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The strongly Bott-Duffin (E, F)-inverse for rectangular matrices

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Abstract. Motivated by the recently introduced strongly Bott-Duffin (e, f)-inverse for elements in a semigroup [M. Drazin, *Linear Multilinear Algebra*, **71**(8) (2023), 1397–1406], the aim of this paper is to investigate this notion in the context of complex rectangular matrices. We provide necessary and sufficient conditions for its existence and derive a general expression involving an arbitrary inner inverse of the matrix. In addition, we obtain a canonical form for this inverse via the classical singular value decomposition. Furthermore, we show that the recently introduced generalized bilateral inverse, as well as several other notions appearing in the recent literature, can be viewed as particular cases of this new inverse.

1. Introduction

1.1. Mise-en-scène

Generalized inverses play a fundamental role in matrix theory, operator theory, and ring theory, unifying and extending classical inverses such as the group inverse, the Drazin inverse, and the Moore-Penrose inverse. Over the last decade, significant progress has been made in establishing frameworks that simultaneously encompass all three of these classical inverses.

In 1972, Rao and Mitra [26] introduced two different types of constraints to extend the concept of the Bott-Duffin inverse [4], and defined a new constrained inverse. In this direction, Mary introduced in 2011 the concept of the inverse along an element d in a semigroup [21], while independently Drazin defined the (b, c)-inverse [8]. These two notions, although formulated differently, are essentially equivalent when $b = c$, and both yield a unique outer inverse, provided it exists. The inverse along d exists if and only if the (d, d)-inverse exists, in which case both coincide.

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However, the (b, c)-inverse allows the greater generality $b \neq c$, a flexibility that is crucial in many recent developments and is not captured by Mary 's formulation. In 2017, Rakić [25] noted that the Rao-Mitra inverse is a direct precursor of the (b, c)-inverse.

These definitions further extend the classical theory and provide a richer structure for analyzing generalized invertibility.

More recently, Drazin introduced a new type of outer inverse for elements in a semigroup, namely, the strongly Bott-Duffin (e, f)-inverse [9]. This inverse can be viewed as a weaker version of the (b, c)-inverse, which in turn reduces to the Bott-Duffin (e, f)-inverse studied by the same author in [8] when $(b, c) = (e, f)$, with e and f both idempotent.

It is worth noting that, although not mentioned in Drazin's work, an equivalent notion had already been introduced in 2005 by Djordjević and Wei [7] in the context of rings: the so-called (p, q)-outer generalized inverse. This inverse is defined using two idempotent elements p and q, which coincide with the strongly Bott-Duffin (e, f)-inverse when $p = e$ and $q = 1 - f$.

As mentioned earlier, the outer inverses and their extensions discussed above have been defined in the context of semigroups or rings. However, the set of complex $m \times n$ matrices does not form a semigroup unless $m = n$.

The aim of this paper is to extend the strongly Bott-Duffin (e, f)-inverse to the setting of complex rectangular matrices. This new framework allows us to unify and generalize several classical and recently introduced inverses, including the Moore-Penrose inverse (when $m \neq n$), Drazin, group, and (b, c)-inverses, as well as other more recent notions appearing in the literature.

The paper is organized as follows. In Section 2, we introduce the strongly Bott-Duffin (E, F)-inverse for rectangular matrices and study its existence and uniqueness by analyzing the solutions of a related matrix system. Several algebraic characterizations and properties are also presented. In Section 3, we establish canonical forms using the singular value decomposition. In Section 4, we show that various recently introduced generalized inverses, including the generalized bilateral and non-bilateral ones, such as the DMP, CMP, MPCEP, core-EP, BT, WG, and GG inverses, among others, can be regarded as particular cases of this new inverse.

1.2. Notation and preliminaries

Let $\mathbb{C}^{m\times n}$ denote the set of complex $m \times n$ matrices and let $A \in \mathbb{C}^{m\times n}$. The conjugate transpose, rank, null space, and column space of A are denoted by A^*, $\text{rank}(A)$, $\mathcal{N}(A)$, and $\mathcal{R}(A)$, respectively. The symbol $P_{\mathcal{M},\mathcal{N}}$ stands for the projector (idempotent) onto the subspace $\mathcal{M}$ along the subspace $\mathcal{N}$. When $\mathcal{N} = \mathcal{M}^\perp$, we write $P_{\mathcal{M}}$.

The Moore-Penrose inverse of A, denoted by $A^\dagger$, is the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying

$$AXA = A, \quad XAX = X, \quad (AX)^* = AX, \quad \text{and} \quad (XA)^* = XA.$$

This inverse is commonly used to represent orthogonal projectors: $P_A := AA^\dagger$ and $Q_A := A^\dagger A$, which project onto $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$, respectively.

A matrix X satisfying $AXA = A$ (or $XAX = X$) is called an inner (or outer) inverse of A, and is denoted by $A^{(1)}$ (or $A^{(2)}$). The sets $A\{1\}$ and $A\{2\}$ consist of all inner and outer inverses of A, respectively.

For $A \in \mathbb{C}^{n\times n}$, the index of A, denoted by $\text{Ind}(A)$, is the smallest nonnegative integer k such that $\text{rank}(A^k) = \text{rank}(A^{k+1})$. Throughout this paper, we assume that $\text{Ind}(A) = k \geq 1$.

The Drazin inverse of $A \in \mathbb{C}^{n\times n}$ with index k is the unique matrix $X \in \mathbb{C}^{n\times n}$ such that

$$XAX = X, \quad AX = XA, \quad \text{and} \quad A^{k+1}X = A^k,$$

and is denoted by A^d. When $k = 1$, the Drazin inverse coincides with the group inverse, denoted by $A^\#$.

