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Weyl's theorem for 2×2 upper triangular operator matrices

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Abstract. Let H and K be complex infinite dimensional separable Hilbert spaces. We denote by $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ a 2×2 upper triangular operator matrix acting on $H \oplus K$, where $A \in B(H)$, $B \in B(K)$ and $C \in B(K, H)$. In this paper, we mainly characterize the equivalent conditions for the 2×2 upper triangular operator matrices such that they satisfy Weyl's theorem using the features of the elements on the diagonal.

1. Introduction

In 1909, Weyl examined the compact perturbations of certain self-adjoint differential operators and found that the intersection of their spectra coincided with the isolated eigenvalues of finite multiplicity. This observation was later abstracted to the assertion that "Weyl's theorem holds". Then there followed a lot of excellent results on it ([8, 10, 13]). With the deepening of the research, a series of deformation has been produced, such as Browder's theorem, property (UW_E), property (R) and so on. As we all know, operator matrices play an important role in many branches of mathematics. In 1994, Du and Jin [5] firstly discussed the existence of C such that M_C is invertible. Then the perturbations of spectra of upper triangular operator matrices have attracted the attention of spectral theorists and many scholars have made fruitful results on this subject [4, 6]. Later, the existence of C such that M_C is respectively an upper semi-Fredholm operator, a Weyl operator, etc has been talked about ([2, 3, 7]). Meanwhile, they talked about the Weyl's theorem of 2×2 upper triangular operator matrices when there are additional conditions on the diagonal elements A and B. Also, the authors in [1] made some researches on the Weyl type theorem of upper triangular operator matrices. Recently, Qiu and Cao give the sufficient and necessary conditions such that M_C satisfies property (UW_E) for all $C \in B(K, H)$ when both A and B are general operators [9]. Besides, Yang and Cao have done some work on the property (R) of 2×2 upper triangular operator matrices [11, 12]. But, as the original theorem, there is no complete discussion on Weyl's theorem of upper triangular operator matrices under the conditions that A and B are general operators. Here, in this paper, we will do some work on this topic. Specifically, we will give the sufficient and necessary conditions for the 2×2 upper triangular

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operator matrices such that they satisfy Weyl's theorem. To begin with, we give some terminology and notations.

Throughout this paper, $\mathbb{C}$ and $\mathbb{N}$ denote the set of all complex numbers and the set of all non-negative integers, respectively. H and K denote the complex infinite dimensional separable Hilbert spaces. We denote by $B(K, H)$ the algebra of all bounded linear operators from K into H. For convenience, we write $B(H)$ for $B(H, H)$. For $A \in B(H)$, $B \in B(K)$, we denote by $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ a 2×2 upper triangular operator matrix acting on $H \oplus K$, where $C \in B(K, H)$. For $T \in B(H)$, $N(T)$ and $R(T)$ stand for the kernel and the range of T, respectively. If $R(T)$ is closed and $n(T) < \infty$, then we say T an upper semi-Fredholm operator; while T is said to be lower semi-Fredholm if $d(T) < \infty$, where $n(T)$ and $d(T)$ denote the dimension of $N(T)$ and the codimension of $R(T)$, respectively. We call $T \in B(H)$ a bounded below operator if T is upper semi-Fredholm with $n(T) = 0$. If T is an upper semi-Fredholm operator or a lower semi-Fredholm operator, then we call T a semi-Fredholm operator. Now, the index of T is defined as $\operatorname{ind}(T) = n(T) - d(T)$. If $\operatorname{ind}(T) = 0$, then T is said to be a Weyl operator. The ascent and descent of T are defined respectively by $\operatorname{asc}(T) = \inf\{n \in \mathbb{N} : N(T^n) = N(T^{n+1})\}$ and $\operatorname{des}(T) = \inf\{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\}$. If the infimum does not exist, we write $\operatorname{asc}(T) = \infty$ (resp. $\operatorname{des}(T) = \infty$). T is called a Browder operator if it is Fredholm of finite ascent and descent.

The spectrum and the Weyl spectrum of T is defined by

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not an invertible operator}\},$$

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a Weyl operator}\}.$$

T is said to satisfy Weyl's theorem if

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T),$$

where $\pi_{00}(T)$ denotes the set of all isolated eigenvalues with finite multiplicity of T. For some subset G of the complex plane, we write iso$G$ for the set of all isolated points of G. For some $x \in H$, we denote by span$\{x\}$ the closed subspace spanned by x.

For the Weyl's theorem of 2×2 upper triangular operator matrices, we get the following results.

Theorem 1.1. *Let $A \in B(H)$ and $B \in B(K)$. Then M_C satisfies Weyl's theorem for every $C \in B(K, H)$ if and only if the following statements hold:*

(1) *M_0 satisfies Weyl's theorem;*

(2) $\mathcal{M}_1 \doteq \{\lambda \in \mathbb{C} : A - \lambda I \text{ is upper semi-Fredholm}, B - \lambda I \text{ is lower semi-Fredholm}, d(A - \lambda I) = n(B - \lambda I) = \infty\} = \emptyset$;

(3) $\mathcal{M}_2 \doteq \{\lambda \in iso\sigma(A) \cap iso\sigma(B) : n(A - \lambda I) < \infty \text{ and } d(A - \lambda I) = n(B - \lambda I) = \infty\} = \emptyset$.

Corollary 1.1. *Let $A \in B(H)$ and $B \in B(K)$. Then M_C satisfies Weyl's theorem for every $C \in B(K, H)$ if and only if the following statements hold:*

(1) *M_{C_0} satisfies Weyl's theorem for some $C_0 \in B(K, H)$;*

(2) $\mathcal{M}_1 \doteq \{\lambda \in \mathbb{C} : A - \lambda I \text{ is upper semi-Fredholm}, B - \lambda I \text{ is lower semi-Fredholm}, d(A - \lambda I) = n(B - \lambda I) = \infty\} = \emptyset$;

(3) $\mathcal{M}_2' \doteq \{\lambda \in iso\sigma(A) \cap iso\sigma(B) : n(A - \lambda I) < \infty, n(A - \lambda I) + n(B - \lambda I) > 0, d(A - \lambda I) = \infty\} = \emptyset$;

(4) $\mathcal{M}_3 \doteq \{\lambda \in \mathbb{C} : A - \lambda I \text{ is bounded below}, B - \lambda I \text{ is surjective}, 0 < d(A - \lambda I) = n(B - \lambda I) < \infty\} = \emptyset$.

