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Self-adjoint block multivalued linear operator matrix and application

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Abstract. It is shown that a self-adjoint block matrix multivalued linear operator (linear relation) is still self-adjoint under diagonally dominant block matrix perturbations. The results obtained generalize the corresponding one for linear operators and relax some of the required conditions. Then, we give some perturbation theorems for matrix linear relations in Banach spaces based on a resolvent approach. Furthermore, we apply the obtained results to investigate the existence of solutions for a system of degenerate partial differential equations.

1. Introduction

Perturbation theory are one of the main topics in both pure and applied mathematics. Since the self-adjoint operators form the most important class of linear operators that appear in applications, the perturbation of self-adjoint operators and the stability of self-adjointness have received lots of attention. In particular, Kato first studied the stability of self-adjointness of closed symmetric operators and showed that the self-adjointness is preserved under relatively bounded perturbations with relative bounds less than [12] . Followed by this work, Behncke, Kissin, etc. extended this work about stability of self-adjointness to results about the stability of the deficiency indices [5, 13]. With further research of operator theory, more and more multi-valued operators and nondensely defined operators have been found. For example, the operators generated by those linear continuous Hamiltonian systems, which do not satisfy the definiteness conditions, and general linear discrete Hamiltonian systems may be multi-valued or not densely defined in their corresponding Hilbert spaces (cf. [16, 17]). So the classical perturbation theory of linear operators is not available in this case. Motivated by this need, von Neumann [15] first introduced linear relations into functional analysis, and then Arens [12] and many other scholars further studied and developed the fundamental theory of linear relations. A liner relation is also called a linear subspace (briefly, subspace). Since the theory of linear relations was established, the related perturbation problems have attracted extensive attention of scholars and some good results have been obtained [2, 6, 9, 18, 19]. It is well known that the self-adjoint relations are the most important class of linear relations that appear in applications. To the best of our knowledge, there seem a few results about the stability of self-adjointness of linear relation under perturbations [18, 19]. But it has not been specifically and thoroughly studied. In the present paper, we shall concentrate on the stability of self-adjointness of linear relations. The results given in the

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present paper not only weaken the conditions of Theorem 4.1 in [18], but also cover the result obtained in [19, Theorem 5.2]. The second aim of this paper is motivated by the need raised during the study of the existence of solutions for some degenerate partial differential system of equations. The results given constitute an abstract theoretical framework of the resolvent approach. It is a question of proving some estimates of the solvent of the linear relations involved allowing to define a semi-group giving the solution of the system.
The rest of the paper is organized as follow. In Section 2, some basic concepts and useful fundamental results about linear relations are introduced. In Section 3, we first show that the deficiency indices of Hermitian relations are invariant under relatively bounded perturbations with relative bounds less than 1 and we display some definitions and properties on adjoint linear relations. In Section 4, as a consequence, stability of self-adjointness of Hermitian relations under bounded and relatively bounded perturbations is obtained. In Section 5, we describe the resolvent approach of perturbing a multivalued linear operator by another possibly multivalued linear operator and satisfying some estimate. In Section 6, we give an application.

2. Preliminaries and auxiliary results

In this section, we shall recall some basic concepts and introduce some fundamental results about linear relations.
Let X and Y be two infinite-dimensional Banach spaces over the some fields $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A multivalued linear operator (or a linear relation) T from X to Y is a mapping from a subspace

$$\mathcal{D}(T) := \{x \in X : Tx \neq \emptyset\}$$

of X, called the domain of T, into $\mathcal{P}(Y)\backslash\{\emptyset\}$ (the collection of non-empty subsets of Y) such that $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all non-zero scalars $\alpha, \beta \in \mathbb{C}$ and $x, y \in \mathcal{D}(T)$. In this section we denote by $\mathcal{L}(X,Y)$ a space of all bounded linear operators from X into Y, by $\mathcal{LR}(X,Y)$ a space of all multivalued linear operators from X into Y. If $X = Y$, then denote $\mathcal{L}(X) := \mathcal{L}(X,Y)$ and $\mathcal{LR}(X) := \mathcal{LR}(X,Y)$.
A linear relation is uniquely determined by its graph, $G(T)$, which is defined by

$$G(T) := \{(x,y) \in X \times Y : x \in \mathcal{D}(T) \ \ and \ \ y \in Tx\}.$$

The inverse of T is the linear relation, T^{-1} defined by

$$G(T^{-1}) := \{(y,x) \in Y \times X : (x,y) \in G(T)\}.$$

The subspace $\mathcal{N}(T) := T^{-1}(0)$ is called the null space of T, and T is called injective if $\mathcal{N}(T) = \{0\}$, that is, if T^{-1} is a single valued linear operator. The range of T is the subspace $\mathcal{R}(T) := T(\mathcal{D}(T))$, and T is called surjective if $\mathcal{R}(T) = Y$. When T is injective and surjective, we say that T is bijective. It is clear that $\mathcal{D}(T^{-1}) = \mathcal{R}(T)$ and $\mathcal{R}(T^{-1}) = \mathcal{D}(T)$.
A linear relation T is said to be closed if its graph $G(T)$ is a closed subspace of $X \times Y$. We denote by $\mathcal{CR}(X,Y)$ the class of all closed linear relations on X into Y and abbreviate $\mathcal{CR}(X,X)$ to $\mathcal{CR}(X)$. The closure of T, denoted by $\overline{T}$, is defined in terms of its graph $G(\overline{T}) := \overline{G(T)}$. We say that T is closable if $\overline{T}$ is an extension of T. A relation $T \in \mathcal{LR}(X,Y)$ from one topological space to another is said to be continuous if for any neighborhood $\mathcal{V} \subset \mathcal{R}(T)$, the inverse image $T^{-1}(\mathcal{V})$ is a neighborhood in $\mathcal{D}(T)$. Let $T \in \mathcal{LR}(X,Y)$. The norm of Tu is denoted as follows

$$\|Tu\| := \inf\{\|y\|_Y : y \in Tu\}, \quad \forall\ u \in \mathcal{D}(T),$$

and

$$\|T\|_{\mathcal{LR}(X,Y)} := \sup_{\|u\|_X \leq 1} \|Tu\|_Y.$$

We note that T is continuous if and only if $\|T\| < +\infty$. We say that T is bounded if it is continuous and everywhere defined.

Proposition 2.1. *Let $S \in \mathcal{LR}(X)$ satisfying $\mathcal{D}(S) = X$ and $\|S\| < 1$, and let $S(0)$ be closed. Then,*
(i) *$(I - S)^{-1} \in \mathcal{LR}(X)$ is everywhere defined and continuous.*
(ii) $\|(I - S)^{-1}\| \leq \dfrac{1}{1 - \|S\|}$.

Proof.
(i) See [7, Proposition 2.4].
(ii) We have $(I - S)^{-1}(S - I) = -I$. Hence, by [6, Proposition I.4.2] $(I - S)^{-1}S + I = (I - S)^{-1}$. So, by [6, Corollary II.3.13]

$$\|(I - S)^{-1}.\| \leq 1 + \|(I - S)^{-1}\|\|S\|.$$

Thus, $\|(I - S)^{-1}\| \leq \dfrac{1}{1 - \|S\|}$. Q.E.D.

