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Characteristic and lower characteristic of bounded linear operators

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Abstract. In this article, we introduce the notion of lower characteristic of a bounded operator by means of the weak non-compactness measure. We prove that if the lower characteristic of T is strictly positive then T is an upper generalized semi-Fredholm. Finally, an application to Markov chains is given.

1. Introduction

Let X and Y be two Banach spaces. We denote by $\mathcal{L}(X,Y)$, the Banach space of all bounded linear operators from X into Y and we denote by $\mathcal{K}(X,Y)$, the subspace of all compact operators of $\mathcal{L}(X,Y)$. If $T \in \mathcal{L}(X,Y)$ then $\alpha(T)$ denotes the dimension of the kernel $N(T)$ and $\beta(T)$ the codimension of $R(T)$ in Y. If $X = Y$, the sets $\mathcal{L}(X,Y)$ and $\mathcal{K}(X,Y)$ are replaced by $\mathcal{L}(X)$ and $\mathcal{K}(X)$, respectively. An operator T is called upper semi-Fredholm from X into Y if $\alpha(T) < \infty$ and $R(T)$ is closed in Y. An operator T is called lower semi-Fredholm from X into Y if $\beta(T) < \infty$ and $R(T)$ is closed in Y. If T is upper or lower semi-Fredholm, then the index of T is defined by $i(T) := \alpha(T) - \beta(T)$.

Let $T \in \mathcal{L}(X)$ be an operator, we denote by $R^\infty(T)$ the set

$$R^\infty(T) = \bigcap_{n=0}^{\infty} R(T^n).$$

We start this section by recalling some notations, results and definitions of weak noncompactness measure given in [10]. If $x \in X$ and $r > 0$, then $B(x,r)$ will denote the closed ball of X with a center at x and a radius r, B_r denotes the closed ball in X centered at 0_X with radius r, and B_X denotes the closed ball in X centered at 0_X with radius 1, and S_X the unit sphere given by

$$S_X = \{x \in X \ : \ \|x\| = 1\}.$$

Let Ω_X be the collection of all nonempty bounded subsets of X, and $\mathcal{K}^w$ be the subset of Ω_X consisting of all weakly compact subsets of X. The notion of the measure of weak noncompactness was introduced by De Blasi in [8]; it is the map $\omega : \Omega_X \longrightarrow [0, +\infty)$ defined by the following:

$$\omega(\mathcal{M}) = \inf\Big\{ r > 0 : \text{ there exists } \mathrm{K} \in \mathcal{K}^w \text{ such that } \mathcal{M} \subset \mathrm{K} + B_r \Big\}, \tag{1}$$

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for all $\mathcal{M} \in \Omega_X$. Let us recall some basic properties of $\omega(.)$ needed below (see, for example, [1–4, 8] see also [5], where an axiomatic approach to the notion of a measure of weak noncompactness is presented).

Lemma 1.1. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two elements of Ω_X. Then, the following conditions are satisfied:*

(1) $\mathcal{M}_1 \subset \mathcal{M}_2$ *implies* $\omega(\mathcal{M}_1) \leq \omega(\mathcal{M}_2)$.

(2) $\omega(\mathcal{M}_1) = 0$ *if, and only if,* $\overline{\mathcal{M}_1}^w \in \mathcal{K}^w$, *where* $\overline{\mathcal{M}_1}^w$ *is the weak closure of the subset* $\mathcal{M}_1$.

(3) $\omega(\overline{\mathcal{M}_1}^w) = \omega(\mathcal{M}_1)$.

(4) $\omega(\mathcal{M}_1 \cup \mathcal{M}_2) = \max\{\omega(\mathcal{M}_1), \omega(\mathcal{M}_2)\}$.

(5) $\omega(\lambda\mathcal{M}_1) = |\lambda|\omega(\mathcal{M}_1)$ *for all* $\lambda \in \mathbb{R}$.

(6) $\omega(co(\mathcal{M}_1)) = \omega(\mathcal{M}_1)$, *where* $co(\mathcal{M}_1)$ *is the convex hull of* $\mathcal{M}_1$.

(7) $\omega(\mathcal{M}_1 + \mathcal{M}_2) \leq \omega(\mathcal{M}_1) + \omega(\mathcal{M}_2)$.