The rest of this paper is organized as follows. In section 2, we shall make some preparation for the proofs of the Theorem 1.1 and Corollary 1.1. Section 3 is devoted to the proof of main results. As an application, we shall describe the Weyl's theorem for upper triangular operator matrices when the main diagonal of the matrix are Toeplitz operators.

2. Perturbations of spectra for upper triangular operator matrices

In order to discuss the Weyl's theorem for operator matrices, in this section, we will give some conclusions about perturbations of spectra of 2×2 upper triangular operator matrices.

Theorem 2.1. *Let $A \in B(H)$ be a bounded below operator and $B \in B(K)$ be a surjective operator. If $d(A) = n(B) = \infty$, then there exists $C \in B(K, H)$ such that M_C is a Weyl operator but not a Browder operator.*

Proof. Suppose $\{h_1, h_3, h_5, \cdots\}$ is an orthonormal basis of $R(A)^\perp$. Let $Ah_1 = h_2$, $Ah_2 = h_4$. Using the fact that $n(A) = 0$, we know that $h_2 \neq 0$, $h_4 \neq 0$. Firstly, we claim $\{h_2,\ h_4\} \subseteq R(A)$ is linearly independent. In fact, if $a_2h_2 + a_4h_4 = 0$, then $A(a_2h_1 + a_4h_2) = 0$. Since A is bounded below, $a_2h_1 + a_4h_2 = 0$. It follows that $a_2h_1 = -a_4h_2 \in R(A)^\perp \cap R(A)$. Thus, we have that $a_2 = a_4 = 0$. The claim is proved.

Suppose $\{e_1, e_3, e_5, \cdots\}$ is an orthonormal basis of $N(B)$. Since B is surjective, there exists $e_2 \in N(B)^\perp$ such that $Be_2 = e_1$. For e_2, there exists $e_4 \in N(B)^\perp$ such that $Be_4 = e_2$. For e_3, there exists $e_6 \in N(B)^\perp$ such that $Be_6 = e_3$. Proceeding sequentially, for any e_n, there exists $e_{2n} \in N(B)^\perp$ such that $Be_{2n} = e_n$. Now, we assert $\{e_{2n}\}_{n=1}^{\infty} \subseteq N(B)^\perp$ is linearly independent. In fact, if $a_2e_2 + a_4e_4 + \cdots + a_{2n}e_{2n} = 0$, then $B(a_2e_2 + a_4e_4 + \cdots + a_{2n}e_{2n}) = 0$, that is, $a_2e_1 + a_4e_2 + a_6e_3 + \cdots + a_{2n}e_n = 0$. It follows that $a_2e_1 + a_6e_3 + a_{10}e_5 + \cdots = -(a_4e_2 + a_8e_4 + \cdots) \in N(B) \cap N(B)^\perp$. Thus, we have

$$\begin{cases} a_2e_1 + a_6e_3 + a_{10}e_5 + \cdots = 0 \\ a_4e_2 + a_8e_4 + a_{12}e_6 + \cdots = 0. \end{cases}$$

It is not difficult to show that $a_2 = a_6 = a_{10} = \cdots = 0$ and $B(a_4e_2 + a_8e_4 + \cdots) = 0$. Similar to the above method in succession, we can obtain $a_{2i} = 0 (i = 1, 2, 3, \cdots, n)$. Therefore, this assertion is proved.

Let $M = \operatorname{span}\{e_1\}$, $N = \operatorname{span}\{h_1\}$, $F = \operatorname{span}\{e_4\}$ and $E = \operatorname{span}\{h_4\}$. Since $\dim(N(B) \ominus M) = \dim(R(A)^\perp \ominus N) = \infty$, there exists an isometric invertible operator $T_1 : N(B) \ominus M \to R(A)^\perp \ominus N$. Define invertible operators $T_2 : F \to N$ satisfying $T_2e_4 = -h_1$ and $T_3 : M \to E$ such that it satisfies $T_3e_1 = h_4$. Let

$$C = \begin{pmatrix} T_1 & 0 & 0 & 0 \\ 0 & T_2 & 0 & 0 \\ 0 & 0 & T_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} N(B) \ominus M \\ F \\ M \\ N(B)^\perp \ominus F \end{pmatrix} \to \begin{pmatrix} R(A)^\perp \ominus N \\ N \\ E \\ R(A) \ominus E \end{pmatrix}.$$

Next, we prove that this C satisfies the conclusion.

(1) M_C is Weyl.

(i) $n(M_C) = 1$.

Let $\begin{pmatrix} x \\ y \end{pmatrix} \in N(M_C)$. Then $\begin{cases} Ax + Cy = 0, \\ By = 0. \end{cases}$ Due to $y \in N(B)$, suppose $y = y_1 + ae_1$, where $a \in \mathbb{C}$ and $y_1 \in N(B) \ominus M$. Then,

$$0 = Ax + Cy = Ax + T_1y_1 + T_3(ae_1) = Ax + T_1y_1 + ah_4 = A(x + ah_2) + T_1y_1,$$

which shows

$$A(x + ah_2) = -T_1y_1 \in R(A) \cap R(A)^\perp.$$

Thus $A(x + ah_2) = 0$ and $T_1y_1 = 0$. Since A is bounded below and T_1 is invertible, we have $x = -ah_2$ and $y_1 = 0$, which shows $y = ae_1$. So

$$N(M_C) \subseteq \operatorname{span}\left\{\begin{pmatrix} -h_2 \\ e_1 \end{pmatrix}\right\}.$$

Moreover, the inclusion $\operatorname{span}\left\{\begin{pmatrix} -h_2 \\ e_1 \end{pmatrix}\right\} \subseteq N(M_C)$ is obvious. Therefore

$$\operatorname{span}\left\{\begin{pmatrix} -h_2 \\ e_1 \end{pmatrix}\right\} = N(M_C),$$

which shows $0 < n(M_C) = 1 < \infty$.

(ii) $R(M_C)$ is closed.

