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A subclass of multivalent starlike functions with fixed second coefficient

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Abstract. We introduce and investigate a subclass of multivalent starlike functions with fixed second coefficient. Such results as an integral representation, an inclusion relationship, a convolution property, coefficient inequalities, a neighborhood property and partial sums for this function class are derived.

1. Introduction and preliminaries

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (p \in \mathbb{N} := \{1, 2, 3, \ldots\}),$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

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Also, let $\mathcal{A}_{p,\alpha}$ denote the class of functions $\mathfrak{f} \in \mathcal{A}_p$ with fixed second coefficient and of the form

$$\mathfrak{f}(z) = z^p + \alpha z^{p+1} + \sum_{k=2}^{\infty} a_{p+k} z^{p+k} \quad (0 \leq \alpha \leq p;\ z \in \mathbb{U}). \tag{1}$$

For two functions $f, g \in \mathcal{A}_p$, the convolution (Hadamard product) of f and g is defined by

$$(f * g)(z) := z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k} \quad (z \in \mathbb{U}),$$

where

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k} \quad (z \in \mathbb{U}).$$

As usual, we denote by $\mathcal{S}_p^*(\rho)$ the subclass of $\mathcal{A}_p$ consisting of functions which are starlike of order ρ $(0 \leq \rho < p)$ in $\mathbb{U}$. For convenience, we write $\mathcal{S}_p^*(0) =: \mathcal{S}_p^*$. For some recent investigations on starlike functions, see [1, 4, 6, 10, 12, 14, 15, 17–23] and the references cited therein.

Let $\mathcal{P}$ denote the class of functions $\mathbf{p}$ given by

$$\mathbf{p}(z) = 1 + \sum_{k=1}^{\infty} p_k z^k, \tag{2}$$

which are analytic in $\mathbb{U}$ and satisfy the condition

$$\Re(\mathbf{p}(z)) > 0 \quad (z \in \mathbb{U}).$$

Assume that $f \in \mathcal{A}_{p,\alpha}$ with $0 \leq \alpha \leq p$, we say that $f \in \mathcal{H}(p, \alpha)$ if it satisfies the condition

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{p(p+\alpha-2)}{2(p+\alpha)} \quad (z \in \mathbb{U}). \tag{3}$$

Recently, Gupta, Nagpal and Ravichandran [3, Corollary 3.2] proved that

$$\mathcal{H}(p, \alpha) \subset \mathcal{S}_p^*(p/2) \subset \mathcal{S}_p^*.$$

Indeed, it is a Marx-Strohhäcker type result for multivalent functions with fixed second coefficient (see also [9, 13, 16]). For more recent investigations on analytic functions with fixed second coefficient, we refer the reader to [5, 7, 8].

Motivated essentially by the above interesting assertion, we shall derive several basic properties of the starlike function class $\mathcal{H}(p, \alpha)$. In order to prove our main results, we need the following two lemmas.

Lemma 1.1. (See [2]) *If a function* $\mathbf{p} \in \mathcal{P}$ *is given by* (2)*, then*

$$|p_k| \leq 2 \quad (k \in \mathbb{N}).$$

Lemma 1.2. *Let* $p \in \mathbb{N}$ *and* $0 \leq \alpha \leq p$*. Suppose also that the sequence* $\{A_{p+k}\}_{k=2}^{\infty}$ *is defined by*

$$A_{p+2} = \frac{p(p+\alpha+2)[p+\alpha(p+1)]}{2(p+2)(p+\alpha)} \tag{4}$$

and

$$A_{p+k+1} = \frac{p(p+\alpha+2)}{(p+\alpha)(k+1)(p+k+1)}\left(p + \alpha(p+1) + \sum_{l=1}^{k-1}(p+k+1-l)A_{p+k+1-l}\right). \tag{5}$$

