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Entropy and renormalized solutions for some parabolic problems with variable exponents and degenerate coercivity

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Abstract. The purpose of this study is to prove the existence of both entropy solutions and renormalized solutions for nonlinear parabolic equations with initial data in L^∞ and degenerate coercivity. The functional setting involves Lebesgue and Sobolev spaces with variable exponents.

1. Introduction

Our main objective is to show that entropy and renormalized solutions exist for the following nonlinear degenerate parabolic problem

$$(P)\qquad \begin{cases} \partial_t u + Au + |u|^{\theta p(x)-2}u|\nabla u|^{p(x)} = |u|^{r(x)-2}u & \text{in } Q_T, \\ u(0,x) = u_0(x) \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_T. \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ $(N > 2)$ with Lipschitz boundary denoted by $\partial\Omega$, T is a positive constant, $u_0 \in L^\infty(\Omega)$, $Q_T = (0,T)\times\Omega$ with the lateral boundary $\Gamma_T = (0,T)\times\partial\Omega$, and A is the operator given by

$$Au = -\text{div}(d(t,x,u)|\nabla u|^{p(x)-2}\nabla u).$$

The function $d : (0,T)\times\mathbb{R}^N\times\mathbb{R} \to \mathbb{R}$ are Carathéodory functions and satisfies for almost every $(t,x) \in Q_T$, $\forall s \in \mathbb{R}$, the following:

$$\frac{\alpha}{(1+|s|)^{\varrho(x)}} \leq d(t,x,s) \leq \beta, \tag{1}$$

where α, β are strictly positive real numbers, $\varrho(x) \in C(\overline{\Omega})$ and $\varrho(x) \geq 0$. The variable exponents $p : \overline{\Omega} \to (1,\infty)$ and $r : \overline{\Omega} \to (1,\infty)$ are continuous functions and let $p^- = \min_{x\in\overline{\Omega}} p(x)$, $p^+ = \max_{x\in\overline{\Omega}} p(x)$ such that

$$\theta p^- + 1 \leq r(x) \leq r^+ = \max_{x\in\overline{\Omega}} r(x) \leq p^-(\theta+1), \quad \text{and} \quad \theta \geq \frac{1}{p^-}. \tag{2}$$

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Differential equations and variational problems with nonstandard growth conditions have garnered significant interest due to their applications in elastic mechanics, electro-theological fluid dynamics, and image processing. Researchers have explored various results for such problems, and the functional spaces used to address them include generalized Lebesgue spaces and generalized Lebesgue Sobolev spaces. The notions of entropy solutions and renormalized solutions have been introduced to deal with these problems, as they require less regularity than traditional weak solutions. Recent studies have focused on the existence and uniqueness of entropy and renormalized solutions for problems with variable exponents and L^1 data, relying on a priori estimates in Marcinkiewicz spaces with variable exponents. DiPerna and Lions [9] first introduced the notion of renormalized solutions to study the Boltzmann equation. This concept was then adapted to analyze certain nonlinear elliptic, parabolic, and fluid mechanics evolution problems. For details, see [4–6, 12]. Concurrently, Bénilan et al. [3] proposed the idea of entropy solutions for nonlinear elliptic problems. This framework was extended to related problems with constant p in [7, 17, 19].

As we have seen, If (1) holds true, the differential operator A is not coercive as u becomes large. Note that the problem (P) includes a parabolic equation which is nonlinear with respect to the gradient of the solution, and with variable exponents of nonlinearity. Thus, it is natural to solve problem (P) under the framework of Sobolev spaces with variable exponents.

In the case of $p(x) = 2$, $\varrho(x) = 0$ and $r(x)$ is a constant, the authors in [1] proved the existence of global weak solutions of problem (P) for non-negative initial data $u_0 \in L^1(\Omega)$.

In the case with $\varrho(x) = 0$ and $p(.) > 2 - \frac{1}{N+1}$, C. Zhang and S. Zhou [22] have established the existence and uniqueness of entropy solutions for the following nonlinear parabolic equation:

$$(1)\qquad \begin{cases} \partial_t u - \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f & \text{in } Q_T = (0,T)\times\Omega \\ u(0,x) = u_0 & \text{in } \Omega \\ u = 0 & \text{on }]0,T[\times\partial\Omega \end{cases}$$

with $f \in L^1(Q_T)$ and $u_0 \in L^1(\Omega)$ such that $u \in L^{q_-}(0,T;W_0^{1,q(.)}(\Omega))$ and $1 \leq q(.) < \frac{N(p(.)+1)-N}{N+1}$. Beside that Bendahmane, Wittbold and Zimmermann in [2], the authors showed the existence and uniqueness of the renormalized solutions to the problem (1). Moreover, they proved that $u \in L^{q^-}\left(0,T;W_0^{1,q(\cdot)}(\Omega)\right)$, when $p^- > 2 - \frac{1}{N+1}$, for all continuous variable exponents $q(.)$ on $\overline{\Omega}$ such that $1 \leq q(x) < \frac{N(p(x)-1)+p(x)}{N+1}$ for all $x \in \overline{\Omega}$. When $p = p(x) \in C(\overline{\Omega})$ satisfies the log-continuity condition, the existence and uniqueness of an entropy solution to problem (1) are proved in [22].