Suppose that $M_C \begin{pmatrix} x_n \\ y_n \end{pmatrix} \to \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ as $n \to \infty$, then $\begin{cases} Ax_n + Cy_n \to u_0 \\ By_n \to v_0 \end{cases}$ $(n \to \infty)$. Let $y_n = \alpha_n + \beta_n$, where $\alpha_n \in N(B)$ and $\beta_n \in N(B)^\perp$. From $By_n = B\beta_n$, we know that $\{B\beta_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Moreover, since

$R(B)$ is closed, it follows that $B|_{N(B)^\perp}$ is invertible, which implies that $\{\beta_n\}_{n=1}^\infty$ is a Cauchy sequence. Suppose $\alpha_n = u_n + a_n e_1$ and $\beta_n = v_n + w_n$, where $u_n \in N(B) \ominus M$, $v_n \in F$ and $w_n \in N(B)^\perp \ominus F$. Then,

$$\begin{aligned} Ax_n + Cy_n &= Ax_n + T_1 u_n + T_3(a_n e_1) + T_2 v_n \\ &= A(x_n + a_n h_2) + (T_1 u_n + T_2 v_n). \end{aligned}$$

Due to $\{A(x_n + a_n h_2)\}_{n=1}^\infty \subseteq R(A)$ and $\{T_1 u_n + T_2 v_n\}_{n=1}^\infty \subseteq R(A)^\perp$, it follows that both $\{A(x_n + a_n h_2)\}_{n=1}^\infty$ and $\{T_1 u_n + T_2 v_n\}_{n=1}^\infty$ are Cauchy sequences. Since $R(A)$ is closed, we can suppose $A(x_n + a_n h_2) \to Ax_0 (n \to \infty)$ for some $x_0 \in H$. Furthermore, from $\{T_1 u_n\}_{n=1}^\infty \subseteq R(A)^\perp \ominus N$ and $\{T_2 v_n\}_{n=1}^\infty \subseteq N$, we obtain both $\{T_1 u_n\}_{n=1}^\infty$ and $\{T_2 v_n\}_{n=1}^\infty$ are Cauchy sequences. It follows that $\{u_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$ are Cauchy sequences because T_1 and T_2 are invertible, so is $\{w_n\}_{n=1}^\infty = \{\beta_n - v_n\}_{n=1}^\infty$. Suppose that $u_n \to a_0 \in N(B) \ominus M$, $v_n \to b_0 \in F$, and $w_n \to c_0 \in N(B)^\perp \ominus F$ as $n \to \infty$. Thus, $Cc_0 = 0$, $Ba_0 = 0$ and

$$\begin{aligned} Ax_n + Cy_n &\to Ax_0 + Ca_0 + Cb_0 (n \to \infty) \\ &= Ax_0 + C(a_0 + b_0 + c_0). \end{aligned}$$

Moreover, $Ba_0 = 0$ shows

$$By_n = B\beta_n = B(u_n + v_n) \to B(b_0 + c_0) = B(a_0 + b_0 + c_0)(n \to \infty).$$

In conclusion, we have

$$M_C \begin{pmatrix} x_n \\ y_n \end{pmatrix} \to M_C \begin{pmatrix} x_0 \\ a_0 + b_0 + c_0 \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix},$$

which means that $R(M_C)$ is closed.

(iii) $d(M_C) = n(M_C^*) = 1$.

To see this, we assume that $\begin{pmatrix} x \\ y \end{pmatrix} \in N(M_C^*)$. Then $\begin{cases} A^* x = 0, \\ C^* x + B^* y = 0. \end{cases}$ Suppose $x = x_1 + x_2 = x_1 + ah_1$, where $x_1 \in R(A)^\perp \ominus N$. Since $C^* x + B^* y = 0$, it follows that

$$T_1^* x_1 = -[T_2^*(ah_1) + B^* y] \in N(B) \cap N(B)^\perp = \{0\}.$$

Observe that T_1^* is invertible, so $x_1 = 0$ and hence $x = ah_1$. By the definition of T_2, $T_2 e_4 = -h_1$, and hence

$$< T_2^* h_1, e_4 >=< h_1, T_2 e_4 >= - < h_1, h_1 >= -\|h_1\|^2 = -\frac{\|h_1\|^2}{\|e_4\|^2} < e_4, e_4 > .$$

Therefore, $T_2^* h_1 = -\frac{\|h_1\|^2}{\|e_4\|^2} e_4$. Suppose that $e_4 = B^* u_0$ since $e_4 \in N(B)^\perp = R(B^*)$. From the fact that $T_2^*(ah_1) + B^* y = 0$, we have $-a\frac{\|h_1\|^2}{\|e_4\|^2} e_4 + B^* y = 0$. Then $-a\frac{\|h_1\|^2}{\|e_4\|^2} B^* u_0 + B^* y = 0$. Combined with the fact that B is surjective, it is known that B^* is injective, which means that $y = a\frac{\|h_1\|^2}{\|e_4\|^2} u_0$. So

$$N(M_C^*) \subseteq \operatorname{span} \left\{ \begin{pmatrix} h_1 \\ \frac{\|h_1\|^2}{\|e_4\|^2} u_0 \end{pmatrix} \right\}.$$

Moreover, the inclusion " $\supseteq$ " is obvious. Therefore

$$\operatorname{span} \left\{ \begin{pmatrix} h_1 \\ \frac{\|h_1\|^2}{\|e_4\|^2} u_0 \end{pmatrix} \right\} = N(M_C^*),$$

which shows $d(M_C) = n(M_C^*) = 1 < \infty$.

Combined with $n(M_C) = 1$, we know that M_C is a Weyl operator.

(2) M_C is not Browder.

