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New estimates of numerical radius inequalities for Hilbert space operators

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Abstract. The aim of this paper is to study numerical radius inequalities for bounded linear operators. Based on the diameter of the numerical range, we establish new inequalities that improve and refine several well-known earlier results.

1. Introduction

Throughout this paper, let H be a complex Hilbert space. We denote by $B(H)$ the C^*-algebra of all bounded linear operators on H. For $T \in B(H)$, the *numerical range* of T is defined by

$$W(T) = \{\langle Tx, x\rangle : x \in H, \ \|x\| = 1\},$$

and the *numerical radius* of T is given by

$$w(T) = \sup\{|z| : z \in W(T)\}.$$

The numerical range $W(T)$, like the spectrum $\sigma(T)$, is a subset of the complex plane. A classical result of Hausdorff and Toeplitz [5, 8] states that $W(T)$ is convex and bounded, and its closure contains the convex hull of $\sigma(T)$. Moreover, $W(T)$ is closed if $\dim(H) < \infty$, but not necessarily when $\dim(H) = \infty$.

It is well known that the numerical radius defines a norm on $B(H)$ equivalent to the operator norm $\|\cdot\|$. Specifically, for all $T \in B(H)$,

$$w(T) \leq \|T\| \leq 2w(T), \tag{1.1}$$

and both inequalities are sharp: equality holds on the left if T is normal, and on the right if $T^2 = 0$.

Let $R(T)$ be the radius of the smallest disk containing $W(T)$. It is known that

$$\frac{1}{2}\operatorname{diam}(W(T)) \leq R(T).$$

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The author of [6] showed that

$$R(T) = \inf_{\lambda \in \mathbb{C}} w(T - \lambda I).$$

In [3], the authors prove that

$$\|T\| \leq w(T) + R(T).$$

We refine this bound by showing

$$\|T\| \leq w(T) + \frac{1}{2} \operatorname{diam}(W(T)).$$

In [1], the authors show that for all $T, S \in B(H)$,

$$w(TS - ST) \leq 4R(T)R(S). \tag{1.2}$$

We further refine this estimate by establishing

$$w(TS - ST) \leq 2\sqrt{R^2(T) + \frac{1}{4} \operatorname{diam}^2(W(T))}\sqrt{R^2(S) + \frac{1}{4} \operatorname{diam}^2(W(S))} \leq 4R(T)R(S).$$

Let $T \in B(H)$. The authors in [3] show that

$$\|TT^* + T^*T\| \leq 2\left(w^2(T) + R^2(T)\right). \tag{1.3}$$

We improve this inequality by proving that

$$\|TT^* + T^*T\| \leq 2\left(w^2(T) + \frac{1}{4} \operatorname{diam}^2(W(T))\right).$$

For a compact set $K \subset \mathbb{C}$, define

$$|K| = \sup_{z \in K} |z|.$$

2. Main results

We need the following lemma.

Lemma 2.1. *[9] Let $T \in B(H)$. Then*

$$w(T) = \sup_{\theta \in \mathbb{R}} \|\Re(e^{i\theta}T)\|.$$

It is clear that from Lemma 2.1,

$$w(T) = \sup_{\theta \in \mathbb{R}} \|\Im(e^{i\theta}T)\|. \tag{2.1}$$

The following lemma is crucial for the development of our main results.

Lemma 2.2. *Let K a compact convex of $\mathbb{C}$. Then there exists $\theta \in [0, \pi]$ such that the projection of $e^{i\theta}K$ on the Ox axis is symmetrical with respect to the origin.*

Proof. Let $P(K)$ the projection of K on the Ox axis. Denote $M(P(K)) = \max P(K)$ and $m(P(K)) = \min P(K)$. We have $M(P(e^{i\pi}K)) = -m(P(K))$ and $m(P(e^{i\pi}K)) = -M(P(K))$. Let $\varphi : [0, \pi] \longrightarrow \mathbb{R}$ defined by $\varphi(\theta) = M(P(e^{i\theta}K)) + m(P(e^{i\theta}K))$. Since the projection is continuous and K is convex, it follows that φ is continuous. We have $\varphi(0) = -\varphi(\pi)$ then there exists $\theta \in [0, \pi]$ such that $\varphi(\theta) = 0$. Thus the projection of $e^{i\theta}K$ on the Ox axis is symmetrical with respect to the origin. $\square$