Given a matrix $A \in \mathbb{C}^{m\times n}$ with $\text{rank}(A) = r$, a subspace $\mathcal{T} \subseteq \mathbb{C}^n$ with $\dim(\mathcal{T}) = s \leq r$, and a subspace $\mathcal{S} \subseteq \mathbb{C}^m$ with $\dim(\mathcal{S}) = m - s$, the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying

$$X = XAX, \quad \mathcal{R}(X) = \mathcal{T}, \quad \text{and} \quad \mathcal{N}(X) = \mathcal{S}$$

is called the outer inverse of A with prescribed range $\mathcal{T}$ and null space $\mathcal{S}$, and is denoted by $A^{(2)}_{\mathcal{T},\mathcal{S}}$. It is well known that such an inverse exists if and only if $A(\mathcal{T}) \oplus \mathcal{S} = \mathbb{C}^m$. Moreover, when it exists, it is the unique matrix X such that

$$XAX = X, \quad XA = P_{\mathcal{T},(A^*(\mathcal{S}^\perp))^\perp}, \quad \text{and} \quad AX = P_{A(\mathcal{T}),\mathcal{S}}. \tag{1}$$

In particular, the Moore-Penrose, Drazin, and group inverses can be represented as:

$$A^\dagger = A^{(2)}_{\mathcal{R}(A^*),\mathcal{N}(A^*)}, \quad A^d = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}(A^k)}, \quad \text{and} \quad A^\# = A^{(2)}_{\mathcal{R}(A),\mathcal{N}(A)}.$$

Let $A, B, C \in \mathbb{C}^{n\times n}$. Rao and Mitra introduced two types of constraints in order to generalize the Bott-Duffin inverse:

Rao-Mitra Constraints	
Type I	**Type II**
c: $\mathcal{R}(X) \subseteq \mathcal{R}(B)$	C: XA is identity on $\mathcal{R}(B)$
r: $\mathcal{R}(X^*) \subseteq \mathcal{R}(C^*)$	R: $(AX)^*$ is identity on $\mathcal{R}(C^*)$

Definition 1.1. *[26] Let $A, B, C \in \mathbb{C}^{n\times n}$. The matrix $X \in \mathbb{C}^{n\times n}$ satisfying constraints c, r, C, and R is called the crCR-inverse of A, and is denoted by $A_{B,C}$.*

It was shown in [26] that $A_{B,C}$ exists if and only if $\operatorname{rank}(CAB) = \operatorname{rank}(C) = \operatorname{rank}(B)$, in which case it is unique and can be expressed as:

$$A_{B,C} = B(CAB)^{(1)}C, \quad \text{with} \quad (CAB)^{(1)} \in CAB\{1\}.$$

Djordjević and Wei studied, in the context of abstract rings, outer generalized inverses with prescribed idempotents. In the matrix setting, the concept can be formulated as follows:

Definition 1.2. *[7] Let $A, P, Q \in \mathbb{C}^{n\times n}$ with P and Q idempotent. A matrix $X \in \mathbb{C}^{n\times n}$ satisfying*

$$XAX = X, \quad XA = P, \quad \textit{and} \quad I_n - AX = Q$$

is called a (P, Q)-outer generalized inverse of A.

Mary introduced the concept of the inverse along an element within the framework of semigroups.

Definition 1.3. *[25] Let $A, D \in \mathbb{C}^{n\times n}$. The inverse of A along D is a matrix $X \in \mathbb{C}^{n\times n}$ such that*

$$XAD = D = DAX, \quad \mathcal{R}(X) \subseteq \mathcal{R}(D), \quad \textit{and} \quad \mathcal{N}(D) \subseteq \mathcal{N}(X).$$

If such X exists uniquely, it is denoted by $A^{\| D}$.

In particular:

$$A^\dagger = A^{\| A^*}, \quad A^d = A^{\| A^k}, \quad \text{and} \quad A^\# = A^{\| A}.$$

Drazin generalized the inverse along an element in the following manner:

Definition 1.4. *[8] Let $A, B, C \in \mathbb{C}^{n\times n}$. The (B, C)-inverse of A is a matrix $X \in \mathbb{C}^{n\times n}$ satisfying*

$$XAB = B, \quad CAX = C, \quad \mathcal{R}(X) \subseteq \mathcal{R}(B), \quad \textit{and} \quad \mathcal{R}(X^*) \subseteq \mathcal{R}(C^*). \tag{2}$$

If such X exists uniquely, it is denoted by $A^{\|(B,C)}$.

As shown in [8], $A^{\|D} = A^{\|(D,D)}$. Moreover, Rakić [25] noted that the *crCR*-inverse is a precursor of the (b,c)-inverse in rings, and in the matrix case, both coincide:

$$A_{B,C} = A^{\|(B,C)}.$$

These definitions are special cases of outer inverses with prescribed idempotents. Recently, Drazin proposed a variant of the Bott-Duffin (e, f)-inverse:

Definition 1.5. *[9] Let $A, E, F \in \mathbb{C}^{n\times n}$ with E and F idempotent. A matrix $X \in \mathbb{C}^{n\times n}$ satisfying*

$$XAX = X, \quad AX = E, \quad \text{and} \quad XA = F \tag{3}$$

is called a strongly Bott-Duffin (E, F)-inverse of A.

2. Characterizations and properties of the strongly Bott-Duffin (E, F)-inverse

In this section, we extend the notion of the strongly Bott–Duffin (E, F)-inverse to the context of rectangular matrices and establish its main characterizations and properties.

We begin by observing that the system of equations defining the strongly Bott–Duffin inverse, as given in (3), admits a natural extension to the rectangular case. In order to analyze this extension and to investigate existence and uniqueness, we first recall two auxiliary lemmas due to Penrose, which will play a fundamental role in the subsequent analysis.

Lemma 2.1. *[24, 26] Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$, $C \in \mathbb{C}^{m\times q}$, $A^{(1)} \in A\{1\}$, and $B^{(1)} \in B\{1\}$. Then, the equation $AXB = C$ is consistent (in X) if and only if $AA^{(1)}CB^{(1)}B = C$, in which case the general solution is*

$$X = A^{(1)}CB^{(1)} + Z - A^{(1)}AZBB^{(1)},$$

where Z is an arbitrary matrix.