(8) *if* $(\mathcal{M}_n)_{n\geq 1}$ *is a decreasing sequence of nonempty, bounded, and weakly closed subsets of X with* $\lim_{n\to\infty} \omega(\mathcal{M}_n) = 0$, *then* $\mathcal{M}_\infty := \cap_{n=1}^{\infty} \mathcal{M}_n$ *is nonempty and* $\omega(\mathcal{M}_\infty) = 0$ *i.e.,* $\mathcal{M}_\infty$ *is relatively weakly compact.*

Remark 1.2. *The measure of weak noncompactness of the unit ball B_X belongs to $\{0, 1\}$ i.e., $\omega(B_X) \in \{0, 1\}$. In fact, it is obvious that $\omega(B_X) \leq 1$. Let $r > 0$ be given such that there is a weakly compact K of X satisfying $B_X \subset K + rB_X$. Then, $\omega(B_X) \leq r\omega(B_X)$. If $\omega(B_X) \neq 0$, then $r \geq 1$. Thus, $\omega(B_X) \geq 1$.*

The paper is organized as follows. In Section 2, we present the main results of this paper. We prove under some hypotheses that $[T]_a > 0$ if, and only if, T is upper semi-Fredholm, where $[\cdot]_a$ is the "lower" characteristic. In Section 3, we present a characterization of the generalized Jeribi essential spectra in Banach space. Finally, in Section 4, an application to Markov chains is given.

2. Main results

Definition 2.1. *Let $T \in \mathcal{L}(X, Y)$, we define the "lower" characteristic*

$$[T]_a = \sup\{k \ : \ k > 0, \ \omega(T(M)) \geq k\omega(M) \text{ for all bounded } M \subset X\} \tag{2}$$

as elements of $[0, \infty]$.

Note that in finite dimensional spaces we have $[T]_a = \infty$. In infinite dimensional spaces, where this characteristic is of more use, we get

$$[T]_a = \inf_{0<\omega(M)<\infty} \frac{\omega(T(M))}{\omega(M)}. \tag{3}$$

Sets with $\omega(M) = 0$ can be left out here, since the continuity of T assures that also $\omega(T(M)) = 0$. This can be seen by considering the following

$$\omega(T(M)) \leq \omega(T(\overline{M})).$$

Proposition 2.2. *[11] Let X, Y, Z be three Banach spaces, T, S $\in \mathcal{L}(X, Y)$ and $R \in \mathcal{L}(Y, Z)$. Then,*

(*i*) $[R]_a[T]_a \leq [RT]_a$.
(*ii*) $[T + S]_a = [T]_a$ *if S is weakly compact.*

Definition 2.3. *We say that X has the property* $(H1)$ *(resp.* $(H2)$*) if every reflexive subspace admits a closed complementary subspace (resp. if every closed subspace with reflexive quotient space admits a closed complementary subspace). We say that X has the property* (H), *if it satisfies both properties* $(H1)$ *and* $(H2)$.

Definition 2.4. *An operator $T \in \mathcal{L}(X, Y)$ is said to be generalized Fredholm if its range is closed and both its kernel and co-kernel are reflexive.*

The upper generalized semi-Fredholm classes is defined by

$$\Phi_{g+}(X, Y) := \{T \in \mathcal{L}(X, Y) \ : \ N(T) \text{ is reflexive and } R(T) \text{ is closed in } Y\},$$

and the lower generalized semi-Fredholm classes is defined as follows

$$\Phi_{g-}(X, Y) := \{T \in \mathcal{L}(X, Y) \ : \ Y/R(T) \text{ is reflexive and } R(T) \text{ is closed in } Y\}.$$

We denote by $\Phi_g(X, Y) := \Phi_{g+}(X, Y) \bigcap \Phi_{g-}(X, Y)$ the set of generalized Fredholm operators in $\mathcal{L}(X, Y)$ and by $\Phi_{g\pm}(X, Y) := \Phi_{g+}(X, Y) \bigcup \Phi_{g-}(X, Y)$ the set of generalized semi-Fredholm operators.

