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Approximate angle in Hilbert C^*-modules

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Abstract. In this paper, we investigate a type of perturbation of angle in Hilbert C^*- modules. When a vector x makes approximate angle θ with a vector y, for some $\theta \in [0, \pi]$, we look for another vector w, close enough to the vector y, so that x makes the exact angle θ with w. This problem has been recently studied for orthogonality by some authors in normed spaces. We have been able to develop the method for angles in the setting of Hilbert C^*-module. We first consider the problem in inner product spaces, and prove some results in this regard. Then, we try to extend it to some classes of normed spaces such as Hilbert C^*- modules and $\mathbf{B}(\mathcal{H})$, the algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$.

1. Introduction

The magnitude of the angle of two vectors, as a real number, in a real inner product space is calculated, very routinely, by invoking the inner product. In a real inner product space $(X, (\cdot, \cdot))$ angle is defined as follows; For two elements x and y of the inner product X, and for some $\theta \in [0, \pi]$, x makes angle θ with y, denoted by $x\angle_\theta y$, if $\cos\theta = \frac{(x,y)}{\|x\|\|y\|}$. Thus tow vectors x and y are orthogonal, $x \perp y$, if $(x, y) = 0$. Also $x\angle_\theta^\varepsilon y$ for some $\varepsilon \in (0, 1)$ and, for some $\theta \in [0, \pi]$ if $|\frac{(x,y)}{\|x\|\|y\|} - \cos\theta| \leq \varepsilon$, which defines the notion of the approximation of angle.
The notion of angle is extended to complex inner product spaces as well. For two elements x and y of a complex inner product space, we say $x\angle_\theta y$ for some $\theta \in [0, \frac{\pi}{2}]$ if

$$\cos\theta = \frac{|(x,y)|}{\|x\|\|y\|},$$

and $x\angle_\theta^\varepsilon y$ for some $\varepsilon \in (0, 1)$ and $\theta \in [0, \frac{\pi}{2}]$, if

$$|\cos\theta - \frac{|(x,y)|}{\|x\|\|y\|}| \leq \varepsilon.$$

This way of considering an angle obviously coincides with our natural understanding of the concept of angle in Euclidean geometry. Generalizing the notion of angle in a normed space in which the norm does

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not necessarily come from an inner product is a challenging task. An interesting approach is that we first consider a type of orthogonality and then define a notion of angle that preserves that orthogonality; meaning that orthogonality corresponds to the case of angle $\frac{\pi}{2}$. Various types of orthogonality are considered by authors in the literature. One of the most famous orthogonality is the one considered by Birkhoff and James; [4, 9]. By definition, x is Birkhoff-James orthogonal to y, denoted by $x\perp_{BJ}y$, if $\|x\| \leq \|x+\lambda y\|$ for all $\lambda \in \mathbb{C}$. A notion of approximate Birkhoff-James orthogonality is defined in [5]. We call $x\perp^{\varepsilon}_{BJ}y$ for x, y in a normed space and $\varepsilon \in (0,1)$, if $\|x+\lambda y\|^2 \geq \|x\|^2 - 2\varepsilon\|x\|\|\lambda y\|$ for all $\lambda \in \mathbb{C}$.

A notion of angle, corresponding to the Birkhoff- James orthogonality, is considered, in [19] as follows:

$$s(x,y) := \inf_{\lambda\in\mathbb{R}} \frac{\|x+\lambda y\|}{\|x\|}$$

for $x \neq 0$ and $s(0,y) := 1$. Thompson [20], Miličić [11] and others gave some alternatives; [2]. Besides these definitions, there are some other definitions for the concept of angle in normed spaces trough other types of orthogonality; for instance isosceles orthogonality[16], Robert orthogonality [17], Pythagorean orthogonality [8]. The notion of angle may vary from one definition to another, however we would logically expect all definitions to correspond to the usual concept of angle in inner product spaces. In Hilbert C^*-modules, due to the existence of the C^*-inner product that somehow creates a structure similar to the structure of Hilbert spaces in such spaces, we can introduce some notions of angles, either trough this type of inner product or trough the norm defined on such spaces and the angle describe above. We deal with the issue in the next section.

Our main interest in this paper is to investigate a type of perturbation of angle in Hilbert C^*- modules. This problem firstly raise by Chmieliński, Stypula and Wójcik, for orthogonality in normed spaces; see [6]. They prove that if X is a real normed space, then for $x, y \in X$ and $\varepsilon \in (0,1)$:

$$x\perp^{\varepsilon}_{BJ}y \Leftrightarrow \exists z \in Lin\{x,y\} : x\perp_{BJ}z, \|z-y\| \leq \varepsilon\|y\|.$$

Moreover, they show that for complex normed spaces the implication $\Leftarrow$ holds true; [6, Theorem 2.2]. Wójcik has, recently, improved these results extending them to complex normed spaces; [22].

Here, we want to generalize these results for angle. We, first, consider the problem in inner product spaces and then try to extend that to general normed spaces. In this part, we could state our results for Hilbert C^*- modules and bounded linear maps defined on a finite Hilbert space. However, we have to state that the problem remains unsolved to us in its general sense.

2. Angle in Hilbert C^*- modules

In this section, we give some notations, terminologies and theorems related to Hilbert C^*-modules.

We denote by $\mathcal{A}^+$ the set of all positive elements of a C^*-algebra $\mathcal{A}$. If $\phi : \mathcal{A} \to \mathcal{B}$ is a linear map between C^*-algebras, ϕ is said to be positive if $\phi(\mathcal{A}^+) \subseteq \mathcal{B}^+$. A state on a C^*-algebra $\mathcal{A}$ is a positive linear functional on $\mathcal{A}$ of norm one. We denote by $S(\mathcal{A})$ the set of all states of $\mathcal{A}$. An extreme point of $S(\mathcal{A})$ is called a pure state and $PS(\mathcal{A})$ stands for the set of all pure states of $\mathcal{A}$.

For a given C^*-algebra $\mathcal{A}$ a pre-Hilbert $\mathcal{A}$-module is a (right) $\mathcal{A}$-module X equipped with an $\mathcal{A}$-valued sesquilinear form $\langle\cdot,\cdot\rangle : X \times X \to \mathcal{A}$ satisfying the following properties:

i $\langle x,x\rangle > 0$ for any $x \in X$;

ii $\langle x,x\rangle = 0$ implies that $x = 0$;

iii $\langle y,x\rangle = \langle x,y\rangle^*$ for any $x, y \in X$;

iv $\langle x,ya\rangle = \langle x,y\rangle a$ for any $x, y \in X$ and any $a \in \mathcal{A}$.