Lemma 2.2. *[3] Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$, $D \in \mathbb{C}^{m\times p}$, and $E \in \mathbb{C}^{n\times q}$. The matrix equations*

$$AX = D \quad \text{and} \quad XB = E, \tag{4}$$

have a common solution if and only if each equation separately has a solution and $AE = DB$. In particular, if $X_0 \in \mathbb{C}^{n\times p}$ is a solution of (4), the general solution is

$$X = X_0 + (I_n - A^{(1)}A)Y(I_p - BB^{(1)}),$$

for arbitrary $A^{(1)} \in A\{1\}$, $B^{(1)} \in B\{1\}$, and $Y \in \mathbb{C}^{n\times p}$.

Theorem 2.3. *Let $A \in \mathbb{C}^{m\times n}$, $E \in \mathbb{C}^{n\times n}$, and $F \in \mathbb{C}^{m\times m}$. The matrix equations*

$$XAX = X, \quad XA = E, \quad \text{and} \quad AX = F. \tag{5}$$

have a common solution if and only if

$$EA^{(1)}A = E, \quad AA^{(1)}F = F, \quad AE = FA, \quad E^2 = E, \quad \text{and} \quad F^2 = F, \tag{6}$$

for some $A^{(1)} \in A\{1\}$, in which case, the unique solution is given by

$$A^{(E,F)} = EA^{(1)}F.$$

Proof. ⇒) Since (5) is consistent, from Lemma 2.2 it follows that each of the matrix equations $XA = E$ and $AX = F$ has a solution in X and verify $AE = FA$. By applying Lemma 2.1 to each of the above equations results that $EA^{(1)}A = E$ and $AA^{(1)}F = F$, respectively, for some $A^{(1)} \in A\{1\}$. Clearly, E and F are idempotent since $XAX = X$. Thus, (6) is satisfied.
⇐) Consider the five conditions given in (6). It suffices to check that the matrix $X := EA^{(1)}F$ satisfies the matrix equations (5). In fact,

$$XAX = EA^{(1)}FAEA^{(1)}F = (EA^{(1)}A)E^2A^{(1)}F = E^3A^{(1)}F = EA^{(1)}F = X.$$

The matrix equations $XA = E$ and $AX = F$ can be verified similarly.
Finally, it remains to prove uniqueness. Let X_1 and X_2 be two matrices satisfying (5). Therefore,

$$X_1 = (X_1A)X_1 = EX_1 = (X_2A)X_1 = X_2(AX_1) = X_2F = X_2(AX_2) = X_2.$$

The proof is complete. □

We now introduce a new type of generalized inverse for rectangular matrices, which extends the recently defined strongly Bott-Duffin (E, F)-inverse to the rectangular case.

Definition 2.4. *Let $A \in \mathbb{C}^{m\times n}$, $E \in \mathbb{C}^{n\times n}$, and $F \in \mathbb{C}^{m\times m}$. A matrix $X \in \mathbb{C}^{n\times m}$ is called the strongly Bott-Duffin (E, F)-inverse of A if it satisfies the system* (5). *If such a matrix X exists, it is denoted by $A^{(E,F)}$.*

Corollary 2.5. *Let $A \in \mathbb{C}^{m\times n}$, $E \in \mathbb{C}^{n\times n}$, and $F \in \mathbb{C}^{m\times m}$. If $A^{(E,F)}$ exists, then*

$$A^{(E,F)} = EX_1 = X_2F,$$

where $X_1, X_2 \in A\{2\}$.

Proof. By Theorem 2.3 we have $A^{(E,F)} = EA^{(1)}F$. Now, we consider $X_1 := A^{(1)}F$. Then, by using the second equation in (6) we obtain

$$X_1AX_1 = A^{(1)}FAA^{(1)}F = A^{(1)}F^2 = A^{(1)}F = X_1.$$

Thus, $A^{(E,F)} = EX_1$ with $X_1 \in A\{2\}$.
Similarly, by taking $X_2 := EA^{(1)}$, from the first equation in (6) we deduce that $X_2 = X_2AX_2$. □

Remark 2.6. *The condition* (6) *in Theorem 2.3 is equivalent to*

$$EA^{(1)}A = E, \quad AA^{(1)}F = F, \quad AE = FA, \quad E^2 = E, \quad \text{and} \quad F^2 = F,$$

for an arbitrary $A^{(1)} \in A\{1\}$.

Theorem 2.7. *Let $A \in \mathbb{C}^{m\times n}$, $E \in \mathbb{C}^{n\times n}$, and $F \in \mathbb{C}^{m\times m}$. The following statements are equivalent:*

(a) $A^{(E,F)}$ *exists;*

(b) $EA^{(1)}A = E$, $AA^{(1)}F = F$, $AE = FA$, $E^2 = E$, *and* $F^2 = F$, *for some* $A^{(1)} \in A\{1\}$;

(c) $\mathcal{N}(A) \subseteq \mathcal{N}(E)$, $\mathcal{R}(F) \subseteq \mathcal{R}(A)$, $AE = FA$, $E^2 = E$, *and* $F^2 = F$;

(d) $\mathcal{R}(E^*) \subseteq \mathcal{R}(A^*)$, $\mathcal{R}(F) \subseteq \mathcal{R}(A)$, $AE = FA$, $E^2 = E$, *and* $F^2 = F$;

(e) $A^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}$ *exists,* $\mathcal{N}(FA) = \mathcal{N}(E)$, *and* $\mathcal{R}(AE) = \mathcal{R}(F)$.

