{B}}(x;z) \in C^B_{[0,\infty)}$ and

$$\epsilon_{\mathfrak{B}}(x;z) = \frac{f(z) - f(x) - (z-x)f'(x) - \frac{(z-x)^2}{2!}f''(x)}{(z-x)^2} \text{ with } \lim_{z\to x}\epsilon_{\mathfrak{B}}(x;z) = 0.$$

Applying $R_n^{(\alpha,\beta)}$ on (15),

$$\begin{aligned} R_n^{(\alpha,\beta)}(f(z);x) &= f(x) + f'(x)\mu_{n,1}(x) + \frac{f''(x)}{2}\mu_{n,2}(x) \\ &\quad + R_n^{(\alpha,\beta)}(\epsilon_{\mathfrak{B}}(x;z)(z-x)^2;x). \end{aligned}$$

$$\begin{aligned} \lim_{n\to\infty} n[R_n^{(\alpha,\beta)}(f(z);x) - f(x)] &= f'(x)\lim_{n\to\infty}(n\mu_{n,1}(x)) \\ &\quad + \frac{f''(x)}{2}\lim_{n\to\infty}(n\mu_{n,2}(x)) \\ &\quad + \lim_{n\to\infty}(nR_n^{(\alpha,\beta)}(\epsilon_{\mathfrak{B}}(x;z)(z-x)^2;x)) \\ &= (1+\alpha)f'(x) + \frac{x(3\beta+1)}{2\beta}f''(x) + \mathcal{R}_n, \end{aligned}$$

where $\mathcal{R}_n = \lim_{n\to\infty}(nR_n^{(\alpha,\beta)}(\epsilon_{\mathfrak{B}}(x;z)(z-x)^2;x))$.
Using the Cauchy-Bunyakovsky-Schwarz inequality, we derive

$$nR_n^{(\alpha,\beta)}(\epsilon_{\mathfrak{B}}(x;z)(z-x)^2;x) \le \sqrt{n^2R_n^{(\alpha,\beta)}(\epsilon^2_{\mathfrak{B}}(x;z);x)}\sqrt{R_n^{(\alpha,\beta)}((z-x)^4;x)}.$$

If n approaches to infinity, then we observe $z \to x$ and $\lim_{z\to x}\epsilon_{\mathfrak{B}}(x;z) = 0$. It follows that

$$\lim_{n\to\infty}(n^2R_n^{(\alpha,\beta)}(\epsilon^2_{\mathfrak{B}}(x;z);x)) = 0 \text{ uniformly with respect to } x \in [0,\infty).$$

Therefore, $\mathcal{R}_n = \lim_{n\to\infty}(nR_n^{(\alpha,\beta)}(\epsilon_{\mathfrak{B}}(x;z)(z-x)^2;x)) = 0$.
Thus, the desired result is proved. □

Theorem 2.5. *For $f \in C^B_{[0,\infty)}$ and $x \in [0,\infty)$, then*

$$|R_n^{(\alpha,\beta)}(f;x) - f(x)| \le 2\omega(f,\delta), \text{ where } \delta = \sqrt{\frac{x(3\beta+1)}{\beta n} + \frac{\alpha^2\beta + 4\alpha\beta + 3\beta + \alpha + 1}{\beta n^2}}.$$

Proof. By the property of the modulus of continuity,

$$
\begin{aligned}
|f(z)-f(x)| &\leq \omega(f,|z-x|)\\
&\leq \left(1+\frac{1}{\delta}|z-x|\right)\omega(f,\delta).
\end{aligned}
$$

$$
\begin{aligned}
|R_n^{(\alpha,\beta)}(f;x)-f(x)| &\leq p_{n,\kappa}(x)\int_0^\infty \mathcal{I}^{\beta}_{\kappa,n}(z)\left|f(z)-f(x)\right|dz\\
&\leq \left(1+\frac{1}{\delta}p_{n,\kappa}(x)\int_0^\infty \mathcal{I}^{\beta}_{\kappa,n}(z)\,|z-x|\,dz\right)\omega(f,\delta),
\end{aligned}
$$

where $p_{n,\kappa}(x)=\exp\left(\frac{-nx}{2}\right)2^{-\alpha-1}\sum_{\kappa=0}^{\infty}2^{-\kappa}L_\kappa^{(\alpha)}\left(\frac{-nx}{2}\right)$.
By the Cauchy-Bunyakovsky-Schwarz inequality, it follows that