Definition 2.1. *Given an operator $T \in \mathcal{LR}(X)$. By $\rho(T) \subset \mathbb{C}$ we denote the resolvent set of T, i.e., $\lambda \in \rho(T)$ if and only if the inverse operator $(\lambda I - T)^{-1}$ is a linear bounded single-valued operator defined on the whole space X. In this case we denote*

$$R(\lambda, T) := (\lambda I - T)^{-1}$$

and call it the resolvent of T.

Lemma 2.1. *Let X and Y be two Banach spaces, $A, A' \in \mathcal{LR}(X)$, $B, B' \in \mathcal{LR}(Y)$, $C, C' \in \mathcal{LR}(Y, X)$ and $D, D' \in \mathcal{LR}(X, Y)$. Then*
(i)

$$\begin{pmatrix} A & C \\ D & B \end{pmatrix}\begin{pmatrix} A' & C' \\ D' & B' \end{pmatrix} \subseteq \begin{pmatrix} AA' + CD' & AC' + CB' \\ DA' + BD' & DC' + BB' \end{pmatrix}$$

(ii) *Moreover, if $A = B = C' = D' = 0$, then we have the equality*

$$\begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix}\begin{pmatrix} A' & 0 \\ 0 & B' \end{pmatrix} = \begin{pmatrix} 0 & CB' \\ DA' & 0 \end{pmatrix}.$$

Proof. (i) [17] Proposition 14 (i).
(ii) Let

$$\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) \in G\left(\begin{pmatrix} 0 & CB' \\ DA' & 0 \end{pmatrix}\right).$$

Then $y_1 \in CB'x_2$ and $y_2 \in DA'x_1$. Hence there exist $z_1 \in A'x_1$ and $z_2 \in B'x_2$ such that $y_1 \in Cz_2$ and $y_2 \in Dz_1$. Hence

$$\left(\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) \in G\left(\begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix}\right) \text{ and } \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) \in G\left(\begin{pmatrix} A' & 0 \\ 0 & B' \end{pmatrix}\right).$$

Thus,

$$\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) \in G\left(\begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix}.\begin{pmatrix} A' & 0 \\ 0 & B' \end{pmatrix}\right).$$

Therefore,

$$\begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix}\begin{pmatrix} A' & 0 \\ 0 & B' \end{pmatrix} = \begin{pmatrix} 0 & CB' \\ DA' & 0 \end{pmatrix}.$$

Q.E.D.

Let X and Y be two Banach spaces. We define on the product space $X \oplus Y$ a norm by letting

$$\|(x, y)\| = \sqrt{\|x\|^2 + \|y\|^2}.$$

In the Banach space $X \oplus Y$, we consider the linear relation $\mathcal{L}$ given by the block matrix of linear relation

$$\mathcal{L} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A \in \mathcal{LR}(X)$, $B \in \mathcal{LR}(Y, X)$, $C \in \mathcal{LR}(X, Y)$ and $D \in \mathcal{LR}(Y)$ are densely defined closable linear relations with its natural domain

$$\mathcal{D}(\mathcal{L}) := \Big(\mathcal{D}(A) \cap \mathcal{D}(C)\Big) \oplus \Big(\mathcal{D}(B) \cap \mathcal{D}(D)\Big)$$

also densely defined, but not necessarily closed. The graph of $\mathcal{L}$ is defined by
$G(\mathcal{L}) := \Big\{\Big((x_1, x_2), (y_1, y_2)\Big) : (x_1, x_2) \in \mathcal{D}(\mathcal{L}),\ y_1 \in Ax_1 + Bx_2 \text{ and } y_2 \in Cx_1 + Dx_2\Big\}.$

Lemma 2.2. *Let X and Y be two Banach spaces, $A \in \mathcal{LR}(X)$, $B \in \mathcal{LR}(Y, X)$, $C \in \mathcal{LR}(X, Y)$ and $D \in \mathcal{LR}(Y)$. Then, for all $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathcal{D}(\mathcal{L})$, we have*
(i) $\|\mathcal{L}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\| \leq \|Av_1\| + \|Bv_2\| + \|Cv_1\| + \|Dv_2\|$.
(ii) *Moreover, if $A = D = 0$, then we have*

$$\|\mathcal{L}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\| = \sqrt{\|Cv_1\|^2 + \|Bv_2\|^2}$$

and if $C = B = 0$, and A, D are continuous, then

$$\|\mathcal{L}\| \leq \sqrt{\|A\|^2 + \|D\|^2}.$$

Proof. (i) By definition, $\|\mathcal{L}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\| = \inf\{\|\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\|_{X\oplus Y} : \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathcal{L}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\}$. Let $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathcal{L}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. Then there exist $y_{1,1} \in Av_1$, $y_{1,2} \in Bv_2$, $y_{2,1} \in Cv_1$ and $y_{2,2} \in Dv_2$ such that $y_1 = y_{1,1} + y_{1,2}$ and $y_2 = y_{2,1} + y_{2,2}$. We have

$$\begin{aligned} \sqrt{\|y_1\|^2 + \|y_2\|^2} &\leq \sqrt{(\|y_{1,1}\| + \|y_{1,2}\|)^2 + (\|y_{2,1}\| + \|y_{2,2}\|)^2} \\ &\leq \|y_{1,1}\| + \|y_{1,2}\| + \|y_{2,1}\| + \|y_{2,2}\|. \end{aligned}$$

Thus,

$$\begin{aligned} \left\|\mathcal{L}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right\| &\leq \|y_{1,1}\| + \|y_{1,2}\| + \|y_{2,1}\| + \|y_{2,2}\| \\ &\leq \|Av_1\| + \|Bv_2\| + \|Cv_1\| + \|Dv_2\|. \end{aligned}$$

(ii) If $A = D = 0$ then

$$\|\mathcal{L}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\| = \inf\{\|\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\|_{X\oplus Y} :\ y_1 \in Bv_2 \text{ and } y_2 \in Cv_1\}.$$

But, $\|Bv_2\| \leq \|y_1\|$, for all $y_1 \in Bv_2$ and $\|Cv_1\| \leq \|y_2\|$, for all $y_2 \in Cv_1$. Thus,

$$\sqrt{\|Cv_1\|^2 + \|Bv_2\|^2} \leq \sqrt{\|y_1\|^2 + \|y_2\|^2},\ \forall y_1 \in Bv_2 \text{ and } y_2 \in Cv_1.$$

This gives the required result. Now, if $C = B = 0$, then by the same way we get

$$\left\|\mathcal{L}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right\| = \sqrt{\|Av_1\|^2 + \|Dv_2\|^2} \leq \sqrt{\|A\|^2 + \|D\|^2}\sqrt{\|v_1\|^2 + \|v_2\|^2}.$$

Hence,

$$\|\mathcal{L}\| \leq \sqrt{\|A\|^2 + \|D\|^2}.$$

Q.E.D.

Definition 2.2. [6, Definition, VII.2.1] *Let $S, T \in \mathcal{LR}(X, Y)$. S is said to be T-bounded if $\mathcal{D}(T) \subset \mathcal{D}(S)$, and there exist non-negative constants a, and b, such that*

$$\|Sx\| \leq a\|x\| + b\|Tx\| \quad \text{for all } x \in \mathcal{D}(T).$$

In that case, the infinimum of all the number b is called the T-bound of S.