Theorem 2.5. *Let X and Y be two Banach space, $T \in \mathcal{L}(X, Y)$. Suppose that $(H1)$ is satisfied and $S_X \bigcap R^\infty(T)$ is relatively compact. Then, $[T]_a > 0$ if, and only if, T is upper generalized semi-Fredholm.*

Proof. Let $T \in \mathcal{L}(X, Y)$ be such that $[T]_a > 0$ and fix $k \in (0, [T]_a)$. Consider M the set

$$\begin{aligned} M &= \{x \in X \text{ such that } Tx = 0 \text{ and } \|x\| = 1\} \\ &= S_X \bigcap N(T), \end{aligned}$$

and $(x_n)_n$ is a bounded sequence of the set M. It is easy to see that

$$N(T) \subset R^\infty(T).$$

Then,

$$M = S_X \bigcap N(T) \subset S_X \bigcap R^\infty(T).$$

The fact that M is closed and $\overline{S_X \bigcap R^\infty(T)}$ is compact, we infer that M is compact. Consequently, $N(T)$ is of finite dimension. We now prove that the range $R(T)$ of T is closed. Indeed, since $\dim N(T) < \infty$, then there exists a closed subspace $X_0 \subset X$ such that $X = X_0 \oplus N(T)$. Let $(y_n)_n$ be a sequence in the range of T, $R(T)$, converging to an element $y \in Y$. Then, there is $(x_n)_n$ in X such that $Tx_n = y_n$. Now, we distinguish two cases. First, assume that $(x_n)_n$ is bounded. With $k > 0$ as before we then obtain

$$\omega(\{x_1, x_2, \cdots, x_n, \cdots\}) \leq \frac{1}{k}\omega(\{y_1, y_2, \cdots, y_n, \cdots\}) = 0.$$

Hence, $\omega(\{x_1, x_2, \cdots, x_n, \cdots\}) = 0$. Thus, $(x_{n_k})_k$ converge weakly to an element x for some subsequence $(x_{n_k})_k$ of $(x_n)_n$ and suitable $x \in X$. Since the operator T is bounded, then $y_{n_k} = Tx_{n_k} \rightharpoonup Tx$. The uniqueness of the limit implies that $Tx = y$, and so $y \in R(T)$. Second, assume that $\|x_n\|$ is unbounded i.e., $\|x_n\| \to \infty$. Set $e_n = \frac{x_n}{\|x_n\|}$ and $E = \{e_1, e_2, \cdots, e_n, \cdots\}$. It is clearly to see that $E \subset \{x \in X \ : \ \|x\| = 1\}$ and

$$Te_n = \frac{Tx_n}{\|x_n\|} = \frac{y_n}{\|x_n\|} \rightharpoonup 0 \text{ as } n \to \infty.$$

Hence, $\omega(T(E)) = 0$. On the other hand, $\omega(T(E)) \geq k\omega(E)$, by (2). So, $\omega(E) = 0$. Whithout loss of generality we may assume that the sequence $(e_n)_n$ converge weakly to some element $e \in \{x \in X_0 \ : \ \|x\| = 1\}$. Hence, $Te = 0$. This, contradict the fact that $X_0 \bigcap N(T) = \{0\}$. Thus, $T \in \Phi_{g+}(X)$.
Inversely, now we can show that the closeness of $R(T)$ and the space $N(T)$ is reflexive imply that $[T]_a > 0$. Assume that $T \in \Phi_{g+}(X)$. Now, from the fact that X satisfies the property $(H1)$, there exists a closed subspace X_0 of X such that $X = N(T) \oplus X_0$. Since $I - P$ is compact, then the projection $P : X \longrightarrow X_0$ satisfies the condition $[P]_a = 1$. Let $\widehat{T} : X_0 \longrightarrow R(T)$ be the canonical isomorphism. Since $T = \widehat{T}P$ and $[\widehat{T}]_a > 0$, then $[T]_a \geq [\widehat{T}]_a [P]_a > 0$. This completes the proof. □

Remark 2.6. *It is noted that the implication $[T]_a > 0$ implies T is upper generalized semi-Fredholm does not require the assumptions $(H1)$ and $S_X \bigcap R^{\infty}(T)$ is relatively compact.*

Lemma 2.7. *[9] Let $T \in \mathcal{L}(X)$. Then, $T \in \Phi_{g+}(X)$ if, and only if, $T^* \in \Phi_{g-}(X^*)$.*

