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Hausdorff type operators and wavelet transform associated with the multidimensional Fourier-Bessel transform

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Abstract. In the present paper, we introduce the Hausdorff operators associated with the multidimensional Fourier-Bessel operator Δ_α; and we prove the boundedness of the multidimensional Fourier-Bessel-Hausdorff operators on the space $L^2_\alpha(\mathbb{R}^d_+)$. We investigate multidimensional Fourier-Bessel wavelet transform, and obtain some useful results. The relation between the multidimensional Fourier-Bessel wavelet transform and multidimensional Fourier-Bessel-Hausdorff operators is also established. The properties of the adjoint multidimensional Fourier-Bessel-Hausdorff operators are further discussed. The results of this paper are illustrated by some examples and figures.

1. Introduction

In this paper, we consider the multidimensional Fourier-Bessel operator [1, 36, 37] defined for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d_+$ by

$$\Delta_\alpha := \sum_{k=1}^{d} \left[\frac{\partial^2}{\partial x_k^2} + \frac{2\alpha_k + 1}{x_k} \frac{\partial}{\partial x_k} \right].$$

This operator has important applications in both pure and applied mathematics and gives rise to a generalization of multi-variable analytic structures like the Fourier-Bessel transform, the Fourier-Bessel convolution and the Fourier-Bessel wavelet transform [2, 5, 6, 23–26, 28–30]. As the harmonic analysis associated with the multidimensional Fourier-Bessel operator has witnessed remarkable development, it is natural to ask whether there exists an analogue of of the Hausdorff operators in the framework of the multidimensional Fourier-Bessel harmonic analysis.

The Hausdorff operators constitute one of the most significant operators in harmonic analysis, and they are used to address various certain classical problems in analysis. This class of operators includes several well-known examples such as Hardy operator, adjoint Hardy operator and the Cesàro operator, among

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many others. Also, the Hardy-Littlewood-Pólya operator and Riemann-Liouville fractional integral can also be derived as particular cases of Hausdorff operator. In the one-dimensional setting, Hausdorff operators on the real line were introduced in [12] and studied on the Hardy space in [15]. The natural generalization in several dimensions was introduced and analyzed in [4, 7, 16]. For a comprehensive overview, the reader is referred to the survey article [17] by Liflyand which presents the main results on Hausdorff operators in various settings and provides an extensive bibliography up to 2013. Recently, Daher and Saadi in [9, 10] investigated the Dunkl-Hausdorff operator on the Lebesgue space $L^1_\alpha(\mathbb{R})$ and on the Hardy space $H^1_\alpha(\mathbb{R})$. Subsequently, Mondal and Poria [19] studied Hausdorff operators associated with the Opdam-Cherednik operator. Another fundamental tool in harmonic analysis is the multidimensional Fourier-Bessel-Hausdorff operators, which is the main object of study in this paper. Precisely, let $L^p_\alpha(\mathbb{R}^d_+)$, $p \in [1, \infty]$, be the space of measurable functions f on $\mathbb{R}^d_+$, for which

$$\|f\|_{L^p_\alpha(\mathbb{R}^d_+)} := \left[\int_{\mathbb{R}^d_+} |f(x)|^p \mathrm{d}\mu_\alpha(x)\right]^{1/p} < \infty,$$
$$\|f\|_{L^\infty_\alpha(\mathbb{R}^d_+)} := \operatorname*{ess\,sup}_{x\in\mathbb{R}^d_+} |f(x)| < \infty,$$

where $x = (x_1, \dots, x_d)$ and

$$\mathrm{d}\mu_\alpha(x) := \prod_{k=1}^{d} \frac{x_k^{2\alpha_k+1}}{2^{\alpha_k}\Gamma(\alpha_k+1)}\mathrm{d}x_k.$$

Specifically, we consider the multidimensional Fourier-Bessel transform

$$\mathscr{F}_\alpha(f)(\lambda) := \int_{\mathbb{R}^d_+} f(x) j_\alpha(\lambda, x)\mathrm{d}\mu_\alpha(x), \quad \lambda \in \mathbb{R}^d_+,$$

where

$$j_\alpha(\lambda, x) := \prod_{k=1}^{d} j_{\alpha_k}(\lambda_k x_k),$$

with j_{α_k} is the normalized Bessel function of the first kind and order α_k (see [34]).
The multidimensional Fourier-Bessel transform can be regarded as a generalization of the Fourier-Bessel transform [3, 11, 20]. Several important results have already been established for the multidimensional Fourier-Bessel transform $\mathscr{F}_\alpha$ (see [2, 23, 25, 28]).

Let $\phi \in L^1(\mathbb{R}_+)$. We define the Hausdorff operator H_ϕ associated with the multidimensional Fourier-Bessel operator Δ_α for $f \in L^1_\alpha(\mathbb{R}^d_+)$ by

$$H_\phi f(x) := \int_{\mathbb{R}_+} f\left(\frac{x}{t}\right) \frac{\phi(t)}{t^{2(\langle\alpha\rangle+d)}}\mathrm{d}t, \quad x \in \mathbb{R}^d_+,$$

where $\langle\alpha\rangle = \alpha_1 + \alpha_2 + \cdots + \alpha_d$.

The main purpose of this paper is to extend some results of the classical Hausdorff operator given in [35] to the framework of multidimensional Fourier-Bessel harmonic analysis, and to investigate the multidimensional Fourier-Bessel wavelet transform. We prove the boundedness of multidimensional Fourier-Bessel-Hausdorff operators on $L^2_\alpha(\mathbb{R}^d_+)$ space. The relation between multidimensional Fourier-Bessel wavelet transform and multidimensional Fourier-Bessel-Hausdorff operators is also established. Next, we introduce the adjoint operator H^*_ϕ on $L^2_\alpha(\mathbb{R}^d_+)$ by

$$H^*_\phi f(x) := \int_{\mathbb{R}_+} f(tx)\phi(t)\mathrm{d}t, \quad x \in \mathbb{R}^d_+.$$

We also give the properties of the adjoint operator H^*_ϕ, including its boundedness on $L^2_\alpha(\mathbb{R}^d_+)$. We also establish a relation between the multidimensional Fourier-Bessel wavelet transform and the adjoint operator H^*_ϕ.

In particular cases, we examine the multidimensional Fourier-Bessel-Cesàro operator C_β, $\beta = 1, 2, \ldots$, and its adjoint C_β^*. We also analyze the multidimensional Fourier-Bessel-Hardy operator $\mathcal{H}$ and its adjoint $\mathcal{H}^*$. Furthermore, we provide numerical and graphical results for these operators.

The paper is organized as follows. In Section 2, we recall some results about the multidimensional Fourier-Bessel harmonic analysis. In Section 3, we introduce the multidimensional Fourier-Bessel-Hausdorff operators H_ϕ and establish their properties. In Section 4, we investigate the multidimensional Fourier-Bessel wavelet transform and derive its relation with the operators H_ϕ and H_ϕ^*. Next, in Section 5, we present numerical and graphical results concerning the multidimensional Fourier-Bessel-Cesàro operator C_β, $\beta = 1, 2, \ldots$, and the multidimensional Fourier-Bessel-Hardy operator $\mathcal{H}$ and its adjoint $\mathcal{H}^*$, in the case $d = 2$ and $\alpha = (\frac{1}{2}, \frac{1}{2})$. In the last section we provide a conclusion and outline future perspectives.

2. Multidimensional Fourier-Bessel harmonic analysis

In this section we recall some basic results related to the multidimensional Fourier-Bessel harmonic analysis [2, 5, 6, 23–26, 28].