The mapping $\langle\cdot,\cdot\rangle$ is called an $\mathcal{A}$-valued inner product or a C^*-inner product. Note that $\|x\| = \|\langle x,x\rangle\|^{\frac{1}{2}}$, defines a norm on X due to the well-known Cauchy-Schwarz inequality, which is valid for the case of $\mathcal{A}$-valued inner products. If X is complete with respect to this norm, then it is called a Hilbert $\mathcal{A}$-module or a

Hilbert C^*-module over $\mathcal{A}$. For every $x \in X$ the positive square root of $\langle x, x\rangle$ is denoted by $|x|$. In an inner product space, the orthogonal complement subspace of vector x is denoted by $x^\perp$ consisting of all vectors y perpendicular to x. We use the same notation for the case of inner product C^*-modules. See [10] for more information about Hilbert C^*-modules. Now we are going to deal with the notion of angle in Hilbert C^*-module. In [1] the authors give a characterization for Birkhoff- James orthogonality in a Hilbert C^*- module defined over a C^*-algebra $\mathcal{A}$. They show that;

Theorem 2.1. *For two elements x and y in a Hilbert $\mathcal{A}$-module X, $x\perp_{BJ}y$ if and only if there is a $\phi \in S(\mathcal{A})$ such that $\phi(\langle x, x\rangle) = \|x\|^2$ and $\phi(\langle x, y\rangle) = 0$.*

This result motivates us to define a notion of angle in Hilbert C^*- modules.

Definition 2.2. *Let x and y be two elements of a Hilbert C^*-module X over a C^*-algebra $\mathcal{A}$. We say that x makes ϕ-angle θ with y for some $\phi \in S(\mathcal{A})$ and some $\theta \in [0, \frac{\pi}{2}]$, denoted by $x\angle_\theta^\phi y$, if $\phi(\langle x, x\rangle) = \|x\|^2 \neq 0$, $\phi(\langle y, y\rangle) \neq 0$ and*

$$\cos\theta = \frac{|\phi(\langle x,y\rangle)|}{\sqrt{\phi(\langle x,x\rangle)}\sqrt{\phi(\langle y,y\rangle)}}.$$

We also write that $x\angle_\theta^{\phi,\varepsilon} y$ for some $\varepsilon \in (0, 1)$, $\phi \in S(\mathcal{A})$ and $\theta \in [0, \frac{\pi}{2}]$ whenever $\phi(\langle x, x\rangle) = \|x\|^2 \neq 0$, $\phi(\langle y, y\rangle) \neq 0$ and

$$\left|\cos\theta - \frac{|\phi(\langle x,y\rangle)|}{\sqrt{\phi(\langle x,x\rangle)}\sqrt{\phi(\langle y,y\rangle)}}\right| \leq \varepsilon.$$

This notion of angle makes sense by virtue of Cauchy- Schwarz inequality. Indeed, for a state ϕ on $\mathcal{A}$ and every $x, y \in X$ we have that $|\phi(\langle x, y\rangle)| \leq \sqrt{\phi(\langle x, x\rangle)}\sqrt{\phi(\langle y, y\rangle)}$. Hence $\frac{|\phi(\langle x,y\rangle)|}{\sqrt{\phi(\langle x,x\rangle)}\sqrt{\phi(\langle y,y\rangle)}} \leq 1$ for all $x, y \in X$ and $\phi \in \mathcal{A}$. Accordingly, in the case where either $\phi(\langle x, x\rangle) = 0$ or $\phi(\langle y, y\rangle) = 0$ occurs, then $\phi(\langle x, y\rangle) = 0$ which means that x and y are parallel. The conclusion of the following proposition is verified very routinely, according to Theorem 2.1.

Proposition 2.3. *Let x and y be two elements of a Hilbert C^*- module X over a C^*- algebra $\mathcal{A}$.*

i *$x\perp_{BJ}y$ if and only if $x\angle_{\frac{\pi}{2}}^\phi y$ for some $\phi \in S(\mathcal{A})$;*

ii *for any scalers $a, b \in \mathbb{C}\backslash\{0\}$, $ax\angle_\theta^\phi by$ if and only if $x\angle_\theta^\phi y$ for some $\theta \in [0, \frac{\pi}{2}]$ and $\phi \in S(\mathcal{A})$.*

Note that in an inner product space, regarded as an inner product $\mathbb{C}$-module, the ϕ-angle is exactly the usual angle of the space whenever ϕ is defined to be $\phi(\lambda) = \lambda$.

In [14] the author introduces a concept of orthogonality between two elements x, y of C^*-algebra $\mathcal{A}$ with respect to absolute value. By definition, x is called $*$-orthogonal to y if $|x + \lambda y|^2 \geq |x|^2$ for all $\lambda \in \mathbb{C}$. Motivated by this definition we express a definition of angle in Hilbert C^*-modules as follows:

Definition 2.4. *Let x, y be two elements of a Hilbert C^*-module X over a C^*-algebra $\mathcal{A}$ and let $|x|$ and $|y|$ be invertible. We say that x has $*$-angle θ with y and denoted by $x\angle_\theta^* y$ for some $\theta \in [0, \pi]$ if*

$$x\perp_{BJ} - \cos\theta\frac{x}{|x|} + \frac{y}{|y|}.$$

Also, we write $x\angle_\theta^{*,\varepsilon} y$ for some $\varepsilon \in (0, 1)$ and $\theta \in [0, \pi]$ whenever $x\angle_{\theta_1}^* y$ and $|\cos\theta - \cos\theta_1| \leq \varepsilon$.

A notion of angle in real or complex normed spaces was introduced in [16] which preserves Birkhoff-James orthogonality. In complex normed spaces this definition is as follows: We say that $x\angle_\theta y$ for some $\theta \in (0, \frac{\pi}{2})$, if

$$x\perp_{BJ}(-\lambda\cos\theta\frac{x}{\|x\|} + \frac{y}{\|y\|}) \tag{1}$$

for some λ with $|\lambda| = 1$. Especially, this definition may be considered for $\mathbf{B}(\mathcal{H})$ the algebra of all bounded linear operators defined on a complex Hilbert space $\mathcal{H}$. In [13] some necessary and sufficient conditions for $A\angle_\theta B$ for some $A, B \in \mathbf{B}(\mathcal{H})$ and $\theta \in [0, \pi]$ are proved. Specially in the case when $\theta = \frac{\pi}{2}$, some necessary and sufficient conditions for $A\angle_{\frac{\pi}{2}} B$ for some $A, B \in \mathbf{B}(\mathcal{H})$ are introduced. in [21]. Given an operator $T \in \mathbf{B}(\mathcal{H})$, M_T is the set of norm attainment of T consisting of all unit vectors ξ for which $\|T\xi\| = \|T\|$. This set is a nonempty set whenever $\mathcal{H}$ is finite dimensional or T is a compact operator.