Theorem 2.8. *Let X and Y be two Banach space, $T \in \mathcal{L}(X, Y)$. Assume that $(H1)$ is satisfied and $S_X \bigcap R^{\infty}(T)$ is relatively compact. Then, $[T^*]_a > 0$ if, and only if, T is lower generalized semi-Fredholm.*

Proof. The result follows by using Lemma 2.7 and Theorem 2.5. □

Theorem 2.9. *Let X be a Banach space having the property $(H1)$ and $T \in \mathcal{L}(X)$. Assume that $S_X \bigcap R^{\infty}(T)$ is relatively compact. If $[T]_a > 0$, then $[T_{|M}]_a > 0$, where $T_{|M}$ is the restriction of T to any closed subspace M of X.*

Proof. If $[T]_a > 0$, then by using Theorem 2.5, $T \in \Phi_{g+}(X)$. We observe that $N(T_{|M}) = N(T) \bigcap M$. Since $N(T)$ is reflexive and M is a closed subspace, the intersection $N(T) \bigcap M$ is also a reflexive subspace. Now, from the fact that X satisfies the property $(H1)$, there exists a closed subspace M_1 of X such that $X = N(T) \oplus M_1$. Clearly, the restriction $T_{|M_1}$ is injective and has closed range, since $R(T) = T(M_1)$. On the other hand, we have $M = N(T) \bigcap M \oplus M_1 \bigcap M$. Which implies $T(M) = T(M_1 \bigcap M)$. Since $T_{|M_1}$ is bounded below, then $T(M_1 \bigcap M)$ is closed. Consequently $T_{|M_1} \in \Phi_{g+}(M)$. The result follows from Theorem 2.5. □

Definition 2.10. *Let $T \in \mathcal{L}(X, Y)$. T is said to have a left weak-Fredholm inverse if there exists $T_l^w \in \mathcal{L}(X, Y)$ such that $I_X - T_l^w T \in \mathcal{W}(X)$.*

Lemma 2.11. *Let X be a Banach space having the property $(H1)$ and $T \in \mathcal{L}(X)$. Suppose that $S_X \bigcap R^{\infty}(T)$ is relatively compact. If $[T]_a > 0$, then T has a left weak-Fredholm inverse.*

Proof. The proof follow from [9] and Theorem 2.5. □

Theorem 2.12. *Let X be a Banach space having the property $(H1)$ and T, $S \in \mathcal{L}(X)$. Suppose that $S_X \bigcap R^{\infty}(T)$ is relatively compact. If $[ST]_a > 0$, then $[T]_a > 0$.*

Proof. If $[ST]_a > 0$, then by using Theorem 2.5, we have $ST \in \Phi_{g+}(X)$. Since X satisfies the property $(H1)$, then by using Lemma 2.11, there exist $W \in \mathcal{W}(X)$ and $A \in \mathcal{L}(X)$ such that $AST = I + W$. Now, we have to prove that $[T]_a > 0$. For a bounded subset D of X, we have

$$\begin{aligned} \omega(D) &= \omega((I + W)(D)) \\ &= \omega(AST(D)) \\ &\leq \|AS\|\omega(T(D)). \end{aligned}$$

Hence

$$[T]_a \geq \frac{1}{\|AS\|}.$$

So, $[T]_a > 0$. □

Theorem 2.13. *Let X be a Banach space having the property $(H1)$ and T, $S \in \mathcal{L}(X)$. Suppose that $S_X \bigcap R^{\infty}(T)$ is relatively compact. If $[T^*S^*]_a > 0$ and X^* satisfies the property $(H1)$, then $[S^*]_a > 0$.*

Proof. Suppose $[T^*S^*]_a > 0$, then according to Theorem 2.5, we get $T^*S^* \in \Phi_{g+}(X^*)$. It follows from the fact that X^* satisfies the property $(H1)$ and reasoning as in the proof of Theorem 2.12, we infer that $S^* \in \Phi_{g+}(X^*)$. Using again Lemma 2.7, we obtain $S \in \Phi_{g-}(X)$. By using Theorem 2.8, we infer that $[S^*]_a > 0$. □

