Filomat 39:34 (2025), 12089–12098
https://doi.org/10.2298/FIL2534089R

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some inequalities for the weighted A-numerical radius

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Abstract. In this paper, we propose an alternative formulation of the weighted A-numerical radius $\omega_{(A,v)}(\cdot)$ for bounded linear operators on semi-Hilbert spaces induced by a positive operator A. Building upon the framework introduced by Gao and Liu [Oper. Matrices, 17 (2023), 343–354], we derive several new inequalities that offer sharper bounds and refined characterizations of this semi-norm. These results enrich the existing theory by extending the analytical tools available for studying operator behavior in semi-Hilbertian contexts.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators acting on a complex Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$ and associated norm $\|\cdot\|$. Let $\mathbb{M}_n(\mathbb{C})$ denote the set of $n \times n$ complex matrices. For $A \in \mathbb{M}_n(\mathbb{C})$, A^* denote the conjugate transpose of A. For $T \in \mathcal{B}(\mathcal{H})$, let

$$\|T\| = \sup\left\{\|Tx\|; \|x\| = 1\right\} \quad \text{and} \quad \omega(T) = \sup\left\{|\langle Tx, x\rangle|; \|x\| = 1\right\}$$

stand for the operator norm and the numerical radius of T, respectively. If $A \in \mathcal{B}(\mathcal{H})$ represents a positive bounded linear operator, i.e., $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$, we write $A \geq 0$. We denote the square root of a positive operator A as $A^{1/2}$. One can verify that a positive operator A induces a semi-inner product $\langle\cdot,\cdot\rangle_A$ on $\mathcal{H}$ defined by $\langle x, y\rangle_A = \langle Ax, y\rangle$ for all $x, y \in \mathcal{H}$, and the semi-inner product $\langle\cdot,\cdot\rangle_A$ induces a semi-norm $\|\cdot\|_A$ on $\mathcal{H}$ defined by $\|x\|_A = \sqrt{\langle Ax, x\rangle}$. It can be seen that $\|\cdot\|_A$ is a norm if and only if A is injective, and $(\mathcal{H}, \|\cdot\|_A)$ is a complete space if and only if the range space $\mathcal{R}(A)$ of A is closed in $\mathcal{H}$. Let $\overline{\mathcal{R}(T)}$ denote the norm closure of $\mathcal{R}(T)$ in $\mathcal{H}$. If there exists a constant $c > 0$ such that $\|Tx\|_A \leq c\|x\|_A$ for all $x \in \overline{\mathcal{R}(A)}$ and $T \in \mathcal{B}(\mathcal{H})$, then A-operator semi-norm of T is given by:

$$\|T\|_A = \sup_{x\in\overline{\mathcal{R}(A)},\ x\neq 0} \frac{\|Tx\|_A}{\|x\|_A} = \sup_{x\in\overline{\mathcal{R}(A)},\ \|x\|_A=1} \|Tx\|_A < +\infty.$$

2020 *Mathematics Subject Classification*. Primary 47A12; 47A05; Secondary 47A30.
Keywords. A-numerical radius; Positive operator inequalities; Semi-inner product.
Received: 02 May 2025; Accepted: 14 October 2025
Communicated by Dragan S. Djordjević
This work is supported by the Natural Science Foundation of Henan (252300421797).
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The collection of all operators $T \in \mathcal{B}(\mathcal{H})$ with $\|T\|_A < +\infty$ is denoted by $\mathcal{B}^A(\mathcal{H})$. In general, $\mathcal{B}^A(\mathcal{H})$ is not a subalgebra of $\mathcal{B}(\mathcal{H})$ (see [7]). An operator $S \in \mathcal{B}(\mathcal{H})$ is called an A-adjoint of $T \in \mathcal{B}(\mathcal{H})$ if $\langle Tx, y\rangle_A = \langle x, Sy\rangle_A$ for all $x, y \in \mathcal{H}$. The existence of an A-adjoint of T is not guaranteed. Let $\mathcal{B}_A(\mathcal{H})$ denote the collection of all operators in $\mathcal{B}(\mathcal{H})$, which admit A-adjoints. By Douglas theorem [4], it follows that

$$\mathcal{B}_A(\mathcal{H}) = \Big\{T \in \mathcal{B}(\mathcal{H}) : \mathcal{R}(T^*A) \subseteq \mathcal{R}(A)\Big\}.$$

For $T \in \mathcal{B}_A(\mathcal{H})$, the operator equation $AX = T^*A$ has a unique solution, denoted by $T^{\sharp_A}$, satisfying $\mathcal{R}(T^{\sharp_A}) \subseteq \overline{\mathcal{R}(A)}$. Also $T^{\sharp_A} = A^{\dagger}T^*A$, where $A^{\dagger}$ is the Moore-Penrose inverse of A. As the author [15] explained that $T^{\sharp_A}$ is not equal to T, in general. For example, putting $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$. Then $T^{\sharp_A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \neq T$. More precisely, $T^{\sharp_A} = T$ if and only if $\mathcal{R}(T) \subseteq \overline{\mathcal{R}(A)}$ and T is $A-$selfadjoint for $T \in \mathcal{B}_A(\mathcal{H})$. Let $\mathcal{B}_{A^{1/2}}(\mathcal{H})$ denote the set of all bounded linear operators that admit $A^{1/2}$-adjoints. Then, by Douglas theorem [4], we have

$$\mathcal{B}_{A^{1/2}}(\mathcal{H}) = \Big\{T \in \mathcal{B}(\mathcal{H}) : \exists\, c > 0 \text{ such that } \|Tx\|_A \le c\|x\|_A, \forall x \in H\Big\}.$$