Proposition 2.1. *Let $T \in B(H)$. Then there exists $\theta \in [0, \pi]$ such that*

$$\|\Re(e^{i\theta}T)\| \leq \frac{1}{2}\, diam(W(T)). \tag{2.2}$$

Proof. Let $K = \overline{W(T)}$. By Lemma 2.2, there exists $\theta \in [0, \pi]$ such that the projection of $e^{i\theta}K$ on the Ox axis is symmetrical with respect to the origin. Hence we get

$$\begin{aligned}
|\Re(e^{i\theta}K)| &= R(\Re(e^{i\theta}K)) \\
&= \frac{1}{2}\,\mathrm{diam}(\Re(e^{i\theta}K)) \\
&\leq \frac{1}{2}\,\mathrm{diam}(e^{i\theta}K) \\
&= \frac{1}{2}\,\mathrm{diam}(K) \\
&= \frac{1}{2}\,\mathrm{diam}(W(T)).
\end{aligned}$$

Since $\|\Re(e^{i\theta}T)\| = |\Re(e^{i\theta}K)|$, then

$$\|\Re(e^{i\theta}T)\| \leq \frac{1}{2}\,\mathrm{diam}(W(T)).$$

□

Theorem 2.1. *Let $T \in B(H)$. Then*

$$\|T\| \leq w(T) + \frac{1}{2}\, diam(W(T)). \tag{2.3}$$

Proof. By Proposition 2.1, there exists $\theta \in [0, \pi]$ such that

$$\|\Re(e^{i\theta}T)\| \leq \frac{1}{2}\,\mathrm{diam}(W(T)).$$

By (2.1), we have

$$\|\Im(e^{i\theta}T)\| \leq w(T).$$

Hence

$$\begin{aligned}
\|T\| &= \|e^{i\theta}T\| \\
&= \|\Re(e^{i\theta}T) + i\Im(e^{i\theta}T)\| \\
&\leq \|\Re(e^{i\theta}T)\| + \|\Im(e^{i\theta}T)\| \\
&\leq \frac{1}{2}\,\mathrm{diam}(K) + w(T).
\end{aligned}$$

Thus

$$\|T\| \leq w(T) + \frac{1}{2}\,\mathrm{diam}(W(T)).$$

□

Remark 2.1. *The inequality* (2.3) *is sharp. Indeed, let* $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. *We have*

$$W(T) = \{z \in \mathbb{C} : |z| \leq \frac{1}{2}\}.$$

Thus

$$\|T\| = \frac{1}{2} + \frac{1}{2} = w(T) + \frac{1}{2}\, diam(W(T)).$$

Corollary 2.1. *Let $T \in B(H)$. Then*

$$\|T\| \leq w(T) + R(T). \tag{2.4}$$

Proof. Since $\operatorname{diam}(W(T)) \leq 2R(T)$, the inequality (2.4) follows from Theorem 2.1. □

Theorem 2.2. *Let $T, S \in B(H)$. Then*

(1) $w(TS) \leq \|TS\| \leq \|S\|\left(w(T) + \frac{1}{2}\, diam(W(T))\right) \leq \|S\|\,(w(T) + R(T)) \leq 2\|S\|w(T)$.