Lemma 2.3. [3, Lemma 3.1] *Let $T \in \mathcal{LR}(X, Y)$, $S \in \mathcal{LR}(X, Y)$ be T-bounded with T-bound $\delta < 1$ satisfies $S(0) \subset T(0)$ and $\mathcal{D}(T) \subset \mathcal{D}(S)$. Then, $T \in \mathcal{CR}(X, Y)$ if, and only if, $T + S \in \mathcal{CR}(X, Y)$.*

Definition 2.3. [3, Definition, 2.1] *The block matrix linear relation $\mathcal{L} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is called diagonally dominant if C is A-bounded with A-bound a real b_1 and B is D-bounded with D-bound a real b_2. In this case the real $r = \max(b_1, b_2)$ is called the order of the domination.*

Theorem 2.1. [3, Theorem, 3.1] *The block matrix linear relation $\mathcal{L}$ is diagonally dominant of order < 1 such that $B(0) \subset A(0)$ and $C(0) \subset D(0)$. Then,*
$\mathcal{L}$ is closable if, and only if, A and D are closable.
In this case, the closure of $\mathcal{L}$ is given by,

$$\overline{\mathcal{L}} = \begin{pmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{pmatrix}.$$

Corollary 2.1. [3, Corollary, 3.1] *The block matrix linear relation $\mathcal{L}$ is closed if $\mathcal{L}$ is diagonally dominant of order < 1, A and D are closed, $B(0) \subset A(0)$ and $C(0) \subset D(0)$.*

2.1. Adjointness of multivalued linear operator

In this subsection, we display some definitions and properties on adjoint linear relations.

Let X be a normed linear space. Then, $X^{'}$ denotes the dual space of X, i.e., the space of all the continuous linear functionals x' defined on X, with the norm

$$\|x'\| = \inf\{\lambda \text{ such that } |x'x| \leq \lambda\|x\| \text{ for all } x \in X\}.$$

If $K \subset X$, and $L \subset X^{'}$, we shall adopt the following notation:

$$\begin{aligned} K^{\perp} &= \{x' \in X^{'} \text{ such that } x'(x) = 0 \text{ for all } x \in K\} \\ L^{\top} &= \{x \in X \text{ such that } x'(x) = 0 \text{ for all } x' \in L\}. \end{aligned}$$

Clearly, $K^{\perp}$ and $L^{\top}$ are closed linear subspaces of $X^{'}$ and X, respectively.
Let $T \in \mathcal{LR}(X)$. The adjoint of T, which is T', is defined by

$$G(T') = G(-T^{-1})^{\perp} \subset X^{'} \times X^{'}$$

where $\Big\langle (y, x), (y', x') \Big\rangle = \langle x, x' \rangle + \langle y, y' \rangle$. This means that

$$(y', x') \in G(T') \text{ if, and only if, } y'y - x'x = 0 \text{ for all } (x, y) \in G(T).$$

Similarly, we have $y'y = x'x$ for all $y \in Tx$, $x \in \mathcal{D}(T)$. Hence, $x' \in T'y$ if, and only if, $y'Tx = x'x$ for all $x \in \mathcal{D}(T)$.

Proposition 2.2. [6, Proposition III.1.3] *Let $T \in \mathcal{LR}(X)$.*

(a) $(\overline{T})' = T'$.

(b) $(T')^{-1} = (T^{-1})'$

(c) $(\lambda T)' = \lambda T'$, $\lambda \neq 0$.

Proposition 2.3. [6, Proposition III.1.4] *Let $T \in \mathcal{LR}(X)$.*

(a) $\mathcal{N}(T') = \mathcal{R}(T)^{\perp}$.

(b) $T'(0) = \mathcal{D}(T)^{\perp}$.

(c) $\mathcal{N}(\overline{T}) = \mathcal{R}(T')^{\top}$.

(d) $\overline{T}(0) = \mathcal{D}(T')^{\top}$.

Proposition 2.4. [6, Proposition III.1.5] *Let $T, S \in \mathcal{LR}(X)$.*

(a) $G(S' + T') \subset G((S + T)')$.

(b) If $\mathcal{D}(T) \subset \mathcal{D}(S)$, and S is continuous, then

$$S' + T' = (S + T)'.$$

(c) $(S + T)'$ is an extension of $S' + T'$ if, and only if, $(\mathcal{D}(S) \cap \mathcal{D}(T))^{\perp} = \mathcal{D}(S)^{\perp} + \mathcal{D}(T)^{\perp}$.

Proposition 2.5. [6, Proposition III.1.6] *Let $T \in \mathcal{LR}(X)$ and Let $S \in \mathcal{LR}(X)$. Then,*

$$G(T'S') \subset G((ST)').$$

Furthermore, if either
(i) $\mathcal{R}(T') = X'$ and $\mathcal{D}(S) \subset \mathcal{R}(T)$, or
(ii) $\mathcal{D}(S') = X'$ and $\mathcal{R}(T) \subset \mathcal{D}(S)$.
Then,

$$(ST)' = T'S'.$$

In the remainder of this section we assume that X is a complex Hilbert space with inner product $< .,. >$. Then, the adjoint of $T \in \mathcal{LR}(X)$ is defined by:

$$T^{'} = \{(y, g) \in X : < g, x >=< y, f > \ \ for\ all\ (x, f) \in T\}.$$

$T \in \mathcal{LR}(X)$ is called a Hermitian relation if $T \subset T^{'}$, and it is called a self-adjoint relation if $T = T^{'}$.

Lemma 2.4. *Let $T \in \mathcal{LR}(X)$. Then T is Hermitian if and only if for all $u, v \in D(T)$, and for all $x \in Tu$, $y \in Tv$ we have $< x, v >=< u, y >$.*

Proof. Let $u, v \in D(T)$, $x \in Tu$, and $y \in Tv$. We have

$$\begin{aligned}(u, x) &\in G(T) \subset G(T');\\ (v, y) &\in G(T) \subset G(T').\end{aligned}$$

As, $(u, x) \in G(T')$ and $(v, y) \in G(T)$, then by the definition of the adjoint we get $< x, v >=< u, y >$ as desired. For the reverse implication, let $(x, y) \in G(T)$ and let $(z, t) \in G(T)$. Thus, $y \in Tx$ and $t \in Tz$. So, by hypothesis we have

$$< y, z >=< x, t > .$$

Hence, $(x, y) \in G(T')$.

Q.E.D.

Lemma 2.5. *Let $T \in \mathcal{LR}(X)$. Then T is Hermitian if and only if for all $x \in D(T)$ and $y \in Tx$ we have $< y, x >$ is real.*

Proof. Let $x \in D(T)$ and $y \in Tx$. We claim that $< y, x >\in \mathbb{R}$. Indeed, as $(x, y) \in G(T)$ and T is Hermitian, then $(x, y) \in G(T')$. So, $< y, x >=< x, y >$. Then, $< y, x >= \overline{< y, x >}$. Therefore, $< y, x >$ is real. For the reverse implication, let $u, v \in D(T)$ and x, y be such that $x \in Tu$ and $y \in Tv$. We shows that $< x, v >=< u, y >$. To do this, we make the following calculation.

$$\begin{aligned} < x, v > - < u, y > &= < x, v - u > + < x, u > - < u - v, y > - < v, y >; \\ &= - < x - y, u - v > + \overline{< v - u, y >} + < v - u, y > + < x, u > \\ &\quad - < v, y > . \end{aligned}$$

But, as, $x \in Tu$, $y \in Tv$ and $x - y \in t(u - v)$, then by hypothesis, we get that $< x, v > - < u, y >$ is real. On another hand we note that the previous calculation remains true if we replace (x, u) by (ix, iu). This shows that

$$< ix, v > - < iu, y >= i(< x, v > - < u, y >)$$

is also real. Hence, we get that $< x, v > - < u, y >= 0$ as claimed. Now, using Lemma 2.4, we get that T is Hermitian. Q.E.D.