Operators in $\mathcal{B}_{A^{1/2}}(\mathcal{H})$ are called A-bounded operators. Note that $\mathcal{B}_A(\mathcal{H})$ and $\mathcal{B}_{A^{1/2}}(\mathcal{H})$ are two subalgebras of $\mathcal{B}(\mathcal{H})$ which are neither closed nor dense in $\mathcal{B}(\mathcal{H})$. Arias et al. [2] showed $\mathcal{B}_A(\mathcal{H}) \subseteq \mathcal{B}_{A^{1/2}}(\mathcal{H})$. An operator $T \in \mathcal{B}_A(\mathcal{H})$ is said to be A-selfadjoint if AT is selfadjoint, that is, $AT = T^*A$. T is said to be A-positive if AT is positive and we write $T \ge_A 0$. Clearly A-positive operator is always A-selfadjoint. For A-positive operators $T, S \in \mathcal{B}_A(\mathcal{H})$, we write $T \ge_A S$ if and only if $T - S \ge_A 0$. In addition, if $T \in \mathcal{B}_A(\mathcal{H})$, then [8] explains that $T^{\sharp_A}T$, $TT^{\sharp_A}$ are A-positive and

$$\left\|T^{\sharp_A}T\right\|_A = \left\|TT^{\sharp_A}\right\|_A = \|T\|_A^2 = \left\|T^{\sharp_A}\right\|_A^2.$$

Observe that for $T, S \in \mathcal{B}_A(\mathcal{H})$, then $(TS)^{\sharp_A} = S^{\sharp_A}T^{\sharp_A}$, $\|Tx\|_A \le \|T\|_A\|x\|_A$ for all $x \in \mathcal{H}$ and $\|TS\|_A \le \|T\|_A\|S\|_A$. Moreover, as many articles mentioned that $T \in \mathcal{B}_A(\mathcal{H})$ implies $T^{\sharp_A} \in \mathcal{B}_A(\mathcal{H})$ and $\left((T^{\sharp_A})^{\sharp_A}\right)^{\sharp_A} = T^{\sharp_A}$.

It is well known that the numerical radius plays an important role in various fields of operator theory and matrix analysis (see [10], [12]). In 2012, Saddi [13] defined the A-numerical radius of an operator $T \in \mathcal{B}(\mathcal{H})$ by

$$\omega_{(A)}(T) = \sup\Big\{|\langle Tx, x\rangle_A|; x \in \mathcal{H}, \|x\|_A = 1\Big\}.$$

Faghih-Ahmadi and Gorjizadeh [5] in 2016 showed that for $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$, we have

$$\|T\|_A = \sup\Big\{|\langle Tx, y\rangle_A|; x, y \in \mathcal{H}, \|x\|_A = \|y\|_A = 1\Big\}.$$

It should be noticed that it may happen that $\|T\|_A$ and $\omega_{(A)}(T)$ are equal to $+\infty$ for some $T \in \mathcal{B}(\mathcal{H})$ [7]. In 2018, Baklouti et al. [3] showed that $\|\cdot\|_A$ and $\omega_{(A)}(\cdot)$ are equivalent semi-norms for $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$, they proved

$$\frac{1}{2}\|T\|_A \le \omega_{(A)}(T) \le \|T\|_A.$$

The above inequalities are sharp, $\frac{1}{2}\|T\|_A = \omega_{(A)}(T)$ if $AT^2 = 0$ and $\omega_{(A)}(T) = \|T\|_A$ if $T^{\sharp_A}T = TT^{\sharp_A}$.

Following by the Crawford number of an operator $T \in \mathcal{B}(\mathcal{H})$ introduced by Gustafson and Rao in [11], we introduce the following notations of $A-$Crawford:

$$c_A(T) = \inf\Big\{|\langle Tx, x\rangle_A| : x \in \mathcal{H}, \|x\|_A = 1\Big\}.$$

As an alternative formula for the numerical radius of $A \in \mathcal{B}(\mathcal{H})$, the identity $w(A) = \sup_{\theta\in\mathbb{R}} \|\Re(e^{i\theta}A)\|$ has been used in many literature. In 2022, Sheikhhosseini, Khosravi and Sababheh [14] defined the weighted numerical radius as

$$w_v(A) = \sup_{\theta\in\mathbb{R}} \left\|\Re_v(e^{i\theta}A)\right\|,$$

where $0 \le v \le 1$ and $\Re_v(A) = vA + (1-v)A^*$. Here, the function $w_v(\cdot) : \mathcal{B}(\mathcal{H}) \to [0,\infty)$ is a norm on $\mathcal{B}(\mathcal{H})$. They [14] also defined $\Im_v(A) = -ivA + i(1-v)A^*$.

In 2019, Zamani [15] gave the definition of A-numerical radius: if $T \in \mathcal{B}_A(\mathcal{H})$, then

$$\omega_A(T) = \sup_{\theta\in\mathbb{R}} \left\|\Re_A(e^{i\theta}T)\right\|_A = \sup_{\theta\in\mathbb{R}} \left\|\Im_A(e^{i\theta}T)\right\|_A,$$

where $\Re_A(T) = \dfrac{T + T^{\sharp A}}{2}$ and $\Im_A(T) = \dfrac{T - T^{\sharp A}}{2i}$, respectively.