Let $\alpha = (\alpha_1, \ldots, \alpha_d) \in [-\frac{1}{2}, \infty)^d$, we denote by Δ_α, the multidimensional Fourier-Bessel operator [1, 36, 37] defined for $x = (x_1, \ldots, x_d) \in \mathbb{R}_+^d$ by

$$\Delta_\alpha := \sum_{k=1}^{d} \left[\frac{\partial^2}{\partial x_k^2} + \frac{2\alpha_k + 1}{x_k} \frac{\partial}{\partial x_k} \right].$$

For any $\lambda \in \mathbb{R}_+^d$, the system

$$\Delta_\alpha u(x) = -|\lambda|^2 u(x), \quad u(0) = 1, \quad \frac{\partial}{\partial x_k} u(x)\Big|_{x_k=0} = 0, \quad k = 1, \ldots, d,$$

admits a unique solution $j_\alpha(\lambda, x)$, given by

$$j_\alpha(\lambda, x) := \prod_{k=1}^{d} j_{\alpha_k}(\lambda_k x_k),$$

where j_{α_k} is the normalized Bessel function of the first kind and order α_k (see [34]) given by

$$j_{\alpha_k}(x_k) := \Gamma(\alpha_k + 1) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n + \alpha_k + 1)} \left(\frac{x_k}{2}\right)^{2n}.$$

For all $x, \lambda \in \mathbb{R}_+^d$, the kernel $j_\alpha(\lambda, x)$ satisfies

$$|j_\alpha(\lambda, x))| \leq 1.$$

The kernel $j_\alpha(\lambda, x)$ gives rise to an integral transform, which is called multidimensional Fourier-Bessel transform on $\mathbb{R}_+^d$, for which many fundamental properties have been established [1]. The multidimensional Fourier-Bessel transform $\mathscr{F}_\alpha$ is defined for $f \in L_\alpha^1(\mathbb{R}_+^d)$ by

$$\mathscr{F}_\alpha(f)(\lambda) := \int_{\mathbb{R}_+^d} f(x) j_\alpha(\lambda, x) \mathrm{d}\mu_\alpha(x), \quad \lambda \in \mathbb{R}_+^d.$$

Moreover if $f \in L_\alpha^1(\mathbb{R}_+^d)$, then

$$\|\mathscr{F}_\alpha(f)\|_{L_\alpha^\infty(\mathbb{R}_+^d)} \leq \|f\|_{L_\alpha^1(\mathbb{R}_+^d)}.$$

Theorem 2.1. (See [2, 23, 25]).

(i) Plancherel formula for $\mathscr{F}_\alpha$. The transform $\mathscr{F}_\alpha$ extends uniquely to an isometric isomorphism on $L^2_\alpha(\mathbb{R}^d_+)$, onto itself. In particular,

$$\|\mathscr{F}_\alpha(f)\|_{L^2_\alpha(\mathbb{R}^d_+)} = \|f\|_{L^2_\alpha(\mathbb{R}^d_+)}.$$

(ii) Parseval formula for $\mathscr{F}_\alpha$. For all $f, g \in L^2_\alpha(\mathbb{R}^d_+)$, we have

$$\langle \mathscr{F}_\alpha(f), \mathscr{F}_\alpha(g)\rangle_{L^2_\alpha(\mathbb{R}^d_+)} = \langle f, g\rangle_{L^2_\alpha(\mathbb{R}^d_+)}.$$

(iii) Inversion formula for $\mathscr{F}_\alpha$. If f and $\mathscr{F}_\alpha(f)$ are both in $L^1_\alpha(\mathbb{R}^d_+)$, then

$$f(x) = \int_{\mathbb{R}^d_+} \mathscr{F}_\alpha(f)(\lambda) j_\alpha(\lambda, x) \mathrm{d}\mu_\alpha(\lambda), \quad \text{a.e} \quad x \in \mathbb{R}^d_+,$$

and

$$\mathscr{F}_\alpha^{-1}(f)(x) = \mathscr{F}_\alpha(f)(x).$$

We denote by $C_*(\mathbb{R}^d_+)$, the space of continuous functions f on $\mathbb{R}^d_+$, even with respect to each variable. For $f \in C_*(\mathbb{R}^d_+)$ and $x, y \in \mathbb{R}^d_+$, we define the multidimensional Fourier-Bessel translation operator (see [2, 26, 36]) by

$$\tau_x f(y) := a_\alpha \int_{(0,\pi)^d} f\left([x_1, y_1]_{\theta_1}, \dots, [x_d, y_d]_{\theta_d}\right) \times \prod_{k=1}^{d} (\sin\theta_k)^{2\alpha_k} \mathrm{d}\theta_1 \dots \mathrm{d}\theta_d, \tag{1}$$

where $[x_i, y_i]_{\theta_i} := \sqrt{x_i^2 + y_i^2 + 2x_i y_i \cos\theta_i}$, $i = 1, \dots, d$ and $a_\alpha := \prod_{k=1}^d \frac{\Gamma(\alpha_k+1)}{\sqrt{\pi}\Gamma(\alpha_k+1/2)}$.

Theorem 2.2. (See [2, 6]).

(i) For suitable function f, for all $x, y \in \mathbb{R}^d_+$, we have

$$\tau_x f(y) = \tau_y f(x) \quad \text{and} \quad \tau_0 f(x) = f(x).$$

(ii) For all $\lambda, x, y \in \mathbb{R}^d_+$, we have the product formula

$$\tau_x(j_\alpha(\lambda, .))(y) = j_\alpha(\lambda, x) j_\alpha(\lambda, y).$$

(iii) For $f \in L^p_\alpha(\mathbb{R}^d_+)$, $p \in [1, \infty]$, and $x \in \mathbb{R}^d_+$, then $\tau_x f \in L^p_\alpha(\mathbb{R}^d_+)$ and

$$\left\|\tau_x f\right\|_{L^p_\alpha(\mathbb{R}^d_+)} \le \|f\|_{L^p_\alpha(\mathbb{R}^d_+)}.$$

(iv) For $f \in L^p_\alpha(\mathbb{R}^d_+)$, $p = 1, 2$ and $x \in \mathbb{R}^d_+$, we have

$$\mathscr{F}_\alpha\left(\tau_x f\right)(\lambda) = j_\alpha(\lambda, x)\mathscr{F}_\alpha(f)(\lambda), \quad \lambda \in \mathbb{R}^d_+.$$

Let $f, g \in L^2_\alpha(\mathbb{R}^d_+)$. The multidimensional Fourier-Bessel convolution product (see [2, 26, 31]) of f and g is defined by

$$f * g(x) := \int_{\mathbb{R}^d_+} f(y)\tau_x g(y) \mathrm{d}\mu_\alpha(y), \quad x \in \mathbb{R}^d_+. \tag{2}$$

The convolution $*$ is commutative, associative and satisfies the Young inequality (see [2]). Let $p, q, r \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Then for $f \in L^p_\alpha(\mathbb{R}^d_+)$ and $g \in L^q_\alpha(\mathbb{R}^d_+)$ we have

$$\|f * g\|_{L^r_\alpha(\mathbb{R}^d_+)} \le \|f\|_{L^p_\alpha(\mathbb{R}^d_+)} \|g\|_{L^q_\alpha(\mathbb{R}^d_+)}.$$

Theorem 2.3. (See [6]).