(2)

$$\begin{aligned} w(TS) &\leq \|TS\| \\ &\leq \left(w(T) + \frac{1}{2}\, diam(W(T))\right)\left(w(S) + \frac{1}{2}\, diam(W(S))\right) \\ &\leq (w(T) + R(T))\,(w(S) + R(S)) \\ &\leq 4w(T)w(S). \end{aligned}$$

Proof. (1) Let $\theta \in [0, \pi]$ satisfy (2.2). Hence

$$\begin{aligned} \|TS\| &= \|(e^{i\theta}T)S\| \\ &= \|(\mathfrak{R}(e^{i\theta}T)S + i\mathfrak{I}(e^{i\theta}T)S)\| \\ &\leq \|S\|(\|\mathfrak{R}(e^{i\theta}T)\| + \|\mathfrak{I}(e^{i\theta}T)\|) \\ &\leq \|S\|\left(w(T) + \frac{1}{2}\operatorname{diam}(W(T))\right) \\ &\leq \|S\|\,(w(T) + R(T)) \\ &\leq 2\|S\|w(T). \end{aligned}$$

(2) Similiar as in (1).

□

Recall that T is called *accretive-dissipative* if both $\mathfrak{R}(T)$ and $\mathfrak{I}(T)$ are positive, that is,

$$\langle \mathfrak{R}(T)x, x\rangle > 0, \quad \langle \mathfrak{I}(T)x, x\rangle > 0, \quad \forall\, x \in H \setminus \{0\}.$$

In [7], the authors proved that for every accretive-dissipative operator T,

$$\|T\| \leq \sqrt{2}\, w(T). \tag{2.5}$$

In the same paper, the authors showed that if either T or S is accretive-dissipative, then

$$w(TS) \leq 2\sqrt{2}\, w(T)w(S). \tag{2.6}$$

If either T or S is accretive-dissipative, then using inequality (2.5), it follows that the first assertion of Theorem (2.2) is a refinement of inequality (2.6).

The following lemma will be useful for the next theorem.

Lemma 2.3. *[1] Let $T, S \in B(H)$. Then*

$$w(TS + ST^*) \leq 2\|T\|w(S).$$

Theorem 2.3. *Let $T, S \in B(H)$. Then*

(1) $w(TS + ST) \leq 2w(S)\left(w(T) + \frac{1}{2}\, diam(W(T))\right) \leq 4w(T)w(S)$.

(2) In particular, if T and S commute then

$$w(TS) \leq w(S)\left(w(T) + \frac{1}{2}\, diam(W(T))\right) \leq 2w(T)w(S).$$

Proof. (1) Let $\theta \in [0, \pi]$ satisfy (2.2). Hence

$$\begin{aligned} w(TS+ST) &= w((e^{i\theta}T)S + S(e^{i\theta}T)) \\ &= w\left(\Re(e^{i\theta}T)S + S\Re(e^{i\theta}T) + i(\Im(e^{i\theta}T)S + S\Im(e^{i\theta}T))\right) \\ &\leq w\left(\Re(e^{i\theta}T)S + S\Re(e^{i\theta}T)\right) + w\left(\Im(e^{i\theta}T)S + S\Im(e^{i\theta}T))\right). \end{aligned}$$

Then by Lemma 2.3, we get

$$w(TS+ST) \leq 2\|\Re(e^{i\theta}T)\|w(S) + 2\|\Im(e^{i\theta}T)\|w(S).$$

Thus by using (2.2), it follows that

$$\begin{aligned} w(TS+ST) &\leq 2w(S)(\|\Re(e^{i\theta}T)\| + \|\Im(e^{i\theta}T)\|) \\ &\leq 2w(S)(\frac{1}{2}\operatorname{diam}(W(T)) + w(T)) \\ &\leq 4w(T)w(S). \end{aligned}$$

(2) It is a simple consequence of (1).
□

Lemma 2.4. *[2] Let $T, S \in B(H)$. Then*

$$w(TS \pm ST) \leq \sqrt{\|TT^* + T^*T\|}\sqrt{\|SS^* + S^*S\|}.$$

In [1], the authors prove that for all $T, S \in B(H)$

$$w(TS+ST) \leq 2\|T\|\sqrt{w^2(S) + R^2(S)} \leq 2\|T\|\sqrt{2}w(S), \tag{2.7}$$

and

$$w(TS+ST) \leq 2\sqrt{w^2(T) + R^2(T)}\sqrt{w^2(S) + R^2(S)} \leq 4w(T)w(S). \tag{2.8}$$