Then

$$A_{p+k} = \frac{p(p+\alpha+2)[p+\alpha(p+1)]}{2(p+2)(p+\alpha)} \prod_{j=2}^{k-1} \frac{(p+j)[(p+\alpha)(p+j)+2p]}{(p+\alpha)(j+1)(p+j+1)} \quad (k \geq 3). \tag{6}$$

Proof. By (5), we see that

$$(p+\alpha)(k+1)(p+k+1)A_{p+k+1} = p(p+\alpha+2)\left(p+\alpha(p+1)+\sum_{l=1}^{k-1}(p+k+1-l)A_{p+k+1-l}\right) \tag{7}$$

and

$$(p+\alpha)k(p+k)A_{p+k} = p(p+\alpha+2)\left(p+\alpha(p+1)+\sum_{l=1}^{k-2}(p+k-l)A_{p+k-l}\right). \tag{8}$$

It follows from (7) and (8) that

$$\frac{A_{p+k+1}}{A_{p+k}} = \frac{(p+k)[(p+\alpha)(p+k)+2p]}{(p+\alpha)(k+1)(p+k+1)}. \tag{9}$$

Therefore, for $k \geq 3$, we deduce from (9) that

$$A_{p+k} = \frac{A_{p+k}}{A_{p+k-1}} \cdot \frac{A_{p+k-1}}{A_{p+k-2}} \cdot \cdots \cdot \frac{A_{p+3}}{A_{p+2}} \cdot A_{p+2} = \frac{p(p+\alpha+2)[p+\alpha(p+1)]}{2(p+2)(p+\alpha)} \prod_{j=2}^{k-1} \frac{(p+j)[(p+\alpha)(p+j)+2p]}{(p+\alpha)(j+1)(p+j+1)}.$$

The proof of Lemma 1.2 is thus completed. □

In this paper, we aim at deriving an integral representation, an inclusion relationship, a convolution property, coefficient inequalities, a neighborhood property and partial sums for the subclass $\mathcal{H}(p,\alpha)$ of multivalent starlike functions.

2. Main results

Firstly, we derive an integral representation for f belonging to the class $\mathcal{H}(p,\alpha)$.

Theorem 2.1. *Let $f \in \mathcal{H}(p,\alpha)$. Then*

$$f(z) = \int_0^z \zeta^{p-1} \cdot \exp\left(\frac{p(p+\alpha+2)}{p+\alpha}\int_0^\zeta \frac{\omega(t)}{t(1-\omega(t))}dt\right)d\zeta, \tag{10}$$

where ω is a Schwarz function with $\omega(0) = 0$ and $|\omega| < 1$.

Proof. Suppose that $f \in \mathcal{H}(p,\alpha)$. It follows from (3) that

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{p + \left(p - \frac{p(p+\alpha-2)}{p+\alpha}\right)z}{1-z} \quad (z \in \mathbb{U}), \tag{11}$$

where " $\prec$ " denotes the familiar subordination in analytic functions theory. We easily find from (11) that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{p + \left(p - \frac{p(p+\alpha-2)}{p+\alpha}\right)\omega(z)}{1-\omega(z)} \quad (\omega(0) = 0;\ |\omega(z)| < 1;\ z \in \mathbb{U}). \tag{12}$$

Now, by (12), we get

$$\frac{(zf'(z))'}{zf'(z)} - \frac{p}{z} = \frac{p(p+\alpha+2)}{p+\alpha} \cdot \frac{\omega(z)}{z(1-\omega(z))} \quad (z \in \mathbb{U}), \tag{13}$$

which, upon integration, yields

$$\log\left(\frac{f'(z)}{z^{p-1}}\right) = \frac{p(p+\alpha+2)}{p+\alpha} \int_0^z \frac{\omega(t)}{t(1-\omega(t))} dt. \tag{14}$$

By (14), we readily get the desired result of Theorem 2.1. □

Next, we give the following inclusion relationship.