(i) For $f, g \in L^2_\alpha(\mathbb{R}^d_+)$, the function $f * g$ belongs to $L^\infty_\alpha(\mathbb{R}^d_+)$, and

$$f * g = \mathscr{F}_\alpha^{-1}(\mathscr{F}_\alpha(f)\mathscr{F}_\alpha(g)).$$

(ii) Let $f, g \in L^2_\alpha(\mathbb{R}^d_+)$. Then $f * g$ belongs to $L^2_\alpha(\mathbb{R}^d_+)$ if and only if $\mathscr{F}_\alpha(f)\mathscr{F}_\alpha(g)$ belongs to $L^2_\alpha(\mathbb{R}^d_+)$, and

$$\mathscr{F}_\alpha(f * g) = \mathscr{F}_\alpha(f)\mathscr{F}_\alpha(g), \quad \text{in the } L^2_\alpha(\mathbb{R}^d_+) - \text{case}.$$

(iii) Let $f, g \in L^2_\alpha(\mathbb{R}^d_+)$. Then

$$\int_{\mathbb{R}^d_+} |f * g(x)|^2 \mathrm{d}\mu_\alpha(x) = \int_{\mathbb{R}^d_+} |\mathscr{F}_\alpha(f)(\lambda)|^2 |\mathscr{F}_\alpha(g)(\lambda)|^2 \mathrm{d}\mu_\alpha(\lambda),$$

where both sides are finite or infinite.

3. Multidimensional Fourier-Bessel-Hausdorff operators

In this section we define and study the Hausdorff operator associated with the multidimensional Fourier-Bessel operator Δ_α. As in the same of the Weinstein case (see [22]) we obtain.

Lemma 3.1. Let $f \in L^2_\alpha(\mathbb{R}^d_+)$, and $t > 0$. The function f_t given by

$$f_t(x) = \frac{1}{t^{2(\langle\alpha\rangle+d)}} f\left(\frac{x}{t}\right), \quad x \in \mathbb{R}^d_+,$$

satisfies

$$\mathscr{F}_\alpha(f_t)(\lambda) = \mathscr{F}_\alpha(f)(t\lambda), \quad \lambda \in \mathbb{R}^d_+, \tag{3}$$

and

$$\|f_t\|_{L^2_\alpha(\mathbb{R}^d_+)} = \frac{1}{t^{\langle\alpha\rangle+d}} \|f\|_{L^2_\alpha(\mathbb{R}^d_+)}. \tag{4}$$

Let $\phi \in L^1(\mathbb{R}_+)$. We define the Hausdorff operator H_ϕ associated with the multidimensional Fourier-Bessel operator Δ_α for $f \in L^1_\alpha(\mathbb{R}^d_+)$ by

$$H_\phi f(x) := \int_{\mathbb{R}_+} f\left(\frac{x}{t}\right) \frac{\phi(t)}{t^{2(\langle\alpha\rangle+d)}} \mathrm{d}t, \quad x \in \mathbb{R}^d_+. \tag{5}$$

Theorem 3.2. Let $\phi \in L^1(\mathbb{R}_+)$. Then for $f \in L^1_\alpha(\mathbb{R}^d_+)$, we have

$$\mathscr{F}_\alpha(H_\phi f)(\lambda) = \int_{\mathbb{R}_+} \mathscr{F}_\alpha(f)(t\lambda)\phi(t)\mathrm{d}t, \quad \lambda \in \mathbb{R}^d_+.$$

Proof. Let $\phi \in L^1(\mathbb{R}_+)$, and let $f \in L^1_\alpha(\mathbb{R}^d_+)$. Then by (5), we have

$$\begin{aligned}\mathscr{F}_\alpha(H_\phi f)(\lambda) &= \int_{\mathbb{R}^d_+} H_\phi f(x) j_\alpha(\lambda, x)\mathrm{d}\mu_\alpha(x) \\ &= \int_{\mathbb{R}^d_+} \left[\int_{\mathbb{R}_+} f\left(\frac{x}{t}\right) \frac{\phi(t)}{t^{2(\langle\alpha\rangle+d)}} \mathrm{d}t\right] j_\alpha(\lambda, x)\mathrm{d}\mu_\alpha(x).\end{aligned}$$

Since

$$\int_{\mathbb{R}^d_+} \int_{\mathbb{R}_+} \left|f\left(\frac{x}{t}\right)\right| \frac{|\phi(t)|}{t^{2(\langle\alpha\rangle+d)}} \mathrm{d}t\mathrm{d}\mu_\alpha(x) \le \|f\|_{L^1_\alpha(\mathbb{R}^d_+)} \|\phi\|_{L^1(\mathbb{R}_+)} < \infty,$$

by Fubini's theorem we obtain

$$\mathscr{F}_\alpha(H_\phi f)(\lambda) = \int_{\mathbb{R}_+} \mathscr{F}_\alpha(f)(t\lambda)\phi(t)\mathrm{d}t, \quad \lambda \in \mathbb{R}^d_+.$$

The theorem is proved. □

Theorem 3.3. Let ϕ be a measurable function on $\mathbb{R}_+$ such that

$$C_{\phi,2} := \int_{\mathbb{R}_+} \frac{|\phi(t)|}{t^{\langle\alpha\rangle+d}}\mathrm{d}t < \infty. \tag{6}$$

Then the Hausdorff operator H_ϕ is bounded on $L^2_\alpha(\mathbb{R}^d_+)$, with

$$\|H_\phi f\|_{L^2_\alpha(\mathbb{R}^d_+)} \leq C_{\phi,2}\|f\|_{L^2_\alpha(\mathbb{R}^d_+)}.$$

Proof. By using Minkowski's inequality for integrals, we have

$$\begin{aligned}
\|H_\phi f\|_{L^2_\alpha(\mathbb{R}^d_+)} &= \left[\int_{\mathbb{R}^d_+} \left|\int_{\mathbb{R}_+} f_t(x)\phi(t)\mathrm{d}t\right|^2 \mathrm{d}\mu_\alpha(x)\right]^{1/2} \\
&\leq \left[\int_{\mathbb{R}^d_+} \left(\int_{\mathbb{R}_+} |f_t(x)||\phi(t)|\mathrm{d}t\right)^2 \mathrm{d}\mu_\alpha(x)\right]^{1/2} \\
&\leq \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}^d_+} |f_t(x)|^2|\phi(t)|^2\mathrm{d}\mu_\alpha(x)\right)^{1/2} \mathrm{d}t \\
&= \int_{\mathbb{R}_+} \|f_t\|_{L^2_\alpha(\mathbb{R}^d_+)}|\phi(t)|\mathrm{d}t.
\end{aligned}$$

Then by (4) we obtain

$$\|H_\phi f\|_{L^2_\alpha(\mathbb{R}^d_+)} \leq C_{\phi,2}\|f\|_{L^2_\alpha(\mathbb{R}^d_+)}.$$

Going back to the definition of

$$\left[\int_{\mathbb{R}^d_+} \left(\int_{\mathbb{R}_+} |f_t(x)||\phi(t)|\mathrm{d}t\right)^2 \mathrm{d}\mu_\alpha(x)\right]^{1/2},$$

we deduce that the integral

$$H_\phi f(x) = \int_{\mathbb{R}_+} f_t(x)\phi(t)\mathrm{d}t,$$

is absolutely convergent for almost all $x \in \mathbb{R}^d_+$, and defines a function $H_\phi f \in L^2_\alpha(\mathbb{R}^d_+)$. □

Remark 3.4. Let ϕ be a measurable function on $\mathbb{R}_+$ such that

$$C_{\phi,p} := \int_{\mathbb{R}_+} \frac{|\phi(t)|}{t^{2(\langle\alpha\rangle+d)(1-\frac{1}{p})}}\mathrm{d}t < \infty,$$

for some $p \in [1,\infty)$. Then the Hausdorff operator H_ϕ is bounded on $L^p_\alpha(\mathbb{R}^d_+)$, with

$$\|H_\phi f\|_{L^p_\alpha(\mathbb{R}^d_+)} \leq C_{\phi,p}\|f\|_{L^p_\alpha(\mathbb{R}^d_+)}.$$

In the next part of this section we define and give the properties of the adjoint operator to H_ϕ, such that the boundedness on $L^2_\alpha(\mathbb{R}^d_+)$, and the relation between the multidimensional Fourier-Bessel transform $\mathscr{F}_\alpha$ and the adjoint operator H^*_ϕ.