In [3] the authors express a necessary and sufficient condition for Birkhoff- James orthogonality of bounded linear operators A and B on finite space $\mathcal{H}$. They show that, A is orthogonal to B in the sense of Birkhoff- James if and only if there exists a unit vector x such that $\|Ax\| = \|A\|$, and $(Ax, Bx) = 0$. Based on this result, we consider the definition of angle in $\mathbf{B}(\mathcal{H})$ when $\mathcal{H}$ is finite dimensional.

Definition 2.5. *Let $A, B \in \mathbf{B}(\mathcal{H})$ and $\mathcal{H}$ be finite dimensional Hilbert space. We write $A\angle_\theta^x B$, if there exists a vector x of unit vectors $\mathcal{H}$ such that $x \in M_A$ and $\frac{|(Ax,Bx)|}{\|Ax\|\|Bx\|} = \cos\theta$. We also say $A\angle_\theta^{x,\varepsilon} B$ for some $\varepsilon \in (0, 1)$ and $\theta \in [0, \pi]$, if $A\angle_{\theta_1} B$ and $|\cos\theta - \cos\theta_1| \leq \varepsilon$.*

Proposition 2.6. *Let $A, B \in \mathbf{B}(\mathcal{H})$. Then the angle between A and B trough Definition 2.5 is the same as that defined trough Relation (1) provided that $M_A \subseteq M_B$.*

Proof. Let a vector x of unit vectors $\mathcal{H}$ exist such that $x \in M_A$, $\frac{|(Ax,Bx)|}{\|Ax\|\|Bx\|} = \cos\theta$ and $M_A \subseteq M_B$. So

$$\cos\theta = \frac{|(Ax,Bx)|}{\|Ax\|\|Bx\|} = \frac{|(Ax,Bx)|}{\|A\|\|B\|}$$

By theorem 2.1 of [13] we have $A\perp_{BJ}(\cos\theta\frac{A}{\|A\|} + \frac{B}{\|B\|})$. The converse is similarly proved using the same theorem. □

Obviously, in light of results of [3], this definition may be extended for infinite dimensional Hilbert spaces, as well. However, we prove our result only for the finite dimensional case.

3. Perturbation of angle in inner product spaces

We, first, consider the problem of perturbation of angle in inner product spaces. In this theorem we in fact extend [6, Theorem 2.2] for approximate angle in inner product spaces.

Note that the case when $\theta = \frac{\pi}{2}$, $x\angle^\varepsilon_{\frac{\pi}{2}} y$ for some x, y in an inner product space and $\varepsilon \in (0, 1)$, is fully responded [6, Theorem 2.2] and [22] for general normed spaces and consequently for inner product spaces. We consider $\theta \in [0, \frac{\pi}{2})$.

Theorem 3.1. *let X be an inner product space and $x, y \in X$ and let $\theta, \theta_1 \in [0, \frac{\pi}{2})$. If $x\angle^\varepsilon_\theta y, x\angle_{\theta_1} y$, for some $\varepsilon \leq \min\{\cos\theta, 1 - \cos\theta\}$, then there exists a $w \in X$ such that $x\angle_\theta w$, $\|w\| = \|y\|$ and*

$$\|w - y\|^2 \leq 4\varepsilon\|y\|^2 \cos\theta,$$

if $\theta < \theta_1$, and

$$\|w - y\|^2 \leq 2\varepsilon\|y\|^2(\cos\theta + \frac{\varepsilon+\cos\theta}{\sqrt{1-(\varepsilon+\cos\theta)^2}}\sin\theta),$$

if $\theta_1 < \theta$. Moreover

$$|(x, w - y)| \leq \varepsilon\|y\|\|x\|$$

Proof. We first prove the theorem in the real inner product spaces. Let $\|y\| = 1$. Since $\cos\theta_1 = \frac{(x,y)}{\|x\|\|y\|}$, so

$$|\cos\theta - \cos\theta_1| \leq \varepsilon.$$

We define $z = -\frac{(x,y)}{\|x\|^2}x + y$. It is easy to see that $(z, x) = 0$. Since $\theta_1 \neq 0$, we have that $z \neq 0$ and

$$\|z\|^2 = (z, y) = -\frac{(x, y)^2}{\|x\|^2} + 1 = \sin^2\theta_1 \tag{2}$$

so $\|z\| = \sin\theta_1$.

Define $w_1 = \sin\theta\frac{z}{\|z\|} + \cos\theta\frac{x}{\|x\|}$. By an easy calculation we have $\|w_1\| = 1$, and $\frac{(x,w_1)}{\|x\|} = \cos\theta$. Hence $x\angle_\theta w_1$. On the other hand

$$\begin{aligned}
&\|w_1 - y\|^2 = (w_1 - y, w_1 - y)\\
&= (\sin\theta\frac{z}{\|z\|} + \cos\theta\frac{x}{\|x\|} - y, \sin\theta\frac{z}{\|z\|} + \cos\theta\frac{x}{\|x\|} - y)\\
&= \sin^2\theta - \sin\theta\frac{(z, y)}{\|z\|} + \cos^2\theta - \cos\theta\frac{(x, y)}{\|x\|} - \sin\theta\frac{(y, z)}{\|z\|} - \cos\theta\frac{(y, x)}{\|x\|} + 1.
\end{aligned}$$

So (2) and the assumption that X is real imply that

$$\|w_1 - y\|^2 = 2 - 2\sin\theta\sin\theta_1 - 2\cos\theta\cos\theta_1. \tag{3}$$

Let $\theta < \theta_1$. The functions sin and cos are differentiable on $\mathbb{R}$ and we apply the mean value theorem. Thus

$$\exists \quad c \in (\theta, \theta_1); \quad \cos(c) = \frac{\sin\theta - \sin\theta_1}{\theta - \theta_1}$$

and

$$\exists \quad c' \in (\theta, \theta_1); \quad -\sin(c') = \frac{\cos\theta - \cos\theta_1}{\theta - \theta_1}.$$