The study aims to prove the existence of entropy and renormalized solutions for (P) and to expand upon the findings in [2, 16, 22] to address degenerate parabolic equations with a source term. The condition of degenerate coercivity indicates that $\frac{\alpha}{(1+|u|)^{\varrho(x)}}$ decreases towards zero as $|u|$ grows. To address this challenge, we will use truncation in d to obtain a coercive differential operator. An additional challenge lies in passing to the limit within the nonlinear terms with variable exponents, $d(t,x,T_n(u_n))|\nabla u_n|^{p(x)-2}\nabla u_n$ and $|u_n|^{\theta p(x)-2}u_n|\nabla u_n|^{p(x)}$. Moreover, the presence of this source term in the equation makes it very difficult to prove the uniqueness of the solution, especially in conjunction with the term $|u|^{\theta p(x)-2}u|\nabla u|^{p(x)}$. Consequently, many studies that address to the uniqueness of the solution simplify the problem by considering the right-hand side of the equation to be a simple function and excludes the lower-order term, as seen in [2, 5, 7, 19, 22] and the related references.

This paper is structured as follows: Section 2 presents basic notations and properties of Sobolev spaces with variable exponents. Section 3 provides the definition of entropy and renormalized solutions to problem (P), along with the main results. Finally, in Section 4, we establish the existence of entropy and renormalized solutions.

2. Mathematical preliminaries

In the following, we'll revisit a few definitions and fundamental characteristics of the generalized Lebesgue-Sobolev spaces $L^{p(\cdot)}(\Omega)$, $W^{1,p(\cdot)}(\Omega)$, and $W_0^{1,p(\cdot)}(\Omega)$, with Ω being an open subset of $\mathbb{R}^N$. For additional properties of the variable exponent Lebesgue-Sobolev spaces, we suggest consulting [13, 14] and related sources.

Set $C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) : \inf_{x\in C(\overline{\Omega})} p(x) > 1.\}$ For any $p \in C(\overline{\Omega})$, we define

$$p^+ = \sup_{x\in C(\overline{\Omega})} p(x), \quad p^- = \inf_{x\in C(\overline{\Omega})} p(x).$$

We define the variable exponent Lebesgue space

$$L^{p(\cdot)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \quad \text{measurable} \mid \int_\Omega |u(x)|^{p(x)} dx < +\infty \right\}.$$

The space $L^{p(\cdot)}(\Omega)$ equipped with the norm

$$\|u\|_{p(\cdot)} = \|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{ \mu > 0 \mid \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \le 1 \right\}.$$

If $p^- > 1$, then $L^{p(\cdot)}(\Omega)$ is reflexive and the dual space of $L^{p(\cdot)}(\Omega)$ can be identified with $L^{p'(\cdot)}(\Omega)$ where $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ the Hölder type inequality

$$\int_\Omega | uv | \, dx \quad \le \quad 2\|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)}, \tag{3}$$

holds true. We may also consider the generalized Lebesgue space

$$\|u\|_{L^{p(\cdot)}(Q_T)} = \inf\left\{ \mu > 0 \mid \int_{Q_T} \left| \frac{u(t,x)}{\mu} \right|^{p(x)} dxdt \le 1 \right\},$$

which, of course, shares the same type of properties as $L^{p(\cdot)}(\Omega)$. We define also the Banach spaces

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega) \right\},$$

$$W_0^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) \mid |\nabla u| \in L^{p(\cdot)}(\Omega) \quad \text{and} \quad u = 0 \quad \text{on} \quad \partial\Omega \right\}.$$

For $u \in W_0^{1,p(\cdot)}(\Omega)$ with $p \in C(\overline{\Omega})$ and $p^- \ge 1$, the poincaré inequality holds [15]

$$\|u\|_{L^{p(\cdot)}(\Omega)} \quad \le \quad C\|\nabla u\|_{L^{p(\cdot)}(\Omega)}, \tag{4}$$

for some constant C, which depends on Ω and the function p.

Remark 2.1. *In general, the smooth functions are in general not dense in* $W_0^{1,p(\cdot)}(\Omega)$, *but if the exponent variable* $p(\cdot) > 1$ *satisfies the log-Hölder continuity condition* (5), *that is* $\exists M > 0$:

$$|p(x) - p(y)| \quad \le \quad -\frac{M}{\ln(|x-y|)} \; \forall x \ne y \in \Omega \text{ such that } |x-y| \le \frac{1}{2}, \tag{5}$$

then the smooth functions are dense in $W^{1,p(\cdot)}(\Omega)$ *and* $W_0^{1,p(\cdot)}(\Omega)$ *(see [8, 13]). As in [2, 21], we do not need these condition to prove our result and will most exclusively work with* $p \in C_+(\overline{\Omega})$.

Remark 2.2. *[2] Let* $p(.) : \Omega \to (1,\infty)$ *be a continuous function. We have the following continuous dense embeddings*

$$L^{p^+}(0,T; L^{p(\cdot)}(\Omega)) \quad \hookrightarrow \quad L^{p(\cdot)}(Q_T) \hookrightarrow L^{p^-}(0,T; L^{p(\cdot)}(\Omega)). \tag{6}$$

3. Statements of results

For $\gamma > 0$, let T_γ be the truncation at levels $-\gamma$ and γ. This function, which is Lipschitz, meets the conditions $T_\gamma(0) = 0$, $|T_\gamma(r)| \le \gamma$, and its primitive function $\Theta_\gamma : \mathbb{R} \to \mathbb{R}^+$ is defined by

$$\Theta_\gamma(r) = \int_0^r T_\gamma(t)dt = \begin{cases} \frac{r^2}{2}, & \text{if;}|r| \le \gamma, \\ \gamma|r| - \frac{\gamma^2}{2}, & \text{if;}|r| > \gamma. \end{cases}$$