Let $f, g \in L^2_\alpha(\mathbb{R}^d_+)$, and let ϕ be a measurable function on $\mathbb{R}_+$ satisfying the condition (6). We define the adjoint operator H^*_ϕ by the relation

$$\int_{\mathbb{R}^d_+} H^*_\phi f(x) g(x) \mathrm{d}\mu_\alpha(x) = \int_{\mathbb{R}^d_+} f(x) H_\phi g(x) \mathrm{d}\mu_\alpha(x).$$

From Theorem 3.3, the operator H^*_ϕ is bounded on $L^2_\alpha(\mathbb{R}^d_+)$, with

$$\|H^*_\phi f\|_{L^2_\alpha(\mathbb{R}^d_+)} \leq C_{\phi,2} \|f\|_{L^2_\alpha(\mathbb{R}^d_+)}, \tag{7}$$

where $C_{\phi,2}$ is the constant given by (6).

Theorem 3.5. Let $f \in L^2_\alpha(\mathbb{R}^d_+)$, and let ϕ be a measurable function on $\mathbb{R}_+$ satisfying the condition (6). Then

$$H^*_\phi f(x) = \int_{\mathbb{R}_+} f(tx)\phi(t) \mathrm{d}t. \tag{8}$$

Proof. Let $f, g \in L^2_\alpha(\mathbb{R}^d_+)$, and let ϕ be a measurable function on $\mathbb{R}_+$ satisfying the condition (6). From (5) and Fubini's theorem we have

$$\begin{aligned}\int_{\mathbb{R}^d_+} f(x) H_\phi g(x) \mathrm{d}\mu_\alpha(x) &= \int_{\mathbb{R}^d_+} f(x) \left[\int_{\mathbb{R}_+} g_t(x)\phi(t)\mathrm{d}t\right] \mathrm{d}\mu_\alpha(x) \\ &= \int_{\mathbb{R}_+} \left[\int_{\mathbb{R}^d_+} f(x) g_t(x) \mathrm{d}\mu_\alpha(x)\right] \phi(t)\mathrm{d}t \\ &= \int_{\mathbb{R}_+} \left[\int_{\mathbb{R}^d_+} f(tx) g(x) \mathrm{d}\mu_\alpha(x)\right] \phi(t)\mathrm{d}t.\end{aligned}$$

Using (4), this calculation is justified by the fact that

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}^d_+} |f(x)||g_t(x)| \mathrm{d}\mu_\alpha(x) |\phi(t)| \mathrm{d}t \leq C_{\phi,2} \|f\|_{L^2_\alpha(\mathbb{R}^d_+)} \|g\|_{L^2_\alpha(\mathbb{R}^d_+)} < \infty.$$

Then according to Fubini's theorem we obtain

$$\begin{aligned}\int_{\mathbb{R}^d_+} f(x) H_\phi g(x) \mathrm{d}\mu_\alpha(x) &= \int_{\mathbb{R}^d_+} \left[\int_{\mathbb{R}_+} f(tx)\phi(t)\mathrm{d}t\right] g(x) \mathrm{d}\mu_\alpha(x) \\ &= \int_{\mathbb{R}^d_+} H^*_\phi f(x) g(x) \mathrm{d}\mu_\alpha(x),\end{aligned}$$

where

$$H^*_\phi f(x) = \int_{\mathbb{R}_+} f(tx)\phi(t) \mathrm{d}t.$$

This calculation is justified by the fact that

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}^d_+} |f(tx)||g(x)| \mathrm{d}\mu_\alpha(x) |\phi(t)| \mathrm{d}t \leq C_{\phi,2} \|f\|_{L^2_\alpha(\mathbb{R}^d_+)} \|g\|_{L^2_\alpha(\mathbb{R}^d_+)} < \infty.$$

This completes the proof of the theorem. □

Theorem 3.6. Let ϕ be a measurable function on $\mathbb{R}_+$ satisfying the condition

$$C_{\phi,\infty} := \int_{\mathbb{R}_+} \frac{|\phi(t)|}{t^{2(\langle\alpha\rangle+d)}} \mathrm{d}t < \infty. \tag{9}$$

Then for $f \in L^1_\alpha(\mathbb{R}^d_+)$, we have

$$\mathscr{F}_\alpha(H^*_\phi f)(\lambda) = \int_{\mathbb{R}_+} \mathscr{F}_\alpha(f)(\frac{\lambda}{t})\frac{\phi(t)}{t^{2(\langle\alpha\rangle+d)}}\mathrm{d}t, \quad \lambda \in \mathbb{R}^d_+.$$

Proof. Let ϕ be a measurable function on $\mathbb{R}_+$ satisfying the condition (9), and let $f \in L^1_\alpha(\mathbb{R}^d_+)$. Then by (8), we have

$$\begin{aligned}\mathscr{F}_\alpha(H^*_\phi f)(\lambda) &= \int_{\mathbb{R}^d_+} H^*_\phi f(x) j_\alpha(\lambda, x)\mathrm{d}\mu_\alpha(x)\\ &= \int_{\mathbb{R}^d_+}\left[\int_{\mathbb{R}_+} f(tx)\phi(t)\mathrm{d}t\right] j_\alpha(\lambda, x)\mathrm{d}\mu_\alpha(x).\end{aligned}$$

Since

$$\int_{\mathbb{R}^d_+}\int_{\mathbb{R}_+} |f(tx)||\phi(t)|\mathrm{d}t\mathrm{d}\mu_\alpha(x) \le C_{\phi,\infty}\|f\|_{L^1_\alpha(\mathbb{R}^d_+)} < \infty,$$

by Fubini's theorem we obtain

$$\mathscr{F}_\alpha(H^*_\phi f)(\lambda) = \int_{\mathbb{R}_+} \mathscr{F}_\alpha(f)(\frac{\lambda}{t})\frac{\phi(t)}{t^{2(\langle\alpha\rangle+d)}}\mathrm{d}t, \quad \lambda \in \mathbb{R}^d_+.$$

The theorem is proved. □

4. Multidimensional Fourier-Bessel wavelet transform

In this section, we first recall some fundamental results on the multidimensional Fourier-Bessel wavelet transform. This transform has been studied extensively in [5, 6, 27, 31] where detailed definitions, illustrative examples, and comprehensive discussions of its properties can be found. By using the harmonic analysis associated with the operator Δ_α, we establish a relation between the multidimensional Fourier-Bessel wavelet transform and the multidimensional Fourier-Bessel-Hausdorff operators.