Theorem 2.2. *Let* $0 \leq \alpha_1 < \alpha_2 \leq p$. *Then*

$$\mathcal{H}(p, \alpha_2) \subset \mathcal{H}(p, \alpha_1).$$

Proof. Suppose that $f \in \mathcal{H}(p, \alpha_2)$. Then

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{p(p+\alpha_2-2)}{2(p+\alpha_2)} \quad (z \in \mathbb{U}).$$

By noting that $0 \leq \alpha_1 < \alpha_2 \leq p$, we obtain

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{p(p+\alpha_2-2)}{2(p+\alpha_2)} > \frac{p(p+\alpha_1-2)}{2(p+\alpha_1)} \quad (z \in \mathbb{U}),$$

which implies that $f \in \mathcal{H}(p, \alpha_1)$. We thus conclude that the assertion of Theorem 2.2 holds. □

Now, we derive a convolution property for the class $\mathcal{H}(p, \alpha)$.

Theorem 2.3. *If* $f \in \mathcal{H}(p, \alpha)$ *and* $\theta \in (0, 2\pi)$, *then*

$$f * \left\{(1-e^{i\theta})z\left(\frac{pz^p + (1-p)z^{p+1}}{(1-z)^2}\right)' - \left[p + \left(p - \frac{p(p+\alpha-2)}{p+\alpha}\right)e^{i\theta}\right]\frac{pz^p + (1-p)z^{p+1}}{(1-z)^2}\right\} \neq 0. \tag{15}$$

Proof. If $f \in \mathcal{H}(p, \alpha)$, we see that (11) holds, it follows that

$$1 + \frac{zf''(z)}{f'(z)} \neq \frac{p + \left(p - \frac{p(p+\alpha-2)}{p+\alpha}\right)e^{i\theta}}{1-e^{i\theta}} \quad (z \in \mathbb{U};\ 0 < \theta < 2\pi). \tag{16}$$

By observing that

$$f(z) = f(z) * \frac{z^p}{1-z}, \tag{17}$$

we get

$$zf'(z) = f(z) * \frac{pz^p + (1-p)z^{p+1}}{(1-z)^2}. \tag{18}$$

By (16) and (18), we deduce that the assertion (15) of Theorem 2.3 is true. □

In what follows, we establish several coefficient inequalities for functions f in the class $\mathcal{H}(p, \alpha)$.

Theorem 2.4. *Let $f \in \mathcal{H}(p,\alpha)$. Then*

$$\left|a_{p+2}\right| \leq \frac{p(p+\alpha+2)[p+\alpha(p+1)]}{2(p+2)(p+\alpha)} \tag{19}$$

and

$$\left|a_{p+k}\right| \leq \frac{p(p+\alpha+2)[p+\alpha(p+1)]}{2(p+2)(p+\alpha)} \prod_{j=2}^{k-1} \frac{(p+j)[(p+\alpha)(p+j)+2p]}{(p+\alpha)(j+1)(p+j+1)} \quad (k \geq 3). \tag{20}$$

Proof. Suppose that

$$\phi(z) := \frac{zf''(z)}{f'(z)} + 1 - \frac{p(p+\alpha-2)}{2(p+\alpha)}. \tag{21}$$

By the definition of the class $\mathcal{H}(p,\alpha)$, we see that $\phi(z)$ is analytic in $\mathbb{U}$ and $\Re\big(\phi(z)\big) > 0$ with

$$\phi(0) = p - \frac{p(p+\alpha-2)}{2(p+\alpha)} = \frac{p(p+\alpha+2)}{2(p+\alpha)} > 0.$$

It follows from (21) that

$$\phi(z)f'(z) = zf''(z) + \left[1 - \frac{p(p+\alpha-2)}{2(p+\alpha)}\right] f'(z). \tag{22}$$