In 2023, motivated by the idea of Sheikhhosseini, Khosravi and Sababheh [14], Gao and Liu [9] gave the weighted definition of $\omega_A(T)$: Let $T \in \mathcal{B}_A(\mathcal{H})$ and $0 \le v \le 1$. Then

$$\omega_{(A,v)}(T) = \sup_{\theta\in\mathbb{R}} \left\|\Re_{(A,v)}(e^{i\theta}T)\right\|_A = \sup_{\theta\in\mathbb{R}} \left\|\Im_{(A,v)}(e^{i\theta}T)\right\|_A, \tag{1}$$

where $\Re_{(A,v)}(T) = T\nabla_v T^{\sharp A} = vT + (1-v)T^{\sharp A}$ is the weighted real part of T and $\Im_{(A,v)}(T) = (-iT)\nabla_v(-iT)^{\sharp A} = v(-iT) + (1-v)(-iT)^{\sharp A}$ is the weighted imaginary part of T, respectively.

It is easy to check that $\omega_{(A,v)}(\cdot)$ is a semi-norm on $\mathcal{B}_A(\mathcal{H})$, and $\omega_{(A,\frac{1}{2})}(T) = \omega_{(A)}(T)$, $\omega_{(A,0)}(T) = \omega_{(A,1)}(T) = \|T\|_A = \left\|T^{\sharp A}\right\|_A$. Then, Gao and Liu showed the following results about the $\omega_{(A,v)}(\cdot)$.

Theorem 1.1 (). *[9] Let $0 \le v \le 1$ and $T \in \mathcal{B}_A(\mathcal{H})$. Then*

$$\frac{1}{2}\|T\|_A \le \max\{v, 1-v\}\|T\|_A \le \omega_{(A,v)}(T) \le \|T\|_A.$$

This paper is organized as follows. We begin by presenting a refined definition of the weighted A-numerical radius $\omega_{(A,v)}(\cdot)$, extending the framework originally introduced by Gao and Liu [9]. An equivalent formulation is established, offering new analytical insight into its structure. We then derive a number of related inequalities that sharpen existing bounds and highlight conditions for equality. Further refinements and generalizations of known results are discussed in detail. The paper also examines the behavior of $\omega_{(A,v)}(\cdot)$ in the context of operator products and adjoints. Several results are provided to illustrate the interaction between the weighting parameter and the A-structure. We conclude with upper bounds and structural estimates that enhance the current understanding of numerical radius theory in semi-Hilbertian spaces. These developments contribute to the growing body of research on operator inequalities and norm estimates in functional analysis.

2. Main results

We begin by introducing an alternative characterization of the weighted A-numerical radius $\omega_{(A,v)}(\cdot)$. This new formulation provides a different perspective on its structure and facilitates further analysis. It complements the original definition by Gao and Liu, offering equivalent but analytically useful expressions. This foundation allows us to establish improved inequalities and sharper bounds in subsequent sections.

Theorem 2.1. *Let $T \in \mathcal{B}_A(\mathcal{H})$ and $0 \le v \le 1$. Then*

$$\omega_{(A,v)}(T) = \frac{1}{2} \sup_{\theta,\varphi\in\mathbb{R}} \left\|\Re_{(A,v)}\big((e^{i\theta} - ie^{i\varphi})T\big)\right\|_A.$$

Proof. For any $T \in \mathcal{B}_A(\mathcal{H})$, we have

$$\begin{aligned}
\omega_{(A,v)}(T) &= \frac{1}{2}\sup_{\theta\in\mathbb{R}} \left\| \Re_{(A,v)}(e^{i\theta}T) + \Re_{(A,v)}(e^{i\theta}T) \right\|_A \\
&= \frac{1}{2}\sup_{\theta\in\mathbb{R}} \left\| \Re_{(A,v)}(e^{i\theta}T) + \Im_{(A,v)}\left(e^{i(\theta+\frac{\pi}{2})}T\right) \right\|_A \\
&\leq \frac{1}{2}\sup_{\theta,\varphi\in\mathbb{R}} \left\| \Re_{(A,v)}(e^{i\theta}T) + \Im_{(A,v)}(e^{i\varphi}T) \right\|_A \\
&= \frac{1}{2}\sup_{\theta,\varphi\in\mathbb{R}} \left\| v(e^{i\theta} - ie^{i\varphi})T + (1-v)\left((e^{i\theta} - ie^{i\varphi})T\right)^{\sharp A} \right\|_A \\
&= \frac{1}{2}\sup_{\theta,\varphi\in\mathbb{R}} \left\| \Re_{(A,v)}\left((e^{i\theta} - ie^{i\varphi})T\right) \right\|_A \\
&\leq \frac{1}{2}\sup_{\theta,\varphi\in\mathbb{R}} \omega_{(A,v)}\left((e^{i\theta} - ie^{i\varphi})T\right) \\
&= \frac{1}{2}\sup_{\theta,\varphi\in\mathbb{R}} |e^{i\theta} - ie^{i\varphi}|\omega_{(A,v)}(T) \\
&= \frac{\omega_{(A,v)}(T)}{2}\sup_{\theta,\varphi\in\mathbb{R}} \sqrt{2 - 2\sin(\theta-\varphi)} \\
&= \omega_{(A,v)}(T),
\end{aligned}$$

as desired. □

Remark 2.2. *The above Theorem provides an equivalent analytical formulation of $\omega_{(A,v)}(T)$, expressed through double supremum expressions as introduced in [1].*

We demonstrate that the second inequality in Theorem 1.1 attains equality under certain conditions. Specifically, we identify cases where the bound is exact, confirming its sharpness. This result highlights the optimality of the inequality within a particular class of operators. Such sharpness conditions enhance the theoretical significance of the established bound.