Note that $\varepsilon \le \min\{\cos\theta, 1 - \cos\theta\}$ and $\|z\| = \sin\theta_1$, so $\theta_1 \in (0, \frac{\pi}{2})$. Now, if $\alpha = \frac{\cos(c)}{\sin(c)'}$, then

$$\begin{aligned}
|\sin\theta - \sin\theta_1| &= \cos(c)|\theta - \theta_1|\\
&= \cos(c)\frac{|\cos\theta - \cos\theta_1|}{|-\sin c'|}\\
&= \frac{\cos(c)}{\sin(c)'}|\cos\theta - \cos\theta_1|\\
&\le \varepsilon\alpha.
\end{aligned}$$

By (3) , we have

$$\begin{aligned}
\|w_1 - y\|^2 &\le 2 + 2\sin\theta(\varepsilon\alpha - \sin\theta) + 2\cos\theta(\varepsilon - \cos\theta)\\
&= 2\varepsilon(\cos\theta + \alpha\sin\theta).
\end{aligned}$$

In the case when $\|y\| \neq 1$, we replace $y_1 = \frac{y}{\|y\|}$ by y. So

$$\|w_1 - y_1\|^2 \quad \le 2\varepsilon(\cos\theta + \alpha\sin\theta).$$

Put $w = w_1\|y\|$, we get

$$\|w - y\|^2 \le 2\varepsilon\|y\|^2(\cos\theta + \alpha\sin\theta).$$

Since $\theta < \theta_1$, we have $0 < \sin\theta < \sin c', 0 < \cos c < \cos\theta$. Hence $\alpha < \frac{\cos\theta}{\sin\theta}$ and

$$\|w - y\|^2 \le 4\varepsilon\|y\|^2\cos\theta.$$

If we assume that $\theta > \theta_1$, then $\alpha < \frac{\cos\theta_1}{\sin\theta_1}$. There exists a real number β less than 1 so that $\cos\theta_1 = \varepsilon\beta + \cos\theta$. Hence

$$\frac{\cos\theta_1}{\sin\theta_1} = \frac{\varepsilon\beta + \cos\theta}{\sqrt{1 - (\varepsilon\beta + \cos\theta)^2}} \le \frac{\varepsilon + \cos\theta}{\sqrt{1 - (\varepsilon + \cos\theta)^2}},$$

and therefore

$$\alpha < \frac{\varepsilon+\cos\theta}{\sqrt{1-(\varepsilon+\cos\theta)^2}},$$

which implies that

$$\|w-y\|^2 \leq 2\varepsilon\|y\|^2(\cos\theta + \frac{\varepsilon+\cos\theta}{\sqrt{1-(\varepsilon+\cos\theta)^2}}\sin\theta).$$

Finally,

$$\begin{aligned}|(x, w_1 - y_1)| &= |(x, \cos\theta\frac{x}{\|x\|} + \sin\theta\frac{z}{\|z\|} - y_1)| \\ &= |\cos\theta\|x\| - (x, y_1)| = |\cos\theta\|x\| - \cos\theta_1\|x\|| \leq \varepsilon\|x\|,\end{aligned}$$

that means $|(x, w-y)| \leq \varepsilon\|x\|\|y\|$ and we are done.

To find w in complex normed space X, we consider the element z to be $-\frac{(x,y)}{\|x\|^2}x + y$. It is readily verified that $(x,z) = 0$ and $(z,z) = (z,y) = \sin^2\theta_1$. If $(x,y) = |(x,y)|e^{i\mu}$ for some $\mu \in \mathbb{R}$ and if assume that $\|y\| = 1$, then let us define w_1 to be

$$\sin\theta\frac{z}{\|z\|} + \cos\theta\frac{x}{\|x\|}e^{-i\mu}.$$

Clearly $x\angle_\theta w_1$ and

$$\begin{aligned}\|w_1 - y\|^2 =&(w_1 - y, w_1 - y) \\ =&(\sin\theta\frac{z}{\|z\|} + \cos\theta\frac{x}{\|x\|}e^{-i\mu} - y, \sin\theta\frac{z}{\|z\|} + \cos\theta\frac{x}{\|x\|}e^{-i\mu} - y) \\ =&\sin^2\theta - \sin\theta\frac{(z,y)}{\|z\|} + \cos^2\theta e^{-i\mu}e^{i\mu} - \cos\theta e^{-i\mu}\frac{(x,y)}{\|x\|} \\ &- \sin\theta\frac{(y,z)}{\|z\|} - \cos\theta e^{i\mu}\frac{(y,x)}{\|x\|} + 1 \\ =&\sin^2\theta - \sin\theta\frac{(z,y)}{\|z\|} + \cos^2\theta - \cos\theta e^{-i\mu}\frac{|(x,y)|e^{i\mu}}{\|x\|} \\ &- \sin\theta\frac{(y,z)}{\|z\|} - \cos\theta e^{i\mu}\frac{|(x,y)|e^{-i\mu}}{\|x\|} + 1 \\ =&2 - 2\sin\theta\sin\theta_1 - 2\cos\theta\cos\theta_1\end{aligned}$$

Now, similar to the real case, if we put $\alpha = \frac{\cos(c)}{\sin(c')}$ where c and c' are obtained between θ and θ_1 according to mean value theorem applying to cos and sin functions ,exactly as what we have done in the real case, then we come to

$$\|w-y\|^2 \leq 2\varepsilon\|y\|^2(\cos\theta + \alpha\sin\theta)$$

and the rest of the proof is a verbatim transcription of the proof for the real case. □

Remark 3.2. *In the recent theorem if we assume $\theta \in (\frac{\pi}{2}, \pi)$, then there exists $w \in X$ such that $x\angle_\theta w$, and if $\theta > \theta_1$, then*

$$\|w-y\|^2 \leq -4\varepsilon\|y\|^2\cos\theta,$$

and if $\theta < \theta_1$ then

$$\|w-y\|^2 \leq 2\varepsilon\|y\|^2(-\cos\theta + \frac{\varepsilon-\cos\theta}{\sqrt{1-(\varepsilon-\cos\theta)^2}}\sin\theta).$$

The converse of Theorem (3.1) is embodied in the next theorem.

Theorem 3.3. *If X is a real inner product space, $x, y \in X$, $\theta \in [0, \frac{\pi}{2})$ and*

$$\exists w; \|w\| = \|y\|, x\angle_\theta w, \|w - y\| \leq \varepsilon\|y\|,$$

then $x\angle_\theta^\varepsilon y$.