Lemma 2.6. [20, Lemma 11] *Let T be a closed and Hermitian linear relation. then, for any $x \in \mathcal{D}(T)$ and any $z \in \mathbb{C}$ with $z = a + ib$ and $a, b \in \mathbb{R}$,*

$$\|(T - zI)(x)\| = \|(T - aI)(x)\| + b\|x\|.$$

Definition 2.4. [17, Definition, 2.3] *Let $T \in \mathcal{LR}(X)$. The subspace $\mathcal{R}(T - \lambda I)^{\perp}$ is called the deficiency space of T and λ, and the number $d_{\lambda}(T) := \dim(\mathcal{R}(T - \lambda I)^{\perp})$ is called the deficiency index of T and λ.*

Lemma 2.7. [20, Lemma 9] *Let $T \in \mathcal{CR}(X)$ and satisfy that there exists $c > 0$ such that*

$$\|T(x)\| \geq c\|x\|, \quad x \in \mathcal{D}(T).$$

Let $S \in \mathcal{LR}(X)$ with $\mathcal{D}(T) \subset \mathcal{D}(S)$ and $S(0) \subset T(0)$, and satisfy

$$\|S(x)\| \leq b\|Tx\|, \quad x \in \mathcal{D}(T),$$

for some constant $0 \leq b < 1$. Then, $T + S$ is closed. Moreover,

$$\dim \mathcal{R}(T + S)^{\perp} = \dim \mathcal{R}(T)^{\perp}.$$

2.2. Main Results

Our goal in this section is to find sufficient conditions ensuring that $\mathcal{L}$ is self-adjoint when the diagonal matrix linear relation $\mathcal{T}$ is self-adjoint. We start our investigation by the following theorem which gives sufficient conditions for $\mathcal{L}$ to be Hermitian. Let us define the block matrix linear relation

$$\mathcal{L} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \mathcal{T} := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \text{ and } \mathcal{S} := \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix},$$

where A, B, C and D be closed linear relations with $\mathcal{D}(A) \subset \mathcal{D}(C)$, $\mathcal{D}(D) \subset \mathcal{D}(B)$.

Theorem 2.2. *Let X and Y be two Hilbert spaces. Suppose that $A \in \mathcal{LR}(X)$, $B \in \mathcal{LR}(Y, X)$, $C \in \mathcal{LR}(X, Y)$ and $D \in \mathcal{LR}(Y)$. If A, D are self-adjoint relations and $C = B^*$, then $\mathcal{L}$ is an Hermitian relation matrix.* ⋄

Proof. Let us assume that $\left[\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} z \\ t \end{pmatrix}\right] \in G(\mathcal{L})$. We propose to show that

$$\left\{\left\langle\begin{pmatrix} z \\ t \end{pmatrix}; \begin{pmatrix} x \\ y \end{pmatrix}\right\rangle : \begin{pmatrix} z \\ t \end{pmatrix} \in \mathcal{L}\begin{pmatrix} x \\ y \end{pmatrix}\right\},$$

takes real values. For this purpose, let $\begin{pmatrix} z \\ t \end{pmatrix} \in \begin{pmatrix} Ax + By \\ Cx + Dy \end{pmatrix}$, then there exists $z_1 \in Ax$, $z_2 \in By$, $t_1 \in Cx$ and $t_2 \in Dy$ such that $z = z_1 + z_2$ and $t = t_1 + t_2$. Hence, we have

$$\begin{aligned} \left\langle\begin{pmatrix} z \\ t \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right\rangle &= \langle z, x\rangle + \langle t, y\rangle \\ &= \langle z_1 + z_2, x\rangle + \langle t_1 + t_2, y\rangle \\ &= \langle z_1, x\rangle + \langle z_2, x\rangle + \langle t_1, y\rangle + \langle t_2, y\rangle. \end{aligned}$$

Since $G(C) = G(B^*)$, then $(x, t_1) \in G(B^*)$. So, by the definition of an adjoint relation on Hilbert spaces we obtain

$$\langle t_1, y\rangle = \langle x, z_2\rangle, \text{ for all } (y, z_2) \in G(B).$$

Thus,

$$\left\langle\begin{pmatrix} z \\ t \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right\rangle = \langle z_1, x\rangle + \overline{\langle t_1, y\rangle} + \langle t_1, y\rangle + \langle t_2, y\rangle. \quad (1)$$

Since A and D are self-adjoint relations, then by using (1) and Lemma 2.5, we conclude that

$$\left\{\left\langle\begin{pmatrix} z \\ t \end{pmatrix}; \begin{pmatrix} x \\ y \end{pmatrix}\right\rangle : \begin{pmatrix} z \\ t \end{pmatrix} \in \mathcal{L}\begin{pmatrix} x \\ y \end{pmatrix}\right\} \subset \mathbb{R}.$$

Now, again by Lemma 2.5, we conclude that $\mathcal{L}$ is Hermitian. Q.E.D.

We then recall the following lemma that will be useful in the proof of the main theorem of this section.

Lemma 2.8. [8, Lemma, 2.5] *Let X be a complex Hilbert space. Let A be an Hermitian linear relation. If there is $\lambda \in \mathbb{C}$ such that $\mathcal{R}(A - \lambda I) = \mathcal{R}(A - \overline{\lambda} I) = X$ then, A is self-adjoint.*

We are now able to state and prove the main theorem.

Theorem 2.3. *Let us define the block matrix linear relations*

$$\mathcal{L} := \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \mathcal{T} := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \text{ and } \mathcal{S} := \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix},$$

where A, B, C and D be closed linear relations with $C = B^*$, $\mathcal{D}(A) \subset \mathcal{D}(C)$, $\mathcal{D}(D) \subset \mathcal{D}(B)$, $B(0) \subset A(0)$ and $C(0) \subset D(0)$.
If $\mathcal{L}$ is diagonally dominant of order $b < \frac{1}{\sqrt{2}}$, then $\mathcal{L}$ is self-adjoint whenever $\mathcal{T}$ is self-adjoint. If Moreover, $C^* = B$ then we have the equivalence.

Proof. Since, $\mathcal{L}$ is diagonally dominant of order $b < \frac{1}{\sqrt{2}}$, it follows from Corollary 2.1, that $\mathcal{L}$ is closed. On another hand, there exists a real $a > 0$ such that

$$\begin{aligned} &\|Bg\| \leq a\|g\| + b\|Dg\| && g \in \mathcal{D}(D) \\ &\|Cf\| \leq a\|f\| + b\|Af\|; && f \in \mathcal{D}(A). \end{aligned}$$

It follows that

$$\begin{aligned} &\|Bg\|^2 \leq 2a^2\|g\|^2 + 2b^2\|Dg\|^2; && g \in \mathcal{D}(D) \\ &\|Cf\|^2 \leq 2a^2\|f\|^2 + 2b^2\|Af\|^2; && f \in \mathcal{D}(A). \end{aligned}$$