We say that a function $g \in L^2_\alpha(\mathbb{R}^d_+)$ is a multidimensional Fourier-Bessel wavelet, if it satisfies for almost all $\lambda \in \mathbb{R}^d_+$, the admissibility condition

$$0 < \omega_g := \int_{\mathbb{R}_+} |\mathscr{F}_\alpha(g)(a\lambda)|^2\frac{\mathrm{d}a}{a} < \infty. \tag{10}$$

Condition (10) is well known in the literature [5, 6, 18, 21, 27, 31], and the constant ω_g is independent of λ.
Example. The function g given by

$$g(x) := \int_{\mathbb{R}^d_+} |\lambda|^2 e^{-|\lambda|^2} j_\alpha(\lambda, x)\mathrm{d}\mu_\alpha(\lambda), \quad x \in \mathbb{R}^d_+,$$

is a multidimensional Fourier-Bessel wavelet and $\omega_g = \frac{1}{8}$.

For a function $g \in L^2_\alpha(\mathbb{R}^d_+)$ and for $(a, b) \in \mathbb{R}^*_+ \times \mathbb{R}^d_+$ we denote by $g_{a,b}$ the function defined on $\mathbb{R}^d_+$ by

$$g_{a,b}(x) := \tau_b g_a(x),$$

where τ_b are the multidimensional Fourier-Bessel translation operators given by (1).

From Theorem 2.2 (iii) and (4) the function $g_{a,b}$ satisfies

$$\|g_{a,b}\|_{L^2_\alpha(\mathbb{R}^d_+)} \le \frac{1}{a^{\langle\alpha\rangle+d}}\|g\|_{L^2_\alpha(\mathbb{R}^d_+)}. \tag{11}$$

Let $g \in L^2_\alpha(\mathbb{R}^d_+)$ be a multidimensional Fourier-Bessel wavelet. We define for $f \in L^2_\alpha(\mathbb{R}^d_+)$, the multidimensional Fourier-Bessel wavelet transform by

$$\Phi_g(f)(a,b) := \int_{\mathbb{R}^d_+} f(x) g_{a,b}(x) \mathrm{d}\mu_\alpha(x), \tag{12}$$

which can also be written in the form

$$\Phi_g(f)(a,b) = f * g_a(b), \tag{13}$$

where $*$ is the multidimensional Fourier-Bessel convolution product given by (2).

From (11) and (12) with Hölder's inequality we have

$$\|\Phi_g(f)(a,.)\|_\infty \le \frac{1}{a^{\langle\alpha\rangle+d}} \|f\|_{L^2_\alpha(\mathbb{R}^d_+)} \|g\|_{L^2_\alpha(\mathbb{R}^d_+)}.$$

From Theorem 2.1 (iii), Theorem 2.3 (i) and (3) we have

$$\Phi_g(f)(a,b) = \int_{\mathbb{R}^d_+} \mathscr{F}_\alpha(f)(\lambda) \mathscr{F}_\alpha(g)(a\lambda) j_\alpha(\lambda, b) \mathrm{d}\mu_\alpha(\lambda). \tag{14}$$

We denote by $L^2_\alpha(\mathbb{R}_+ \times \mathbb{R}^d_+)$, the space of measurable functions f on $\mathbb{R}_+ \times \mathbb{R}^d_+$, such that

$$\|f\|_{L^2_\alpha(\mathbb{R}_+ \times \mathbb{R}^d_+)} := \left[\int_{\mathbb{R}_+} \int_{\mathbb{R}^d_+} |f(a,b)|^2 \mathrm{d}\mu_\alpha(b) \frac{\mathrm{d}a}{a} \right]^{1/2} < \infty.$$

Theorem 4.1. Let $g \in L^2_\alpha(\mathbb{R}^d_+)$ be a multidimensional Fourier-Bessel wavelet.

(i) Plancherel formula for Φ_g. For $f \in L^2_\alpha(\mathbb{R}^d_+)$ we have

$$\|f\|^2_{L^2_\alpha(\mathbb{R}^d_+)} = \frac{1}{\omega_g} \|\Phi_g(f)\|^2_{L^2_\alpha(\mathbb{R}_+ \times \mathbb{R}^d_+)}.$$

(ii) Parseval formula for Φ_g. For $f, h \in L^2_\alpha(\mathbb{R}^d_+)$ we have

$$\langle f, h \rangle_{L^2_\alpha(\mathbb{R}^d_+)} = \frac{1}{\omega_g} \langle \Phi_g(f), \Phi_g(h) \rangle_{L^2_\alpha(\mathbb{R}_+ \times \mathbb{R}^d_+)}.$$

Proof. (i) Using Fubini's theorem, Theorem 2.3 (iii), and the relation (13), we obtain

$$\begin{aligned}
\frac{1}{\omega_g} \|\Phi_g(f)\|^2_{L^2_\alpha(\mathbb{R}_+ \times \mathbb{R}^d_+)} &= \frac{1}{\omega_g} \int_{\mathbb{R}_+} \int_{\mathbb{R}^d_+} |f * g_a(b)|^2 \mathrm{d}\mu_\alpha(b) \frac{\mathrm{d}a}{a} \\
&= \frac{1}{\omega_g} \int_{\mathbb{R}_+} \int_{\mathbb{R}^d_+} |\mathscr{F}_\alpha(f)(\lambda)|^2 |\mathscr{F}_\alpha(g_a)(\lambda)|^2 \mathrm{d}\mu_\alpha(\lambda) \frac{\mathrm{d}a}{a} \\
&= \int_{\mathbb{R}^d_+} |\mathscr{F}_\alpha(f)(\lambda)|^2 \left(\frac{1}{\omega_g} \int_{\mathbb{R}_+} |\mathscr{F}_\alpha(g)(a\lambda)|^2 \frac{\mathrm{d}a}{a} \right) \mathrm{d}\mu_\alpha(\lambda).
\end{aligned}$$

By relation (10) we have

$$\frac{1}{\omega_g} \int_{\mathbb{R}_+} |\mathscr{F}_\alpha(g)(a\lambda)|^2 \frac{\mathrm{d}a}{a} = 1.$$

Then we deduce the desired result from Theorem 2.1 (i).

(ii) The result is easily deduced from (i). □

We define the operator Φ_g^* for $F \in L^2_\alpha(\mathbb{R}_+ \times \mathbb{R}^d_+)$ by

$$\Phi_g^*(F)(x) := \int_{\mathbb{R}_+ \times \mathbb{R}^d_+} F(a,b)\overline{g_{a,b}(x)}\mathrm{d}\mu_\alpha(b)\frac{\mathrm{d}a}{a}, \quad x \in \mathbb{R}^d_+. \tag{15}$$

For $f \in L^2_\alpha(\mathbb{R}^d_+)$ and $F \in L^2_\alpha(\mathbb{R}_+ \times \mathbb{R}^d_+)$ we have Φ_g^* by

$$\langle \Phi_g(f), F\rangle_{L^2_\alpha(\mathbb{R}_+\times\mathbb{R}^d_+)} = \langle f, \Phi_g^*(F)\rangle_{L^2_\alpha(\mathbb{R}^d_+)}.$$

Theorem 4.2. Let $g \in L^2_\alpha(\mathbb{R}^d_+)$ be a multidimensional Fourier-Bessel wavelet. For $F \in L^2_\alpha(\mathbb{R}_+ \times \mathbb{R}^d_+)$ we have

$$\Phi_g\Phi_g^*(F)(a,b) = \int_{\mathbb{R}_+ \times \mathbb{R}^d_+} F(a',b')W_g((a,b);(a',b'))\mathrm{d}\mu_\alpha(b')\frac{\mathrm{d}a'}{a'},$$

where

$$W_g((a,b);(a',b')) = \int_{\mathbb{R}^d_+} g_{a,b}(x)\overline{g_{a',b'}(x)}\mathrm{d}\mu_\alpha(x).$$