Throughout this paper, $\langle .,. \rangle$ denotes the duality between X and X^*(dual space of X, the set of continuous linear functional on X). We will then use the following result:

$$\int_0^T \langle \partial_t v, T_\gamma(v) \rangle dt = \int_\Omega \Theta_\gamma(v(T)) - \int_\Omega \Theta_\gamma(v(0)). \tag{7}$$

and

$$\gamma|r| - \frac{\gamma^2}{2} \le \Theta_\gamma(r) \le \gamma|r|, \quad \forall r \in \mathbb{R}. \tag{8}$$

Set

$$\tau_0^{1,p(.)}(Q_T) = \Big\{ u : (0,T] \times \Omega \to \mathbb{R} \quad \text{measurable}, \quad T_\gamma(u) \in L^{p^-}(0,T; W_0^{1,p(.)}(\Omega)), \\ \text{with} \quad \nabla T_\gamma(u) \in (L^{p(.)}(Q_T))^N, \text{ for every} \quad \gamma > 0 \Big\}.$$

Next, we define the weak gradient of a measurable function $u \in \tau_0^{1,p(.)}(Q_T)$. The proof follows from ([3], Lemma 2.1) due to the fact that $W_0^{1,p(.)}(\Omega) \subset W_0^{1,p^-}(\Omega)$ where $p^- = \min_{x \in \overline{\Omega}} p(x)$.

Proposition 3.1. *For every measurable function $u \in \tau_0^{1,p(.)}(Q_T)$, there exists a unique measurable function $v : Q_T \longrightarrow \mathbb{R}$ such that*

$$\nabla T_\gamma(u) = v\chi_{\{|u| \le \gamma\}}, \text{ a.e. in } Q_T, \ \forall \gamma > 0,$$

where χ_A denotes the characteristic function of a measurable set A. The functions v are called the weak gradient of u and are still denoted by ∇u. Moreover, if u belongs to $L^1(0,T; W_0^{1,1}(\Omega))$, then v coincides with the weak gradient of u, that is, $v = \nabla u$.

Definition 3.1. *Let $u_0 \in L^\infty(\Omega)$. Assume that* (1) *holds true. We will call an entropy solution of* (P) *a function $u \in \tau_0^{1,p(.)}(Q_T) \cap C([0,T]; L^1(\Omega))$ such that for every $\gamma > 0$,*

$$\int_\Omega \Theta_\gamma(u - \varphi)(T)dx - \int_\Omega \Theta_\gamma(u - \varphi)(0)dx + \int_0^T \langle \partial_t \varphi, T_\gamma(u - \varphi) \rangle dt \\ + \int_{Q_T} d(t,x,u)|\nabla u|^{p(x)-2} \nabla u \, \nabla T_\gamma(u - \varphi)dxdt \\ + \int_{Q_T} u^{\theta p(x)-1} |\nabla u|^{p(x)} T_\gamma(u - \varphi)dxdt \le \int_{Q_T} u^{r(x)-1} T_\gamma(u - \varphi), \tag{9}$$

where $u^{\theta p(x)-1}|\nabla u|^{p(x)} \in L^1(Q_T)$, $u^{r(x)-1} \in L^1(Q_T)$ and $\varphi \in L^{p^-}(0,T; W_0^{1,p(.)}(\Omega)) \cap L^\infty(Q_T)$ with $\partial_t \varphi \in L^{p'^-}(0,T; W^{-1,p'(.)}(\Omega)) + L^1(Q_T)$.

Definition 3.2. *Let $u_0 \in L^\infty(\Omega)$. A measurable function u is a renormalized solution of the parabolic problem* (P) *if $T_\gamma(u) \in \tau_0^{1,p(.)}(Q_T)$ for all $\gamma > 0$,*

$$u \in C([0,T]; L^1(\Omega)), \quad \lim_{\gamma \longrightarrow +\infty} \int_{B_\gamma} d(t,x,u)|\nabla u|^{p(x)}dxdt = 0$$

and

$$\int_0^T \langle \partial_t u, S'(u)\varphi\rangle dt + \int_{Q_T} d(t,x,u)|\nabla u|^{p(x)-2}\nabla u(S''(u)\varphi\nabla u + S'(u)D_i\varphi)dxdt$$
$$+ \int_{Q_T} u^{\theta p(x)-1}|\nabla u|^{p(x)}S'(u)\varphi dxdt = \int_{Q_T} u^{r(x)-1}S'(u)\varphi dxdt, \quad (10)$$

where $B_\gamma = \{\gamma \le |u(t,x)| \le \gamma+1\}, \gamma > 0$, $u^{\theta p(x)-1}|\nabla u|^{p(x)}$, $u^{r(x)-1} \in L^1(Q_T)$, $\varphi \in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)) \cap L^\infty(Q_T)$ *and any renormalization* $S(.) \in C^\infty(\mathbb{R})$ *such that* $suppS'(.) \in [-M,M]$ *for some constant* $M > 0$.

Now we state our main results.

Theorem 3.2. *If* u_0 *belongs to* $L^\infty(Q_T)$ *and* (1)-(2) *is satisfied, then problem* (P) *possesses at least one an entropy solution.*

The Theorem's proof has multiple steps. Initially, we approximate problem (P) with a sequence of problems (P_n) that have smooth solutions u_n and we obtain uniform estimates of u_n. Then, we take the limit as n goes to infinity. Lastly, we prove the existence of an entropy solution.