Hence, we obtain for $z^t = (f, g)^t \in \mathcal{D}(A) \oplus \mathcal{D}(D) = \mathcal{D}(\mathcal{T})$

$$\begin{aligned}
\|\mathcal{S}z\|^2 &= \|Bg\|^2 + \|Cf\|^2 \\
&\leq 2a^2\|g\|^2 + 2b^2\|Dg\|^2 + 2a^2\|f\|^2 + 2b^2\|Af\|^2 \\
&\leq 2a^2 \left\| \begin{pmatrix} f \\ g \end{pmatrix} \right\|^2 + 2b^2(\|Dg\|^2 + \|Af\|^2) \\
&\leq 2a^2\ \|z\|^2 + 2b^2\ \|\mathcal{T}z\|^2 \\
&\leq 2b^2\left(\frac{a^2}{b^2}\|z\|^2 + \|\mathcal{T}z\|^2\right) \\
&\leq \alpha(\beta\|z\|^2 + \|\mathcal{T}z\|^2),
\end{aligned}$$

with $\alpha = 2b^2$ and $\beta = \frac{a^2}{b^2}$. In addition, since $\mathcal{T}$ is closed and Hermitian, by Lemma 2.6, we obtain

$$\|(\mathcal{T} \pm i\beta I)(z)\| = \|\mathcal{T}(z)\| + \beta\|z\|.$$

Therefore,

$$\|\mathcal{S}z\|^2 \leq \alpha(\beta\|z\|^2 + \|\mathcal{T}z\|^2) = \alpha\|(\mathcal{T} \pm i\beta I)(z)\|,$$

$$\|\mathcal{S}z\| \leq \alpha\|(\mathcal{T} \pm i\beta I)(z)\|.$$

Since $0 < \alpha < 1$, applying Lemma 2.7 to $\mathcal{T} \pm i\beta I$ and $\mathcal{S}$ we get that

$$\dim \mathcal{R}(\mathcal{L} \pm i\beta I)^{\perp} = \dim \mathcal{R}(\mathcal{T} \pm i\beta I)^{\perp}.$$

But as $\mathcal{T}$ is self adjoint, then by Remark 12 in [1], the spectrum of $\mathcal{T}$ is real and consequently $\dim \mathcal{R}(\mathcal{T} \pm i\beta I)^{\perp} = 0$. Hence, $\dim \mathcal{R}(\mathcal{L} \pm i\beta I)^{\perp} = 0$. On another hand, by Theorem 2.2, $\mathcal{L}$ is Hermitian and therefore it is the same for $\mathcal{L} \pm i\beta I$. Applying Lemma 2.8, we get that $\mathcal{L} \pm i\beta I$ and hence $\mathcal{L}$ is self-adjoint. The proof is complete.
Q.E.D.

Theorem 2.4. *Let us define the block matrix linear relation*

$$\mathcal{L} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A \in \mathcal{LR}(X)$, $B \in \mathcal{LR}(Y, X)$, $C \in \mathcal{LR}(X, Y)$ and $D \in \mathcal{LR}(Y)$ are densely defined closable linear relations. Let the following conditions be satisfied:

1. *For some $\eta \in (0, 1]$,*

$$\begin{aligned}
\|R(\lambda, A)\| &\leq K_1|\lambda|^{-\eta},\ for\ sufficiently\ large\ \lambda \in \Gamma. \\
\|R(\lambda, D)\| &\leq K_2|\lambda|^{-\eta},\ for\ sufficiently\ large\ \lambda \in \Gamma,
\end{aligned}$$

where Γ is an unbounded set of the complex plane contained in $\rho(A) \cap \rho(D)$.

2. *$\mathcal{D}(A) \subseteq \mathcal{D}(C)$, $\mathcal{D}(D) \subseteq \mathcal{D}(B)$, $B(0)$ and $C(0)$ are closed and*

$$\begin{aligned}
\|Cx\| &\leq K_3\|Ax\|^{\frac{\eta}{2}}\|x\|^{1-\frac{\eta}{2}},\ for\ all\ x \in \mathcal{D}(A). \\
\|Bx\| &\leq K_4\|Dx\|^{\frac{\eta}{2}}\|x\|^{1-\frac{\eta}{2}},\ for\ all\ x \in \mathcal{D}(D).
\end{aligned}$$

Then, for every sufficiently large $\lambda \in \Gamma$, the multivalued linear operator $(\lambda I - \mathcal{L})^{-1}$ is defined on the whole space $X \oplus Y$ and

$$\|(\lambda I - \mathcal{L})^{-1}\| \leq K|\lambda|^{-\eta},$$

for sufficiently large $\lambda \in \Gamma$ and some constant K.
Moreover, if for sufficiently large $\lambda \in \Gamma$, $(I - R(\lambda, T)S)^{-1}$ is single valued then $(\lambda I - \mathcal{L})^{-1}$ is also single valued for sufficiently large $\lambda \in \Gamma$.

Proof. Let us define the block matrix linear relations

$$\mathcal{T} := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \text{ and } \mathcal{S} := \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.$$

We note that, for $\lambda \in \rho(\mathcal{T})$, $I+T(0) = (\lambda-\mathcal{T})R(\lambda,\mathcal{T}) = \lambda R(\lambda,\mathcal{T}) - \mathcal{T}R(\lambda,\mathcal{T})$. Hence, we get that $-I+\lambda R(\lambda,\mathcal{T}) = \mathcal{T}R(\lambda,\mathcal{T})$ for all $\lambda \in \rho(\mathcal{T})$. Then, for sufficiently large $\lambda \in \Gamma$, by condition 1.,

$$\|\mathcal{T}R(\lambda,\mathcal{T})\| \leq 1 + |\lambda|\|R(\lambda,\mathcal{T})\|.$$

Thus,

$$\|AR(\lambda,A)\| \leq 2K_1|\lambda|^{1-\eta}.$$

and

$$\|DR(\lambda,D)\| \leq 2K_2|\lambda|^{1-\eta}.$$

Hence, by conditions 1. 2., Lemmas 2.1 and 2.2, for sufficiently large $\lambda \in \Gamma$ and for any $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in X \oplus Y$ we have

$$\begin{aligned} \|\mathcal{S}R(\lambda,\mathcal{T})v\| &\leq \left\| \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \begin{pmatrix} (\lambda) - A^{-1} & 0 \\ 0 & (\lambda - D)^{-1} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\|, \\ &\leq ((K_3\|A(\lambda - A)^{-1}v_1\|^{\frac{\eta}{2}}\|(\lambda - A)^{-1}v_1\|^{1-\frac{\eta}{2}})^2 + \\ &\quad (K_4\|D(\lambda - D)^{-1}v_2\|^{\frac{\eta}{2}}\|(\lambda - D)^{-1}v_2\|^{1-\frac{\eta}{2}})^2)^{\frac{1}{2}}. \end{aligned}$$

On another hand, for any $\epsilon > 0$ we have

$$\begin{aligned} K_3\|A(\lambda - A)^{-1}v_1\|^{\frac{\eta}{2}}\|(\lambda - A)^{-1}v_1\|^{1-\frac{\eta}{2}} &\leq \frac{\epsilon^2}{2}\|A(\lambda - A)^{-1}v_1\|^{\eta}\|(\lambda - A)^{-1}v_1\|^{1-\eta} + \\ &\quad \frac{K_3^2}{2\epsilon^2}\|(\lambda - A)^{-1}v_1\| \\ &\leq (\frac{\epsilon^2}{2}K_1 2^{\eta} + \frac{K_3^2}{2\epsilon^2}K_1|\lambda|^{-\eta})\|v_1\| \end{aligned}$$

and

$$\begin{aligned} K_4\|D(\lambda - D)^{-1}v_2\|^{\frac{\eta}{2}}\|(\lambda - D)^{-1}v_2\|^{1-\frac{\eta}{2}} &\leq \frac{\epsilon^2}{2}\|D(\lambda - D)^{-1}v_2\|^{\eta}\|(\lambda - D)^{-1}v_2\|^{1-\eta} + \\ &\quad \frac{K_4^2}{2\epsilon^2}\|(\lambda - D)^{-1}v_2\| \\ &\leq (\frac{\epsilon^2}{2}K_2 2^{\eta} + \frac{K_4^2}{2\epsilon^2}K_2|\lambda|^{-\eta})\|v_2\|. \end{aligned}$$