By observing that

$$\varphi(z) := \frac{\phi(z)}{\frac{p(p+\alpha+2)}{2(p+\alpha)}} \in \mathcal{P},$$

if we set

$$\phi(z) = p_0 + \sum_{k=1}^{\infty} p_k z^k \quad \left(p_0 = \frac{p(p+\alpha+2)}{2(p+\alpha)}\right),$$

by Lemma 1.1, we get

$$\left|p_k\right| \leq \frac{p(p+\alpha+2)}{p+\alpha} \quad (k \in \mathbb{N}). \tag{23}$$

Now, we find from (22) that

$$\begin{aligned}
&\left(p_0 + \sum_{k=1}^{\infty} p_k z^k\right)\left(pz^{p-1} + \alpha(p+1)z^p + \sum_{k=2}^{\infty}(p+k)a_{p+k}z^{p+k-1}\right) \\
&\quad = p(p-1)z^{p-1} + \alpha p(p+1)z^p + \sum_{k=2}^{\infty}(p+k)(p+k-1)a_{p+k}z^{p+k-1} \\
&\qquad + \left(1 - \frac{p(p+\alpha-2)}{2(p+\alpha)}\right)\left(pz^{p-1} + \alpha(p+1)z^p + \sum_{k=2}^{\infty}(p+k)a_{p+k}z^{p+k-1}\right).
\end{aligned} \tag{24}$$

Comparing the coefficients of z^{p+1} and z^{p+k} $(k \geq 2)$ of both sides of (24), respectively, we obtain

$$pp_2 + \alpha(p+1)p_1 + p_0(p+2)a_{p+2} = (p+1)(p+2)a_{p+2} + \left(1 - \frac{p(p+\alpha-2)}{2(p+\alpha)}\right)(p+2)a_{p+2} \tag{25}$$

and

$$
\begin{aligned}
&p_0(p+k+1)a_{p+k+1} + pp_{k+1} + \alpha(p+1)p_k + \sum_{l=1}^{k-1}(p+k+1-l)p_l a_{p+k+1-l} \\
&\quad = (p+k)(p+k+1)a_{p+k+1} + \left(1 - \frac{p(p+\alpha-2)}{2(p+\alpha)}\right)(p+k+1)a_{p+k+1} \quad (k \geq 2).
\end{aligned} \tag{26}
$$

By (23) and (25), we readily get (19). Furthermore, we find from (23) and (26) that

$$
\left|a_{p+k+1}\right| \leq \frac{p(p+\alpha+2)}{(p+\alpha)(k+1)(p+k+1)}\left[p + \alpha(p+1) + \sum_{l=1}^{k-1}(p+k+1-l)\left|a_{p+k+1-l}\right|\right]. \tag{27}
$$

Now, we consider the sequence $\{A_{p+k}\}_{k=2}^{\infty}$ defined in Lemma 1.2. To prove

$$
\left|a_{p+k}\right| \leq A_{p+k} \quad (k \geq 2),
$$

we shall make use of the principle of mathematical induction. By noting that

$$
\left|a_{p+2}\right| \leq A_{p+2} = \frac{p(p+\alpha+2)[p+\alpha(p+1)]}{2(p+2)(p+\alpha)},
$$

moreover, we assume that

$$
\left|a_{p+l}\right| \leq A_{p+l} \quad (l = 2, 3, \ldots, k;\ k \geq 2).
$$

Combining (4), (5) and (27), we get

$$
\begin{aligned}
\left|a_{p+k+1}\right| &\leq \frac{p(p+\alpha+2)}{(p+\alpha)(k+1)(p+k+1)}\left(p + \alpha(p+1) + \sum_{l=1}^{k-1}(p+k+1-l)\left|a_{p+k+1-l}\right|\right) \\
&\leqq \frac{p(p+\alpha+2)}{(p+\alpha)(k+1)(p+k+1)}\left(p + \alpha(p+1) + \sum_{l=1}^{k-1}(p+k+1-l)A_{p+k+1-l}\right) \\
&= A_{p+k+1} \quad (k \geq 2).
\end{aligned}
$$