Theorem 3.3. *Let* $u_0 \in L^\infty(\Omega)$. *Under the assumptions* (1)-(2), *the entropy solution of parabolic problem* (P) *is also a renormalized solution*

4. The proofs of main results

Proof of Theorem 3.2

Let $n \in \mathbb{N}$ be arbitrary, let us consider the following approximated problem

$$(P_n^*) \quad \begin{cases} \partial_t u_n - \operatorname{div}(d(t,x,T_n(u_n))|\nabla u_n|^{p(x)-2}\nabla u_n) + |u_n|^{\theta p(x)-2}u_n|\nabla u_n|^{p(x)} = T_n(|u_n|^{r(x)-2}u_n), & \text{in } Q_T \\ u_n(0,x) = u_0(x), & \text{in } \Omega \\ u_n = 0, & \text{on } \Gamma_T. \end{cases}$$

Note that by (1), we have

$$d(t,x,T_n(u_n)) \ge \frac{\alpha}{(1+|T_n(u_n)|)^{\varrho(x)}} \ge \frac{\alpha}{(1+n)^{\varrho^+}},$$

so that the operator $B : v \mapsto \operatorname{div}(d(t,x,T_n(v_n)|\nabla v_n|^{p(x)-2}\nabla v_n)$ is coercive. Thus, the fact that $|T_n(|u_n|^{r(x)-2}u_n)| \le n$, the existence of the approximate solution $u_n \in L^{p^-}(0,T;W_0^{1,p(.)}(\Omega)) \cap C\left([0,T];L^2(\Omega)\right)$ is proved as in [11]. Due to the fact that the lower-order term has the same sign of the solution, it is easy to prove by taking u_n^- as test function in the weak formulation of problem (P_n^*) that $u_n \ge 0$. Consequently, u_n solves the problem

$$\begin{cases} \partial_t u_n - \operatorname{div}(d(t,x,T_n(u_n))|\nabla u_n|^{p(x)-2}\nabla u_n) + u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)} = T_n(u_n^{r(x)-1}), & \text{in } Q_T \\ u_n(0,x) = u_0(x), & \text{in } \Omega \\ u_n = 0, & \text{on } \Gamma_T. \end{cases} \quad (11)$$

Lemma 4.1. *Let* u_n *be the solutions to problems* (11). *Then we have for all* $\gamma > 0$

$$\frac{\alpha}{(1+\gamma)^{\varrho^+}} \int_0^T \int_\Omega |\nabla T_\gamma(u_n)|^{p(x)}dxdt \le \gamma \int_0^T \int_\Omega u_n^{r(x)-1}dxdt + \gamma\|u_0\|_{L^1(\Omega)}. \quad (12)$$

Proof. We choose $T_\gamma(u_n)$ as test function in (11) and the fact that $|T_n(u_n^{r(x)-1})| \le u_n^{r(x)-1}$, we obtain

$$\begin{aligned}\int_\Omega \Theta_\gamma(u_n)(T)dx + \int_0^T\int_\Omega d(t,x,T_n(u_n))|\nabla u_n|^{p(x)-2}\nabla u_n \nabla(T_\gamma(u_n))dxdt \\ + \int_0^T\int_\Omega u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)}T_\gamma(u_n)dxdt \\ \le \int_0^T\int_\Omega u_n^{r(x)-1}T_\gamma(u_n)dxdt + \int_\Omega \Theta_\gamma(u_n)(0)dx. \end{aligned} \tag{13}$$

Since $\Theta_\gamma \ge 0$ and $u_n^{\theta p(x)-1}T_\gamma(u_n) \ge 0$, so after dropping non-negative terms and using (8), we obtain

$$\int_0^T\int_\Omega d(t,x,T_n(u_n))|\nabla u_n|^{p(x)-2}\nabla u_n \nabla(T_\gamma(u_n))dxdt \le \int_0^T\int_\Omega u_n^{r(x)-1}|T_\gamma(u_n)|dxdt + \gamma\|u_0\|_{L^1(\Omega)}. \tag{14}$$

According to the conditions (1) and for $n > \gamma > 0$, we get (12). □

We shall denote by C or C_j various constants depending only on the structure of p^-, θ, γ, T, u_0, $|\Omega|$, for $j \in \mathbb{N}$.

Lemma 4.2. *Let u_n be a solution of problem* (11) *and suppose that* (1), (2) *hold true. Then, we have*

- *the sequence $\{u_n\}$ is bounded in $L^\infty(Q_T)$,*
- *the sequence $\{T_\gamma(u_n)\}$ is bounded in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$.*

Moreover. there exist a positive constant C such that

$$\|u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)}\|_{L^1(Q_T)} \le C.$$

Proof. Choosing $\varphi = u_n$ as test function in the approximated problem (11), we obtain, using (1)

$$\begin{aligned}\int_0^T \langle \partial_t u_n, u_n\rangle dt + \alpha\int_0^T\int_\Omega \frac{|\nabla u_n|^{p(x)}}{(1+|T_n(u_n)|)^{\varrho(x)}}dxdt + \int_0^T\int_\Omega u_n^{\theta p(x)}|\nabla u_n|^{p(x)}dxdt \\ \le \int_0^T\int_\Omega |T_n(u_n^{r(x)-1})|u_n dxdt.\end{aligned} \tag{15}$$

Using that $|T_n(u_n^{r(x)-1})|u_n \le u_n^{r(x)}$ and

$$\int_0^T \langle \partial_t u_n, u_n\rangle dt = \frac{1}{2}\int_\Omega (u_n(T,x))^2dx - \frac{1}{2}\int_\Omega (u_n(0,x))^2dx,$$

dropping positive term, we get this inequality

$$\int_0^T\int_\Omega u_n^{p(x)\theta}|\nabla u_n|^{p(x)}dxdt \le \int_0^T\int_\Omega u_n^{r(x)}dxdt + \frac{1}{2}\int_\Omega u_0^2dx.$$