It is suffices to verify that $N(M_C^p) \neq N(M_C^{p+1})$ for any $p > 2$. Take $z_0 = \begin{pmatrix} 0 \\ e_{2^p} \end{pmatrix}$ and write $x_0 = 0$, $y_0 = e_{2^p}$. By the definition, we know

$$Cy_0 = 0, By_0 = e_{2^{p-1}}, B^2y_0 = e_{2^{p-2}}, \cdots, B^{p-2}y_0 = e_{2^2} = e_4,\ B^{p-1}y_0 = e_2 \text{ and } B^py_0 = e_1.$$

So

$$Cy_0 = CBy_0 = \cdots = CB^{p-3}y_0 = CB^{p-1}y_0 = 0.$$

Hence,

$$\begin{aligned} & A^{p+1}x_0 + A^pCy_0 + \cdots + A^2CB^{p-2}y_0 + ACB^{p-1}y_0 + CB^py_0 \\ = \ & A^2Ce_4 + Ce_1 = A^2(T_2e_4) + T_3e_1 \\ = \ & A^2(-h_1) + h_4 = -Ah_2 + Ah_2 = 0 \end{aligned}$$

and $B^{p+1}y_0 = B(B^py_0) = Be_1 = 0$. It means that $z_0 \in N(M_C^{p+1})$. We know that $z_0 \notin N(M_C^p)$ because $B^py_0 = e_1 \neq 0$. Therefore, $N(M_C^p) \neq N(M_C^{p+1})$ for any $p > 2$ and so $\mathrm{asc}(M_C) = \infty$. In short, we give a complete proof. □

Theorem 2.2. *Let $A \in B(H)$ be an upper-semi Fredholm operator and $B \in B(K)$ be a lower-semi Fredholm operator. If $d(A) = n(B) = \infty$, then there exists $C \in B(K, H)$ such that M_C is a Weyl operator with $n(M_C) = d(M_C) = n(A) + d(B) + 1$.*

Proof. Suppose $d(B) = t$ and $n(A) = m$. Suppose that $\{e_i\}_{i=1}^{t+1} \subseteq N(B)$ is orthonormal and let $M = \mathrm{span}\{e_1, e_2, \cdots, e_{t+1}\}$. Let $R(A)^\perp = N \oplus W$, where N is a closed subspace of $R(A)^\perp$ with $\dim N = n(A) + 1 = m + 1$. And let F be a subspace of $N(B)^\perp$ with $\dim F = m + 1$. Now, we take $\{v_i\}_{i=1}^{m+1}$ and $\{h_k\}_{k=1}^{m+1}$ as the orthonormal bases of N and F satisfying $h_k = B^*g_k$, where $g_k \in N(B^*)^\perp$, respectively. Define an invertible operator T_1 from $N(B) \ominus M$ onto $R(A)^\perp \ominus N$, and define an invertible operator $T_2 : F \to N$ so that it satisfies $T_2h_i = v_i (i = 1, 2, \cdots, m + 1)$. It is clear that $\dim R(A) = \infty$. Therefore, there is a closed subspace $E \subseteq R(A)$ such that $\dim E = \dim M = t + 1$ and suppose that $\{Au_1, Au_2, \cdots, Au_{t+1}\}$ is an orthornormal basis of E. Define an invertible linear operator $T_3 : M \to E$ satisfying $T_3e_i = Au_i (i = 1, 2, \cdots, t + 1)$. Let

$$C = \begin{pmatrix} T_1 & 0 & 0 & 0 \\ 0 & T_2 & 0 & 0 \\ 0 & 0 & T_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} N(B) \ominus M \\ F \\ M \\ N(B)^\perp \ominus F \end{pmatrix} \to \begin{pmatrix} R(A)^\perp \ominus N \\ N \\ E \\ R(A) \ominus E \end{pmatrix}.$$

Next, we prove that this C satisfies the conclusion.

(i) $n(M_C) = m + t + 1$.

Let $\begin{pmatrix} x \\ y \end{pmatrix} \in N(M_C)$. Then $\begin{cases} Ax + Cy = 0, \\ By = 0. \end{cases}$ Due to $y \in N(B)$, suppose $y = y_1 + a_1e_1 + a_2e_2 + \cdots + a_{t+1}e_{t+1}$, where $y_1 \in N(B) \ominus M$. Then,

$$\begin{aligned} 0 = Ax + Cy &= Ax + T_1y_1 + T_3(a_1e_1 + a_2e_2 + \cdots + a_{t+1}e_{t+1}) \\ &= Ax + T_1y_1 + A(a_1u_1 + a_2u_2 + \cdots + a_{t+1}u_{t+1}) \\ &= A(x + a_1u_1 + a_2u_2 + \cdots + a_{t+1}u_{t+1}) + T_1y_1, \end{aligned}$$

which shows

$$A(x + a_1u_1 + a_2u_2 + \cdots + a_{t+1}u_{t+1}) = -T_1y_1 \in R(A) \cap R(A)^\perp.$$

Thus $A(x + a_1u_1 + a_2u_2 + \cdots + a_{t+1}u_{t+1}) = 0$ and $T_1y_1 = 0$. Since T_1 is invertible, we have $y_1 = 0$, which shows $y = a_1e_1 + a_2e_2 + \cdots + a_{t+1}e_{t+1}$. Moreover, we get $x + a_1u_1 + a_2u_2 + \cdots + a_{t+1}u_{t+1} \in N(A)$. It follows that $x \in -(a_1u_1 + a_2u_2 + \cdots + a_{t+1}u_{t+1}) + N(A)$. So

$$N(M_C) \subseteq \mathrm{span}\left\{\begin{pmatrix} -u_i \\ e_1 \end{pmatrix}\right\}_{i=1}^{t+1} \oplus [N(A) \oplus \{0\}].$$

Moreover, the inverse inclusion clearly holds. Therefore

$$N(M_C) = \text{span}\left\{\begin{pmatrix} -u_i \\ e_1 \end{pmatrix}\right\}_{i=1}^{t+1} \oplus [N(A) \oplus \{0\}].$$

which shows $0 < n(M_C) = m + t + 1 < \infty$.

(ii) Similar to the theorem 2.1, we can prove that $R(M_C)$ is closed.