Thus, by the principle of mathematical induction, we see that

$$
\left|a_{p+k}\right| \leq A_{p+k} \quad (k \geq 2) \tag{28}
$$

as desired. By (6) and (28), we deduce that the assertion (20) of Theorem 2.4 holds. □

Theorem 2.5. *If $f \in \mathcal{A}_{p,\alpha}$ with $0 \leq \alpha < \beta$ satisfies the condition*

$$
\sum_{k=2}^{\infty}\left(\frac{2p}{p+\alpha} + p + 2k\right)(p+k)\left|a_{p+k}\right| \leq p^2\left(\frac{2}{p+\alpha} + 1\right) - \alpha(p+1)\left(\frac{4p+2\alpha}{p+\alpha} + p\right), \tag{29}
$$

where

$$
\beta := \frac{p}{2(p+1)}\left(\sqrt{p^2+8p+8} - p - 2\right),
$$

then $f \in \mathcal{H}(p, \alpha)$.

Proof. To prove $f \in \mathcal{H}(p,\alpha)$, it suffices to show that

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right| < \left|1+\frac{zf''(z)}{f'(z)}-\left(\frac{p(p+\alpha-2)}{p+\alpha}-p\right)\right| = \left|1+\frac{zf''(z)}{f'(z)}+\frac{2p}{p+\alpha}\right|.$$

By (29), we find that

$$(p+1)\alpha+\sum_{k=2}^{\infty}k(p+k)\left|a_{p+k}\right| \leq p^2\left(\frac{2}{p+\alpha}+1\right)-\left(\frac{3p+\alpha}{p+\alpha}+p\right)(p+1)\alpha-\sum_{k=2}^{\infty}\left(\frac{2p}{p+\alpha}+p+k\right)(p+k)\left|a_{p+k}\right|. \quad (30)$$

Now, by the maximum modulus principle, we deduce from (1) and (30) that

$$\begin{aligned}
&\left|\frac{1+\frac{zf''(z)}{f'(z)}-p}{1+\frac{zf''(z)}{f'(z)}+\frac{2p}{p+\alpha}}\right| \\
=&\left|\frac{(p+1)\alpha z^p+\sum\limits_{k=2}^{\infty}k(p+k)a_{p+k}z^{p+k-1}}{p^2\left(\frac{2}{p+\alpha}+1\right)z^{p-1}+\left(\frac{3p+\alpha}{p+\alpha}+p\right)(p+1)\alpha z^p+\sum\limits_{k=2}^{\infty}\left(\frac{2p}{p+\alpha}+p+k\right)(p+k)a_{p+k}z^{p+k-1}}\right| \\
<&\frac{(p+1)\alpha+\sum\limits_{k=2}^{\infty}k(p+k)\left|a_{p+k}\right|}{p^2\left(\frac{2}{p+\alpha}+1\right)-\left(\frac{3p+\alpha}{p+\alpha}+p\right)(p+1)\alpha-\sum\limits_{k=2}^{\infty}\left(\frac{2p}{p+\alpha}+p+k\right)(p+k)\left|a_{p+k}\right|} \\
\leq&1.
\end{aligned}$$

This completes the proof of Theorem 2.5. □

In view of the earlier works (based upon the familiar concept of a neighborhood of analytic functions) by Goodman [2] and Ruscheweyh [11], we begin by introducing here the δ-neighborhood of a function $f \in \mathcal{A}_{p,\alpha}$ of the form (1) by means of the following definition:

$$\begin{aligned}
\mathcal{N}_\delta(f) := \Bigg\{&g \in \mathcal{A}_{p,\alpha} : g(z) = z^p+\alpha z^{p+1}+\sum_{k=2}^{\infty}b_{p+k}z^{p+k} \text{ and} \\
&\sum_{k=1}^{\infty}\frac{\left(\frac{2p}{p+\alpha}+p+2k\right)(p+k)}{p^2\left(\frac{2}{p+\alpha}+1\right)-\alpha(p+1)\left(\frac{4p+2\alpha}{p+\alpha}+p\right)}\left|a_{p+k}-b_{p+k}\right| \leq \delta \quad (\delta \geq 0;\ 0 \leq \alpha < \beta)\Bigg\}.
\end{aligned} \quad (31)$$