We improve the left-hand side of inequalities (2.7) and (2.8) as follows:

Theorem 2.4. *Let $T, S \in B(H)$. Then*

1. $w(TS \pm ST) \leq 2\|T\|\sqrt{w^2(S) + \frac{1}{4}\, diam^2(W(S))} \leq 2\|T\|\sqrt{w^2(S) + R^2(S)}$.
2. $$\begin{aligned} w(TS \pm ST) &\leq 2\sqrt{w^2(T) + \frac{1}{4}\, diam^2(W(T))}\sqrt{w^2(S) + \frac{1}{4}\, diam^2(W(S))} \\ &\leq 2\sqrt{w^2(T) + R^2(T)}\sqrt{w^2(S) + R^2(S)}. \end{aligned}$$

Proof. We prove the second assertion; the first assertion is proved similarly. Let θ and α such that $\|\Re(e^{i\theta}T)\| \leq \frac{1}{2}\operatorname{diam}(W(T))$ and $\|\Re(e^{i\alpha}S)\| \leq \frac{1}{2}\operatorname{diam}(W(S))$. By Lemma 2.4 and (2.4), we get

$$\begin{aligned} w(TS \pm ST) &\leq \sqrt{\|TT^* + T^*T\|}\sqrt{\|SS^* + S^*S\|} \\ &= \sqrt{\|(e^{i\theta}T)(e^{i\theta}T)^* + (e^{i\theta}T)^*(e^{i\theta}T)\|}\sqrt{\|(e^{i\alpha}S)(e^{i\alpha}S)^* + (e^{i\alpha}S)^*(e^{i\alpha}S)\|} \\ &= \sqrt{\|2\Re^2(e^{i\theta}T) + 2\Im^2(e^{i\theta}T)\|}\sqrt{\|2\Re^2(e^{i\alpha}S) + 2\Im^2(e^{i\alpha}S)\|} \\ &\leq \sqrt{2(\|\Re(e^{i\theta}T)\|^2 + \|\Im(e^{i\theta}T)\|^2)}\sqrt{\|2(\Re(e^{i\alpha}S)\|^2 + \|\Im(e^{i\alpha}S)\|^2)} \\ &= 2\sqrt{\|\Re(e^{i\theta}T)\|^2 + \|\Im(e^{i\theta}T)\|^2}\sqrt{\|\Re(e^{i\alpha}S)\|^2 + \|\Im(e^{i\alpha}S)\|^2} \\ &\leq 2\sqrt{w^2(T) + \frac{1}{4}\operatorname{diam}^2(W(T))}\sqrt{w^2(S) + \frac{1}{4}\operatorname{diam}^2(W(S))}. \end{aligned}$$

□

Remark 2.2. *As in the proof of Theorem 2.4, we get*

$$\|TT^* + T^*T\| \leq 2\left(w^2(T) + \frac{1}{4}\, diam^2(W(T))\right). \tag{2.9}$$

Hence, inequality (2.9) *is a refinement of inequality* (1.3).

The following Theorem gives a refinement of inequality (1.2).

Theorem 2.5. *Let $T, S \in B(H)$. Then*

$$w(TS - ST) \leq 2\sqrt{R^2(T) + \frac{1}{4}\, diam^2(W(T))}\sqrt{R^2(S) + \frac{1}{4}\, diam^2(W(S))} \leq 4R(T)R(S).$$

Proof. Fix $t, s \in \mathbb{C}$ such that $R(T) = w(T - tI)$ and $R(S) = w(S - sI)$. By Theorem 2.4, we have

$$\begin{aligned} w(TS - ST) &= w((T - tI)(S - sI) - (S - sI)(T - tI)) \\ &\leq 2\sqrt{w^2(T - tI) + \frac{1}{4}\,\mathrm{diam}^2(W(T - tI))}\sqrt{w^2(S - sI) + \frac{1}{4}\,\mathrm{diam}^2(W(S - sI))} \\ &= 2\sqrt{R^2(T) + \frac{1}{4}\,\mathrm{diam}^2(W(T))}\sqrt{R^2(S) + \frac{1}{4}\,\mathrm{diam}^2(W(S))}. \\ &\leq 4R(T)R(S). \end{aligned}$$