Moreover, in this case

$$A^{(E,F)} = EA^{\dagger}F = A^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}. \tag{7}$$

Proof. (a) ⇔ (b) Follows from Theorem 2.3.
(b) ⇔ (c) Note that the equation $EA^{(1)}A = E$ is equivalent to $\mathcal{R}(I_n - A^{(1)}A) \subseteq \mathcal{N}(E)$ which in turn is equivalent to $\mathcal{N}(A^{(1)}A) = \mathcal{N}(A) \subseteq \mathcal{N}(E)$. Similarly, $AA^{(1)}F = F$ is true if and only if $\mathcal{R}(F) \subseteq \mathcal{R}(A)$.
(c) ⇔ (d) It is consequence of the classical property $\mathcal{N}(A) = \mathcal{R}(A^*)^\perp$.
(a) ⇒ (e) Let $X := A^{(E,F)}$. As X is an outer inverse we have

$$\mathcal{N}(X) = \mathcal{N}(AX) = \mathcal{N}(F) \quad \text{and} \quad \mathcal{R}(X) = \mathcal{R}(XA) = \mathcal{R}(E).$$

Therefore,

$$A^{(E,F)} = A^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}. \tag{8}$$

From equivalence (a) ⇔ (d) we get

$$\mathcal{R}(F) = \mathcal{R}(F^2) = F\mathcal{R}(F) \subseteq F\mathcal{R}(A) = \mathcal{R}(FA) = \mathcal{R}(AE) \subseteq \mathcal{R}(F).$$

Similarly, as (a) ⇔ (c) we obtain

$$\mathcal{R}(E^*) = \mathcal{R}((E^*)^2) = E^*\mathcal{R}(E^*) \subseteq E^*\mathcal{R}(A^*) = \mathcal{R}((AE)^*) = \mathcal{R}((FA)^*),$$

which implies $\mathcal{N}(FA) \subseteq \mathcal{N}(E) \subseteq \mathcal{N}(AE) = \mathcal{N}(FA)$.
(e) ⇒ (a) Let $X := A^{(2)}_{\mathcal{R}(E),\mathcal{N}(F)}$ be such that $\mathcal{N}(FA) = \mathcal{N}(E)$ and $\mathcal{R}(AE) = \mathcal{R}(F)$. From (1), it follows that $XAX = X$,

$$\begin{aligned} XA &= P_{\mathcal{R}(E),(A^*(\mathcal{N}(F)^\perp))^\perp} \\ &= P_{\mathcal{R}(E),\mathcal{R}((FA)^*)^\perp} \\ &= P_{\mathcal{R}(E),\mathcal{N}(FA)} \\ &= P_{\mathcal{R}(E),\mathcal{N}(E)} = E, \end{aligned}$$

and

$$\begin{aligned} AX &= P_{A(\mathcal{R}(E)),\mathcal{N}(F)} \\ &= P_{\mathcal{R}(AE),\mathcal{N}(F)} \\ &= P_{\mathcal{R}(F),\mathcal{N}(F)} = F. \end{aligned}$$

Finally, the first equality in (7) follows from Remark 2.6, while the second equality is due to (8). □

An example of how to calculate the strongly Bott-Duffin (E, F)-inverse is shown below.

Example 2.8. *Consider the matrix*

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad a \in \mathbb{C} \setminus \{0\},$$

in conjunction with the projectors

$$E = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

These matrices satisfy $EA^\dagger A = E$, $AA^\dagger F = F$, $AE = FA$, $E^2 = E$, and $F^2 = F$, which is a guarantee for the existence of the strongly Bott-Duffin (E, F)-inverse of A given by

$$A^{(E,F)} = EA^\dagger F = \begin{bmatrix} \frac{1}{2a} & \frac{1}{2a} & 0 & 0 \\ \frac{1}{2a} & \frac{1}{2a} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now, we show that the *crCR*-inverse is a particular case of the strongly Bott-Duffin (E,F)-inverse. Before, we prove a new characterization of the *crCR*-inverse.

Theorem 2.9. *Let $A,B,C \in \mathbb{C}^{n\times n}$. If $A_{B,C}$ exists, then it is the unique matrix $X \in \mathbb{C}^{n\times n}$ such that*

$$XAX = X, \quad XA = P_{\mathcal{R}(B),\mathcal{N}(CA)}, \quad \text{and} \quad AX = P_{\mathcal{R}(AB),\mathcal{N}(C)}. \tag{9}$$

Proof. Let $X := A_{B,C}$ satisfying (2). It follows that $\mathcal{R}(X) = \mathcal{R}(B)$ and $\mathcal{N}(X) = \mathcal{N}(C)$. Since $\mathcal{R}(X) \subseteq \mathcal{R}(B)$, there exists some matrix D such that

$$X = BD = (XAB)D = XA(BD) = XAX.$$

Consequently,

$$\mathcal{R}(XA) = \mathcal{R}(X) = \mathcal{R}(B) \quad \text{and} \quad \mathcal{N}(AX) = \mathcal{N}(X) = \mathcal{N}(C). \tag{10}$$

Now, according to (10) we have

$$\mathcal{R}(AX) = \mathcal{R}(AB) \quad \text{and} \quad \mathcal{N}(XA) = \mathcal{N}(CA).$$

Since $XAX = X$, clearly XA and AX are idempotent. Therefore, from the uniqueness of an oblique projector we obtain the last two equations in (9).
Finally, the uniqueness follows similarly to the uniqueness proof of Theorem 2.3. □

Corollary 2.10. *Let $A,B,C \in \mathbb{C}^{n\times n}$. If $A_{B,C}$ exists, then $A_{B,C} = A^{(E,F)}$, where $E = P_{\mathcal{R}(B),\mathcal{N}(CA)}$ and $F = P_{\mathcal{R}(AB),\mathcal{N}(C)}$.*

Proof. Assume that $A_{B,C}$ exists. By Theorem 2.9 we have that $X := A_{B,C}$ verifies $XAX = X$, $XA = E$, and $AX = F$. The rest of the equalities are a consequence of Theorem 2.3 and Definition 2.4. □

Corollary 2.11. *Let $A,E,F \in \mathbb{C}^{n\times n}$. If $A^{(E,F)}$ exists, then $A_{E,F}$ exists and $A^{(E,F)} = A_{E,F}$.*

Proof. If $A^{(E,F)}$ exists, by Theorem 2.7, $A^{(2)}_{R(E),N(F)}$ exists and $A^{(E,F)} = A^{(2)}_{R(E),N(F)}$. Then, from Theorem 1.5 in [25], $A_{E,F}$ exists and $A^{(2)}_{R(E),N(F)} = A_{E,F}$. Thus $A^{(E,F)} = A_{E,F}$. □

The following example illustrates the previous corollary.