Proof. Let $F \in L^2_\alpha(\mathbb{R}_+ \times \mathbb{R}^d_+)$. From (12), (15) and Fubini's theorem we have

$$\begin{aligned}\Phi_g\Phi_g^*(F)(a,b) &= \int_{\mathbb{R}^d_+} \Phi_g^*(F)(x)g_{a,b}(x)\mathrm{d}\mu_\alpha(x)\\ &= \int_{\mathbb{R}^d_+}\left(\int_{\mathbb{R}_+ \times \mathbb{R}^d_+} F(a',b')\overline{g_{a',b'}(x)}\mathrm{d}\mu_\alpha(b')\frac{\mathrm{d}a'}{a'}\right)g_{a,b}(x)\mathrm{d}\mu_\alpha(x)\\ &= \int_{\mathbb{R}_+ \times \mathbb{R}^d_+} F(a',b')\left(\int_{\mathbb{R}^d_+} g_{a,b}(x)\overline{g_{a',b'}(x)}\mathrm{d}\mu_\alpha(x)\right)\mathrm{d}\mu_\alpha(b')\frac{\mathrm{d}a'}{a'}.\end{aligned}$$

Therefore we obtain

$$\Phi_g\Phi_g^*(F)(a,b) = \int_{\mathbb{R}_+ \times \mathbb{R}^d_+} F(a',b')W_g((a,b);(a',b'))\mathrm{d}\mu_\alpha(b')\frac{\mathrm{d}a'}{a'},$$

where

$$W_g((a,b);(a',b')) = \int_{\mathbb{R}^d_+} g_{a,b}(x)\overline{g_{a',b'}(x)}\mathrm{d}\mu_\alpha(x).$$

The theorem is proved. □

We obtain a relation between the multidimensional Fourier-Bessel wavelet transform and the multidimensional Fourier-Bessel-Hausdorff operators.

Theorem 4.3. Let $g \in L^2_\alpha(\mathbb{R}^d_+)$ be a multidimensional Fourier-Bessel wavelet, and let ϕ be a measurable function on $\mathbb{R}_+$ satisfying the condition (6). Then for $f \in L^2_\alpha(\mathbb{R}^d_+)$ we have

$$\Phi_g(H_\phi f)(a,b) = \int_{\mathbb{R}_+} \Phi_g(f)(\frac{a}{t},\frac{b}{t})\frac{\phi(t)}{t^{2(\langle\alpha\rangle+d)}}\mathrm{d}t.$$

Proof. Let $g \in L^2_\alpha(\mathbb{R}^d_+)$ be a multidimensional Fourier-Bessel wavelet, and let $f \in L^2_\alpha(\mathbb{R}^d_+)$. From Theorem 3.3 we have $H_\phi f \in L^2_\alpha(\mathbb{R}^d_+)$. Then by (5) and (12), we get

$$\begin{aligned}\Phi_g(H_\phi f)(a,b) &= \int_{\mathbb{R}^d_+} H_\phi f(x)g_{a,b}(x)\mathrm{d}\mu_\alpha(x)\\ &= \int_{\mathbb{R}^d_+}\left[\int_{\mathbb{R}_+} f_t(x)\phi(t)\mathrm{d}t\right]g_{a,b}(x)\mathrm{d}\mu_\alpha(x)\\ &= \int_{\mathbb{R}_+}\left[\int_{\mathbb{R}^d_+} f_t(x)g_{a,b}(x)\mathrm{d}\mu_\alpha(x)\right]\phi(t)\mathrm{d}t\\ &= \int_{\mathbb{R}_+} \Phi_g(f_t)(a,b)\phi(t)\mathrm{d}t.\end{aligned}$$

According to (14) we have

$$\Phi_g(f_t)(a,b) = \frac{1}{t^{2(\langle\alpha\rangle+d)}}\Phi_g(f)(\frac{a}{t},\frac{b}{t}). \tag{16}$$

Therefore we deduce that

$$\Phi_g(H_\phi f)(a,b) = \int_{\mathbb{R}_+} \Phi_g(f)(\frac{a}{t},\frac{b}{t})\frac{\phi(t)}{t^{2(\langle\alpha\rangle+d)}}\mathrm{d}t.$$

Using (4) and (11), this calculation is justified by the fact that

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}^d_+} |f_t(x)||g_{a,b}(x)|\mathrm{d}\mu_\alpha(x)|\phi(t)|\mathrm{d}t \le \frac{C_{\phi,2}}{a^{\langle\alpha\rangle+d}}\|f\|_{L^2_\alpha(\mathbb{R}^d_+)}\|g\|_{L^2_\alpha(\mathbb{R}^d_+)} < \infty.$$

This ends the proof of the theorem. □

We obtain a relation between the multidimensional Fourier-Bessel wavelet transform and the adjoint operator H^*_ϕ.

Theorem 4.4. Let $g \in L^2_\alpha(\mathbb{R}^d_+)$ be a multidimensional Fourier-Bessel wavelet, and let ϕ be a measurable function on $\mathbb{R}_+$ satisfying the condition (6). Then for $f \in L^2_\alpha(\mathbb{R}^d_+)$ we have

$$\Phi_g(H^*_\phi f)(a,b) = \int_{\mathbb{R}_+} \Phi_g(f)(ta,tb)\phi(t)\mathrm{d}t.$$

Proof. Let $g \in L^2_\alpha(\mathbb{R}^d_+)$ be a multidimensional Fourier-Bessel wavelet, and let $f \in L^2_\alpha(\mathbb{R}^d_+)$. From (7) we have $H^*_\phi f \in L^2_\alpha(\mathbb{R}^d_+)$. Then by (8) and (12), we get

$$\begin{aligned}\Phi_g(H^*_\phi f)(a,b) &= \int_{\mathbb{R}^d_+} H^*_\phi f(x) g_{a,b}(x)\mathrm{d}\mu_\alpha(x)\\ &= \int_{\mathbb{R}^d_+}\left[\int_{\mathbb{R}_+} f(tx)\phi(t)\mathrm{d}t\right] g_{a,b}(x)\mathrm{d}\mu_\alpha(x)\\ &= \int_{\mathbb{R}_+}\left[\int_{\mathbb{R}^d_+} f_{\frac{1}{t}}(x) g_{a,b}(x)\mathrm{d}\mu_\alpha(x)\right]\frac{\phi(t)}{t^{2(\langle\alpha\rangle+d)}}\mathrm{d}t\\ &= \int_{\mathbb{R}_+} \Phi_g(f_{\frac{1}{t}})(a,b)\frac{\phi(t)}{t^{2(\langle\alpha\rangle+d)}}\mathrm{d}t.\end{aligned}$$

According to (16) we have

$$\Phi_g(f_{\frac{1}{t}})(a,b) = t^{2(\langle\alpha\rangle+d)}\Phi_g(f)(ta,tb).$$

Therefore we deduce that

$$\Phi_g(H^*_\phi f)(a,b) = \int_{\mathbb{R}_+} \Phi_g(f)(ta,tb)\phi(t)\mathrm{d}t.$$

Using (4) and (11), this calculation is justified by the fact that

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}^d_+} |f_{\frac{1}{t}}(x)||g_{a,b}(x)|\mathrm{d}\mu_\alpha(x)\frac{|\phi(t)|}{t^{2(\langle\alpha\rangle+d)}}\mathrm{d}t \le \frac{C_{\phi,2}}{a^{\langle\alpha\rangle+d}}\|f\|_{L^2_\alpha(\mathbb{R}^d_+)}\|g\|_{L^2_\alpha(\mathbb{R}^d_+)} < \infty.$$

This ends the proof of the theorem. □

5. Numerical applications

If we choose $\phi(t) = \beta(1-t)^{\beta-1}\chi_{(0,1)}(t)$, $\beta = 1, 2, \ldots$, where $\chi_{(0,1)}$ is the characteristic function of the interval $(0, 1)$; we obtain the multidimensional Fourier-Bessel-Cesàro operator of order β denoted by C_β and given by (Figures 1–4)

$$C_\beta f(x) := \beta \int_0^1 f\left(\frac{x}{t}\right) \frac{(1-t)^{\beta-1}}{t^{2(\langle\alpha\rangle+d)}} \mathrm{d}t.$$

A brief history of the study of Cesàro operator can be found in [13].