Proof. Let $\cos\theta_1 = \frac{(x,y)}{\|x\|\|y\|}$. Then

$$\begin{aligned} |\cos\theta - \cos\theta_1| &= \left|\frac{(x,w)}{\|x\|\|w\|} - \frac{(x,y)}{\|x\|\|y\|}\right| \\ &= \frac{|(x, w-y)|}{\|x\|\|y\|} \leq \frac{\|w-y\|}{\|y\|} \leq \varepsilon. \end{aligned}$$

□

In the next corollary, we reexpress a version of Theorem (3.1) for bounded linear operators defined on a finite dimensional Hilbert space.

Corollary 3.4. *Let $A, B \in \mathbf{B}(\mathcal{H})$ and $\mathcal{H}$ be a finite dimensional real or complex Hilbert space. If $A\angle_\theta^{x,\varepsilon} B$ for some $\theta \in (0, \pi/2)$ and $|(Ax, Bx)| \neq \|Ax\|\|Bx\|$ for some unit vector x of $\mathcal{H}$, then there exists an operator $W \in \mathbf{B}(\mathcal{H})$ such that*

$$\|Wx - Bx\|^2 \leq 4\varepsilon\|Bx\|^2\cos\theta,$$

if $A\angle_{\theta_1} B$ and $\theta_1 > \theta$. And

$$\|Wx - Bx\|^2 \leq 2\varepsilon\|Bx\|^2\left(\cos\theta + \tfrac{\varepsilon+\cos\theta}{\sqrt{1-(\varepsilon+\cos\theta)^2}}\sin\theta\right),$$

if $\theta_1 < \theta$.

Proof. Let $A\angle_{\theta_1}^x B$. So $\|Ax\| = \|A\|$, $\frac{|(Ax,Bx)|}{\|Ax\|\|Bx\|} = \cos\theta_1$ for some unit vector x, and $|\cos\theta - \cos\theta_1| \leq \varepsilon$. Assume that $\theta_1 > \theta$. So $Ax\angle_{\theta_1} Bx$. By Theorem (3.1) there exists a vector w such that $\|w\| = \|Bx\|$, $Ax\angle_\theta w$ and

$$\|w - Bx\|^2 \leq 4\varepsilon\|Bx\|^2\cos\theta,$$

Now, let W be an operator in $\mathbf{B}(\mathcal{H})$ defined to be

$$W(h) = \begin{cases} \alpha w, & \text{if } h = \alpha x \text{ for some } \alpha \in \mathbb{C} \\ B(h), & \text{if } h \in x^\perp. \end{cases}$$

for every $h \in \mathcal{H}$. By an easy calculation we have

$$\begin{aligned} (Ax, Wx) &= (Ax, w) = \cos\theta\|Ax\|\|w\| \\ \|Wx\| &= \|w\|. \end{aligned}$$

Thus $(Ax, Wx) = \cos\theta\|Ax\|\|Wx\|$. Since $\|Ax\| = \|A\|$ and $Wx = w$, we have $A\angle_\theta^x W$ and

$$\|Wx - Bx\|^2 \leq 4\varepsilon\|Bx\|^2\cos\theta.$$

Similarly, for the case when, $\theta_1 < \theta$ we could obtain

$$\|Wx - Bx\|^2 \leq 2\varepsilon\|Bx\|^2\left(\cos\theta + \tfrac{\varepsilon+\cos\theta}{\sqrt{1-(\varepsilon+\cos\theta)^2}}\sin\theta\right).$$

□

4. Approximate angle in Hilbert C^*- modules

In this section, we try to extend the results of the previous section for approximate angle to Hilbert C^*-modules.

Theorem 4.1. *Let X be a Hilbert C^*- module over a unital C^*- algebra $\mathcal{A}$ and let $x, y \in X$. If $x\angle_{\theta}^{\phi,\varepsilon} y$ and $x\angle_{\theta_1}^{\phi} y$ for some $\varepsilon \in (0,1)$, and $\phi \in S(\mathcal{A})$, $\theta, \theta_1 \in (0, \frac{\pi}{2})$, $\phi(\langle x, y\rangle) \neq \|x\|\|y\|$, then there exists a $w \in X$ such that $x\angle_{\theta}^{\phi} w$ and if $\theta_1 > \theta$, then*

$$\phi(|w-y|^2) \leq 4\varepsilon\phi(|y|^2)\cos\theta \tag{4}$$

if $\theta_1 < \theta$, then

$$\phi(|w-y|^2) \leq 2\varepsilon\phi(|y|^2)(\cos\theta + \frac{\varepsilon+\cos\theta}{\sqrt{1-(\varepsilon+\cos\theta)^2}}\sin\theta). \tag{5}$$

Moreover $|\phi(\langle x, w-y\rangle)| \leq \varepsilon\|y\|\|x\|$.

Proof. First, we assume that $\phi(|y|^2) = 1$. Put $z = \frac{-\phi(\langle x,y\rangle)}{\phi(|x|^2)}x + y$. It is clear that $\phi(\langle x, z\rangle) = 0$. If $\phi(\langle x, y\rangle) = |\phi(\langle x, y\rangle)|e^{i\mu}$ for some μ, then

$$\phi(\langle y, z\rangle) = \phi(\langle z, z\rangle) = 1 - \frac{|\phi(\langle x, y\rangle)|^2}{\phi(\langle x, x\rangle)\phi(\langle y, y\rangle)}.$$

Now, we define $w = e^{i\mu}\cos\theta\frac{x}{\sqrt{\phi(|x|^2)}} + \sin\theta\frac{z}{\sqrt{\phi(|z|^2)}}$. By an easy calculation we have

$$\phi(\langle w, w\rangle) = 1, \phi(\langle x, w\rangle) = e^{i\mu}\cos\theta\|x\|.$$

Hence $x\angle_{\theta}^{\phi} w$. Linearity of ϕ and (6) imply that

$$\begin{aligned}
\phi(|w-y|^2) &= \phi(\langle w-y, w-y\rangle)\\
&= \phi(\langle w, w\rangle) - \phi(\langle y, w\rangle) - \phi(\langle w, y\rangle) + \phi(\langle y, y\rangle)\\
&= 2 - \frac{e^{i\mu}\cos\theta\phi(\langle y, x\rangle)}{\sqrt{\phi(\langle x, x\rangle}} - \sin\theta\frac{\phi(\langle y, z\rangle)}{\sqrt{\phi(\langle z, z\rangle)}} - \frac{e^{-i\mu}\cos\theta\phi(\langle x, y\rangle)}{\sqrt{\phi(\langle x, x\rangle)}} - \sin\theta\frac{\phi(\langle z, y\rangle)}{\sqrt{\phi(\langle z, z\rangle)}}\\
&= 2 - 2\cos\theta\cos\theta_1 - 2\sin\theta\sin\theta_1.
\end{aligned}$$