Remark that $u_n^{\theta p^-}|\nabla u_n|^{p^-} - u_n^{\theta p^-} - |\nabla u_n|^{p^-} \le u_n^{\theta p(x)}|\nabla u_n|^{p(x)}$ and $u_0 \in L^\infty(\Omega)$. Then, we have

$$\int_0^T\int_\Omega u_n^{p^-\theta}|\nabla u_n|^{p^-}dxdt \le \int_0^T\int_\Omega u_n^{r(x)}dxdt + \int_0^T\int_\Omega u_n^{\theta p^-}dxdt + \int_0^T\int_\Omega |\nabla u_n|^{p^-}dxdt + C_0, \tag{16}$$

Since $\theta p^- \le r(x)$ and by (12), we have that

$$\int_0^T\int_\Omega u_n^{p^-\theta}|\nabla u_n|^{p^-}dxdt \le C_1\int_0^T\int_\Omega u_n^{r(x)}dxdt + C_2.$$

Applying Poincaré inequality and since $r(x) \leq r^+ < p^-(\theta+1)$, it follows from Young inequality that

$$\int_0^T \int_\Omega u_n^{p^-(\theta+1)} dxdt \leq C_3.$$

which implies that the boundedness of (u_n) in $L^{p^-(\theta+1)}(Q_T)$. Now, we prove that the sequence $(u_n^{r(x)-1})_n$ is bounded in $L^1(Q_T)$. To this end, we choose $u_n^{\nu+1}$ as test function in (11) where $\nu > 0$, we find

$$\alpha(\nu+1)\int_0^T \int_\Omega \frac{|\nabla u_n|^{p(x)}}{(1+|T_n(u_n)|)^{\varrho(x)}} u_n^\nu dxdt + \int_0^T \int_\Omega u_n^{\theta p(x)+\nu} |\nabla u_n|^{p(x)} dxdt \leq \int_0^T \int_\Omega |T_n(u_n^{r(x)-1})| u_n^{\nu+1} dxdt + C.$$

Dropping the non-negative term and with the same previous calculations, we find

$$\int_0^T \int_\Omega u_n^{p^-(\theta+1)+\nu} dxdt \leq \int_0^T \int_\Omega u_n^{r^+ +\nu} dxdt + C. \tag{17}$$

Now, we can choose $\nu = p^-(\theta+1) - r^+$ ($\nu > 0$ since (2)), so $p^-(\theta+1) + \nu = 2p^-(\theta+1) - r^+$. Then by (17), we obtain that u_n is bounded in $L^{2p^-(\theta+1)-r^+}(Q_T)$. Consequently, an iterating procedure gives us that (u_n) is bounded in $L^m(Q_T)$ for all $m < +\infty$. Indeed, if we consider $\nu_1 > 0$ such that $r^+ + \nu_1 = 2p^-(\theta+1) - r^+$, by (17) and the fact that (u_n) is bounded in $L^{2p^-(\theta+1)-r^+}(Q_T)$, then it is bounded in $L^{4p^-(\theta+1)-3r^+}$. Now consider $\nu_2 > 0$ such that $r^+ + \nu_2 = 4p^-(\theta+1) - 3r^+$ and deduce that (u_n) is bounded in $L^{8p^-(\theta+1)-7r^+}(Q_T)$. Hence we can obtain that (u_n) is bounded in $L^{2^s p^-(\theta+1)-(2^s-1)r^+}(Q_T)$ for all $s \in \mathbb{N}$. Since

$$2^s p^-(\theta+1) - (2^s-1)r^+ = 2^s(p^-(\theta+1) - r^+) + r^+ \to +\infty, \text{ as } s \to +\infty,$$

we deduce that (u_n) is bounded in $L^m(Q_T)$ for all $m < +\infty$. Because there is $s' > 0$ such that $\frac{s'(p^-(\theta+1)-r^+)+r^+}{r^+-1} > \frac{N}{p^-} + 1$,

$$(u_n^{r^+-1}) \text{ is bounded in } L^m(Q_T) \text{ for some } m > \frac{N}{p^-} + 1. \tag{18}$$

Standard parabolic estimates, performed using only the principal part of the operator (see for example [10]), and taking advantage of the nonnegativity of the lower order gradient term, then imply that $(u_n)_n$ is bounded in $L^\infty(Q_T)$. Since $r(x) - 1 \leq r^+ - 1$ we have $(u_n^{r(x)-1})$ is bounded in $L^1(Q_T)$. Now, to prove that the sequence $(T_\gamma(u_n))_n$ is bounded in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$, we get using (12),(18) and the fact that $u_0 \in L^\infty(\Omega)$, there exist a positive constant C such that

$$\|T_\gamma(u_n)\|_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))} \leq C, \tag{19}$$

Therefore, we have

$$\int_{Q_T} |\nabla u_n|^{p(x)} dxdt = \int_{\{u_n \leq \gamma\}} |\nabla T_\gamma(u_n)|^{p(x)} dxdt + \int_{\{u_n > \gamma\}} |\nabla u_n|^{p(x)} dxdt,$$

by L^∞ estimate of u_n and using the dominated convergence theorem

$$\chi_{\{u_n > \gamma\}} |\nabla u_n|^{p(x)} \longrightarrow 0 \quad \text{as} \quad \gamma \to +\infty.$$

Thus, by the estimate (19) we obtain

$$\|u_n\|_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))} \leq C. \tag{20}$$

Moreover, if we take $\frac{T_\gamma(u_n)}{\gamma}$ as test function in (11) and dropping the non-negative terms, we obtain

$$\int_0^T\int_\Omega u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)}\frac{T_\gamma(u_n)}{\gamma}dxdt \le \int_0^T\int_\Omega u_n^{r(x)-1}\left|\frac{T_\gamma(u_n)}{\gamma}\right|dxdt + \frac{1}{\gamma}\int_\Omega |\Theta_\gamma(u_n)(0)|dx.$$