Theorem 2.6. *If $f \in \mathcal{H}(p,\alpha)$ satisfies the condition*

$$\frac{f(z)+\varepsilon z^p}{1+\varepsilon} \in \mathcal{H}(p,\alpha) \quad (\varepsilon \in \mathbb{C},\ |\varepsilon| < \delta;\ \delta > 0), \quad (32)$$

then

$$\mathcal{N}_\delta(f) \subset \mathcal{H}(p,\alpha). \quad (33)$$

Proof. By observing that the condition (3) is equivalent to

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}-p}{1+\frac{zf''(z)}{f'(z)}+\frac{2p}{p+\alpha}}\right| < 1 \quad (z \in \mathbb{U}), \quad (34)$$

we readily find from (34) that a function $g \in \mathcal{H}(p,\alpha)$ if and only if

$$\frac{1+\frac{zg''(z)}{g'(z)}-p}{1+\frac{zg''(z)}{g'(z)}+\frac{2p}{p+\alpha}} \neq \sigma \quad (z \in \mathbb{U};\ \sigma \in \mathbb{C},\ |\sigma|=1), \tag{35}$$

which can be rewritten as follows:

$$\frac{(g*h)(z)}{z^p} \neq 0 \quad (z \in \mathbb{U}), \tag{36}$$

where

$$h(z) = z^p + \alpha z^{p+1} + \sum_{k=2}^{\infty} c_{p+k} z^{p+k} \quad (0 \leq \alpha \leq p;\ z \in \mathbb{U}) \tag{37}$$

with

$$c_{p+k} := \frac{k(p+k) - \left(\frac{2p}{p+\alpha}+p+k\right)(p+k)\sigma}{\left[p^2\left(\frac{2}{p+\alpha}+1\right) - \alpha(p+1)\left(\frac{4p+2\alpha}{p+\alpha}+p\right)\right]\sigma}.$$

By (37), we find that

$$\begin{aligned} \left|c_{p+k}\right| &= \left|\frac{k(p+k) - \left(\frac{2p}{p+\alpha}+p+k\right)(p+k)\sigma}{\left[p^2\left(\frac{2}{p+\alpha}+1\right) - \alpha(p+1)\left(\frac{4p+2\alpha}{p+\alpha}+p\right)\right]\sigma}\right| \\ &\leq \frac{k(p+k) + \left(\frac{2p}{p+\alpha}+p+k\right)(p+k)\,|\sigma|}{\left[p^2\left(\frac{2}{p+\alpha}+1\right) - \alpha(p+1)\left(\frac{4p+2\alpha}{p+\alpha}+p\right)\right]|\sigma|} = \frac{\left(\frac{2p}{p+\alpha}+p+2k\right)(p+k)}{p^2\left(\frac{2}{p+\alpha}+1\right) - \alpha(p+1)\left(\frac{4p+2\alpha}{p+\alpha}+p\right)} \quad (|\sigma|=1). \end{aligned} \tag{38}$$

If $f \in \mathcal{A}_{p,\alpha}$ satisfies the condition (32), we deduce from (36) that

$$\frac{(f*h)(z)}{z^p} \neq -\varepsilon \quad (|\varepsilon| < \delta;\ \delta > 0),$$

or equivalently,

$$\left|\frac{(f*h)(z)}{z^p}\right| \geq \delta \quad (\delta > 0;\ z \in \mathbb{U}). \tag{39}$$

We now assume that

$$q(z) = z^p + \alpha z^{p+1} + \sum_{k=2}^{\infty} d_{p+k} z^{p+k} \in \mathcal{N}_\delta(f).$$