$$
\begin{aligned}
|R_n^{(\alpha,\beta)}(f;x)-f(x)| &\leq 1+\left(\frac{1}{\delta}\sqrt{p_{n,\kappa}(x)\int_0^\infty \mathcal{I}^{\beta}_{\kappa,n}(z)dz}\sqrt{p_{n,\kappa}(x)\int_0^\infty \mathcal{I}^{\beta}_{\kappa,n}(z)(z-x)^2dz}\right)\\
&\quad\times\omega(f,\delta)\\
&=\left(1+\frac{1}{\delta}\sqrt{(R_n^{(\alpha,\beta)}(f;x)((z-x)^2;x)}\right)\omega(f,\delta)\\
&=\left(1+\frac{1}{\delta}\sqrt{\frac{x(3\beta+1)}{\beta n}+\frac{\alpha^2\beta+4\alpha\beta+3\beta+\alpha+1}{\beta n^2}}\right)\omega(f,\delta)\\
&=2\omega(f,\delta).
\end{aligned}
$$

□

Now, the rate of convergence of $R_n^{(\alpha,\beta)}(f;x)$ by the Lipschitz class $Lip_K(\tau)$, $\tau>0$ is obtained. If $f\in Lip_K(\tau)$, then the function f satisfies the inequality

$$
|f(z)-f(x)|\leq K|z-x|^{\tau},\quad x,z\in[0,\infty), \tag{16}
$$

where K is a positive constant.

Theorem 2.6. *If $x\in[0,\infty)$ and $f\in C^B_{[0,\infty)}$ belongs to the class $Lip_K(\tau)$, then*

$$
|R_n^{(\alpha,\beta)}(f;x)-f(x)|\leq K(\mu_{n,2}(x))^{\frac{\mu}{2}},
$$

where $\mu_{n,2}(x)=\frac{x(3\beta+1)}{\beta n}+\frac{\alpha^2\beta+4\alpha\beta+3\beta+\alpha+1}{\beta n^2}$.

Proof. By (16),

$$
\begin{aligned}
|R_n^{(\alpha,\beta)}(f;x)-f(x)| &\leq R_n^{(\alpha,\beta)}(|f(z)-f(x)|;x)\\
&\leq R_n^{(\alpha,\beta)}(K|z-x|^{\tau};x).
\end{aligned}
$$

By the Cauchy-Bunyakovsky-Schwarz inequality, we obtain

$$
|R_n^{(\alpha,\beta)}(f;x)-f(x)|\leq K[R_n^{(\alpha,\beta)}((z-x)^2;x)]^{\frac{\tau}{2}}.
$$

Using Lemma 1.4,

$$
|R_n^{(\alpha,\beta)}(f;x)-f(x)|\leq K\left[\frac{x(3\beta+1)}{\beta n}+\frac{\alpha^2\beta+4\alpha\beta+3\beta+\alpha+1}{\beta n^2}\right]^{\frac{\tau}{2}}=K(\mu_{n,2}(x))^{\frac{\tau}{2}}.
$$

□

3. Weighted approximation

The goal of this section is to find approximation properties of the operators $R_n^{(\alpha,\beta)}$ corresponding to weighted spaces of continuous functions.
Let $\mathcal{B}_\zeta([0,\infty))$ be a normed space defined by

$$\mathcal{B}_\zeta([0,\infty)) = \{f : [0,\infty) \to \mathbb{R} : |f(x)| \le \mathcal{M}_f \zeta(x),\ x \in [0,\infty)\},$$

endowed with the norm

$$\|f\|_2 = \sup_{x\in[0,\infty)} \frac{|f(x)|}{\zeta(x)},$$

where $\zeta(x) = 1 + x^2$, and the constant $\mathcal{M}_f > 0$ depends on the function f.
Additionally, we define the following spaces,

i.) $C_\zeta([0,\infty)) = \{f \in \mathcal{B}_\zeta([0,\infty)) : f$ is continuous function on $[0,\infty)\}$,

ii.) $C_\zeta^*([0,\infty)) = \{f \in C_\zeta([0,\infty)) : \lim\limits_{x\to\infty} \frac{f(x)}{\zeta(x)}$ exists in $\mathbb{R}\}$.