Theorem 2.3. *Let $T \in \mathcal{B}_A(\mathcal{H})$ and let $0 \leq v \leq 1$. If $T = T^{\sharp A}$, then $\omega_{(A,v)}(T^{\sharp A}) = \|T\|_A$.*

Proof. Since $T = T^{\sharp A}$, we have $\Re_{(A,v)}(e^{i\theta}T) = ve^{i\theta}T + (1-v)e^{-i\theta}T$. Thus

$$\begin{aligned}
\omega_{(A,v)}(T) &= \sup_{\theta\in\mathbb{R}} \left\| \Re_{(A,v)}(e^{i\theta}T) \right\|_A \\
&= \sup_{\theta\in\mathbb{R}} \left\| ve^{i\theta}T + (1-v)e^{-i\theta}T \right\|_A \\
&= \sup_{\theta\in\mathbb{R}} |ve^{i\theta} + (1-v)e^{-i\theta}| \left\| T \right\|_A \\
&= \sup_{\theta\in\mathbb{R}} \sqrt{1 - 4v(1-v)\sin^2\theta}\, \|T\|_A \\
&= \|T\|_A.
\end{aligned}$$

□

In this section, we present further refinements of Theorem 1.1, aiming to improve the established bounds. These enhancements yield tighter inequalities and provide a more accurate estimation of the weighted A-numerical radius. Moreover, they uncover additional structural insights into the behavior of operators in semi-Hilbertian spaces.

Theorem 2.4. *Let $T \in \mathcal{B}_A(\mathcal{H})$ and $0 \le v \le 1$. Then*

$$v\|T\|_A \le \sqrt{\left\|v^2 TT^{\sharp_A} + (1-v)^2 T^{\sharp_A} T\right\|_A + 2v(1-v)c_A(T^2)} \le \omega_{(A,v)}(T).$$

Proof. Let $x \in \mathcal{H}$ with $\|x\|_A = 1$. Suppose that $\left|\left\langle T^{\sharp_A}T^{\sharp_A}x, x\right\rangle_A\right| = e^{-2i\theta}\left\langle T^{\sharp_A}T^{\sharp_A}x, x\right\rangle_A$ for some real number θ, then

$$e^{2i\theta}\left\langle (T^{\sharp_A})^{\sharp_A}(T^{\sharp_A})^{\sharp_A}x, x\right\rangle_A = e^{2i\theta}\overline{\langle T^{\sharp_A}T^{\sharp_A}x, x\rangle_A} = \left|\left\langle T^{\sharp_A}T^{\sharp_A}x, x\right\rangle_A\right| = \left|\left\langle x, T^2x\right\rangle_A\right|.$$

Thus

$$e^{-2i\theta}\langle T^{\sharp_A}T^{\sharp_A}x, x\rangle_A = |\langle x, T^2x\rangle_A| = e^{2i\theta}\left\langle (T^{\sharp_A})^{\sharp_A}(T^{\sharp_A})^{\sharp_A}x, x\right\rangle_A. \tag{2}$$