If $\phi(|y|^2) \neq 1$, we replace y by $y_1 = \frac{y}{\sqrt{\phi(|y|^2)}}$. By an easy calculation, we have

$$\phi(|y_1|^2) = 1, \frac{|\phi(\langle x,y_1\rangle)|}{\sqrt{\phi(\langle x,x\rangle)}\sqrt{\phi(\langle y_1,y_1\rangle)}} = \frac{|\phi(\langle x,y\rangle)|}{\sqrt{\phi(\langle x,x\rangle)}\sqrt{\phi(\langle y,y\rangle)}} = \cos\theta_1.$$

So $x\angle_{\theta}^{\varepsilon,\phi} y_1$ and using the previous step there exists an element w_1 such that $\phi(|w_1|^2) = 1$ and $x\angle_{\theta}^{\phi} w_1$ moreover $\phi(|w_1 - y_1|^2) \leq 2 - 2\cos\theta\cos\theta_1 - 2\sin\theta\sin\theta_1$.
Put $w = \sqrt{\phi(|y|^2)}w_1$. Hence $\phi(|w|^2) = \phi(|y|^2)$ and

$$\begin{aligned}
\phi(|w-y|^2) &= \phi(|y|^2)\phi(|w_1-y_1|^2) \leq \|y\|^2\phi(|w_1-y_1|^2)\\
&\leq \|y\|^2(2 - 2\cos\theta\cos\theta_1 - 2\sin\theta\sin\theta_1).
\end{aligned}$$

Now, similar to the proof of Theorem (3.1), according to the mean value theorem, applying to the sin and cos functions, we could find parameter α, exactly as what introduced in the proof of Theorem (3.1), and after some routine calculation we find an upper bound for

$$2 - 2\cos\theta\cos\theta_1 - 2\sin\theta\sin\theta_1$$

which yields (4) and (5) for the cases raised in the theorem.

Finally, if $\cos\theta_1 = \frac{|\phi(\langle x,y\rangle)|}{\sqrt{\phi(\langle x,x\rangle)}\sqrt{\phi(\langle y,y\rangle)}}$, then

$$\begin{aligned}|\phi(\langle x, w-y\rangle)| &= |\phi(\langle x, \sqrt{\phi(|y|^2)}e^{i\mu}\cos\theta\frac{x}{\sqrt{\phi(|x|^2)}} + \sqrt{\phi(|y|^2)}\sin\theta\frac{z}{\sqrt{\phi(|z|^2)}} - \sqrt{\phi(|y|^2)}y_1\rangle)| \\ &= |\sqrt{\phi(|y|^2)}|\phi(\langle x, e^{i\mu}\cos\theta\frac{x}{\sqrt{\phi(|x|^2)}}\rangle) - \phi(\langle x, y_1\rangle)| \\ &= |\sqrt{\phi(|y|^2)}\left|e^{i\mu}\cos\theta\|x\| - e^{i\mu}|\phi(\langle x, y_1\rangle)|\right| \\ &= |\sqrt{\phi(|y|^2)}|e^{i\mu}\cos\theta\|x\| - e^{i\mu}\cos\theta_1\|x\|| \le \varepsilon\|y\|\|x\|.\end{aligned}$$

□

Corollary 4.2. *Let X be a Hilbert C^*- module over C^*- algebra $\mathcal{A}$, $x, y \in X$, $|y|^2 = 1$ and let $\phi \in S(\mathcal{A})$.*

$$\exists w; |w|^2 = 1, x\angle_\theta^\phi w, |\phi(\langle x, w-y\rangle)| \le \varepsilon\|x\| \Longleftrightarrow x\angle_\theta^{\phi,\varepsilon} y$$

with $\theta \in (0, \frac{\pi}{2})$.

Proof. First, suppose that there exists a w such that $|w|^2 = 1, x\angle_\theta^\phi w$ and $\phi(\langle x, w-y\rangle) \le \varepsilon\|x\|$. Put $\cos\theta_1 = \frac{|\phi(\langle x,y\rangle)|}{\sqrt{\phi(\langle x,x\rangle)}\sqrt{\phi(\langle y,y\rangle)}}$. Without loss of generality we assume $\cos\theta \le \frac{|\phi(\langle x,y\rangle)|}{\sqrt{\phi(\langle x,x\rangle)}\sqrt{\phi(\langle y,y\rangle)}}$. Then

$$\begin{aligned}\cos\theta_1 - \cos\theta =& \frac{|\phi(\langle x,w\rangle)|}{\sqrt{\phi(\langle x,x\rangle)}\sqrt{\phi(\langle y,y\rangle)}} - \frac{|\phi(\langle x,y\rangle)|}{\sqrt{\phi(\langle x,x\rangle)}\sqrt{\phi(\langle y,y\rangle)}} \\ &\le \left|\frac{\phi(\langle x,w\rangle) - \phi(\langle x,y\rangle)}{\|x\|}\right| = \left|\frac{\phi(\langle x, w-y\rangle)}{\|x\|}\right| \le \varepsilon.\end{aligned}$$

Therefore $|\cos\theta_1 - \cos\theta| \le \varepsilon$, because $\cos\theta \le \frac{|\phi(\langle x,y\rangle)|}{\sqrt{\phi(\langle x,x\rangle)}\sqrt{\phi(\langle y,y\rangle)}}$. The implication $\Longleftarrow$ has just proved in the previous theorem. □

Example 4.3. *Let $M_{2\times2}$ be the C^*-algebra of all 2×2 Matrices. One can regard it as the Hilbert $\mathcal{M}_{2\times2}$- module equipped with C^*-inner product*

$$\langle A, B\rangle = A^*B,$$

for all $A, B \in M_{2\times2}$. Assume that

$$A_\theta = \begin{pmatrix}\sin\theta & \cos\theta\\ \cos\theta & -\sin\theta\end{pmatrix} \text{and } B = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \tag{6}$$

Let v be the unit vector $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Define $\phi : A \longmapsto (Av, v)$, where $(\cdot,\cdot)$ is the usual inner product of $\mathbb{C}^2$. Obviously ϕ defines a state on $M_{2\times2}$. We see that A_θ is a self-adjoint unitary matrix, so $B^*A_\theta = A_\theta$, $\phi(A_\theta^*A_\theta) = \phi(B^*B) = 1$ and

$$B^*A_\theta v = \tfrac{1}{\sqrt{2}}(\cos\theta + \sin\theta, \cos\theta - \sin\theta).$$

Therefore

$$\phi(B^*A_\theta) = (B^*A_\theta v, v) = \tfrac{1}{2}\cos\theta + \sin\theta + \cos\theta - \sin\theta) = \cos\theta$$

which means $B\angle_\theta^\phi A_\theta$. Now, let θ_1 and θ_2 be in $(0, \frac{\pi}{2})$ so that $|\cos\theta_1 - \cos\theta_2| < \varepsilon$. Then

$$|\phi(\langle B, A_{\theta_1} - A_{\theta_2}\rangle)| = |\cos\theta_1 - \cos\theta_2| < \varepsilon.$$

The next theorem deals with the issue via the definition of angle $\angle^*_\theta$.