It was shown in [17] that for any $f \in C_\zeta^*([0,\infty))$, the weighted modulus of continuity is denoted by

$$\Omega(f;\delta) = \sup_{0\le\epsilon<\delta, x\in[0,\infty)} \frac{|f(x+\epsilon) - f(x)|}{(1+\epsilon^2)\zeta(x)}. \tag{17}$$

Lemma 3.1. *If $f \in C_\zeta([0,\infty))$, then*

(i) $\Omega(f;\delta)$ *is a monotonically increasing function of* δ,
(ii) $\Omega(f;\lambda\delta) \le 2(1+\lambda)(1+\delta^2)\Omega(f;\delta)$, $\lambda > 0$,
(iii) $\Omega(f;\delta) \to 0$ *as* $\delta \to 0$.

The weighted modulus of the continuity definition enables us to write

$$|f(z) - f(x)| \le \zeta(x)(1 + (z-x)^2)\Omega(f;|z-x|). \tag{18}$$

Theorem 3.2. *For $f \in C^B_{[0,\infty)}$ and $x \in [0,\infty)$, then*

$$|R_n^{(\alpha,\beta)}(f;x) - f(x)| \le 2\left(1+\frac{1}{n}\right)\zeta(x)\Omega\left(f;\frac{1}{\sqrt{n}}\right)\left(1 + \mathcal{E}_1\frac{x(3\beta+1)}{\beta n} + \sqrt{\mathcal{E}_1\frac{x(3\beta+1)}{\beta}}\right.$$
$$\left.\times\left(1 + \frac{1}{n}\sqrt{\mathcal{E}_2\frac{x^2(3\beta+1)^2}{\beta^2}}\right)\right).$$

Proof. For $x, z \in [0,\infty)$. From (18) and Lemma 3.1, we get

$$\begin{aligned}|f(z) - f(x)| &\le (1 + (z-x)^2)\zeta(x)\Omega\left(f;\frac{|z-x|\delta}{\delta}\right)\\ &\le 2(1+\delta^2)\zeta(x)\left(1 + \frac{|z-x|}{\delta}\right)(1 + (z-x)^2)\Omega(f;\delta).\end{aligned} \tag{19}$$

Applying $R_n^{(\alpha,\beta)}$ on (19),

$$\begin{aligned}|R_n^{(\alpha,\beta)}(f;x)-f(x)| &\leq 2(1+\delta^2)\zeta(x)\Omega(f;\delta)R_n^{(\alpha,\beta)}\left[\left(1+\frac{|z-x|}{\delta}\right)(1+(z-x)^2);x\right]\\ &= 2(1+\delta^2)\zeta(x)\Omega(f;\delta)\left[1+\mu_{n,2}(x)+\frac{1}{\delta}R_n^{(\alpha,\beta)}(|z-x|;x)\right.\\ &\quad\left.+\frac{1}{\delta}R_n^{(\alpha,\beta)}(|z-x|(z-x)^2;x)\right].\end{aligned}$$

By the Cauchy-Bunyakovsky-Schwarz inequality, we get

$$\begin{aligned}|R_n^{(\alpha,\beta)}(f;x)-f(x)| &\leq 2(1+\delta^2)\zeta(x)\Omega(f;\delta)\left[1+\mu_{n,2}(x)\right.\\ &\quad\left.+\frac{1}{\delta}(\mu_{n,2}(x))^{1/2}+\frac{1}{\delta}\left(\mu_{n,2}(x)\right)^{1/2}\left(\mu_{n,4}(x)\right)^{1/2}\right].\end{aligned} \tag{20}$$