(iii) We also can prove that $d(M_C) = n(M_C^*) = m + t + 1$. To see this, we assume that $\begin{pmatrix} x \\ y \end{pmatrix} \in N(M_C^*)$. Then

$$\begin{cases} A^*x = 0, \\ C^*x + B^*y = 0. \end{cases}$$

Note that $x \in N(A^*) = R(A)^\perp$. Suppose that $x = x_1 + x_2$, where $x_1 \in R(A)^\perp \ominus N$ and $x_2 \in N$. Due to $C^*x + B^*y = 0$, it follows that

$$T_1^*x_1 = -(T_2^*x_2 + B^*y) \in N(B) \cap N(B)^\perp = \{0\}.$$

Observe that T_1^* is invertible, so $x_1 = 0$ and hence $x = x_2$. Assume that $y = y_1 + y_2$, where $y_1 \in N(B^*)^\perp$ and $y_2 \in N(B^*)$. Then, $T_2^*x_2 = -B^*y_1 \in F$. Suppose that $x = x_2 = a_1v_1 + a_2v_2 + \cdots + a_{m+1}v_{m+1}$. Then,

$$\begin{aligned} T_2^*x_2 &= a_1h_1 + a_2h_2 + \cdots + a_{m+1}h_{m+1} \\ &= a_1B^*g_1 + a_2B^*g_2 + \cdots + a_{m+1}B^*g_{m+1} \\ &= B^*(a_1g_1 + a_2g_2 + \cdots + a_{m+1}g_{m+1}). \end{aligned}$$

Hence, $B^*(y_1 + a_1g_1 + a_2g_2 + \cdots + a_{m+1}g_{m+1}) = 0$. Combined with $y_1, g_1, \cdots, g_{m+1} \in N(B^*)^\perp$ and injectivity of $B^*|_{N(B^*)^\perp}$, we have $y_1 = -(a_1g_1 + \cdots + a_{m+1}g_{m+1})$. So, $y \in -(a_1g_1 + a_2g_2 + \cdots + a_{m+1}g_{m+1}) + N(B^*)$. So,

$$N(M_C^*) \subseteq \text{span}\left\{\begin{pmatrix} v_i \\ -g_i \end{pmatrix}\right\}_{i=1}^{m+1} \oplus [\{0\} \oplus N(B^*)].$$

Moreover, the inclusion

$$\text{span}\left\{\begin{pmatrix} v_i \\ -g_i \end{pmatrix}\right\}_{i=1}^{m+1} \oplus [\{0\} \oplus N(B^*)] \subseteq N(M_C^*)$$

is obvious. Therefore

$$N(M_C^*) = \text{span}\left\{\begin{pmatrix} v_i \\ -g_i \end{pmatrix}\right\}_{i=1}^{m+1} \oplus [\{0\} \oplus N(B^*)].$$

which shows $d(M_C) = n(M_C^*) = m + t + 1$. Combined with $n(M_C) = m + t + 1$, we know that M_C is a Weyl operator with $n(M_C) = d(M_C) = n(A) + d(B) + 1$. □

3. Weyl's theorem for upper triangular operator matrices

Weyl's theorem may or may not hold for a direct sum of operators for which Weyl's theorem holds. In the section, we will consider the sufficient and necessary conditions such that M_C satisfy the Weyl's theorem for all $C \in B(K, H)$. The following lemma is very useful to give this result.

Lemma 3.1. *Let $A \in B(H)$ and $B \in B(K)$ with $n(A) < \infty$, $n(A) + n(B) > 0$ and $d(A) = \infty$. If $0 \in iso\sigma(A) \cap iso\sigma(B)$, then there exists $C \in B(K, H)$ such that $0 \in \pi_{00}(M_C)$.*

Proof. It is clear that $0 \in \text{iso}\sigma(M_C)$ for any $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. We can claim that $R(A)$ is not closed. In fact, if not, A is upper semi-Fredholm. From $0 \in \text{iso}\sigma(A)$, we get A is a Browder operator, which is in contradiction to $d(A) = \infty$. If $n(B) < \infty$, taking $M_C = M_0$, the conclusion obviously holds. If $n(B) = \infty$, by [11, Lemma 3.1], the conclusion holds, too. □

Based on this, we give the proofs of the main results.

Proof of Theorem 1.1. Necessity. (2) If there is $\lambda_0 \in \mathcal{M}_1$, then by using Theorem 2.2 we can find $C \in B(K, H)$ such that $M_C - \lambda_0 I$ is a Weyl operator. Since M_C satisfies Weyl's theorem, it follows that $M_C - \lambda_0 I$ is Browder, and hence $\mathrm{asc}(A - \lambda_0 I) < \infty$ and $\mathrm{des}(B - \lambda_0 I) < \infty$. Thus, by perturbation theorem of semi-Fredholm operator, there exists $\varepsilon > 0$ such that $A - \lambda I$ is bounded below, $B - \lambda I$ is surjective, $\mathrm{ind}(A - \lambda I) = \mathrm{ind}(A - \lambda_0 I)$, $\mathrm{ind}(B - \lambda I) = \mathrm{ind}(B - \lambda_0 I)$ and $d(A - \lambda I) < \infty \Leftrightarrow n(B - \lambda I) < \infty$. From Theorem 2.1, there exists $C_0 \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $M_{C_0} - \lambda I$ is Weyl but not Browder if $d(A - \lambda I) = n(B - \lambda I) = \infty$, which is in contradiction to the fact that M_C satisfies Weyl's theorem for every $C \in B(K, H)$. This means that $d(A - \lambda I) < \infty$ and $n(B - \lambda I) < \infty$ if $0 < |\lambda - \lambda_0| < \varepsilon$. Thus $A - \lambda_0 I$ and $B - \lambda_0 I$ are Fredholm operators, a desired contradiction.

(3) If there is $\lambda_0 \in \mathcal{M}_2$, then by using Lemma 3.1 we can find $C \in B(K, H)$ such that $\lambda_0 \in \pi_{00}(M_C)$. Since M_C satisfies Weyl's theorem, it follows that $M_C - \lambda_0 I$ is Browder, and hence $A - \lambda_0 I$ is upper semi-Fredholm and $B - \lambda_0 I$ is lower semi-Fredholm. Combined with $\lambda_0 \in \mathrm{iso}\sigma(A) \cap \mathrm{iso}\sigma(B)$, we have $A - \lambda_0 I$ and $B - \lambda_0 I$ are Browder operator, which shows a contradiction.