Theorem 2.14. *Let X be a Banach space having the property $(H1)$ and T, $S \in \mathcal{L}(X)$. Suppose that $S_X \bigcap R^{\infty}(T)$ is relatively compact. If $ST \in \Phi_g(X)$ and X^* satisfies the property $(H1)$, then $[T]_a > 0$ and $[S^*]_a > 0$.*

Proof. The result follows from Theorems 2.12 and 2.13. □

Theorem 2.15. *Let X, Y and Z be three Banach spaces and let $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y, Z)$. If $[T]_a > 0$ and $[S]_a > 0$, then $[ST]_a > 0$.*

Proof. Assume that $[T]_a > 0$ and $[S]_a > 0$. Then, from Proposition 2.2, $[ST]_a > 0$. □

Theorem 2.16. *Let X, Y and Z be three Banach spaces and let $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y, Z)$. Suppose that X^* satisfies the property (H1). If $[T^*]_a > 0$ and $[S^*]_a > 0$, then $ST \in \Phi_{g-}(X)$.*

Proof. Assume that $[T^*]_a > 0$ and $[S^*]_a > 0$. Since X^* satisfies the property (H1) and by Theorem 2.15, we infer that $[T^*S^*]_a > 0$. Thus, $ST \in \Phi_{g-}(X)$. □

We begin by recalling the definition of a Tauberian operator introduced by Kalton and Wilansky in [7].

Definition 2.17. *An operator $T \in \mathcal{L}(X, Y)$ is called Tauberian if $T^{**-1}(Y) \subset X$.*

The class of Tauberian operators from X into Y is denoted by $\mathcal{T}(X, Y)$. Now, let us recall some important properties of Tauberian operators.

Proposition 2.18. *[7] Let X, Y and Z be three Banach spaces and let $T, W \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y, Z)$. Then we have*

(*i*) *If T and S are Tauberian, then ST is Tauberian.*
(*ii*) *If ST is Tauberian, then T is Tauberian.*
(*iii*) *If T is Tauberian and weakly compact if and only if X is reflexive.*
(*iv*) *If $T \in \mathcal{T}(X, Y)$ and $W \in \mathcal{W}(X, Y)$, then $T + W \in \mathcal{T}(X, Y)$.*
(*v*) *T is Tauberian, then N(T) is reflexive.*

Definition 2.19. *An operator $T \in \mathcal{L}(X, Y)$ is said to be co-Tauberian when its conjugate T^* is Tauberian.*

The classes of co-Tauberian operators from X into Y is denoted by $\mathcal{T}^d(X, Y)$.

Lemma 2.20. *Let X and Y be two Banach spaces and $Z \subset X$. For an operator $T \in \mathcal{L}(X, Y)$, the following statements are equivalent*

(*i*) *T is co-Tauberian.*
(*ii*) *Every operator $S \in \mathcal{L}(Y, Z)$ is weakly compact whenever ST is weakly compact.*

Theorem 2.21. *Let X, Y and Z be three Banach spaces. Suppose that X satisfies the property (H1). If $[T]_a > 0$ and $W \in \mathcal{W}(X)$, then $[T + W]_a > 0$.*

Proof. Since X has the property (H1), $[T]_a > 0$ and $W \in \mathcal{W}(X)$, then by using assertions (*iii*) and (*v*) from Proposition 2.18, we get $[T + W]_a = [T]_a > 0$. It follows that $[T + W]_a > 0$. □

3. Generalized Jeribi essential spectra

We introduced the generalized Jeribi essential spectra of a bounded linear operator by

$$\sigma_{ej,g}(T) := \{\lambda \in \mathbb{C} \text{ such that } [\lambda - T]_a = 0\} := \mathbb{C}\backslash\Phi_{g+}(T),$$

and

$$\sigma_{eje,g}(T) := \{\lambda \in \mathbb{C} \text{ such that } [\lambda - T]_a = 0 \text{ or } [\overline{\lambda} - T^*]_a = 0\} := \mathbb{C}\backslash\Phi_g(T).$$