Example 2.12. *Consider the matrix*

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad a,b \in \mathbb{C} \setminus \{0\},$$

in conjunction with the projectors

$$E = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} 1 & \frac{a}{b} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

These matrices satisfy $EA^{\dagger}A = E$, $AA^{\dagger}F = F$, $AE = FA$, $E^2 = E$, and $F^2 = F$, which is a guarantee for the existence of the strongly Bott-Duffin (E,F)-inverse of A given by

$$A^{(E,F)} = EA^{\dagger}F = \begin{bmatrix} \frac{1}{a} & \frac{1}{b} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

On the other hand, the crCR-inverse is given by

$$A_{E,F} = E(FAE)^{\dagger}F = \begin{bmatrix} \frac{1}{a} & \frac{1}{b} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We finish this section with a result about when the strongly Bott-Duffin (E, F)-inverse is an inner inverse of the matrix.

Theorem 2.13. *Let $A \in \mathbb{C}^{m\times n}$, $E \in \mathbb{C}^{n\times n}$, and $F \in \mathbb{C}^{m\times m}$. If $A^{(E,F)}$ exists, then the following statements are equivalent:*

(a) $A^{(E,F)} \in A\{1\}$;

(b) $A = AE$;

(c) $A = FA$;

(d) $\mathcal{R}(A) \subseteq \mathcal{R}(F)$;

(e) $\mathcal{N}(E) \subseteq \mathcal{N}(A)$.

Proof. (a) ⇒ (b). By (5) we get $A = AA^{(E,F)}A = AE$.
(b) ⇒ (a). From Theorem 2.7 we have $AA^{(E,F)}A = AEA^{\dagger}FA = AA^{\dagger}A = A$.
(b) ⇔ (c). Follows from Theorem 2.7.
(c) ⇔ (d). Note that $A = FA$ holds if and only if $(I_m - F)A = 0$, which in turn is equivalent to $\mathcal{R}(A) \subseteq \mathcal{N}(I_m - F) = \mathcal{R}(F)$.
(d) ⇔ (b). Clearly, $A = AE$ holds if and only if $\mathcal{N}(E) = \mathcal{R}(I_n - E) \subseteq \mathcal{N}(A)$.
The proof is complete. □

3. Canonical form of the strongly Bott-Duffin (E, F)-inverse

In this section we exhibit an interesting result about canonical form of the strongly Bott-Duffin (E, F)-inverse.

We recall the classical Singular Value Decomposition (SVD).

Theorem 3.1. *(SVD) Let $A \in \mathbb{C}^{m\times n}$ be a nonnull matrix of rank $r > 0$ and let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ be the singular values of A. Then there exist unitary matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ such that $A = U\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}V^*$, where $\Sigma = diag(\sigma_1, \sigma_2, \ldots, \sigma_r)$. In particular, the Moore-Penrose inverse of A is given by*

$$A^{\dagger} = V\begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*.$$

Theorem 3.2. *Let $A \in \mathbb{C}^{m\times n}$ be written as in Theorem 3.1 and let*

$$E = V\begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix}V^* \in \mathbb{C}^{n\times n}, \quad F = U\begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix}U^* \in \mathbb{C}^{m\times m}. \tag{11}$$

Then the following statements are equivalent:

(a) $A^{(E,F)}$ *exists;*

(b) $E_1^2 = E_1$, $F_1^2 = F_1$, $\Sigma E_1 = F_1\Sigma$, $E_3E_1 = E_3$, $F_1F_2 = F_2$, $E_2 = 0$, $E_4 = 0$, $F_3 = 0$, *and* $F_4 = 0$;

(c) $E_1^2 = E_1$, $F_1^2 = F_1$, $\Sigma E_1 = F_1\Sigma$, $\mathcal{N}(E_1) \subseteq \mathcal{N}(E_3)$, $\mathcal{R}(F_2) \subseteq \mathcal{R}(F_1)$, $E_2 = 0$, $E_4 = 0$, $F_3 = 0$, *and* $F_4 = 0$.

In this case,

$$A^{(E,F)} = V\begin{bmatrix} E_1\Sigma^{-1} & E_1\Sigma^{-1}F_2 \\ E_3\Sigma^{-1} & E_3\Sigma^{-1}F_2 \end{bmatrix}U^*. \tag{12}$$

Proof. (a) ⇒ (b). Since $A^{(E,F)}$ exists, from Definition 2.4 and Remark 2.6 we have

$$EQ_A = E, \quad P_A F = F, \quad AE = FA, \quad E^2 = E, \quad \text{and} \quad F^2 = F. \tag{13}$$

The first equality in (13) is equivalent to $E_2 = 0$ and $E_4 = 0$. While the second equality is equivalent to $F_3 = 0$ and $F_4 = 0$. Thus,

$$E = V \begin{bmatrix} E_1 & 0 \\ E_3 & 0 \end{bmatrix} V^* \in \mathbb{C}^{n\times n}, \quad F = U \begin{bmatrix} F_1 & F_2 \\ 0 & 0 \end{bmatrix} U^* \in \mathbb{C}^{m\times m}. \tag{14}$$