Sufficiency. For any $C \in B(K, H)$, let $\lambda_0 \in \sigma(M_C) \setminus \sigma_w(M_C)$, then $n(A - \lambda_0 I) + n(B - \lambda_0 I) > 0$, $A - \lambda_0 I$ is an upper semi-Fredholm operator, $B - \lambda_0 I$ is a lower semi-Fredholm operator and $d(A - \lambda_0 I) < \infty \Leftrightarrow n(B - \lambda_0 I) < \infty$. From (2) we get $A - \lambda_0 I$ and $B - \lambda_0 I$ are Fredholm and hence $M_0 - \lambda_0 I$ is Fredholm and

$$\mathrm{ind}(M_0 - \lambda_0 I) = \mathrm{ind}(A - \lambda_0 I) + \mathrm{ind}(B - \lambda_0 I) = \mathrm{ind}(M_C - \lambda_0 I) = 0,$$

that is $M_0 - \lambda_0 I$ is Weyl. Since M_0 satisfies Weyl's theorem, we know that $M_0 - \lambda_0 I$ is Browder. It follows that both $A - \lambda_0 I$ and $B - \lambda_0 I$ are Browder. Therefore, $M_C - \lambda_0 I$ is Browder. We get that $\lambda_0 \in \pi_{00}(M_C)$.

On the other hand, for any $\lambda_0 \in \pi_{00}(M_C)$, then $n(A - \lambda_0 I) < \infty$, $n(A - \lambda_0 I) + n(B - \lambda_0 I) > 0$, and there exists some $\varepsilon > 0$ such that $A - \lambda I$ is bounded below and $B - \lambda I$ is surjective for any $0 < |\lambda - \lambda_0| < \varepsilon$. Besides, $d(A - \lambda I) < \infty$ if and only if $n(B - \lambda I) < \infty$. From (2) we know $d(A - \lambda I) < \infty$ and $n(B - \lambda I) < \infty$, and so $A - \lambda I$ and $B - \lambda I$ are Fredholm. Therefore, $M_0 - \lambda I$ is Fredholm and $\mathrm{ind}(M_0 - \lambda I) = \mathrm{ind}(M_C - \lambda I) = 0$, that is $M_0 - \lambda I$ is Weyl. So, $M_0 - \lambda I$ is Browder. It follows that $A - \lambda I$ and $B - \lambda I$ are Browder. Since $A - \lambda I$ is bounded below and $B - \lambda I$ is surjective, it shows that both $A - \lambda I$ and $B - \lambda I$ are invertible. That is to say, $\lambda_0 \in \mathrm{iso}\sigma(A) \cup \rho(A)$ and $\lambda_0 \in \mathrm{iso}\sigma(B) \cup \rho(B)$. Next, we will divide it into the following cases.

Case 1. $\lambda_0 \in \rho(A)$.

Now, we claim that $n(B - \lambda_0 I) < \infty$. In fact, if not, we can take an orthornormal set $\{y_n\}_{n=1}^{\infty} \subseteq N(B - \lambda_0 I)$. There exists a sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{H}$ such that $(A - \lambda_0 I)x_n = Cy_n$ for any $n \geq 1$ because $A - \lambda_0 I$ is surjective. Thus, $\left\{\begin{pmatrix} -x_n \\ y_n \end{pmatrix}\right\}_{n=1}^{\infty} \subseteq N(M_C - \lambda_0 I)$. It is easy to show that $\left\{\begin{pmatrix} -x_n \\ y_n \end{pmatrix}\right\}_{n=1}^{\infty} \subseteq N(M_C - \lambda_0 I)$ is linearly independent, which contradicts with $n(M_C - \lambda_0 I) < \infty$. Therefore, $n(B - \lambda_0 I) < \infty$. This shows that $\lambda_0 \in \pi_{00}(M_0)$. Since M_0 satisfies Weyl's theorem, it follows that $M_0 - \lambda_0 I$ is Browder, and so both $A - \lambda_0 I$ and $B - \lambda_0 I$ are Browder. Therefore, $M_C - \lambda_0 I$ is Browder. So, $\lambda_0 \in \sigma(M_C) \setminus \sigma_w(M_C)$.

Case 2. $\lambda_0 \in \rho(B)$.

Now, $0 < n(A - \lambda_0 I) < \infty$. Then $\lambda_0 \in \pi_{00}(M_0)$. Similar to case 1, we get that $\lambda_0 \in \sigma(M_C) \setminus \sigma_w(M_C)$.

Case 3. $\lambda_0 \in \mathrm{iso}\sigma(A) \cap \mathrm{iso}\sigma(B)$.

Since $\mathcal{M}_2 \doteq \emptyset$, it follows that $n(B - \lambda_0 I) < \infty$ or $d(A - \lambda_0 I) < \infty$. If $n(B - \lambda_0 I) < \infty$, then $0 < n(A - \lambda_0 I) + n(B - \lambda_0 I) < \infty$, that is $\lambda_0 \in \pi_{00}(M_0)$. It follows that $M_C - \lambda_0 I$ is Browder. Now, suppose that $d(A - \lambda_0 I) < \infty$. This shows that $A - \lambda_0 I$ is Fredholm. Since $\lambda_0 \in \mathrm{iso}\sigma(A)$, it follows that $A - \lambda_0 I$ is Browder. We can claim $n(B - \lambda_0 I) < \infty$. In fact, if not, then there are two cases to consider.

(i) $\dim C(N(B - \lambda_0 I)) < \infty$.

Then there exists a linearly independent sequence $\{y_n\}_{n=1}^{\infty} \subseteq N(B - \lambda_0 I)$ such that $Cy_n = 0$ for any $n \geq 1$. It is clear that $\left\{\begin{pmatrix} 0 \\ y_n \end{pmatrix}\right\}_{n=1}^{\infty} \subseteq N(M_C - \lambda_0 I)$, which gets that $n(M_C - \lambda_0 I) = \infty$, a contradiction.

(ii) $\dim C(N(B - \lambda_0 I)) = \infty$.

Since $A - \lambda_0 I$ is Browder, it follows that $\mathcal{H} = R(A - \lambda_0 I) \oplus R(A - \lambda_0 I)^{\perp}$ and $d(A - \lambda_0 I) < \infty$, and hence $\dim C(N(B - \lambda_0 I)) \cap R(A - \lambda_0 I) = \infty$. Thus there exists $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{H}$ such that $(A - \lambda_0 I)x_n = Cy_n$, where $\{y_n\}_{n=1}^{\infty}$ is an orthonormal set of $N(B - \lambda_0 I)$. It is easy to see that $\left\{\begin{pmatrix} x_n \\ -y_n \end{pmatrix}\right\}_{n=1}^{\infty}$ is a linearly independent set of $N(M_C - \lambda_0 I)$, a contradiction.