For $\theta \in \mathbb{R}$, we have the following chain of inequalities

$$\begin{aligned}
\omega^2_{(A,v)}(T) &\ge \left\|ve^{i\theta}T + (1-v)(e^{i\theta}T)^{\sharp_A}\right\|_A^2 \\
&= \left\|v(e^{i\theta}T)^{\sharp_A} + (1-v)((e^{i\theta}T)^{\sharp_A})^{\sharp_A}\right\|_A^2 \\
&\quad \left(\text{since } \|R\|_A = \left\|R^{\sharp_A}\right\|_A \text{ for every } R \in \mathcal{B}_A(\mathcal{H})\right) \\
&= \left\|\left(v(e^{i\theta}T)^{\sharp_A} + (1-v)((e^{i\theta}T)^{\sharp_A})^{\sharp_A}\right)\left(v((e^{i\theta}T)^{\sharp_A})^{\sharp_A} + (1-v)(e^{i\theta}T)^{\sharp_A}\right)\right\|_A \\
&\quad \left(\text{since } \|R\|_A^2 = \left\|RR^{\sharp_A}\right\|_A \text{ and } ((R^{\sharp_A})^{\sharp_A})^{\sharp_A} = R^{\sharp_A} \text{ for every } R \in \mathcal{B}_A(\mathcal{H})\right) \\
&= \Big\|v^2T^{\sharp_A}(T^{\sharp_A})^{\sharp_A} + (1-v)^2(T^{\sharp_A})^{\sharp_A}T^{\sharp_A} + v(1-v)e^{-2i\theta}T^{\sharp_A}T^{\sharp_A} \\
&\qquad + v(1-v)e^{2i\theta}(T^{\sharp_A})^{\sharp_A}(T^{\sharp_A})^{\sharp_A}\Big\|_A \\
&\ge \Big|\Big\langle\Big(v^2T^{\sharp_A}(T^{\sharp_A})^{\sharp_A} + (1-v)^2(T^{\sharp_A})^{\sharp_A}T^{\sharp_A} + v(1-v)e^{-2i\theta}T^{\sharp_A}T^{\sharp_A} \\
&\qquad + v(1-v)e^{2i\theta}(T^{\sharp_A})^{\sharp_A}(T^{\sharp_A})^{\sharp_A}\Big)x, x\Big\rangle_A\Big| \\
&= \Big|\Big\langle\Big(v^2T^{\sharp_A}(T^{\sharp_A})^{\sharp_A} + (1-v)^2(T^{\sharp_A})^{\sharp_A}T^{\sharp_A}\Big)x, x\Big\rangle_A \\
&\qquad + v(1-v)e^{-2i\theta}\left\langle T^{\sharp_A}T^{\sharp_A}x, x\right\rangle_A + v(1-v)e^{2i\theta}\left\langle (T^{\sharp_A})^{\sharp_A}(T^{\sharp_A})^{\sharp_A}x, x\right\rangle_A\Big| \\
&= \left|\left\langle\left(v^2T^{\sharp_A}(T^{\sharp_A})^{\sharp_A} + (1-v)^2(T^{\sharp_A})^{\sharp_A}T^{\sharp_A}\right)x, x\right\rangle_A + 2v(1-v)\left|\left\langle x, T^2x\right\rangle_A\right|\right| \\
&\quad \big(\text{by } (2)\big) \\
&\ge \left\langle\left(v^2T^{\sharp_A}(T^{\sharp_A})^{\sharp_A} + (1-v)^2(T^{\sharp_A})^{\sharp_A}T^{\sharp_A}\right)x, x\right\rangle_A + 2v(1-v)c_A(T^2).
\end{aligned}$$

It follows that

$$\sqrt{\left\langle\left(v^2T^{\sharp_A}(T^{\sharp_A})^{\sharp_A} + (1-v)^2(T^{\sharp_A})^{\sharp_A}T^{\sharp_A}\right)x, x\right\rangle_A + 2v(1-v)c_A(T^2)} \le \omega_{(A,v)}(T).$$

Taking the supremum over $x \in \mathcal{H}$ with $\|x\|_A = 1$ in the above inequality, we have

$$\sqrt{\left\|v^2TT^{\sharp_A} + (1-v)^2T^{\sharp_A}T\right\|_A + 2v(1-v)c_A(T^2)} \le \omega_{(A,v)}(T). \tag{3}$$

Since $T^{\sharp_A}T$ is an A-positive operator, we have $\left\|v^2TT^{\sharp_A} + (1-v)^2T^{\sharp_A}T\right\|_A \ge v^2\left\|TT^{\sharp_A}\right\|_A = v^2\|T\|_A^2$, it follows that

$$\sqrt{\left\|v^2TT^{\sharp_A} + (1-v)^2T^{\sharp_A}T\right\|_A + 2v(1-v)c_A(T^2)} \ge \sqrt{v^2\left\|TT^{\sharp_A}\right\|_A} \ge v\|T\|_A. \tag{4}$$

Combination (3) and (4), we obtain

$$v\|T\|_A \leq \sqrt{\left\|v^2TT^{\sharp A} + (1-v)^2T^{\sharp A}T\right\|_A + 2v(1-v)c_A(T^2)} \leq \omega_{(A,v)}(T).$$

□

In the following theorem, we provide a refinement of the inequality $v\|T\|_A \leq \omega_{(A,v)}(T)$. This result offers a tighter lower bound by incorporating additional operator terms. It improves the estimate of the weighted A-numerical radius beyond the basic inequality. Such refinement contributes to a more precise understanding of norm behavior in this setting.

Theorem 2.5. *Let $T \in \mathcal{B}_A(\mathcal{H})$, $0 \leq v \leq 1$, $m_1 = \max\left\{\|\mathfrak{R}_{(A,v)}(T)\|_A, v\|T\|_A\right\}$, $m_2 = \max\left\{\|\mathfrak{I}_{(A,v)}(T)\|_A, v\|T\|_A\right\}$, $d_1 = \left|\|\mathfrak{R}_{(A,v)}(T)\|_A - v\|T\|_A\right|$, $d_2 = \left|\|\mathfrak{I}_{(A,v)}(T)\|_A - v\|T\|_A\right|$. Then we have*

$$v\|T\|_A + \frac{1}{4}(d_1 + d_2) + \frac{1}{2}|m_1 - m_2| \leq \omega_{(A,v)}(T).$$

Proof. By equations (1) and Theorem 1.1, we have $\omega_{(A,v)}(T) \geq \|\mathfrak{R}_{(A,v)}(T)\|_A, \omega_{(A,v)}(T) \geq \|\mathfrak{I}_{(A,v)}(T)\|_A$, and $\omega_{(A,v)}(T) \geq v\|T\|_A$. Hence

$$\begin{aligned}
\omega_{(A,v)}(T) \geq \max\{m_1, m_2\} &= \frac{(m_1 + m_2) + |m_1 - m_2|}{2} \\
&= \frac{\|\mathfrak{R}_{(A,v)}(T)\|_A + \|\mathfrak{I}_{(A,v)}(T)\|_A + 2v\|T\|_A + d_1 + d_2}{4} + \frac{|m_1 - m_2|}{2} \\
&\geq \frac{1}{4}\|\mathfrak{R}_{(A,v)}(T) + i\mathfrak{I}_{(A,v)}(T)\|_A + \frac{v\|T\|_A}{2} + \frac{d_1 + d_2}{4} + \frac{|m_1 - m_2|}{2} \\
&= \frac{v}{2}\|T\|_A + \frac{v\|T\|_A}{2} + \frac{d_1 + d_2}{4} + \frac{|m_1 - m_2|}{2} \\
&= v\|T\|_A + \frac{d_1 + d_2}{4} + \frac{|m_1 - m_2|}{2}.
\end{aligned}$$

□

The following presents an upper bound for $\omega_{(A,v)}(T)$, providing a constraint on its maximum value.