□

Definition 1. *Let $T \in \mathcal{B}(H)$, set*

$$\Gamma(T) = \{A \in \mathcal{B}(H) : AT - TA^* = 0\}.$$

It is clear that $\Gamma(T)$ is a closed real linear subspace of $B(H)$ in the uniform operator topology. Moreover, $\Gamma(T) = \Gamma(T^*)$. Let $dist(T, i\Gamma(T)) = \inf\{\|T - iC\| : C \in \Gamma(T)\}$.

Theorem 2.6. *Let $T, S \in \mathcal{B}(H)$. Then*

$$\begin{aligned} w(TS + ST^*) &\leq 2dist(T, i\Gamma(S))w(S) \\ &\leq 2\|T\|w(S). \end{aligned}$$

Proof. Fix $C \in \Gamma(S)$. We have

$$w(TS + ST^*) = w((T - iC)S + S(T - iC)^*).$$

From Lemma 2.3, we obtain

$$w(TS + ST^*) \leq 2\|T - iC\|w(S).$$

Thus, by taking the infimum over all $C \in \Gamma(S)$, we deduce that

$$\begin{aligned} w(TS + ST^*) &\leq 2dist(T, \Gamma(S))w(S) \\ &\leq 2\|T\|w(S). \end{aligned}$$

□

Remark 2.3. *If we replace T by iT in Theorem 2.6, we obtain*

$$\begin{aligned} w(TS - ST^*) &\leq 2dist(T, i\Gamma(S))w(S) \\ &\leq 2\|T\|w(S). \end{aligned}$$

Theorem 2.7. *Let $T, S \in \mathcal{B}(H)$. Then*

$$\begin{aligned} w(TS+ST) &\le 2\left(dist(\Re(e^{i\theta}T), i\Gamma(S)) + dist(\Im(e^{i\theta}T), i\Gamma(S))\right) w(S) \\ &\le 2\left(w(T) + \frac{1}{2}\, diam(W(T))\right) w(S) \\ &\le 2\,(w(T) + R(T))\, w(S). \end{aligned}$$

If $TS = ST$ then

$$\begin{aligned} w(TS) &\le \left(dist(\Re(e^{i\theta}T), i\Gamma(S)) + dist(\Im(e^{i\theta}T), i\Gamma(S))\right) w(S) \\ &\le \left(w(T) + \frac{1}{2}\, diam(W(T))\right) w(S) \\ &\le (w(T) + R(T))\, w(S). \end{aligned}$$

Proof. We have

$$\begin{aligned} w(TS+ST) &= w\left(S\Re(e^{i\theta}T) + \Re(e^{i\theta}T)S + i\left(S\Im(e^{i\theta}T) + \Im(e^{i\theta}T)S\right)\right) \\ &\le w\left(S\Re(e^{i\theta}T) + \Re(e^{i\theta}T)S\right) + w\left(S\Im(e^{i\theta}T) + \Im(e^{i\theta}T)S\right). \end{aligned}$$

Applying Theorem 2.6, we get

$$\begin{aligned} w(TS+ST) &\le 2\left(dist(\Re(e^{i\theta}T), i\Gamma(S)) + dist(\Im(e^{i\theta}T), i\Gamma(S))\right) w(S) \\ &\le 2\left(\|\Re(e^{i\theta}T), \| + \|\Im(e^{i\theta}T)\|\right) w(S) \\ &\le 2\left((w(T) + \frac{1}{2}\,\mathrm{diam}(W(T))\right) w(S) \\ &\le 2\,(w(T) + R(T))\, w(S). \end{aligned}$$

Thus, the proof is complete. □

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