Hence,

$$\begin{aligned} \|\mathcal{S}R(\lambda,\mathcal{T})v\| &\leq \sqrt{(\frac{\epsilon^2}{2}K_1 2^{\eta} + \frac{K_3^2}{2\epsilon^2}K_1|\lambda|^{-\eta})^2\|v_1\|^2 + (\frac{\epsilon^2}{2}K_2 2^{\eta} + \frac{K_4^2}{2\epsilon^2}K_2|\lambda|^{-\eta})^2\|v_2\|^2} \\ &\leq (\frac{\epsilon^2}{2}2^{\eta} + \frac{\max(K_3,K_4)^2}{2\epsilon^2}|\lambda|^{-\eta})\max(K_1,K_2)\|v\|. \end{aligned}$$

Therefore, for $q \in]0,1[$ given and $\epsilon > 0$ fixed,

$$\|\mathcal{S}R(\lambda,\mathcal{T})\| \leq q < 1, \; for\ sufficiently\ large\ \lambda \in \Gamma.$$

Then, by using Proposition 2.1, *i*), the multivalued linear operator $(I - \mathcal{S}R(\lambda,\mathcal{T}))^{-1}$ is everywhere defined and continuous and moreover by Proposition 2.1, *ii*), the multivalued linear operator has an uniformly bounded norm

$$\|(I - \mathcal{S}R(\lambda,\mathcal{T}))^{-1}\| \leq \frac{1}{q-1j}, \; for\ sufficiently\ large\ \lambda \in \Gamma. \tag{2}$$

Further, for $\lambda \in \Gamma$,

$$(I - \mathcal{S}R(\lambda,\mathcal{T}))(\lambda I - \mathcal{T}) \subseteq \lambda I - \mathcal{T} - \mathcal{S}R(\lambda,\mathcal{T})(\lambda I - \mathcal{T}) = \lambda I - \mathcal{T} - \mathcal{S} = \lambda I - \mathcal{L}. \tag{3}$$

Then, by the definition of the inverse operator of multivalued linear operators we get that

$$\left[(I - \mathcal{S}R(\lambda,\mathcal{T}))(\lambda I - \mathcal{T})\right]^{-1} \subseteq (\lambda I - \mathcal{L})^{-1}.$$

From this, taking into account, e.g., [14, Proposition A.1.1 (a) page 280] and using [6, Formula I.1.3 (6) page 3] , we get, for sufficiently large $\lambda \in \Gamma$,

$$(\lambda I - \mathcal{T})^{-1}(I - \mathcal{S}R(\lambda,\mathcal{T}))^{-1} \subseteq (\lambda I - \mathcal{L})^{-1}. \tag{4}$$

This says that $(\lambda I - \mathcal{L})^{-1}$ is defined on the whole space $X \oplus Y$. Indeed, by (3), the range of $\lambda I - \mathcal{L}$ contains that of $(I - \mathcal{S}R(\lambda,\mathcal{T}))(\lambda I - \mathcal{T})$ which is $X \oplus Y$. Then, the claimed result follows from (5). On the other hand, by (2), (5), [6, Corollary II.3.13 page 38] and condition 1., we obtain

$$\begin{aligned} \|(\lambda I - \mathcal{L})^{-1}\| &\leq \|(I - \mathcal{T})^{-1}\|.\|(I - \mathcal{S}R(\lambda,\mathcal{T}))^{-1}\|, \\ &\leq K|\lambda|^{-\eta}, \end{aligned}$$

for sufficiently large $\lambda \in \Gamma$, where $K = \frac{1}{1-q}\sqrt{K_1^2 + K_2^2}$. Let us see that $(\lambda I - \mathcal{L})^{-1}$ is single valued whenever $(I - R(\lambda,\mathcal{T})\mathcal{S})^{-1}$ in single valued. From $I \subset (\lambda I - \mathcal{T})R(\lambda,\mathcal{T})$ and [14, Proposition A.1.1 (d) and (e) page 280], for sufficiently large $\lambda \in \Gamma$

$$\begin{aligned} \lambda I - \mathcal{L} &\subseteq \lambda I - \mathcal{T} - \mathcal{S}; \\ &\subseteq \lambda I - \mathcal{T} - (\lambda I - \mathcal{T})R(\lambda,\mathcal{T})\mathcal{S}, \\ &\subseteq (\lambda I - \mathcal{T})(I - R(\lambda,\mathcal{T})\mathcal{S}). \end{aligned}$$

Then, by the definition of the inverse operator of multivalued linear operators (see, e.g., [6, Preliminaries I.1.1 page 1]) and e.g., [14, Proposition A.1.1 (a) page 280],

$$(\lambda I - \mathcal{L})^{-1} \subseteq (I - R(\lambda,\mathcal{T})\mathcal{S})^{-1}(\lambda I - \mathcal{T})^{-1} = (I - R(\lambda,\mathcal{T})\mathcal{S})^{-1}R(\lambda,\mathcal{T}). \tag{5}$$

Since the right-hand side operator is single valued, for sufficiently large $\lambda \in \Gamma$, then $(\lambda I - \mathcal{L})^{-1}$ is also single valued for the same λ. We get that these λ belongs to $\rho(\mathcal{L})$ and $(\lambda I - \mathcal{L})^{-1} = R(\lambda,\mathcal{L})$. Q.E.D.

Remark 2.1. *The condition $(I - R(\lambda,\mathcal{T})\mathcal{S})^{-1}$ in single valued for sufficiently large $\lambda \in \Gamma$ is satisfied when one of the following conditions is fulfilled.*
i) B and C are singfe valued bounded operators.
ii) $R(C) \subset D(0)$ and B is single valued.
iii) $R(B) \subset A(0)$ and C is single valued.

Theorem 2.5. *Let us define the block matrix linear relation*

$$\mathcal{L} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A \in \mathcal{LR}(X)$, $B \in \mathcal{LR}(Y, X)$, $C \in \mathcal{LR}(X, Y)$ *and* $D \in \mathcal{LR}(Y)$ *are densely defined closable linear relations. Let the following conditions be satisfied:*

1. *For some* $\eta \in (0, 1]$,

$$\begin{aligned} \|R(\lambda, A)\| &\leq K_1|\lambda|^{-\eta}, \ for\ sufficiently\ large\ \lambda \in \Gamma. \\ \|R(\lambda, D)\| &\leq K_2|\lambda|^{-\eta}, \ for\ sufficiently\ large\ \lambda \in \Gamma, \end{aligned}$$

where Γ *is an unbounded set of the complex plane contained in* $\rho(A) \cap \rho(D)$.