In consequence, from $E^2 = E$ we obtain that $E_1^2 = E_1$ and $E_3E_1 = E_3$. Similarly, from $F^2 = F$ we have $F_1^2 = F_1$ and $F_1F_2 = F_2$. Now, by using the expressions for E and F given in (14) and the condition $AE = FA$ we obtain $\Sigma E_1 = F_1\Sigma$.
(b) ⇒ (a). It easy to check that the matrix given in (12) satisfies the three conditions in (5).
(b) ⇔ (c). Note that under assumption $E_1^2 = E_1$, the equality $E_3E_1 = E_3$ is equivalent to $\mathcal{N}(E_1) \subseteq \mathcal{N}(E_3)$. In fact, $E_3E_1 = E_3$ holds if and only if $E_3(I_r - E_1) = 0$ which in turn is equivalent to $\mathcal{N}(E_1) \subseteq \mathcal{N}(E_3)$. Similarly, $F_1F_2 = F_2$ is equivalent to $\mathcal{R}(F_2) \subseteq \mathcal{R}(F_1)$ provided $F_1^2 = F_1$.
Finally, in order to prove (12) we use the representation of the strongly Bott-Duffin (E, F)-inverse of A obtained in (7) and the fact $F_1 = \Sigma E_1 \Sigma^{-1}$, $E_3E_1 = E_3$, and $E_1^2 = E_1$. □

Example 3.3. *Consider the matrix*

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad a \in \mathbb{C} \setminus \{0\}, b \in \mathbb{C}.$$

in conjunction with the projectors

$$E = \begin{bmatrix} 1 & 0 & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{bc}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that the matrix A is written as in Theorem 3.1 with $U = I_4$, $V = I_3$, and $\Sigma = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. So, from (11) *one can see that $E_1 = \begin{bmatrix} 1 & 0 \\ c & 0 \end{bmatrix}$, $E_2 = 0$, $E_3 = 0$, $E_4 = 0$, $F_1 = \begin{bmatrix} 1 & 0 \\ \frac{bc}{a} & 0 \end{bmatrix}$, and $F_2 = F_3 = F_4 = 0$. These matrices satisfy the conditions in (b) of Theorem 3.2, which is a guarantee for the existence of the strongly Bott-Duffin (E, F)-inverse of A. Thus, from* (12) *it is easy to check*

$$A^{(E,F)} = \begin{bmatrix} \frac{1}{a} & 0 & 0 & 0 \\ \frac{c}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

4. Redefining some recent generalized inverses

In recent years, a variety of generalized inverses for complex matrices have been introduced, each capturing different algebraic and geometric properties relevant to applications in matrix theory and operator analysis. These inverses, including the Moore–Penrose, Drazin and core inverses, together with numerous extensions, are often characterized by algebraic equations involving the matrix and certain projectors, or by prescribed range and null-space conditions.

In this section, we revisit several notable generalized inverses, highlighting their defining properties and interrelations. We then demonstrate how these various inverses can be unified and reinterpreted within the framework of the strongly Bott–Duffin (E, F)-inverse, thereby providing a common perspective.

It is well known that the Moore-Penrose inverse $A^\dagger$ of $A \in \mathbb{C}^{m\times n}$ is the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying

$$XAX = X, \quad XA = P_{\mathcal{R}(A^*),\mathcal{N}(A)}, \quad \text{and} \quad AX = P_{\mathcal{R}(A),\mathcal{N}(A^*)}.$$

Similarly, the Drazin inverse A^d of $A \in \mathbb{C}^{n\times n}$ is the unique $X \in \mathbb{C}^{n\times n}$ such that

$$XAX = X \quad \text{and} \quad AX = XA = P_{\mathcal{R}(A^k),\mathcal{N}(A^k)}.$$

In particular, the group inverse of a matrix A of index 1 is the unique matrix X that satisfies

$$XAX \quad \text{and} \quad AX = XA = P_{\mathcal{R}(A),\mathcal{N}(A)}.$$

The core inverse for a square matrix was introduced in [1] by Baksalary and Trenkler as recently as 2010. Since then a considerable amount of research has been added to advance the theory of this inverse [10, 17, 20, 28]. For a given matrix $A \in \mathbb{C}^{n\times n}$, the core inverse of A is defined to be a matrix $X \in \mathbb{C}^{n\times n}$ satisfying the conditions $AX = P_A$ and $\mathcal{R}(X) \subseteq \mathcal{R}(A)$. The authors proved that A is core invertible if and only if $\text{Ind}(A) = 1$. In this case, the core inverse (or GMP) of A is the unique matrix given by $A^{\#\!\!\!\bigcirc} = A^\# AA^\dagger$. It is easy to see that the core inverse can be characterized by the following conditions

$$XAX = X, \quad XA = P_{\mathcal{R}(A),\mathcal{N}(A)}, \quad \text{and} \quad AX = P_{\mathcal{R}(A),\mathcal{N}(A^*)}.$$

As the core inverse exists only for matrices of index at most 1, in 2014 three kinds of generalizations of the core inverse were defined for complex square matrices of an arbitrary index. We recall its definitions. Let $A \in \mathbb{C}^{n\times n}$ be with $\text{Ind}(A) = k$. Then the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying

$$XAX = X \quad \text{and} \quad \mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k),$$

is called the core-EP inverse (or CEP) of A and is denoted by $A^{\dagger\!\!\!\bigcirc}$ [18]. In the same way, the dual core-EP (or ∗CEP) was defined as the unique matrix satisfying

$$XAX = X \quad \text{and} \quad \mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}((A^k)^*),$$

and is denoted by $A_{\dagger\!\!\!\bigcirc}$.

The DMP inverse of A is the unique matrix $X := A^d AA^\dagger$ that satisfies

$$XAX = X, \qquad XA = A^d A, \quad \text{and} \quad A^k X = A^k A^\dagger,$$

is called and is represented by $A^{d,\dagger}$ [19]. The associated dual inverse is given by the matrix $A^{\dagger,d} = A^\dagger AA^d$ and is called ∗DMP (or MPD) inverse of A.

The unique matrix given by $A^\diamond = (AP_A)^\dagger$ is called the BT inverse of A [2].