Using the fact that $\left|\frac{T_\gamma(u_n)}{\gamma}\right| \le 1$ and (8), we have

$$\int_0^T\int_\Omega u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)}\frac{T_\gamma(u_n)}{\gamma}dxdt \le \int_0^T\int_\Omega u_n^{r(x)-1}dxdt + \|u_0\|_{L^1(\Omega)}.$$

Letting γ tend to 0 and by Fatou's Lemma, we deduce

$$\|u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)}\|_{L^1(Q_T)} \le C. \tag{21}$$

□

Since $(u_n)_n$ is bounded in $L^\infty(Q_T)$ and taking n large enough, we get $T_n(u_n^{r(x)-1}) = u_n^{r(x)-1}$ and $T_n(u_n) = u_n$, so that we conclude that u_n is a weak solution of

$$(P_n) \qquad \begin{cases} \partial_t u_n - \operatorname{div}(d(t,x,u_n)|\nabla u_n|^{p(x)-2}\nabla u_n) + u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)} = u_n^{r(x)-1}, & \text{in } Q_T \\ u_n(0,x) = u_0(x) \ge 0, & \text{in } \Omega \\ u_n = 0, & \text{on } \Gamma_T. \end{cases}$$

Passage to the limit.

Lemma 4.3.

$$\nabla T_\gamma(u_n) \longrightarrow \nabla T_\gamma(u) \ \textit{strongly in} \ (L^{p(.)}(Q_T))^N, \quad \textit{as} \quad n \longrightarrow +\infty, \quad \textit{for every} \quad \gamma > 0. \tag{22}$$

Proof. By the estimate (20), there exist a function $u \in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$ and a subsequence, still denoted by $\{u_n\}$, such that

$$u_n \rightharpoonup u \quad \text{weakly in} \quad L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)). \tag{23}$$

Moreover, going back to (P_n), the sequence $(\partial_t u_n)$ is bounded in the space

$$L^{p'^-}(0,T;W^{-1,p'(x)}(\Omega)) + L^1(Q_T), \quad p'(x) = \frac{p(x)}{p(x)-1}.$$

Using compactness argument in ([20] Corollary 4), we obtain that

$$u_n \to u \quad \text{strongly in} \quad L^1(Q_T), \text{ and a.e. in } Q_T. \tag{24}$$

We now introduce a time-regularized truncation $T_\gamma(u)$ that belongs to the space $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$. With $\nu > 0$, we define

$$(T_\gamma(u))_\nu(t,x) := \nu\int_{-\infty}^t e^{\nu(s-t)}T_\gamma(u(s,x))ds + e^{-\nu t}T_\gamma(u_0), \tag{25}$$

where $T_\gamma(u(s,x))$ is the zero extension of u for $s<0$ (See [22]), we have the following properties:

- $((T_\gamma(u))_\nu)_t = \nu(T_\gamma(u) - (T_\gamma(u))_\nu)$
- $((T_\gamma(u))_\nu)(0,x) = T_\gamma(u_0)$
- $|((T_\gamma(u))_\nu)| \le \gamma$

- $(\nabla T_\gamma(u))_\nu \longrightarrow \nabla T_\gamma(u)$ strongly in $(L^{p(\cdot)}(Q_T))^N$ as $\nu \longrightarrow +\infty$.

Let $h > \gamma$ and let us take

$$w_n = T_{2\gamma}(u_n - T_h(u_n) + T_\gamma(u_n) - (T_\gamma(u))_\nu),$$

as test function in (P_n), we have

$$\int_0^T \langle \partial_t u_n, w_n \rangle dt + \int_0^T \int_\Omega d(t,x,u_n)(|\nabla u_n|^{p(x)-2}\nabla u_n)\nabla w_n dxdt \\ + \int_0^T \int_\Omega u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)} w_n dxdt = \int_0^T \int_\Omega u_n^{r(x)-1} w_n dxdt. \tag{26}$$

We will also denote by $\tau(n,\nu,h)$ any quantity I such that

$$\lim_{h\longrightarrow+\infty}\lim_{\nu\longrightarrow+\infty}\lim_{n\longrightarrow+\infty} I = 0.$$

By the same technique of [18], we can obtain that

$$\int_0^T \langle \partial_t u_n, w_n \rangle dt \geq \tau(n,\nu,h).$$

If we set $M = 4\gamma + h > h > \gamma$, then it is easy to see that $\nabla w_n = 0$ on the set $\{|u_n| > M\}$. Therefore, from the above estimate, we can write the inequality (26) as

$$\int_0^T \int_\Omega d(t,x,u_n)|\nabla T_M(u_n)|^{p(x)-2}\nabla T_M(u_n)\nabla w_n dxdt + \int_0^T \int_\Omega u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)} w_n dxdt \\ \leq \int_0^T \int_\Omega u_n^{r(x)-1} w_n dxdt - \tau(n,\nu,h). \tag{27}$$

Splitting the integral in the left-hand side on the sets where $|u_n| \leq \gamma$ and where $|u_n| > \gamma$, we find

$$\int_0^T \int_\Omega d(t,x,u_n)|\nabla T_M(u_n)|^{p(x)-2}\nabla T_M(u_n)\nabla w_n dxdt \\ \geq \frac{\alpha}{(1+\gamma)^{\varrho^+}} \int_0^T \int_{\{|u_n|\leq\gamma\}} |\nabla T_\gamma(u_n)|^{p(x)-2}\nabla T_\gamma(u_n)\nabla(T_\gamma(u_n) - (T_\gamma(u))_\nu)dxdt \\ -\beta \int_0^T \int_{\{|u_n|>\gamma\}} ||\nabla T_M(u_n)|^{p(x)-2}\nabla T_M(u_n)||\nabla(T_\gamma(u))_\nu|dxdt.$$