Thus, $\lambda_0 \in \pi_{00}(M_0)$. Similarly, we can prove that $\lambda_0 \in \sigma(M_C) \setminus \sigma_w(M_C)$. $\square$

In Theorem 1.1, the set $\mathcal{M}_2$ can also be replaced by $\{\lambda \in \text{iso}\sigma(A) \cap \text{iso}\sigma(B) : n(A-\lambda I) < \infty, n(A-\lambda I) + n(B-\lambda I) > 0, d(A-\lambda I) = \infty\}$ such that the theorem holds.

Under the condition A is a bounded below operator and B is a surjective operator, by Theorem 2.1, we know that there exists $C \in B(K, H)$ such that M_C is a Weyl operator but not a Browder operator if $d(A) = n(B) = \infty$. Meanwhile, it is obvious that there exists $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is a Weyl operator but not a Browder operator if $0 < d(A) = n(B) < \infty$.

Proof of Corollary 1.1. By Theorem 1.1 and the previous analysis, we only need to prove sufficiency. And we only need to prove that Weyl's theorem holds for M_0.

Let $\lambda_0 \in \sigma(M_0) \setminus \sigma_w(M_0)$, then $M_{C_0} - \lambda_0 I$ is Weyl. We get that $M_{C_0} - \lambda_0 I$ is Browder because M_{C_0} satisfies Weyl's theorem. Thus there exists $\varepsilon > 0$ such that $M_{C_0} - \lambda I$ is invertible if $0 < |\lambda - \lambda_0| < \varepsilon$ and hence $A - \lambda I$ is bounded below, $B - \lambda I$ is surjective and $d(A - \lambda I) = n(B - \lambda I)$. From (2) and (4) we get both $A - \lambda I$ and $B - \lambda I$ are invertible and so $M_0 - \lambda I$ is invertible, that is $\lambda_0 \in \pi_{00}(M_0)$.

On the other hand, for any $\lambda_0 \in \pi_{00}(M_0)$, then $n(A - \lambda_0 I) < \infty$ and $0 < n(A - \lambda_0 I) + n(B - \lambda_0 I) < \infty$. We know that $\lambda_0 \in \text{iso}\sigma(M_{C_0})$ and $n(M_{C_0} - \lambda_0 I) \leq n(A - \lambda_0 I) + n(B - \lambda_0 I) < \infty$. Clearly, $\lambda_0 \in \text{iso}\sigma(A) \cup \rho(A)$ and $\lambda_0 \in \text{iso}\sigma(B) \cup \rho(B)$. Next, we continue to divide it into the following cases.

Case 1. $\lambda_0 \in \rho(A)$.

In this case, $0 < n(B - \lambda_0 I) < \infty$. Suppose $\{e_1, e_2, \cdots, e_k\}$ is an orthonormal basis of $N(B - \lambda_0 I)$, where $k = \dim N(B - \lambda_0 I)$. There exists a sequence $\{x_n\}_{n=1}^k \subseteq \mathcal{H}$ such that $(A - \lambda_0 I)x_n = C_0 e_n$ for any $1 \leq n \leq k$ because $A - \lambda_0 I$ is surjective. Thus $\left\{ \begin{pmatrix} -x_n \\ e_n \end{pmatrix} \right\}_{n=1}^k \subseteq N(M_{C_0} - \lambda_0 I)$. Thus $n(M_{C_0} - \lambda_0 I) > 0$. It follows that $\lambda_0 \in \pi_{00}(M_{C_0})$. We can get that $B - \lambda_0 I$ is Browder because M_{C_0} satisfies Weyl's theorem. Combined with $\lambda_0 \in \rho(A)$, $M_0 - \lambda_0 I$ is Browder, i.e., $\lambda_0 \in \sigma(M_0) \setminus \sigma_w(M_0)$.

Case 2. $\lambda_0 \in \rho(B)$.

Now, $0 < n(A - \lambda_0 I) < \infty$. Combined with $n(A - \lambda_0 I) \leq n(M_{C_0} - \lambda_0 I) \leq n(A - \lambda_0 I) + n(B - \lambda_0 I)$, we know that $\lambda_0 \in \pi_{00}(M_{C_0})$. Similar to case 1, we get that $\lambda_0 \in \sigma(M_0) \setminus \sigma_w(M_0)$.

Case 3. $\lambda_0 \in \text{iso}\sigma(A) \cap \text{iso}\sigma(B)$.

Since $\mathcal{M}_2' \doteq \emptyset$, it follows that $d(A - \lambda_0 I) < \infty$. Due to $\lambda_0 \in \text{iso}\sigma(A)$, it follows that $A - \lambda_0 I$ is Browder and $n(A - \lambda_0 I) > 0$. Then $0 < n(A - \lambda_0 I) \leq n(M_{C_0} - \lambda_0 I) \leq n(A - \lambda_0 I) + n(B - \lambda_0 I)$, that is $\lambda_0 \in \pi_{00}(M_{C_0})$. Again we can show that $\lambda_0 \in \sigma(M_0) \setminus \sigma_w(M_0)$. $\square$

Remark 3.2. Comparing the conditions in Theorem 1.1 and Corollary 1.1, it is clear that $\mathcal{M}_2 = \{\lambda \in \text{iso}\sigma(A) \cap \text{iso}\sigma(B) : n(A - \lambda I) < \infty, d(A - \lambda I) = n(B - \lambda I) = \infty\} \subseteq \mathcal{M}_2' = \{\lambda \in \text{iso}\sigma(A) \cap \text{iso}\sigma(B) : n(A - \lambda I) < \infty, n(A - \lambda I) + n(B - \lambda I) > 0, d(A - \lambda I) = \infty\}$. But in Corollary 3.1 "$\mathcal{M}_2' = \emptyset$" cannot be changed by "$\mathcal{M}_2 = \emptyset$".