Proposition 3.1. *Let X be a Banach space having the property (H1) and let T and S be two bounded linear operators on X. Then,*

$$\sigma_{ej,g}(T) = \bigcap_{W \in \mathcal{W}(X)} \sigma_{eje,g}(T + W).$$

Proof. Suppose that $\lambda \notin \sigma_{ej,g}(T)$, then $[\lambda - T]_a > 0$. From the fact that X satisfy the property $(H1)$ and $[\lambda - T]_a > 0$, then by Theorem 2.21 we deduce that

$$[\lambda - T - W_1]_a > 0, \tag{4}$$

where $W_1 \in W(X)$. It remains to show that $(\lambda - T - W_1)$ is co-Tauberian. Let $A \in \mathcal{L}(X)$ such that $A(\lambda - T - W_1)$ is weakly compact. By Lemma 2.20, it is enough to show that A is weakly compact. We have

$$\overline{\omega}(A(\lambda - T - W_1)) \geq [\lambda - T - W_1]_a \overline{\omega}(A). \tag{5}$$

Since $[\lambda - T - W_1]_a > 0$, and taking into account that $\overline{\omega}(A(\lambda - T - W_1)) = 0$, then the use of Equation (5) leads to $\overline{\omega}(A) = 0$. Hence, A is weakly compact, and then $(\lambda - T - W_1) \in \mathcal{T}^d(X)$. From Equation (4), we obtain that $R(\lambda - T - W_1)$ is closed. Thus

$$[\overline{\lambda} - T^* - W_1^*]_a > 0. \tag{6}$$

From Equation (4) and (6), we conclude that $\lambda \notin \bigcap_{W \in \mathcal{W}(X)} \sigma_{eje,g}(T + W)$. Conversely, let $\lambda \notin \bigcap_{W \in \mathcal{W}(X)} \sigma_{eje,g}(T + W)$, then there exists $W \in \mathcal{W}(X)$ such that $\lambda \notin \sigma_{eje,g}(T + W)$. Thus, $[\lambda - T - W]_a > 0$ and $[\overline{\lambda} - T^* - W^*]_a > 0$. Since X satisfy the property $(H1)$ and $[\lambda - T - W]_a > 0$, then by Lemma 2.11 there exist $T_0 \in \mathcal{L}(X)$ and $W_1 \in \mathcal{W}(X)$ such that

$$T_0(\lambda - T - W) = I + W_1.$$

It follows that $T_0(\lambda - T) = I + W_1 + T_0 W$. By using Theorem 2.12, we infer that $[\lambda - T]_a > 0$ and consequently $\lambda \notin \sigma_{ej,g}(T)$. □

In the next result a refinement of Proposition 3.1 is presented.

Theorem 3.2. *Let X be a Banach space having the property (H) and T and S be two bounded linear operators on X. Then,*

$$\sigma_{ej,g}(T) = \bigcap_{W \in \mathcal{W}_n(X)} \sigma_{eje,g}(T + W),$$

where $\mathcal{W}_n(X) = \{T \in \mathcal{L}(X)$ such that $\overline{\omega}((TS)^n) < 1$ for all $S \in \mathcal{L}(X)\}$.

Proof. Taking into account that $\mathcal{W}(X) \subset \mathcal{W}_n(X)$, we deduce by Proposition 3.1 that

$$\bigcap_{W \in \mathcal{W}_n(X)} \sigma_{eje,g}(T + W) \subset \sigma_{ej,g}(T).$$

Conversely, let $\lambda \notin \bigcap_{W \in \mathcal{W}_n(X)} \sigma_{eje,g}(T + W)$, then there exists $W \in \mathcal{W}_n(X)$ such that $\lambda - T - W \in \Phi_g(X)$. Since X satisfies the property (H) and $W \in \mathcal{W}_n(X)$, then by using Proposition 3.4.1, we deduce that $[\lambda - T]_a > 0$ and $[\overline{\lambda} - T^*]_a > 0$. Consequently, $\lambda \notin \sigma_{ej,g}(T)$, which completes the proof. □

4. Application to Markov chains

The aim of this section is to apply the results obtained to characterize Markov chains having an upper generalized semi-Fredholm action on a Banach space. To get this done; we recall some framework on Markov chains. Let (E, ε) be a measurable space of finite measure σ. Let M be a Banach space of the set of functions on (E, ε) which are additive with bounded total variation and $P : f \in M \longrightarrow g \in M$ be the operator defined by

$$Pf(t) := g(t) = \int_E P(x, t) f \, dx. \tag{7}$$

The mapping $P(x, dy)$ from (E, ε) to (E, ε) determines an operator P, given in (7). P is called a transition probability on (E, ε).