Theorem 2.6. *Let $T \in \mathcal{B}_A(\mathcal{H})$ and $0 \leq v \leq 1$. Then*

$$\omega_{(A,v)}(T) \leq \sqrt{\|(1-v)^2TT^{\sharp A} + v^2T^{\sharp A}T\|_A + 2v(1-v)\omega_{(A)}(T^2)}.$$

Proof. Let $\theta \in \mathbb{R}$. We have

$$\begin{aligned}
\omega_{(A,v)}(T) &= \sup_{\theta\in\mathbb{R}} \|\mathfrak{R}_{(A,v)}(e^{i\theta}T)\|_A \\
&= \sup_{\theta\in\mathbb{R}} \left\|ve^{i\theta}T + (1-v)(e^{i\theta}T)^{\sharp A}\right\|_A \\
&= \sup_{\theta\in\mathbb{R}} \left\|v(e^{i\theta}T)^{\sharp A} + (1-v)((e^{i\theta}T)^{\sharp A})^{\sharp A}\right\|_A \\
&\quad \left(\text{since } \|R\|_A = \|R^{\sharp A}\|_A \text{ for every } R \in \mathcal{B}_A(\mathcal{H})\right) \\
&= \sup_{\theta\in\mathbb{R}} \left(\left\|\left(v(e^{i\theta}T)^{\sharp A} + (1-v)((e^{i\theta}T)^{\sharp A})^{\sharp A}\right)\left(v((e^{i\theta}T)^{\sharp A})^{\sharp A} + (1-v)(e^{i\theta}T)^{\sharp A}\right)\right\|_A\right)^{\frac{1}{2}} \\
&\quad \left(\text{since } \|R\|_A^2 = \|RR^{\sharp A}\|_A \text{ for every } R \in \mathcal{B}_A(\mathcal{H})\right)
\end{aligned}$$

$$= \sup_{\theta\in\mathbb{R}}\left(\left\|v^2T^{\sharp_A}(T^{\sharp_A})^{\sharp_A} + (1-v)^2(T^{\sharp_A})^{\sharp_A}T^{\sharp_A}\right.\right.$$
$$\left.\left. + v(1-v)\left((e^{i\theta}T)^{\sharp_A}\right)^2 + v(1-v)\left(((e^{i\theta}T)^{\sharp_A})^{\sharp_A}\right)^2\right\|_A\right)^{\frac{1}{2}}$$
$$\leq \sup_{\theta\in\mathbb{R}}\left(\left\|v^2T^{\sharp_A}(T^{\sharp_A})^{\sharp_A} + (1-v)^2(T^{\sharp_A})^{\sharp_A}T^{\sharp_A}\right\|_A\right.$$
$$\left. + \left\|v(1-v)\left((e^{i\theta}T)^{\sharp_A}\right)^2 + v(1-v)\left(((e^{i\theta}T)^{\sharp_A})^{\sharp_A}\right)^2\right\|_A\right)^{\frac{1}{2}}$$
$$= \sup_{\theta\in\mathbb{R}}\sqrt{\left\|(1-v)^2TT^{\sharp_A} + v^2T^{\sharp_A}T\right\|_A + 2v(1-v)\left\|\frac{e^{2i\theta}T^2 + (e^{2i\theta}T^2)^{\sharp_A}}{2}\right\|_A}$$
$$= \sqrt{\left\|(1-v)^2TT^{\sharp_A} + v^2T^{\sharp_A}T\right\|_A + \sup_{\theta\in\mathbb{R}}2v(1-v)\left\|\frac{e^{2i\theta}T^2 + (e^{2i\theta}T^2)^{\sharp_A}}{2}\right\|_A}$$
$$= \sqrt{\left\|(1-v)^2TT^{\sharp_A} + v^2T^{\sharp_A}T\right\|_A + 2v(1-v)\omega_{(A)}(T^2)},$$

as desired. □

It has been demonstrated in [6] that the following inequality holds:

$$\frac{1}{4}\left\|T^{\sharp_A}T + TT^{\sharp_A}\right\|_A \leq \omega^2_{(A)}(T) \leq \frac{1}{2}\left\|T^{\sharp_A}T + TT^{\sharp_A}\right\|_A \tag{5}$$

for $T \in \mathcal{B}_A(\mathcal{H})$. In this section, we aim to present a generalization and further enhancement of the first inequality in (5).