2. *B and C are single valued operators,* $\mathcal{D}(A) \subseteq \mathcal{D}(C)$, $\mathcal{D}(D) \subseteq \mathcal{D}(B)$, $B(0)$ *and* $C(0)$ *are closed and*

$$\begin{aligned} \|Cx\| &\leq K_3\|Ax\|^{\frac{\eta}{2}}\|x\|^{1-\frac{\eta}{2}}, \ for\ all\ x \in \mathcal{D}(A). \\ \|Bx\| &\leq K_4\|Dx\|^{\frac{\eta}{2}}\|x\|^{1-\frac{\eta}{2}}, \ for\ all\ x \in \mathcal{D}(D). \end{aligned}$$

3. *There exist two Banach spaces* $\mathcal{Z}_1$ *and* $\mathcal{Z}_2$ *satisfying* $\mathcal{D}(A) \subset \mathcal{Z}_1 \subset \mathcal{D}(C)$, $\mathcal{D}(D) \subset \mathcal{Z}_2 \subset \mathcal{D}(B)$ *such that, for some* $\theta \in (0, 1]$,

$$\|R(\lambda, A)\|_{\mathcal{L}(X,\mathcal{Z}_1)} \leq K_1|\lambda|^{-\theta},$$

$$\|R(\lambda, D)\|_{\mathcal{L}(Y,\mathcal{Z}_2)} \leq K_2|\lambda|^{-\theta},$$

for sufficiently large $\lambda \in \Gamma$.

Then, every sufficiently large $\lambda \in \Gamma$ *belongs to* $\rho(\mathcal{L})$ *and*

$$\|R(\lambda, \mathcal{L})\| \leq K|\lambda|^{-\eta},$$

for some constant K.

Proof. Let us define the block matrix linear relation

$$\mathcal{T} := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \ and \ \mathcal{S} := \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.$$

From conditions 1. and 2. we get condition 1. and 2. of Theorem 2.4. Therefore, the result of Theorem 2.4 is true. Thus, in order to get the assertion of Theorem 2.5, it is now enough to show that the operator $(\lambda I - \mathcal{L})^{-1}$ is single valued.

Since $\mathcal{S}$ is single valued then, for sufficiently large $\lambda \in \Gamma$, $R(\lambda, \mathcal{T})\mathcal{S}$ is also single valued. On the other hand, as $\mathcal{S}$ is closable and $\mathcal{Z} := \mathcal{Z}_1 \oplus \mathcal{Z}_2$ is a Banach space contained in $\mathcal{D}(\mathcal{S})$ then $\|\mathcal{S}\|_{\mathcal{L}(\mathcal{Z},X\oplus Y})$ is finite. Hence, by condition 3.,

$$\|R(\lambda, \mathcal{T})\mathcal{S}\|_{\mathcal{L}(\mathcal{Z})} \leq \|R(\lambda, \mathcal{T})\|_{\mathcal{L}(X\oplus Y,\mathcal{Z})}\|\mathcal{S}\|_{\mathcal{L}(\mathcal{Z},X\oplus Y)} \leq K|\lambda|^{-\theta} < 1,$$

for sufficiently large $\lambda \in \Gamma$. Therefore, $I - R(\lambda, \mathcal{T})\mathcal{S}$ has a bounded single valued inverse operator $(I - R(\lambda, \mathcal{T})\mathcal{S})^{-1}$ in $\mathcal{L}(\mathcal{Z})$. Then, since the image $\mathcal{R}((\lambda I - \mathcal{L})^{-1}) = \mathcal{D}(\mathcal{T})$ and the image $\mathcal{R}(R(\lambda, \mathcal{T})) = \mathcal{D}(\mathcal{T})$, we obtain, by (5), that the operator $(\lambda I - \mathcal{L})^{-1}$ is single valued (in fact, $(\lambda I - \mathcal{L})^{-1} = (I - R(\lambda, \mathcal{T})\mathcal{S})^{-1}R(\lambda, \mathcal{T})$, for sufficiently large $\lambda \in \Gamma$). Q.E.D.

Remark 2.2. *The conclusion of Theorem 2.5 remains valid if we replace the condition 2 by one of the following two conditions.*
i) B and C are single valued operators, $\mathcal{D}(A) \subseteq \mathcal{D}(C)$, $\mathcal{D}(D) \subseteq \mathcal{D}(B)$, and for any $\varepsilon > 0$, there exists $K_1(\varepsilon) > 0$ and $K_2(\varepsilon) > 0$ such that

$$\begin{aligned} \|Cx\| &\leq \varepsilon\|Ax\|^{\eta}.\|x\|^{1-\eta} + K_1(\varepsilon)\|x\|, \ \textit{for all } x \in \mathcal{D}(A). \\ \|By\| &\leq \varepsilon\|Dy\|^{\eta}.\|y\|^{1-\eta} + K_2(\varepsilon)\|y\|, \ \textit{for all } y \in \mathcal{D}(D). \end{aligned}$$

ii) For some $\sigma > 0$, $\|\mathcal{S}R(\lambda, \mathcal{T})\| \leq K|\lambda|^{-\sigma}$ for sufficiently large $\lambda \in \Gamma$. Indeed, the condition 2 in Theorem 2.5 is needed in order to apply Theorem 2.4. But, in Theorem 2.4 the condition 2 has been only used for proving that $\|\mathcal{S}R(\lambda, \mathcal{T})\| \leq q < 1$, for sufficiently large $\lambda \in \Gamma$.

3. Application

Consider an initial boundary value problem for an elliptic-parabolic system of coupled equations of the type

$$\left\{ \begin{aligned} \tfrac{\partial}{\partial t}(m_1(x)u_1(x,t)) &= \nabla \bullet (a(x)\nabla u_1(x,t)) + \sum_{i=1}^{n} a_i(x)\frac{\partial(m_2(x)u_2(x,t))}{\partial x_i} \\ &\quad + c_1(x)u_1(x,t) + f_1(x,t), \quad (x,t) \in \Omega \times (0,\tau], \\ \tfrac{\partial}{\partial t}(m_2(x)u_2(x,t)) &= \nabla \bullet (b(x)\nabla u_2(x,t)) + \sum_{i=1}^{n} b_i(x)\frac{\partial(m_1(x)u_1(x,t))}{\partial x_i} \\ &\quad + c_2(x)u_2(x,t) + f_2(x,t), \quad (x,t) \in \Omega \times (0,\tau], \\ \begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (x,t) \in \partial\Omega \times (0,\tau], \\ \begin{pmatrix} m_1(x)u_1(x,0) \\ m_2(x)u_2(x,0) \end{pmatrix} &= \begin{pmatrix} v_{0,1}(x) \\ v_{0,2}(x) \end{pmatrix}, \quad x \in \Omega. \end{aligned} \right. \tag{6}$$

Here, Ω is a bounded domain in $\mathbb{R}^n$ of class C^2, $n = 2, 3$; ∇ denotes the gradient vector with respect to x variable; $m_1(x) > 0$ and $m_2(x) \geq 0$ on $x \in \overline{\Omega}$ and $m_1(.), m_2(.) \in C^2(\overline{\Omega})$; $a(x), b(x), a_i(x), b_i(x), c_1(x), c_2(x)$ are real-valued smooth functions on $\overline{\Omega}$, there exists $\delta_1 > 0, \delta_2 > 0$ such that $a(x) \geq \delta_1 > 0$, $b(x) \geq \delta_2 > 0$, $c_1(x) \leq 0$ and $c_2(x) \leq 0$ for all $x \in \overline{\Omega}$.
Moreover, m_1 and m_2 are ρ-regular in the sense that $m_1, m_2 \in C^1(\overline{\Omega})$ and for some $\rho \in (0,1)$ for $n = 2$ and $\rho \in (\frac{2}{3}, 1)$ for $n = 3$,

$$\begin{aligned} \|\nabla m_1(x)\| &\leq C_1\, m_1(x)^{\rho}, \quad x \in \overline{\Omega}, \\ \|\nabla m_2(x)\| &\leq C_2\, m_2(x)^{\rho}, \quad x \in \overline{\Omega}. \end{aligned}$$