The adjoint of the multidimensional Fourier-Bessel-Cesàro operator is given by

$$C_\beta^* f(x) := \beta \int_0^1 f(tx)(1-t)^{\beta-1} \mathrm{d}t.$$

If $d = 2$, $\alpha = (\frac{1}{2}, \frac{1}{2})$ and $f(x_1, x_2) = e^{-\sqrt{x_1^2+x_2^2}}$, we obtain

$$C_\beta f(x_1, x_2) := \beta \int_0^1 e^{-\frac{1}{t}\sqrt{x_1^2+x_2^2}} \frac{(1-t)^{\beta-1}}{t^6} \mathrm{d}t,$$

and

$$C_\beta^* f(x_1, x_2) := \beta \int_0^1 e^{-t\sqrt{x_1^2+x_2^2}} (1-t)^{\beta-1} \mathrm{d}t.$$

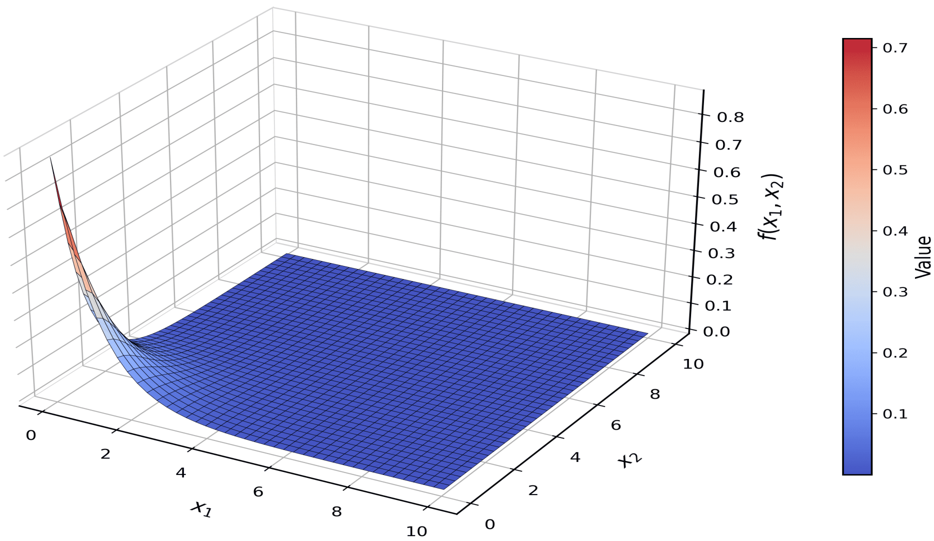

Figure 1: Plot of $f(x_1, x_2) = e^{-\sqrt{x_1^2+x_2^2}}$ when $(x_1, x_2) \in [0, 10] \times [0, 10]$.

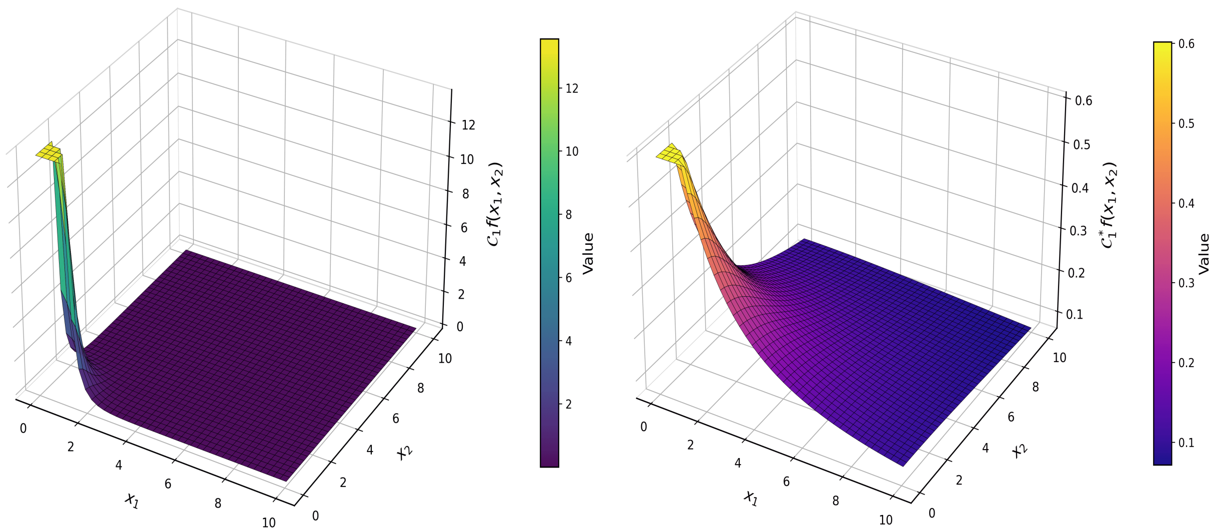

Figure 2: Plots of $C_1 f$ and $C_1^* f$ when $\alpha = (\frac{1}{2}, \frac{1}{2})$ and $(x_1, x_2) \in [0, 10] \times [0, 10]$.

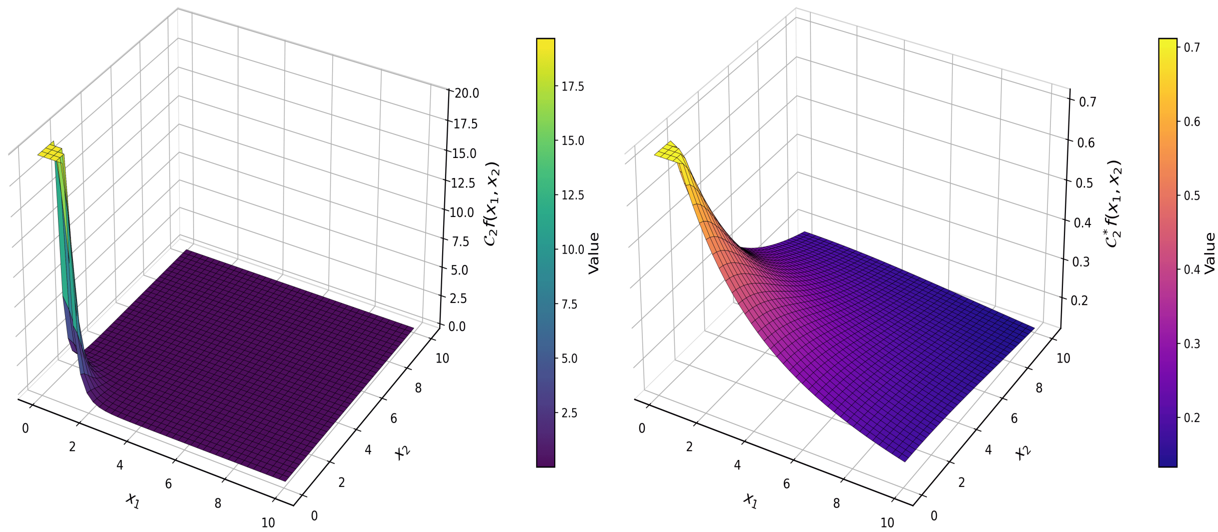

Figure 3: Plots of $C_2 f$ and $C_2^* f$ when $\alpha = (\frac{1}{2}, \frac{1}{2})$ and $(x_1, x_2) \in [0, 10] \times [0, 10]$.