Let $(X_n)_n$ be a sequence of positive contractions on $L^1(E,\varepsilon)$ which preserves the norm on

$$L^1_+ = \{f \in L^1(E,\varepsilon) \ : \ f \geq 0 \text{ a.e.}\},$$

i.e., $\|T\| = 1$, $Tf \geq 0$ a.e. and $\|Tf\|_{L^1} = \|f\|_{L^1}$ are maintained each time $f \geq 0$ a.e. This sequence $(X_n)_n$ on (E,ε) associated with P is called a Markov chain.

Consider w from E to $[1,+\infty[$ a measurable function. Consider C_w the Banach space of measurable functions with complex values on E satisfying the following condition

$$\sup_{x\in E} \frac{|f(x)|}{w(x)} < +\infty,$$

with the following norm

$$\|f\|_w = \sup_{x\in E} \frac{|f(x)|}{w(x)}.$$

Definition 4.1. *Let $(X_n)_n$ be a Markov chain with state space E and transition probability P. $(X_n)_n$ is said to be w-geometrically ergodic if there exists an invariant distribution $\mathcal{V}$ on E such that $\mathcal{V}(w) < +\infty$ and certain constants $r < 1$ and $R \in \mathbb{R}_+$ such that for each $f \in C_w$, we have*

$$\|P^n f - \mathcal{V}(f)1_E\| \leq Rr^n \|f\|_w,$$

where P is given in (7).

Our goal is to give an interesting case of lower transition probability characteristic, where $(X_n)_n$ is w-geometrically ergodic Markov chain. It is easy, by Definition 4.1, that the transition probability satisfies $[I-P]_a > 0$. A periodic Markov chain satisfying the Doeblin condition is a geometric ergodicity with a bounded function w. In the following, an example is given by the functional autoregressive model.

(i) Let X_0 be random variable on $\mathbb{R}^d$ and $(Y_n)_{n\geq 1}$ be an independent and identically distributed sequence of random variables on $\mathbb{R}^d$, independent of X_0. Consider

$$X_n = \varphi(X_{n-1}) + Y_n,$$

where φ is a function in $\mathbb{R}^d$. Suppose that the random variables Y_n have a common density $p > 0$ and a moment of order δ_0 and φ continuous such that

$$\limsup \left(\frac{\|\varphi(x)\|}{\|x\|} \right) < 1$$

when $\|x\| \to \infty$. Let's put $w(x) = (1 + \|x\|)^{\delta_0}$. Since $p > 0$, then it is clear to see that $(X_n)_{n\geq 0}$ is aperiodic and Lebesgue-irreducible. Furthermore, we have

$$\limsup \left(\frac{Pw(x)}{w(x)} \right) < 1$$

when $\|x\| \to \infty$. Using dominated convergence theorem, we infer the following

$$(Pf)(x) = \int_{\mathbb{R}^d} f(\varphi(x) + y) p(y)\, dy.$$

Thanks to the results of [6], it is clear to see that $(X_n)_{n\geq 0}$ is w-geometrically ergodic with $P(x,dy) = K(x,y)dy$, $K(x,y) = p(y - \varphi(x))$. This implies that $I - P$ is an upper generalized semi-Fredholm. Then, in view of Theorem 2.5, we deduce that $[I-P]_a > 0$.

(*ii*) Let X_0 be random variable on $\mathbb{R}^d$ and $(Z_n)_{n\geq 1}$ be an independent and identically distributed sequence of real-valued random variables, independent of X_0, such that $m = \mathbb{E}(|Z_1|) < +\infty$, the state space is $E = \mathbb{R}$ and $P(x, A) = \mathbb{E}(1_A(\alpha x + Z_1))$. Another simple example is provided by the linear model

$$X_n = \alpha X_{n-1} + Z_n,$$

with $\alpha \in]-1, 1[$. Let $w(y) = 1+|y|$, $y \in \mathbb{R}$. If Z_1 has positive density everywhere, then $(X_n)_n$ is w-geometrically ergodic Markov chain. Therefore, $I - P$ is an upper generalized semi-Fredholm. Thus, by applying the Theorem 2.5, we obtain the associated transition probability $[I - P]_a > 0$.

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