It follows from (31) that

$$\begin{aligned} \left|\frac{((q-f)*h)(z)}{z^p}\right| &= \left|\sum_{k=2}^{\infty}(d_{p+k}-a_{p+k})c_{p+k}z^k\right| \\ &\leq |z| \sum_{k=1}^{\infty} \frac{\left(\frac{2p}{p+\alpha}+p+2k\right)(p+k)}{p^2\left(\frac{2}{p+\alpha}+1\right) - \alpha(p+1)\left(\frac{4p+2\alpha}{p+\alpha}+p\right)} \left|d_{p+k}-a_{p+k}\right| \\ &< \delta. \end{aligned} \tag{40}$$

By (38) and (40), we find that

$$\left|\frac{(q*h)(z)}{z^p}\right| = \left|\frac{([f+(q-f)]*h)(z)}{z^p}\right| \geq \left|\frac{(f*h)(z)}{z^p}\right| - \left|\frac{((q-f)*h)(z)}{z^p}\right| > 0, \tag{41}$$

it follows that

$$\left|\frac{(q*h)(z)}{z^p}\right| \neq 0 \quad (z \in \mathbb{U}).$$

Thus, we deduce that

$$q(z) \in \mathcal{N}_\delta(f) \subset \mathcal{H}(p, \alpha).$$

The proof of Theorem 2.6 is completed. □

Finally, we derive the partial sums of functions f belong to the class $\mathcal{H}(p) := \mathcal{H}(p, 0)$.

Theorem 2.7. *Let $f \in \mathcal{A}_{p,0}$ be given by* (1) *and define the partial sums $f_n(z)$ of f by*

$$f_n(z) = z^p + \sum_{k=2}^{n} a_{p+k} z^{p+k} \quad (n \in \mathbb{N};\ n \geq 2). \tag{42}$$

If

$$\sum_{k=2}^{\infty} \frac{(p+2k+2)\,(p+k)}{p\,(p+2)} \left|a_{p+k}\right| \leq 1, \tag{43}$$

then

1. $f \in \mathcal{H}(p)$;
2. $$\Re\left(\frac{f(z)}{f_n(z)}\right) \geq \frac{[p+2(n+1)+2]\,(p+n+1) - p\,(p+2)}{[p+2(n+1)+2]\,(p+n+1)} \tag{44}$$

   and

   $$\Re\left(\frac{f_n(z)}{f(z)}\right) \geq \frac{[p+2(n+1)+2]\,(p+n+1)}{[p+2(n+1)+2]\,(p+n+1) + p\,(p+2)}. \tag{45}$$

The bounds of (44) *and* (45) *are sharp.*

Proof. (i) Assume that $f_1(z) = z^p \in \mathcal{H}(p)$, it follows that

$$\frac{f_1(z) + \varepsilon z^p}{1+\varepsilon} = z^p \in \mathcal{H}(p).$$

By (43), we easily find that

$$\sum_{k=2}^{\infty} \frac{(p+2k+2)\,(p+k)}{p\,(p+2)} \left|a_{p+k} - 0\right| \leq 1, \tag{46}$$

which implies that $f \in \mathcal{N}_1(z)$. By Theorem 2.6, we conclude that

$$f \in \mathcal{N}_1(z) \subset \mathcal{H}(p).$$

(ii) It is easy to check that

$$\frac{[p+2(n+1)+2](p+n+1)}{p(p+2)} > \frac{(p+2n+2)(p+n)}{p(p+2)} > 1 \quad (n \geq 2).$$

Thus, we know that

$$\sum_{k=2}^{n} |a_{p+k}| + \frac{[p+2(n+1)+2](p+n+1)}{p(p+2)} \sum_{k=n+1}^{\infty} |a_{p+k}| \leq \sum_{k=2}^{\infty} \frac{(p+2k+2)(p+k)}{p(p+2)} |a_{p+k}| \leq 1. \tag{47}$$