For example, let $A, B \in B(\ell^2)$ be defined by

$$A(x_1, x_2, x_3, \cdots) = (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots), B(x_1, x_2, x_3, \cdots) = (x_2, \frac{x_3}{2}, \frac{x_4}{3}, \cdots).$$

Since $\sigma(A) = \sigma(B) = \{0\}$, it follows that $\sigma(M_C) = \sigma_w(M_C) = \{0\}$ for all $C \in B(\ell^2)$ and $\mathcal{M}_2 = \emptyset$, $0 \in \mathcal{M}_2'$. Meanwhile, $\mathcal{M}_1 = \mathcal{M}_3 = \emptyset$. Define $C_0 \in B(\ell^2)$ by

$$C_0(x_1, x_2, x_3, \cdots) = (x_1, 0, 0, \cdots).$$

Then $\pi_{00}(M_{C_0}) = \emptyset$ and so M_{C_0} satisfies Weyl's theorem. But since $\sigma(M_0) \setminus \sigma_w(M_0) = \emptyset$ and $\pi_{00}(M_0) = \{0\}$, it follows that M_0 does not satisfy Weyl's theorem.

In the following, we shall give an application of Theorem 1.1. Let $H^2(\mathbb{T})$ denote the Hardy space of the unit circle $\mathbb{T}$ in the plane. Give $\varphi \in L^\infty(\mathbb{T})$, the Toeplitz operator with symbol φ is the operator on $H^2(\mathbb{T})$ defined by

$$T_\varphi : f \to P(\varphi f),$$

where $f \in H^2(\mathbb{T})$ and P is the orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. The spectrum of a Toeplitz operator is always connected and $\sigma(T_\varphi) = \sigma_w(T_\varphi)$, this implies that Weyl's theorem holds for every Toeplitz operator. We write $C(\mathbb{T})$ for the algebra of all complex-valued continuous functions on $\mathbb{T}$. For $\varphi \in C(\mathbb{T})$, we know $T_\varphi - \lambda I$ is Fredholm if and only if $T_\varphi - \lambda I$ is upper semi-Fredholm and the essential spectrum $\sigma_e(T_\varphi)$ of T_φ is $\sigma_e(T_\varphi) = \varphi(\mathbb{T})$. We can see that if A or B is Toeplitz operator, both the set $\mathcal{M}_1$ and $\mathcal{M}_2$ in Theorem 1.1 are empty set. Then:

Example 3.1. Suppose that T_φ and T_ϕ are Toeplitz operator with $\varphi, \phi \in C(\mathbb{T})$, then Weyl's theorem holds for $M_C = \begin{pmatrix} T_\varphi & C \\ 0 & T_\phi \end{pmatrix}$ for every $C \in B(H^2(\mathbb{T}))$ if and only if Weyl's theorem holds for $M_0 = \begin{pmatrix} T_\varphi & 0 \\ 0 & T_\phi \end{pmatrix}$.

Using the property of Toeplitz operator, we also get that Weyl's theorem holds for $M_C = \begin{pmatrix} T_\varphi & C \\ 0 & T_\phi \end{pmatrix}$ for every $C \in B(H^2(\mathbb{T}))$ if and only if $\sigma_w(M_0) = \sigma_w(T_\varphi) \cup \sigma_w(T_\phi)$.

Let $\mathrm{wn}(\varphi)$ denote the winding number of φ with respect to the origin. We have the fact that if T_φ is Frehdolm, then $\mathrm{ind}(T_\varphi) = -\mathrm{wn}(\varphi)$.

If $\varphi, \phi \in C(\mathbb{T})$ satisfies

$$\mathrm{wn}(\varphi - \lambda) \cdot \mathrm{wn}(\phi - \lambda) \geq 0 \text{ for each } \lambda \in \mathbb{C} \backslash [\varphi(\mathbb{T}) \cup \phi(\mathbb{T})],$$

then Weyl's theorem holds for $M_C = \begin{pmatrix} T_\varphi & C \\ 0 & T_\phi \end{pmatrix}$ for every $C \in B(H^2(\mathbb{T}))$.

References

[1] Q. M. Bai, J. J. Huang, A Chen, *Weyl type theorem of* 2×2 *upper triangular operator matrices*, J. Math. Anal. Appl. **434** (2016), 1065–1076.

[2] X. H. Cao, M. Z. Guo, B. Meng, *Semi-Fredholm spectrum and Weyl's theorem for operator matrices*, Acta Math. Sin. (Engl. Ser.) **22** (2006), 169–178.

[3] X. H. Cao, B. Meng, *Essential approximate point spectra and Weyl's theorem for operator matrices*, J. Math. Anal. Appl. **304** (2005), 759–771.

[4] D. S. Djordjevic, *Perturbations of spectra of operator matrices*, J. Operator Theory, **48** (2002), 467–486.

[5] H. K. Du, P. Jin. *Perturbation of spectrums of* 2×2 *operator matrices*, Proc. Amer. Math. Soc. **121** (1994), 761–766.

[6] J. K. Han, H. Y. Lee, W. Y. Lee, *Invertible completions of 2x2 upper triangular operator matrices*, Proc. Amer. Math. Soc. **128** (2000), 119–123.

[7] W. Y. Lee, *Weyl spectra of operator matrices*, Proc. Amer. Math. Soc. **129** (2001), 131–138.

[8] C. G. Li, S. Zhu, Y. Feng. *Weyl's theorem for functions of operators and approximation*, Integr. Equat. Oper. Th. **67** (2010), 481–497.

[9] S. N. Qiu, X. H. Cao. *Property* (UW_E) *for operators and operator matrices*, J. Math. Anal. Appl. **509** (2022), 125951.

[10] C. Wang, X. Zhao, *Weyl's theorem for Toeplitz operators with polynomial symbols on the harmonic Bergman space*, J. Math. Anal. Appl. **495** (2020), 124770.

[11] L. L. Yang, X. H. Cao, *Perturbations of spectra and property (R) for upper triangular operator matrices*, J. Math. Anal. Appl. **519** (2023), 126797.

[12] L. L. Yang, X. H. Cao, *Property (R) for upper triangular operator matrices*. Acta Math. Sin. (Engl. Ser.) **39** (2023), 523–532.

[13] S. Zhu, *Weyl's theorem for complex symmetric operators*, J. Math. Anal. Appl. **474** (2019), 1470–1480.