Theorem 2.7. *Let $T \in \mathcal{B}_A(\mathcal{H})$ and $0 \leq v \leq 1$. Then*

$$v(1-v)\left\|T^{\sharp_A}T + TT^{\sharp_A}\right\|_A$$
$$\leq \frac{1}{4}\left(\left\|(\mathfrak{R}_{(A,v)}(T) + \mathfrak{I}_{(A,v)}(T))\right\|_A^2 + \left\|(\mathfrak{R}_{(A,v)}(T) - \mathfrak{I}_{(A,v)}(T))\right\|_A^2\right)$$
$$\leq \omega^2_{(A,v)}(T).$$

Proof. We have the following inequalities

$$v(1-v)\left\|T^{\sharp_A}T + TT^{\sharp_A}\right\|_A$$
$$= \frac{1}{4}\left\|4v(1-v)\left(T^{\sharp_A}T + TT^{\sharp_A}\right)\right\|_A$$
$$= \frac{1}{4}\left\|\left(\mathfrak{R}_{(A,v)}(T) + \mathfrak{I}_{(A,v)}(T)\right)^2 + \left(\mathfrak{R}_{(A,v)}(T) - \mathfrak{I}_{(A,v)}(T)\right)^2\right\|_A$$
$$\leq \frac{1}{4}\left(\left\|\left(\mathfrak{R}_{(A,v)}(T) + \mathfrak{I}_{(A,v)}(T)\right)^2\right\|_A + \left\|\left(\mathfrak{R}_{(A,v)}(T) - \mathfrak{I}_{(A,v)}(T)\right)^2\right\|_A\right)$$
$$\leq \frac{1}{4}\left(\left\|\left(\mathfrak{R}_{(A,v)}(T) + \mathfrak{I}_{(A,v)}(T)\right)\right\|_A^2 + \left\|\left(\mathfrak{R}_{(A,v)}(T) - \mathfrak{I}_{(A,v)}(T)\right)\right\|_A^2\right)$$
$$\leq \frac{1}{4}\left(2\left(\|\mathfrak{R}_{(A,v)}(T)\|_A^2 + \|\mathfrak{I}_{(A,v)}(T)\|_A^2\right)\right)$$
$$\leq \frac{1}{2}\left(\omega^2_{(A,v)}(T) + \omega^2_{(A,v)}(T)\right)$$
$$= \omega^2_{(A,v)}(T).$$

□

At the conclusion of this paper, we provide several upper bounds for the A-numerical radius of the product of two operators. Specifically, we focus on operators T and S within the space $\mathcal{B}_A(\mathcal{H})$. These bounds offer insight into the behavior of the A-numerical radius in the context of operator products. Our results contribute to a deeper understanding of the relationships between such operators. We also discuss the implications of these bounds in various operator theory contexts. Finally, these findings extend previous work on the A-numerical radius.

Lemma 2.8. *[9] Let $T \in \mathcal{B}_A(\mathcal{H})$ and $v \in [0,1]$. Then $\omega_{(A,v)}(T) = \omega_{(A,v)}(T^{\sharp_A}) = \omega_{(A,1-v)}(T)$.*

Theorem 2.9. *Let $T, S \in \mathcal{B}_A(\mathcal{H})$ and $0 \le v \le 1$. Then we have*

$$\omega_{(A,v)}\Big((TS)^{\sharp_A} \pm T^{\sharp_A}S\Big) \le 2\omega_{(A,v)}(T)\|S\|_A.$$

Proof. Let $\theta \in \mathbb{R}$. Since $\Big((R^{\sharp_A})^{\sharp_A}\Big)^{\sharp_A} = R^{\sharp_A}$ and $\|R\|_A = \|R^{\sharp_A}\|_A$ for $R \in \mathcal{B}_A(\mathcal{H})$, by Lemma 2.8, we have

$$\begin{aligned}
&\omega_{(A,v)}\Big((TS)^{\sharp_A} + T^{\sharp_A}S\Big)\\
&= \omega_{(A,v)}\Big(\Big((TS)^{\sharp_A} + T^{\sharp_A}S\Big)^{\sharp_A}\Big)\\
&= \sup_{\theta\in\mathbb{R}}\Big\|v\Big(e^{i\theta}\Big((TS)^{\sharp_A} + T^{\sharp_A}S\Big)^{\sharp_A}\Big) + (1-v)\Big(e^{i\theta}\Big((TS)^{\sharp_A} + T^{\sharp_A}S\Big)^{\sharp_A}\Big)^{\sharp_A}\Big\|_A\\
&= \sup_{\theta\in\mathbb{R}}\Big\|v\Big(e^{i\theta}(T^{\sharp_A})^{\sharp_A}(S^{\sharp_A})^{\sharp_A} + e^{i\theta}S^{\sharp_A}(T^{\sharp_A})^{\sharp_A}\Big)\\
&\quad + (1-v)\Big(e^{-i\theta}S^{\sharp_A}T^{\sharp_A} + e^{-i\theta}T^{\sharp_A}(S^{\sharp_A})^{\sharp_A}\Big)\Big\|_A\\
&= \sup_{\theta\in\mathbb{R}}\Big\|\Big(ve^{i\theta}(T^{\sharp_A})^{\sharp_A} + (1-v)e^{-i\theta}T^{\sharp_A}\Big)(S^{\sharp_A})^{\sharp_A}\\
&\quad + S^{\sharp_A}\Big(ve^{i\theta}(T^{\sharp_A})^{\sharp_A} + (1-v)e^{-i\theta}T^{\sharp_A}\Big)\Big\|_A\\
&\le \sup_{\theta\in\mathbb{R}}\Big\|ve^{i\theta}(T^{\sharp_A})^{\sharp_A} + (1-v)e^{-i\theta}T^{\sharp_A}\Big\|_A\Big(\Big\|(S^{\sharp_A})^{\sharp_A}\Big\|_A + \Big\|S^{\sharp_A}\Big\|_A\Big)\\
&= \omega_{(A,v)}\Big((T^{\sharp_A})^{\sharp_A}\Big)\Big(\Big\|(S^{\sharp_A})^{\sharp_A}\Big\|_A + \Big\|S^{\sharp_A}\Big\|_A\Big)\\
&= \omega_{(A,v)}(T)\Big(\|S\|_A + \|S\|_A\Big)\\
&= 2\omega_{(A,v)}(T)\|S\|_A.
\end{aligned}$$