Theorem 4.4. *Let x, y be in Hilbert $\mathcal{A}$- module X and $x\angle^{*,\varepsilon}_\theta y$ for some $\varepsilon \in (0,1)$ and $\theta \in (0, \frac{\pi}{2})$, and let $|y|$ and $|x|$ be invertible such that $|\langle x, y\rangle| \neq |x||y|$. Then there exist an element $w \in X$ and a state ϕ such that*

$$x\angle^*_\theta w, |\phi(\langle x, w - y\rangle)| \leq \varepsilon\phi(|y|)\|x\|.$$

Proof. First, we, assume that $|y| = 1$. Since $x\perp_{BJ} - \cos\theta_1 \frac{x}{|x|} + y$, by Theorem 2.7 of [1] we have

$$\exists\phi \in S(\mathcal{A}); \phi(\langle x, x\rangle) = \|x\|^2, \phi(\langle x, y\rangle) = \cos\theta_1\phi(|x|).$$

Put $z = -x\frac{\langle x,y\rangle}{|x|^2} + y$. Since $\langle z, z\rangle \neq 0$ so $|z|$ is invertible also

$$\langle x, z\rangle = -\langle x, x\rangle|x|^{-2}\langle x, y\rangle + \langle x, y\rangle = 0.$$

Define $w := \cos\theta\frac{x}{|x|} + \sin\theta\frac{z}{|z|}$. By an easy calculation we have $|w|^2 = \langle w, w\rangle = 1$, therefore $|w| = 1$ and $\phi(\langle x, -\cos\theta\frac{x}{|x|} + \frac{w}{|w|}\rangle) = 0$, that means $x\angle^*_\theta w$.

Moreover, $\phi(\langle x, w\rangle) = \cos\theta\phi(|x|)$. On the other hand $\phi(\langle x, y\rangle) = \cos\theta_1\phi(|x|)$, so $|\phi(\langle x, w - y\rangle)| \leq \varepsilon\|x\|$. In the case when $|y| \neq 1$, we replace $y_1 = \frac{y}{|y|}$ by y. Hence $x\angle^{*,\varepsilon}_\theta y_1$ and there exists an element w_1 such that $x\angle^*_\theta w_1$ and $|\phi(\langle x, w_1 - y_1\rangle)| \leq \varepsilon\|x\|$ for some ϕ. Let $w = w_1|y|$, hence

$$|\phi(\langle x, w - y\rangle)| = |\phi(\langle x, w_1|y| - y_1|y|\rangle)| \leq |\phi(\langle x, w_1 - y_1\rangle)|\left|\phi(|y|)\right| \leq \varepsilon\phi(|y|)\|x\|$$

□

Now, we consider the problem for bounded linear operators defined over a Hilbert space $\mathcal{H}$; the vectors of $\mathbf{B}(\mathcal{H})$.

Corollary 4.5. *Let $A, B \in \mathbf{B}(\mathcal{H})$, $A\angle^{*,\varepsilon}_\theta B$, $|A|$, $|B|$ be invertible. By previouse theorem there exist a $W \in \mathbf{B}(\mathcal{H})$, $\phi \in S(\mathbf{B}(\mathcal{H}))$ such that $A\angle^*_\theta W$ and*

$$\phi(A^*W) - \phi(A^*B) \leq \varepsilon\|A\|\phi(|B|).$$

The concepts of $*$-angle and ϕ-angle are not the same as that defined by (1). Let X be a Hilbert C^*-module over a C^*- algebra $\mathcal{A}$. According to [1, Theorem 2.7] an element $x \in X$ is Birkhoff- James orthogonal to $y \in X$ if

$$\exists\phi \in S(\mathcal{A}); \phi(\langle x, x\rangle) = \|x\|^2, \phi(\langle x, y\rangle) = 0.$$

If we confine ourselves to pure states of $\mathcal{A}$, we then reach a weaker definition of Birkhoff- James orthogonality as follows:

$$x\perp^w_{BJ}y \Leftrightarrow \exists\phi \in PS(\mathcal{A}); \phi(\langle x, x\rangle) = \|x\|^2, \phi(\langle x, y\rangle) = 0.$$

Obviously, for every $x, y \in X$ we have that

$$x\perp^w_{BJ}y \Rightarrow x\perp_{BJ}y. \tag{7}$$

We could consider a notion of angle with respect to weak Birkhoff- James orthogonality. Let x, y be two elements of a Hilbert $\mathcal{A}$- module X. We say that $x\angle^w_\theta y$ for some $\theta \in [0, \pi]$, if

$$x\perp^w_{BJ} - \cos\theta e^{i\mu}\frac{x}{\|x\|} + \frac{y}{\|y\|}.$$

We also say $x\angle^{w,\varepsilon}_\theta y$ for some $\varepsilon \in (0,1)$ and $\theta \in (0, \frac{\pi}{2})$ whenever $x\angle^w_{\theta_1} y$ for some $\theta_1 \in (0, \frac{\pi}{2})$ and

$$|\cos\theta - \cos\theta_1| \leq \varepsilon.$$

Now, we prove Theorem (3.1) for this notion of angle in Hilbert C^*- modules.

Theorem 4.6. *Let X be a Hilbert C^*- module over a commutative C^*- algebra $\mathcal{A}$, $x, y \in X$ and let $|x|, |y|$ be invertible. If $x\angle_{\theta}^{w,\varepsilon} y$ and $x\angle_{\theta_1}^{w} y$for some $\varepsilon \in (0,1)$ and $\theta, \theta_1 \in (0, \frac{\pi}{2})$, then there exist $w \in X$ and a pure state ϕ such that $x\angle_{\theta}^{w} w$ and*

$$\phi(|w-y|^2) \leq 4\varepsilon\phi(|y|^2)\cos\theta,$$

if $\theta_1 > \theta$, and

$$\phi(|w-y|^2) \leq 2\varepsilon\phi(|y|^2)(\cos\theta + \frac{\varepsilon+\cos\theta}{\sqrt{1-(\varepsilon+\cos\theta)^2}}\sin\theta),$$

if $\theta_1 < \theta$.