From Lemma 1.4, therefore

$$\mu_{n,2}(x)\leq\frac{\mathcal{E}_1 x(3\beta+1)}{\beta n},$$

and

$$\mu_{n,4}(x)\leq\frac{\mathcal{E}_2 x^2(3\beta+1)^2}{\beta^2 n^2},$$

where $\mathcal{E}_1>1$ and $\mathcal{E}_2>1$ are constants.
Using the above inequalities in (20) and choosing $\delta=\frac{1}{\sqrt{n}}$, we get

$$\begin{aligned}|R_n^{(\alpha,\beta)}(f;x)-f(x)| &\leq 2\left(1+\frac{1}{n}\right)\zeta(x)\Omega\left(f;\frac{1}{\sqrt{n}}\right)\left(1+\mathcal{E}_1\frac{x(3\beta+1)}{\beta n}+\sqrt{\mathcal{E}_1\frac{x(3\beta+1)}{\beta}}\right.\\ &\quad\left.\times\left(1+\frac{1}{n}\sqrt{\mathcal{E}_2\frac{x^2(3\beta+1)^2}{\beta^2}}\right)\right).\end{aligned}$$

□

Theorem 3.3. *For each $f\in C_\zeta^*([0,\infty))$, then*

$$\lim_{n\to\infty}\|R_n^{(\alpha,\beta)}(f;.)-f\|_2=0.$$

Proof. Using [13], to prove this theorem, it is sufficient to verify the following conditions

$$\lim_{n\to\infty}\|R_n^{(\alpha,\beta)}(\varrho_j;x)-x^j\|_2=0,\ j=0,1,2. \tag{21}$$

Since $R_n^{(\alpha,\beta)}(\varrho_0;x)=1$, so for $j=0$ (21) holds.
By Lemma 1.3,

$$\begin{aligned}\|R_n^{(\alpha,\beta)}(\varrho_1;x)-x\|_2 &= \sup_{x\in[0,\infty)}\frac{|R_n^{(\alpha,\beta)}(\varrho_1;x)-x|}{\zeta(x)}\\ &= \frac{1}{n}\sup_{x\in[0,\infty)}\left(\frac{1+\alpha}{\zeta(x)}\right)\to 0 \text{ as } n\to\infty.\end{aligned}$$

The condition (21) holds for $\jmath = 1$.
Again by Lemma 1.3,

$$\begin{aligned}\|R_n^{(\alpha,\beta)}(\gamma_2;x)-x^2\|_2 &= \sup_{x\in[0,\infty)}\frac{|R_n^{(\alpha,\beta)}(\gamma_2;x)-x^2|}{\zeta(x)}\\ &= \frac{1}{\beta n^2}\sup_{x\in[0,\infty)}\left(\frac{nx(2\alpha\beta+5\beta+1)+\beta(\alpha^2+4\alpha+3)+\alpha+1}{\zeta(x)}\right).\end{aligned}$$

Clearly, $\|R_n^{(\alpha,\beta)}(\varrho_2;x)-x^2\|_2\to 0$ as $n\to\infty$, the condition (21) holds for $\jmath = 2$.
Hence, the theorem is proved. $\square$

Theorem 3.4. *For each $f\in C_\zeta([0,\infty))$ and $\ell>0$, then*

$$\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{|R_n^{(\alpha,\beta)}(f;x)-f(x)|}{(\zeta(x))^{\ell+1}}=0.$$

Proof. For any fixed $x_0>0$,

$$\begin{aligned}\sup_{x\in[0,\infty)}\frac{|R_n^{(\alpha,\beta)}(f;x)-f(x)|}{(\zeta(x))^{\ell+1}} &\le \sup_{x\le x_0}\frac{|R_n^{(\alpha,\beta)}(f;x)-f(x)|}{(\zeta(x))^{\ell+1}}+\sup_{x\ge x_0}\frac{|R_n^{(\alpha,\beta)}(f;x)-f(x)|}{(\zeta(x))^{\ell+1}}\\ &\le \|R_n^{(\alpha,\beta)}(f;.)-f\|_{C[0,x_0]}+\|f\|_2\sup_{x\ge x_0}\frac{|R_n^{(\alpha,\beta)}(1+z^2;x)|}{(\zeta(x))^{\ell+1}}\\ &\quad+\sup_{x\ge x_0}\frac{|f(x)|}{(\zeta(x))^{\ell+1}}.\end{aligned}$$

By Theorem 3.3, the second term from the left in the above inequality tends to 0 as $n\to\infty$, and for the fixed x_0, if we choose x_0 large enough, then terms $\|f\|_2\sup_{x\ge x_0}\frac{|R_n^{(\alpha,\beta)}(1+z^2;x)|}{(\zeta(x))^{\ell+1}}$ and $\sup_{x\ge x_0}\frac{|f(x)|}{(\zeta(x))^{\ell+1}}$ can be made small enough. Thus, the desired proof is proved. $\square$

4. Graphical and numerical analysis

In this section, the graphical representations present the behavior of the proposed operators $R_n^{(\alpha,\beta)}$ to f for the different values of parameters α, β, and $n=25,50,75,100$.