That is

$$\omega_{(A,v)}\Big((TS)^{\sharp_A} + T^{\sharp_A}S\Big) \le 2\omega_{(A,v)}(T)\|S\|_A. \tag{6}$$

Replacing S by $-iS$ in inequality (6), we obtain

$$\omega_{(A,v)}\Big((TS)^{\sharp_A} - T^{\sharp_A}S\Big) \le 2\omega_{(A,v)}(T)\|S\|_A.$$

□

Another result concerning the A-numerical radius of the product of two operators is stated as follows.

Theorem 2.10. *Let $T, S \in \mathcal{B}_A(\mathcal{H})$ and $0 \le v \le 1$. Then*

$$v\omega_{(A,v)}(TS) \le \omega_{(A,v)}(T)\|S\|_A + (1-v)\omega_{(A,v)}\Big(S^{\sharp_A}T^{\sharp_A} \pm T^{\sharp_A}S\Big).$$

Proof. Let $\theta \in \mathbb{R}$. By Theorem 1.1, we have

$$\begin{aligned}
& v\left\| ve^{i\theta}TS + (1-v)(e^{i\theta}TS)^{\sharp_A} \right\|_A \\
& \leq \omega_{(A,v)}\Big(ve^{i\theta}TS + (1-v)(e^{i\theta}TS)^{\sharp_A}\Big) \\
& = \omega_{(A,v)}\Big(\Big(ve^{i\theta}T + (1-v)(e^{i\theta}T)^{\sharp_A}\Big)S + (1-v)e^{-i\theta}\Big(S^{\sharp_A}T^{\sharp_A} - T^{\sharp_A}S\Big)\Big) \\
& \leq \omega_{(A,v)}\Big(\Big(ve^{i\theta}T + (1-v)(e^{i\theta}T)^{\sharp_A}\Big)S\Big) + \omega_{(A,v)}\Big(\Big(1-v)e^{-i\theta}(S^{\sharp_A}T^{\sharp_A} - T^{\sharp_A}S\Big)\Big) \\
& \leq \left\|\Big(ve^{i\theta}T + (1-v)(e^{i\theta}T)^{\sharp_A}\Big)S\right\|_A + (1-v)\omega_{(A,v)}\Big(S^{\sharp_A}T^{\sharp_A} - T^{\sharp_A}S\Big) \\
& \leq \left\|ve^{i\theta}T + (1-v)(e^{i\theta}T)^{\sharp_A}\right\|_A \|S\|_A + (1-v)\omega_{(A,v)}\Big(S^{\sharp_A}T^{\sharp_A} - T^{\sharp_A}S\Big) \\
& \leq \omega_{(A,v)}(T)\|S\|_A + (1-v)\omega_{(A,v)}\Big(S^{\sharp_A}T^{\sharp_A} - T^{\sharp_A}S\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
v\omega_{(A,v)}(TS) &= \sup_{\theta\in\mathbb{R}} v\left\|ve^{i\theta}TS + (1-v)(e^{i\theta}TS)^{\sharp_A}\right\|_A \\
&\leq \omega_{(A,v)}(T)\|S\|_A + (1-v)\omega_{(A,v)}\Big(S^{\sharp_A}T^{\sharp_A} - T^{\sharp_A}S\Big).
\end{aligned} \tag{7}$$

Replacing S by $-iS$ in inequality (7), we obtain

$$v\omega_{(A,v)}(TS) \leq \omega_{(A,v)}(T)\|S\|_A + (1-v)\omega_{(A,v)}\Big(S^{\sharp_A}T^{\sharp_A} + T^{\sharp_A}S\Big).$$

□

3. Conclusions

In this work, we introduced an alternative formulation of the weighted A-numerical radius $\omega_{(A,v)}(\cdot)$ for operators on semi-Hilbert spaces. We then established several related inequalities that provide refinements and bounds for this functional. These results extend and complement the framework developed by Gao and Liu [9]. In particular, we demonstrated sharpness conditions, presented equivalent characterizations, and derived improved estimates under specific operator assumptions. Our approach offers deeper insight into the behavior of weighted numerical radii and their dependence on the weighting parameter v. Furthermore, we explored upper and lower bounds, as well as operator products, in the context of the A-numerical radius. These findings contribute to the broader theory of semi-Hilbertian operator analysis. The results may have further implications for spectral theory and applications in matrix analysis. Overall, this work supplements and expands upon existing literature with new perspectives on weighted operator norms.

Acknowledgment

The authors would like to express their sincere gratitude to the anonymous reviewers for their valuable comments, constructive suggestions, and careful reading, which greatly helped to improve the quality and presentation of this paper.

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