Proof. First, we assume that $|y|^2 = 1$, so $|y| = 1$, and $\phi \in PS(\mathcal{A})$ is a pure state on a C^*- algebra $\mathcal{A}$ such that $\phi(\langle x, x\rangle) = \|x\|^2$ and $\cos\theta_1 = \frac{|\phi(\langle x,y\rangle)|}{\|x\|\|y\|}$. Put $z = \frac{-x\langle x,y\rangle}{|x|^2} + y$. So we have

$$\begin{aligned}\langle x, z\rangle &= \langle x, \frac{-x\langle x,y\rangle}{|x|^2} + y\rangle = \langle x, \frac{-x\langle x,y\rangle}{|x|^2}\rangle + \langle x, y\rangle \\ &= -\langle x, x\rangle\langle x, y\rangle|x|^{-2} + \langle x, y\rangle = 0.\end{aligned}$$

Since ϕ is a pure state and $\mathcal{A}$ is a commutative C^*- algebra, ϕ is a character. Assume that $\phi(\langle x, y\rangle) = |\phi(\langle x, y\rangle)|e^{i\mu}$ for some μ. Then

$$\begin{aligned}\phi(\langle y, z\rangle) &= \phi(\langle z, z\rangle) = 1 - \frac{\phi(\langle x,y\rangle)\phi(\langle y,x\rangle)}{\phi(|x|^2)} = 1 - \frac{\phi(\langle x,y\rangle)\phi(\langle y,x\rangle)}{\phi(|x|^2)} \\ &= 1 - \frac{|\phi(\langle x,y\rangle)|^2}{\|x\|^2} = 1 - \cos^2\theta_1 = \sin^2\theta_1. \end{aligned} \tag{8}$$

Now, we define $w = e^{i\mu}\cos\theta\frac{x}{|x|} + \sin\theta\frac{z}{|z|}$. By an easy calculation we come to $\langle w, w\rangle = 1$. Therefore, $\|w\| = 1$. Moreover

$$\begin{aligned}\phi(\langle x, w\rangle) &= e^{i\mu}\cos\theta\phi(\langle x, \frac{x}{|x|}\rangle) + \sin\theta\phi(\langle x, \frac{z}{|z|}\rangle) \\ &= e^{i\mu}\cos\theta\phi(\langle x, x\rangle\frac{1}{|x|}) + \sin\theta\phi(\langle x, \frac{z}{|z|}\rangle) \\ &= e^{i\mu}\cos\theta\phi(\langle x, x\rangle)\frac{\phi(1)}{\phi(|x|)} + \sin\theta\frac{\phi(\langle x,z\rangle)}{\phi(|z|)} \\ &= e^{i\mu}\cos\theta\|x\|,\end{aligned}$$

hence $x\angle_{\theta}^{w} w$. On the other hand, by (8) we have

$$\begin{aligned}\phi(\langle w-y, w-y\rangle) =& \phi(\langle e^{i\mu}\cos\theta\frac{x}{|x|} + \sin\theta\frac{z}{|z|} - y, e^{i\mu}\cos\theta\frac{x}{|x|} + \sin\theta\frac{z}{|z|} - y\rangle) \\ =& \sin^2\theta - \sin\theta\phi(\frac{\langle z,y\rangle}{|z|}) + \cos^2\theta e^{-i\mu}e^{i\mu} - \cos\theta e^{-i\mu}\phi(\frac{\langle x,y\rangle}{|x|}) \\ & - \sin\theta\phi(\frac{\langle y,z\rangle}{|z|}) - \cos\theta e^{i\mu}\phi(\frac{\langle y,x\rangle}{|x|}) + 1 \\ =& \sin^2\theta - \sin\theta\frac{\phi(\langle z,y\rangle)}{\phi(|z|)} + \cos^2\theta - \cos\theta e^{-i\mu}\frac{|\phi(\langle x,y\rangle)|e^{i\mu}}{\phi(|x|)} \\ & - \sin\theta\frac{\phi(\langle y,z\rangle)}{\phi(|z|)} - \cos\theta e^{i\mu}\frac{|\phi(\langle x,y\rangle)|e^{-i\mu}}{\phi(|x|)} + 1 \\ =& 2 - 2\sin\theta\sin\theta_1 - 2\cos\theta\cos\theta_1.\end{aligned}$$

In the case when $|y|^2 \neq 1$, we replace $y_1 = \frac{y}{|y|}$ by y. Hence $x\angle_\theta^{w,\varepsilon} y_1$ and there exists an element w_1 such that $x\angle_\theta^{w} w_1$ and $\phi(\langle w_1 - y_1, w_1 - y_1\rangle) \leq (2 - 2\sin\theta\sin\theta_1 - 2\cos\theta\cos\theta_1)$ for some ϕ. Let $w = w_1|y|$, hence

$$\phi(\langle w - y, w - y\rangle) = \phi(\langle w_1|y| - y_1|y|, w_1|y| - y_1|y|\rangle) = \phi(\langle w_1 - y_1, w_1 - y_1\rangle)\phi(|y|^2) \leq (2 - 2\sin\theta\sin\theta_1 - 2\cos\theta\cos\theta_1)\phi(|y|^2)$$

For the rest of the proof we act similar to that of Theorem (3.1) in finding a bound for the right hand side of the above inequality. □

Corollary 4.7. *Let $A, B \in \mathbf{B}(\mathcal{H})$, $A\angle_\theta^{w,\varepsilon} B$, $A\angle_{\theta_1}^{w,} B$ for some $\theta, \theta_1 \in (0, \frac{pi}{2})$ and $\varepsilon \in (0,1)$ also $|A|$, $|B|$ be invertible. By previouse theorem there exist a $W \in \mathbf{B}(\mathcal{H})$, $\phi \in PS(\mathbf{B}(\mathcal{H}))$ such that $A\angle_\theta^{w} W$ and*

$$\phi(|W - B|^2) \leq 4\varepsilon\phi(|B|^2)\cos\theta.$$

if $\theta_1 > \theta$, and

$$\phi(|W - B|^2) \leq 2\varepsilon\phi(|B|^2)(\cos\theta + \frac{\varepsilon + \cos\theta}{\sqrt{1 - (\varepsilon + \cos\theta)^2}}\sin\theta),$$

if $\theta_1 < \theta$.

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