Example 4.1. Let us consider test function $f(x)=x^3-2x^2+3$. For $\alpha=0.5$ and $\beta=2$, the operators $R_n^{(\alpha,\beta)}$ converge uniformly to f in Figure 1. The absolute error $\epsilon_n^{(\alpha,\beta)}$, defined as $\epsilon_n^{(\alpha,\beta)}=|R_n^{(\alpha,\beta)}(f;x)-f(x)|$ on the interval $[0,2]$, is calculated for the various values of $\alpha=0.5,2$ and $\beta=2,0.5$ in Table 1. For $\alpha=0.5$ and $\beta=2$, we observe that the approximation by $R_n^{(\alpha,\beta)}$ improves as n increases. Figure 2 visualizes the error associated with $R_n^{(\alpha,\beta)}$ and depicts the convergence behavior of the proposed operators.

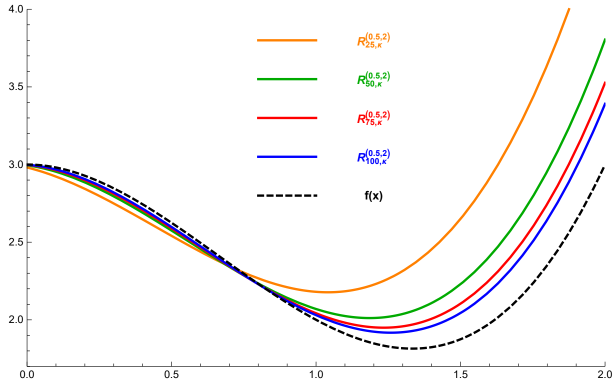

Figure 1: Approximation of $R_n^{(\alpha,\beta)}$ with $\alpha = 0.5,\ \beta = 2$, and $n = 25, 50, 75, 100$.

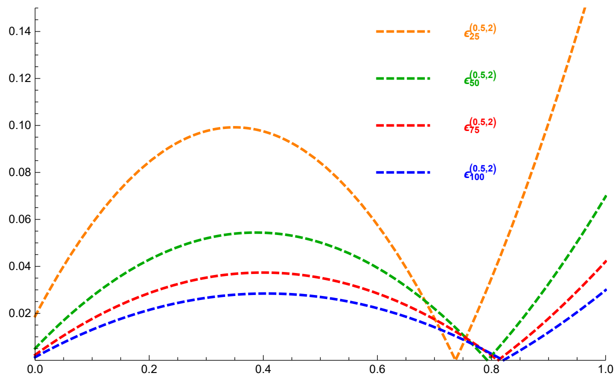

Figure 2: Graph of error $\epsilon_n^{(\alpha,\beta)}$ with $\alpha = 0.5,\ \beta = 2$, and $n = 25, 50, 75, 100$.

Table 1: Table for absolute error of the operators $R_n^{(\alpha,\beta)}$.

$\alpha = 0.5,\ \beta = 2$				
x	$n = 25$	$n = 50$	$n = 75$	$n = 100$
0	0.01876800	0.00504600	0.00229511	0.00130575
0.5	0.08416800	0.05014600	0.03511730	0.02695580
1.0	0.18043200	0.06975400	0.04206040	0.02989420
1.5	0.77503200	0.35465400	0.22923800	0.16924400
2.0	1.69963000	0.80455400	0.52641600	0.39109400
$\alpha = 2,\ \beta = 0.5$				
x	$n = 25$	$n = 50$	$n = 75$	$n = 100$
0	0.06316800	0.01869600	0.00873956	0.00503700
0.5	0.10636800	0.08449600	0.06242840	0.04898700
1.0	0.45043200	0.14970400	0.08388270	0.05706300
1.5	1.60723000	0.68390400	0.43019400	0.31311300
2.0	3.36403000	1.51810000	0.97650500	0.71916300

Example 4.2. The approximation for test function $f(x) = xe^{-5x}$ by operators $R_n^{(\alpha,\beta)}$ for $\alpha = 1$ and $\beta = 0.98$, is illustrated in Figure 3. In Table 2, the absolute error $\epsilon_n^{(\alpha,\beta)}$ to f on the interval $[0, 2]$ is calculated for the various values of $\alpha = 1, 5$ and $\beta = 0.98, 10$. The results in Table 2 demonstrate that the selection of suitable values for the parameters α and β significantly influences the performance of the operators $R_n^{(\alpha,\beta)}$. Figure 4

visualizes the error associated with $R_n^{(\alpha,\beta)}$ with large n and depicts the convergence behavior of the proposed operators.