Take the space $X := L^2(\Omega) \oplus L^2(\Omega)$, and the. operator $\mathcal{M}$ defined on X by

$$\mathcal{M}u = \begin{pmatrix} \mathcal{M}_1 & 0 \\ 0 & \mathcal{M}_2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} := \begin{pmatrix} \mathcal{M}_1 u_1 \\ \mathcal{M}_2 u_2 \end{pmatrix} := \begin{pmatrix} m_1(.)u_1 \\ m_2(.)u_2 \end{pmatrix}.$$

Define the matrix linear operators :

$$\mathcal{G}u := \begin{pmatrix} \mathcal{G}_1 & 0 \\ 0 & \mathcal{G}_2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} := \begin{pmatrix} \nabla.(a(.)\nabla u_1) + c_1(.)u_1 \\ \nabla.(b(.)\nabla u_2) + c_2(.)u_2 \end{pmatrix}$$

with $\mathcal{D}(\mathcal{G}) := \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \oplus \left(H^2(\Omega) \cap H_0^1(\Omega)\right)$ and

$$\mathcal{S}u := \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} := \begin{pmatrix} \sum_{i=1}^{n} a_i(.)\frac{\partial u_2}{\partial x_i} \\ \sum_{i=1}^{n} b_i(.)\frac{\partial u_1}{\partial x_i} \end{pmatrix}$$

with $\mathcal{D}(\mathcal{S}) := H^1(\Omega) \oplus H^1(\Omega)$.
Introduce the new unknown function $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} := \mathcal{M}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and set

$$\mathcal{T} = \mathcal{G}\mathcal{M}^{-1} = \begin{pmatrix} \mathcal{G}_1\mathcal{M}_1^{-1} & 0 \\ 0 & \mathcal{G}_2\mathcal{M}_2^{-1} \end{pmatrix}$$

defined on $\mathcal{M}_1(\Big(H^2(\Omega) \cap H_0^1(\Omega)\Big)) \oplus \mathcal{M}_2(\Big(H^2(\Omega) \cap H_0^1(\Omega)\Big))$.
Then the system (6) is reduced to the inclusion system

$$\frac{\partial}{\partial t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathcal{L}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} f_1(x,t) \\ f_2(x,t) \end{pmatrix} \tag{7}$$

with initial condition

$$v_1(0) = v_{0,1}, \; v_2(0) = v_{0,2} \tag{8}$$

where

$$\mathcal{L} := \mathcal{T} + \mathcal{S} = \begin{pmatrix} \mathcal{G}_1\mathcal{M}_1^{-1} & B \\ C & \mathcal{G}_2\mathcal{M}_2^{-1} \end{pmatrix}.$$

In order to apply Theorem 2.5 to the system defined by the matrix relation $\mathcal{L}$, we will first check its assumptions. In view of Remark 2.2, and since the operators B and C are continuous, it is sufficient to check the conditions 1 and 3. To do this, we start by noting that,

$$\|R(\lambda, \mathcal{G}_i\mathcal{M}_i^{-1})\|_{\mathcal{L}(X)} = \|\mathcal{M}_i(\lambda\mathcal{M}_i - \mathcal{G}_i)^{-1}\|_{\mathcal{L}(X)} \text{ for } i \in \{1,2\}.$$

On another hand, by [10], (1.5) we have the estimate

$$\|\mathcal{M}_i(\lambda\mathcal{M}_i - \mathcal{G}_i)^{-1}\|_{\mathcal{L}(X)} \quad \leq \quad K_i(1+|\lambda|)^{-\frac{1}{(2-\rho)}}; \tag{9}$$

holds in the sector $\Sigma_i := \{\lambda \in \mathbb{C} : Re\lambda \geq -C_i(1+|Im\lambda|)\}$ for some $C_i > 0$. Thus, the condition 1 of Theorem 2.5 is fulfilled with $\eta = \frac{1}{(2-\rho)}$.
Further, arguing as in [7] and introduce the Hilbert space $\mathcal{Z} := H^{\frac{n}{2}+\varepsilon}(\Omega)$, $0 < \varepsilon < \frac{1}{2}$.
It can be shown that

$$\|R(\lambda, \mathcal{G}_i)\|_{L(X,\mathcal{Z})} \leq C_i(1+|\lambda|)^{\frac{n}{4}+\frac{\varepsilon}{2}-\frac{1}{(2-\rho)}};$$

in the sector Σ_i. Then for any $\rho \in (0,1)$ if $n = 2$ and for $\rho \in (\frac{2}{3}, 1)$ if $n = 3$, there exists $\varepsilon > 0$ such that the last exponent will be negative. Therefore, condition 3 of Theorem 2.5, is fulfilled with $\theta = -\frac{n}{4} - \frac{\varepsilon}{2} + \frac{1}{(2-\rho)} > 0$.
Noting now that $\mathcal{Z} \subset H^1(\Omega)$ and $\mathcal{S}$ is bounded from $H^1(\Omega) \oplus H^1(\Omega)$ into $L^2(\Omega) \oplus L^2(\Omega)$. Therefore, the Theorem 2.5 can be applied and we conclude that

$$\|R(\lambda, \mathcal{T} + \mathcal{S})\|_{L(X)} \leq C|\lambda|^{-\frac{1}{(2-\rho)}}, \tag{10}$$

for any sufficiently large λ in $\Sigma_1 \cap \Sigma_2$. In addition, a change of unknown function in (6) to $u(x,t) = e^{-kt}w(t,x)$ with sufficiently large k yelds that (10) can be assumed to hold for all λ in the sector $\Sigma_1 \cap \Sigma_2$ and not only for $|\lambda|$ big enough. Hence, by Theorem 3.1 in [11], we conclude that $\mathcal{L} = \mathcal{T} + \mathcal{S}$ generates an infinitely differentiable semigroup $e^{t\mathcal{L}}$ on X with $\frac{\partial}{\partial t}e^{t\mathcal{L}} \subset \mathcal{L}e^{t\mathcal{L}}$, $t > 0$. Then in view of Theorem 3.7 in [11], we get that the function defined by

$$\begin{pmatrix} v_1(x,t) \\ v_2(x,t) \end{pmatrix} = e^{t\mathcal{L}}\begin{pmatrix} v_{0,1}(x) \\ v_{0,2}(x) \end{pmatrix} + \int_0^t e^{(t-\tau)\mathcal{L}}\begin{pmatrix} f_1(x,\tau) \\ f_2(x,\tau) \end{pmatrix} d\tau$$

is the unique strict solution to (7) and (8). This then yields that for any $f \in C^\sigma([0,\tau); L^2(\Omega) \oplus L^2(\Omega))$, $\frac{1-\rho}{2-\rho} < \sigma \leq 1$, and $\begin{pmatrix} v_{0,1} \\ v_{0,2} \end{pmatrix} \in L^2(\Omega) \oplus L^2(\Omega)$, there exists a unique strict solution of the system (6).

Data availability statement : Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Conflict of interest statement

The authors declare that they have no conflict of interest.

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