It follows from the above inequality that

$$\frac{\alpha}{(1+\gamma)^{\varrho^+}} \int_0^T \int_{\{|u_n|\leq\gamma\}} |\nabla T_\gamma(u_n)|^{p(x)-2}\nabla T_\gamma(u_n)\nabla(T_\gamma(u_n) - (T_\gamma(u))_\nu)dxdt \\ + \int_0^T \int_\Omega u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)} w_n dxdt \leq \beta \int_0^T \int_{\{|u_n|>\gamma\}} |\nabla T_M(u_n)|^{p(x)-1}|\nabla(T_\gamma(u))_\nu|dxdt \\ + \int_0^T \int_\Omega u_n^{r(x)-1} w_n dxdt - \tau(n,\nu,h).$$

Furthermore, we have

$$\begin{aligned}
&\frac{\alpha}{(1+\gamma)^{\varrho^+}}\int_0^T\int_\Omega[|\nabla T_\gamma(u_n)|^{p(x)-2}\nabla T_\gamma(u_n)-|\nabla T_\gamma(u)|^{p(x)-2}\nabla T_\gamma(u)]\\
&\qquad\times\nabla(T_\gamma(u_n)-(T_\gamma(u))_\nu)dxdt\\
&\leq \underbrace{\beta\int_0^T\int_{\{|u_n|>\gamma\}}|\nabla T_M(u_n)|^{p(x)-1}|\nabla(T_\gamma(u))_\nu|dxdt}_{I_1}\\
&\underbrace{-\frac{\alpha}{(1+\gamma)^{\varrho^+}}\int_0^T\int_\Omega|\nabla T_\gamma(u)|^{p(x)-2}\nabla T_\gamma(u)\nabla(T_\gamma(u_n)-(T_\gamma(u))_\nu)dxdt}_{I_2}\\
&\underbrace{-\int_0^T\int_\Omega u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)}w_ndxdt}_{I_3}+\underbrace{\int_0^T\int_\Omega u_n^{r(x)-1}w_ndxdt}_{I_4}-\tau(n,\nu,h).
\end{aligned}$$

Limit of I_1.We observe that $|\nabla T_M(u_n)|^{p(x)-1}$ is bounded in $L^{p'(.)}(Q_T)$ and using the fact that

$$(\nabla T_\gamma(u))_\nu\longrightarrow\nabla T_\gamma(u)\quad\text{strongly in}\quad(L^{p(.)}(Q_T))^N\text{ as}\quad\nu\longrightarrow+\infty,$$

by the dominated convergence theorem

$$\chi_{\{|u_n|>\gamma\}}|\nabla(T_\gamma(u))_\nu|\longrightarrow\chi_{\{|u|\geq\gamma\}}|\nabla T_\gamma(u)|\quad\text{strongly in}\quad L^{p(.)}(Q_T),$$

which is zero, as n and ν tends to infinity. Thus, we obtain

$$\lim_{\nu\longrightarrow+\infty}\lim_{n\longrightarrow+\infty}I_1\ =\ 0. \tag{28}$$

Limit of I_2. Using the boundedness of $\nabla T_\gamma(u_n)$ in $(L^{p(x)}(Q_T))^N$ (Since (19)), we draw a subsequence (still denoted by $\{u_n\}$) from $\{u_n\}$ such that

$$\nabla T_\gamma(u_n)\ \rightharpoonup\ \nabla T_\gamma(u)\quad\text{weakly in}\quad(L^{p(x)}(Q_T))^N. \tag{29}$$

and the boundedness of $|\nabla T_\gamma(u)|^{p(x)-2}\nabla T_\gamma(u)$ in $L^{p'(.)}(Q_T)$, we obtain

$$\lim_{\nu\longrightarrow+\infty}\lim_{n\longrightarrow+\infty}I_2=0. \tag{30}$$

Limit of I_3 Using that $u_n^{\theta p(x)-1}\leq u_n^{\theta p^+-1}+1$, we have

$$\begin{aligned}
-\int_0^T\int_\Omega u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)}w_ndxdt&\leq-\int_0^T\int_{\{0\leq u_n\leq\gamma\}}u_n^{\theta p(x)-1}|\nabla T_\gamma(u_n)|^{p(x)}w_ndxdt\\
&\leq\int_0^T\int_\Omega u_n^{\theta p^+-1}|\nabla T_\gamma(u_n)|^{p(x)}|w_n|dxdt+\int_0^T\int_\Omega|\nabla T_\gamma(u_n)|^{p(x)}|w_n|dxdt\\
&\leq(C_{\gamma,\theta}^{p^+}+1)\int_0^T\int_\Omega|\nabla T_\gamma(u_n)|^{p(x)}|w_n|dxdt.
\end{aligned}\tag{31}$$

where $C_{\gamma,\theta}^{p^+}$ is a positive constant, such that $C_{\gamma,\theta}^{p^+}=\max_{u_n\in[0,\gamma]}u_n^{\theta p(x)-1}$. Since (19) and the fact that

$$|\nabla T_\gamma(u_n)|^{p(x)}|w_n|\leq2\gamma\|\nabla T_\gamma(u_n)\|_{L^{p(x)}(Q_T)}\in L^1(Q_T),$$

by Lebesgue dominated convergence theorem,

$$\lim_{h\longrightarrow+\infty}\lim_{\nu\longrightarrow+\infty}\lim_{n\longrightarrow+\infty} I_3 = 0.$$