Now, we suppose that

$$\begin{aligned}\phi(z) :=& \frac{[p+2(n+1)+2](p+n+1)}{p(p+2)} \left\{ \frac{f(z)}{f_n(z)} - \frac{[p+2(n+1)+2](p+n+1) - p(p+2)}{[p+2(n+1)+2](p+n+1)} \right\} \\ =& 1 + \frac{\frac{[p+2(n+1)+2](p+n+1)}{p(p+2)} \sum_{k=n+1}^{\infty} a_{p+k} z^k}{1 + \sum_{k=2}^{n} a_{p+k} z^k}.\end{aligned} \tag{48}$$

It follows from (47) and (48) that

$$\left| \frac{\phi(z)-1}{\phi(z)+1} \right| \leq \frac{\frac{[p+2(n+1)+2](p+n+1)}{p(p+2)} \sum_{k=n+1}^{\infty} |a_{p+k}|}{2 - 2\sum_{k=2}^{n} |a_{p+k}| - \frac{[p+2(n+1)+2](p+n+1)}{p(p+2)} \sum_{k=n+1}^{\infty} |a_{p+k}|} \leq 1 \quad (z \in \mathbb{U}),$$

which implies that

$$\Re(\phi(z)) \geq 0 \quad (z \in \mathbb{U}). \tag{49}$$

By (48) and (49), we deduce that the assertion (44) of Theorem 2.7 holds.

Furthermore, if we take

$$f(z) = z^p + \frac{p(p+2)}{[p+2(n+1)+2](p+n+1)} z^{p+n+1} \quad (n \geq 1), \tag{50}$$

then for $z = re^{i\pi/(n+1)}$, we get

$$\frac{f(z)}{f_n(z)} = 1 + \frac{p(p+2)}{[p+2(n+1)+2](p+n+1)} z^{n+1} = \frac{[p+2(n+1)+2](p+n+1) - p(p+2)}{[p+2(n+1)+2](p+n+1)} \quad (r \to 1^-),$$

which shows that the bound in (44) is the best possible for each $n \in \mathbb{N}$.

Similarly, we suppose that

$$\begin{aligned}\psi(z) :=& \frac{[p+2(n+1)+2](p+n+1) + p(p+2)}{p(p+2)} \left\{ \frac{f_n(z)}{f(z)} - \frac{[p+2(n+1)+2](p+n+1)}{[p+2(n+1)+2](p+n+1) + p(p+2)} \right\} \\ =& 1 - \frac{\frac{[p+2(n+1)+2](p+n+1)+p(p+2)}{p(p+2)} \sum_{k=n+1}^{\infty} a_{p+k} z^k}{1 + \sum_{k=2}^{\infty} a_{p+k} z^k}.\end{aligned} \tag{51}$$

By (47) and (51), we deduce that

$$\left|\frac{\psi(z)-1}{\psi(z)+1}\right| \leq \frac{\frac{\left[p+2(n+1)+2\right](p+n+1)+p(p+2)}{p(p+2)} \sum_{k=n+1}^{\infty} \left|a_{p+k}\right|}{2-2\sum_{k=2}^{n}\left|a_{p+k}\right| - \frac{\left[p+2(n+1)+2\right](p+n+1)-p(p+2)}{p(p+2)} \sum_{k=n+1}^{\infty}\left|a_{p+k}\right|} \leq 1 \quad (z \in \mathbb{U}),$$

which shows that

$$\Re\Big(\psi(z)\Big) \geq 0 \quad (z \in \mathbb{U}). \tag{52}$$

By (51) and (52), we readily get the assertion (45) of Theorem 2.7. The bound in (45) is sharp for the extremal function f given by (50). □

Remark 2.8. When $0 < \alpha \leq p$, results on the partial sums of class $\mathcal{H}(p, \alpha)$ tend to be far more complex. Interested readers are encouraged to explore this further.

Conflicts of interest. The authors declare that they have no conflict of interest.
Data availability statement. Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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