In 2018, the CMP inverse of a square matrix was presented by Mehdipour and Salemi [22] as the unique matrix $X := A^\dagger AA^d AA^\dagger$ (denoted by $A^{c,\dagger}$) that satisfies

$$XAX = X, \quad AXA = AA^d A, \quad XA = A^\dagger AA^d A, \quad \text{and} \quad AX = AA^d AA^\dagger.$$

In the same year, Wang and Chen [27] introduced the WG inverse of a matrix $A \in \mathbb{C}^{n\times n}$ as the unique matrix $X := A^{\text{Ⓦ}}$ satisfying

$$AX^2 = X \quad \text{and} \quad AX = A^{\dagger\!\!\!\bigcirc} A.$$

If $\text{Ind}(A) = 1$, the WG inverse and the group inverse coincide.

Similarly, by using the core-EP and dual core-EP inverses, Chen et al. [5] defined the MPCEP and ∗CEPMP inverses of A as the matrices $A^{\dagger,\textcircled{\dagger}} = A^{\dagger}AA^{\textcircled{\dagger}}$ and $A_{\textcircled{\dagger},\dagger} = A_{\textcircled{\dagger}}AA^{\dagger}$, respectively.

To extend and unify most of the above mentioned definitions of generalized inverses, the OMP, MPO, and MPOMP inverses were defined in [23] by composing an arbitrary outer inverse and the Moore-Penrose inverse. More precisely, the OMP inverse of $A \in \mathbb{C}^{m\times n}$ is defined as the unique matrix $X := A^{(2)}_{\mathcal{T},\mathcal{S}}AA^{\dagger} \in \mathbb{C}^{n\times m}$ such that

$$XAX = X, \quad XA = A^{(2)}_{\mathcal{T},\mathcal{S}}A, \quad \text{and} \quad AX = AA^{(2)}_{\mathcal{T},\mathcal{S}}AA^{\dagger},$$

and is denoted by $A^{(2),\dagger}_{\mathcal{T},\mathcal{S}}$. Here, $A^{(2)}_{\mathcal{T},\mathcal{S}}$ denotes the outer inverse of A with prescribed range $\mathcal{T}$ and null space $\mathcal{S}$. Clearly, the core, DMP, and ∗CEPMP inverses are particular cases of the OMP inverse. Dually, the MPO (or ∗OMP) inverse of A is the matrix $A^{\dagger,(2)}_{\mathcal{T},\mathcal{S}} := A^{\dagger}AA^{(2)}_{\mathcal{T},\mathcal{S}}$, which extends the dual core (or MPG), MPD, and MPCEP inverses.

On the other hand, the MPOMP inverse of A given by the matrix $A^{\dagger,(2),\dagger}_{\mathcal{T},\mathcal{S}} := A^{\dagger}AA^{(2)}_{\mathcal{T},\mathcal{S}}AA^{\dagger}$ generalizes the CMP and Moore-Penrose inverses. Notice that the MPOMP inverse of A can be rewritten in terms of the OMP (or MPO) inverse as $A^{\dagger,(2),\dagger}_{\mathcal{T},\mathcal{S}} = A^{\dagger}AA^{(2),\dagger}_{\mathcal{T},\mathcal{S}} = A^{\dagger,(2)}_{\mathcal{T},\mathcal{S}}AA^{\dagger}$.

Motived by the way in which some of these inverses were defined, recently Kheirandish and Salemi introduced the notion of generalized bilateral inverse as a unified approach to such inverses.

The following definition is a slight modification of [16, Definition 2.1] according to characterizations presented by the authors in Theorems 2.5 and 2.6.

Definition 4.1. *Let $A \in \mathbb{C}^{m\times n}$ and let $X_1, X_2 \in \mathbb{C}^{n\times m}$ be such that $X_1 \in A\{2\}$ and $X_2 \in A\{1\}$. Then X_1AX_2 (or X_2AX_1) is called generalized bilateral inverse of A.*

Remark 4.2. *By Theorems 2.5 and 2.6 in [16], we know that the generalized bilateral inverse of a matrix $A \in \mathbb{C}^{m\times n}$ always exists and is unique.*

Next, we show that the generalized bilateral inverse of a matrix can be obtained as a particular case of the strongly Bott-Duffin (E, F)-inverse.

Theorem 4.3. *Let $A \in \mathbb{C}^{m\times n}$ and let $X_1, X_2 \in \mathbb{C}^{n\times m}$ be such that $X_1 \in A\{2\}$ and $X_2 \in A\{1\}$. Then $A^{(E,F)} = X_1AX_2$, where $E = X_1A$ and $F = AX_1AX_2$.*

Proof. Let $X := X_1AX_2$. From [16, Theorems 2.5] we deduce that X is the unique matrix such that

$$XAX = X, \quad XA = X_1A = E, \quad \text{and} \quad AX = AXAX = AX_1AX_2 = F. \tag{15}$$

Therefore, $A^{(E,F)}$ exists and $X = A^{(E,F)}$. □

Applying the same method as in Theorem 4.3 and by using [16, Theorems 2.6] instead of [16, Theorems 2.5], we obtain the following result.

Theorem 4.4. *Let $A \in \mathbb{C}^{m\times n}$ and let $X_1, X_2 \in \mathbb{C}^{n\times m}$ be such that $X_1 \in A\{2\}$ and $X_2 \in A\{1\}$. Then $A^{(E,F)} = X_2AX_1$, where $E = X_2AX_1A$ and $F = AX_1$.*

We would like to point out that the BT, CEP, and WG inverses are not generalized bilateral inverses.

In the following tables we illustrate that both generalized bilateral and non-bilateral inverses are particular cases of the strongly Bott-Duffin (E, F)-inverse. For the sake of completeness, we also add definitions and notations of some more generalized inverses studied recently in the literature.