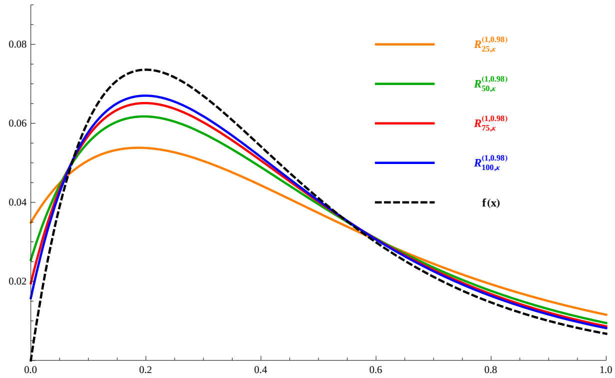

Figure 3: Approximation of $R_n^{(\alpha,\beta)}$ with $\alpha = 1,\ \beta = 0.98$, and $n = 25, 50, 75, 100$.

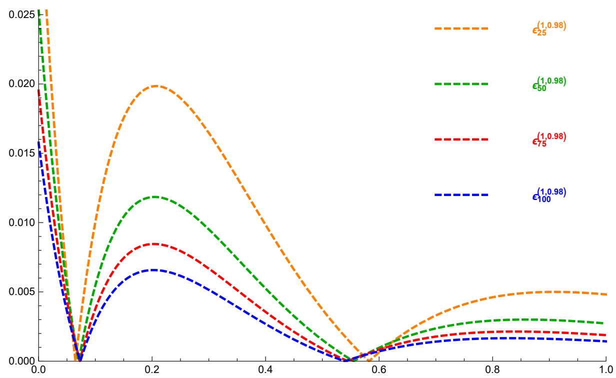

Figure 4: Graph of error $\epsilon_n^{(\alpha,\beta)}$ with $\alpha = 1,\ \beta = 0.98$, and $n = 25, 50, 75, 100$.

Table 2: Table for absolute error of the operators $R_n^{(\alpha,\beta)}$.

$\alpha = 1,\ \beta = 0.98$				
x	$n = 25$	$n = 50$	$n = 75$	$n = 100$
0	0.034901800	0.025423800	0.019518100	0.015763700
0.5	0.003740950	0.001533680	0.000898391	0.000617602
1.0	0.004815770	0.002722240	0.001880080	0.001433030
1.5	0.002000540	0.000915633	0.000583079	0.000425828
2.0	0.000534200	0.000197393	0.000115992	0.000081252
$\alpha = 5,\ \beta = 10$				
x	$n = 25$	$n = 50$	$n = 75$	$n = 100$
0	0.060720100	0.057087900	0.048077200	0.040729100
0.5	0.016013200	0.009493930	0.006739870	0.005223240
1.0	0.000626519	0.000237580	0.000133640	0.000089514
1.5	0.000436015	0.000248134	0.000171949	0.000131331
2.0	0.000150698	0.000071107	0.000045981	0.000033881

5. Conclusion

This study explores the innovative aspects of the integral type of construction of operators adopting modified Laguerre polynomials and the Păltănea basis function. It covers various aspects of the operators, including approximation properties, convergence rate, Voronovskaja-type asymptotic formula, and weighted approximation. The adaptability and convergence of the proposed operators are vital aspects of the study and depend on the choice of parameters α and β, respectively. It explores how different values of these parameters impact the performance of operators. The graphs are also used to illustrate the performance of the operators under various selections of parameters. Graphs provide a more instinctive understanding of how these parameters affect the behavior of proposed operators, which can be especially helpful for conveying findings to others.

Declarations

The authors declare that there are no conflicts of interest.

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