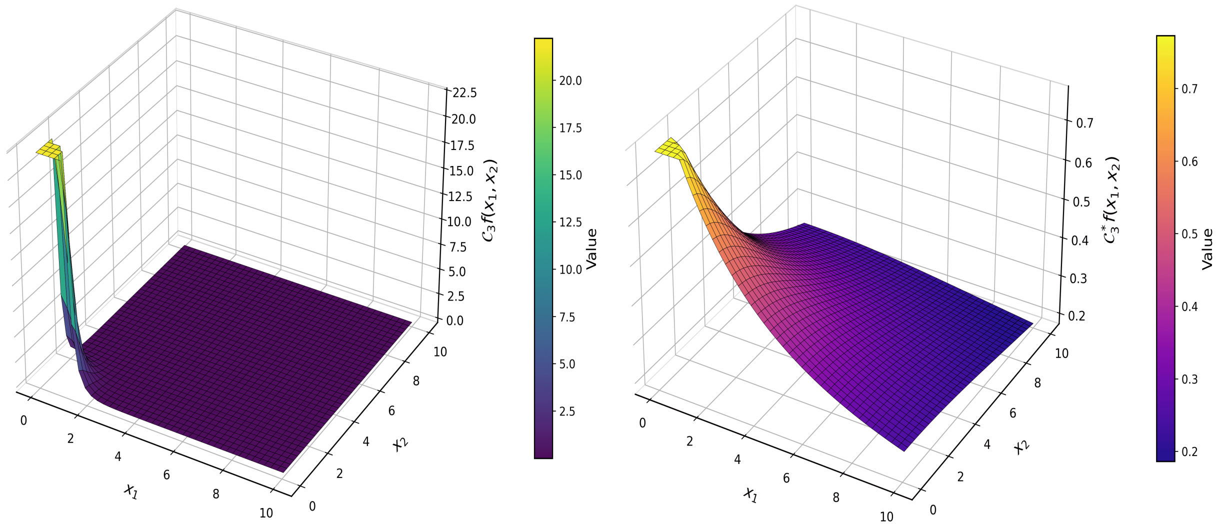

Figure 4: Plots of $C_3 f$ and $C_3^* f$ when $\alpha = (\frac{1}{2}, \frac{1}{2})$ and $(x_1, x_2) \in [0, 10] \times [0, 10]$.

If we choose $\phi(t) = \frac{1}{t}\chi_{(1,\infty)}(t)$, we obtain the multidimensional Fourier-Bessel-Hardy operator denoted by $\mathcal{H}$ and given by (Figure 5)

$$\mathcal{H}f(x) := \int_1^\infty f\left(\frac{x}{t}\right)\frac{dt}{t^{2\langle\alpha\rangle+2d+1}}.$$

It is well known that Hardy operators are important operators in harmonic analysis, for instance, see [8, 14].

The adjoint of the multidimensional Fourier-Bessel-Hardy operator is given by

$$\mathcal{H}^* f(x) := \int_1^\infty f(tx)\frac{dt}{t}.$$

If $d = 2$, $\alpha = (\frac{1}{2}, \frac{1}{2})$ and $f(x_1, x_2) = e^{-\sqrt{x_1^2+x_2^2}}$, we obtain

$$\mathcal{H}f(x_1, x_2) := \int_1^\infty e^{-\frac{1}{t}\sqrt{x_1^2+x_2^2}}\frac{dt}{t^7},$$

and

$$\mathcal{H}^* f(x_1, x_2) := \int_1^\infty e^{-t\sqrt{x_1^2+x_2^2}}\frac{dt}{t}.$$

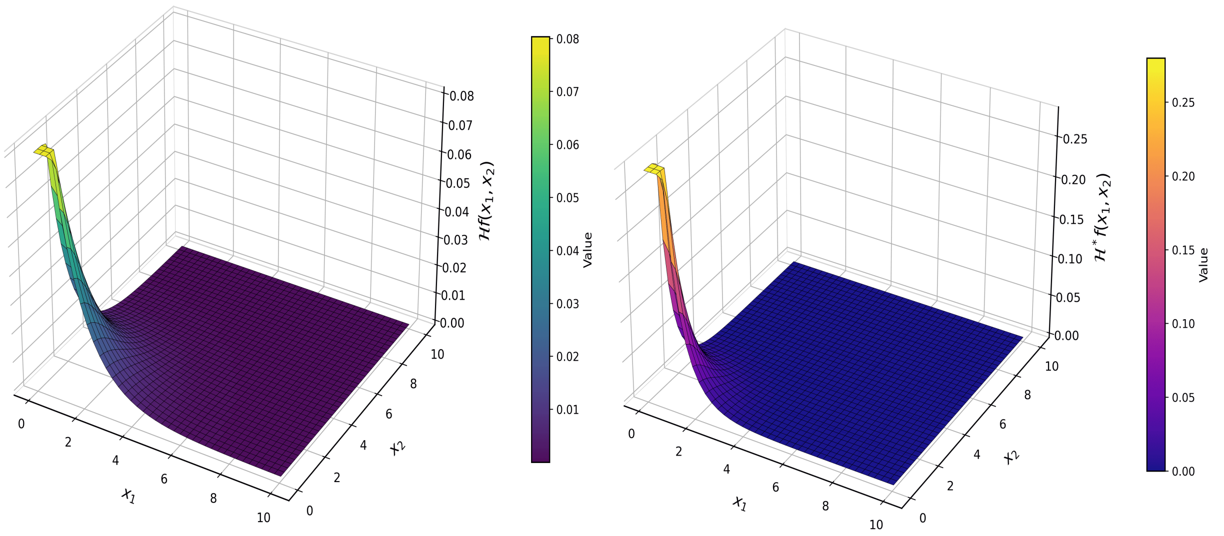

Figure 5: Plots of $\mathcal{H}f$ and $\mathcal{H}^*f$ when $\alpha = (\frac{1}{2}, \frac{1}{2})$ and $(x_1, x_2) \in [0, 10] \times [0, 10]$.

6. Conclusion and perspective

In this work we have succeeded in generalizing the results of Móricz for the classical Hausdorff operators [35], Upadhyay et al. for the Fourier-Bessel-Hausdorff operators [32, 33] and Daher et al. for the Dunkl-Hausdorff operators [9, 10] to the setting of the multidimensional Fourier-Bessel harmonic analysis. In this paper, we have studied the multidimensional Fourier-Bessel-Hausdorff operators and their adjoint on the Lebesgue space $L^2_\alpha(\mathbb{R}^d_+)$. Several other questions arise naturally, namely, the multidimensional Fourier-Bessel-Hausdorff operators on the Sobolev space, the Hardy space, the homogeneous weighted Herz space and the Herz-type Hardy space.

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