Generalized bilateral inverses	Name	$XA = E$	$AX = F$	Reference
$A^{\textcircled{\#}} = A^{\#}P_A$	GMP	$A^{\#}A$	P_A	[1]
$A_{\textcircled{\#}} = Q_A A^{\#}$	MPG	Q_A	$AA^{\#}$	[18]
$A^{d,\dagger} = A^d P_A$	DMP	$A^d A$	$AA^d P_A$	[19]
$A^{\dagger,d} = Q_A A^d$	MPD	$Q_A A^d A$	AA^d	[19]
$A^{c,\dagger} = A^{\dagger,d} P_A$	CMP	$Q_A A^d A$	$AA^d P_A$	[22]
$A^{\dagger,\textcircled{\dagger}} = Q_A A^{\textcircled{\dagger}}$	MPCEP	$Q_A A^{\textcircled{\dagger}} A$	$AA^{\textcircled{\dagger}}$	[5]
$A_{\textcircled{\dagger},\dagger} = A_{\textcircled{\dagger}} P_A$	*CEPMP	$A_{\textcircled{\dagger}} A$	$AA_{\textcircled{\dagger}} P_A$	[5]
$A^{\textcircled{w},\dagger} = A^{\textcircled{w}} P_A$	WGMP	$A^{\textcircled{w}} A$	$AA^{\textcircled{w}} P_A$	[12]
$A^{\dagger,\textcircled{w}} = Q_A A^{\textcircled{w}}$	MPWG	$Q_A A^{\textcircled{w}} A$	$AA^{\textcircled{w}}$	[12]
$A^{(2),\dagger}_{\mathcal{T},\mathcal{S}} = A^{(2)}_{\mathcal{T},\mathcal{S}} P_A$	OMP	$A^{(2)}_{\mathcal{T},\mathcal{S}} A$	$AA^{(2)}_{\mathcal{T},\mathcal{S}} P_A$	[23]
$A^{\dagger,(2)}_{\mathcal{T},\mathcal{S}} = Q_A A^{(2)}_{\mathcal{T},\mathcal{S}}$	MPO	$Q_A A^{(2)}_{\mathcal{T},\mathcal{S}} A$	$AA^{(2)}_{\mathcal{T},\mathcal{S}}$	[23]
$A^{\dagger,(2),\dagger}_{\mathcal{T},\mathcal{S}} = A^{\dagger,(2)}_{\mathcal{T},\mathcal{S}} P_A$	MPOMP	$Q_A A^{(2)}_{\mathcal{T},\mathcal{S}} A$	$AA^{(2)}_{\mathcal{T},\mathcal{S}} P_A$	[23]

Table 1: Generalized bilateral inverses

Non generalized bilateral inverses	Name	$XA = E$	$AX = F$	Reference
$A^{\diamond} = (AP_A)^{\dagger}$	BT	$(AP_A)^{\dagger}A$	P_{A^2}	[2, 13]
$A^{\textcircled{\dagger}} = A^d P_{A^k}$	CEP	$A^d P_{A^k} A$	P_{A^k}	[11, 18]
$A_{\textcircled{\dagger}} = Q_{A^k} A^d$	*CEP	Q_{A^k}	$AQ_{A^k} A^d$	[23]
$A^{\textcircled{w}} = (A^{\textcircled{\dagger}})^2 A$	WG	$(A^{\textcircled{\dagger}})^2 A^2$	$A(A^{\textcircled{\dagger}})^2 A$	[27]
$A^{\textcircled{w}_2} = (A^{\textcircled{\dagger}})^3 A^2$	GG	$(A^{\textcircled{\dagger}})^3 A^3$	$A(A^{\textcircled{\dagger}})^3 A^2$	[14]
$A^{\textcircled{w}_m} = (A^{\textcircled{\dagger}})^{m+1} A^m$	m-WG	$(A^{\textcircled{\dagger}})^{m+1} A^{m+1}$	$A(A^{\textcircled{\dagger}})^{m+1} A^m$	[29]
$A^{\textcircled{\#}_m} = A^{\textcircled{w}_m} P_{A^m}$	m-WC	$A^{\textcircled{w}_m} P_{A^m} A$	$AA^{\textcircled{w}_m} P_{A^m}$	[15]
$A^{k,(2),\dagger}_{\mathcal{T},\mathcal{S}} = A^{(2)}_{\mathcal{T},\mathcal{S}} P_{A^k}$	k-OMP	$A^{(2)}_{\mathcal{T},\mathcal{S}} P_{A^k} A$	$AA^{(2)}_{\mathcal{T},\mathcal{S}} P_{A^k}$	[23]
$A^{k,\dagger,(2)}_{\mathcal{T},\mathcal{S}} = Q_{A^k} A^{(2)}_{\mathcal{T},\mathcal{S}}$	k-MPO	$Q_{A^k} A^{(2)}_{\mathcal{T},\mathcal{S}} A$	$AQ_{A^k} A^{(2)}_{\mathcal{T},\mathcal{S}}$	[23]

Table 2: Non generalized bilateral inverses

Conclusions and future work

We have established necessary and sufficient conditions for the existence of the strongly Bott–Duffin (E, F)-inverse for complex rectangular matrices, formulated in terms of ranges and kernels. Moreover, we provided an explicit formula for this inverse that is independent of the particular generalized inverse $A^{(1)}$ chosen. These results extend the theory of generalized inverses associated with pairs of projections and supply practical tools for problems in matrix and operator analysis.

We point out several directions for further research:

(i) Extension to bounded linear operators on infinite-dimensional Hilbert spaces.

(ii) Computational aspects and numerical stability: development and analysis of algorithms for computing $A^{(E,F)}$.

(iii) Perturbation analysis: derivation of bounds for the variation of $A^{(E,F)}$ under additive and multiplicative perturbations of A, E, and F, including structured perturbations and sensitivity of the existence conditions.

Pursuing these directions should deepen both the theoretical understanding and the practical applicability of the strongly Bott–Duffin inverse in more general settings.

Declarations

Conflict of interest The authors have no conflicts of interest.

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