Limit of I_4 By the properties of $(T_\gamma(u))_\nu$ and (18) and we have

$$|u_n^{r(x)-1}w_n| \le 2\gamma(\|u_n^{r^+-1}\|_{L^\infty(Q_T)} + 1) \in L^1(Q_T),$$

by Lebesgue dominated convergence theorem,

$$\lim_{h\longrightarrow+\infty}\lim_{\nu\longrightarrow+\infty}\lim_{n\longrightarrow+\infty} I_4 = 0.$$

Now, passing to the limits in (28) as n, ν, h tend to infinity, we deduce that

$$\lim_{n\longrightarrow+\infty} I_n^\gamma = 0,$$

where

$$I_n^\gamma = \int_0^T\int_\Omega [|\nabla T_\gamma(u_n)|^{p(x)-2}\nabla T_\gamma(u_n) - |\nabla T_\gamma(u)|^{p(x)-2}\nabla T_\gamma(u)]\nabla(T_\gamma(u_n) - T_\gamma(u))dxdt.$$

As [22]. We recall the following well-known inequalities: for any two real vectors ξ, η and for every $\varepsilon \in [0.1]$,:

$$|\xi-\eta|^p \le \begin{cases} c(p)(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi-\eta), & \text{if } p \ge 2,\\ k(p)\varepsilon^{(p-2)/p}(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi-\eta) + \varepsilon|\eta|^p, & \text{if } 1 < p < 2,\end{cases}$$

where $c(p) = \frac{p-1}{2^{1-p}}$ and $k(p) = \frac{3^{2-p}}{p-1}$. Therefore, we have

$$\frac{1}{c(p^+)}\int_0^T\int_{\{x\in\Omega, p(x)\ge2\}} |\nabla(T_\gamma(u_n) - T_\gamma(u))|^{p(x)}dxdt \le I_n^\gamma,$$

and

$$\int_0^T\int_{\{x\in\Omega,1<p(x)<2\}} |\nabla(T_\gamma(u_n) - T_\gamma(u))|^{p(x)}\,dxdt \le k(p^-)\varepsilon^{(p^--2)/p^-} I_n^\gamma + \varepsilon\int_0^T\int_\Omega |\nabla T_\gamma(u)|^{p(x)}dxdt$$

Since $I_n^\gamma \longrightarrow 0$ as $n \longrightarrow +\infty$, then using the arbitrariness of ε and $\nabla T_\gamma(u)$ is bounded in $(L^{p(x)}(Q_T))^N$ (see (19) and (29)), we obtain

$$\lim_{n\longrightarrow+\infty}\int_0^T\int_\Omega |\nabla(T_\gamma(u_n) - T_\gamma(u))|^{p(x)}dxdt = 0.$$

From this result, we can deduce that (22) holds, and consequently (up to subsequences).

$$\nabla u_n \longrightarrow \nabla u \quad \text{almost everywhere in} \quad Q_T. \tag{32}$$

□

Lemma 4.4. *u is an entropy solution of* (P).

Proof. To prove that u is an entropy solution of problem (P) in Q_T, several facts are needed:

1. $u^{r(x)-1} \in L^1(Q_T)$ and $u^{\theta p(x)-1}|\nabla u|^{p(x)} \in L^1(Q_T)$,
2. $u \in C([0,T]; L^1(\Omega))$,
3. the entropic formulation (9) holds.

The first condition is a consequence of proving that

$$u_n^{r(x)-1} \longrightarrow u^{r(x)-1} \quad \text{strongly in} \quad L^1(Q_T), \tag{33}$$

and

$$u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)} \longrightarrow u^{\theta p(x)-1}|\nabla u|^{p(x)} \quad \text{strongly in} \quad L^1(Q_T). \tag{34}$$

Thanks to (24), we only need to prove that the sequence $(u_n^{r(x)-1})$ is uniformly integrable in Q_T. For this purpose, we use $T_1(u_n)$ as test function in (P_n), eliminate non-negative terms, and utilize the inequality $u_n^{\theta p^-}|\nabla u_n|^{p^-} - u_n^{\theta p^-} - |\nabla u_n|^{p^-} \le u_n^{\theta p(x)}|\nabla u_n|^{p(x)}$, (12) and $\theta p^- \le r(x) - 1$, (since (2)), it follows that

$$\int_0^T \int_\Omega u_n^{\theta p^- -1}|\nabla u_n|^{p^-} T_1(u_n)dxdt \le C \int_0^T \int_\Omega u_n^{r(x)-1} T_1(u_n)dxdt + 2\|u_0\|_{L^1(\Omega)}.$$

Since

$$\int_{Q_T} u_n^{\theta p^- -1}|\nabla u_n|^{p^-} T_1(u_n)dxdt \ge \int_{\{u_n>1\}\cap Q_T} u_n^{\theta p^- -1}|\nabla u_n|^{p^-} dxdt,$$

and $r(x) \le r^+$, $u_0 \in L^\infty(\Omega)$, we have

$$\begin{aligned}\int_{\{u_n>1\}\cap Q_T} u_n^{\theta p^- -1}|\nabla u_n|^{p^-} dxdt &\le C \int_{Q_T} u_n^{r^+ -1} T_1(u_n)dxdt + C_0 \\ &\le C \int_{\{u_n\le 1\}\cap Q_T} u_n^{r^+} dxdt + C \int_{\{u_n>1\}\cap Q_T} u_n^{r^+ -1} dxdt + C_0 \\ &\le C.meas(Q_T) + C_0 + C_1 \int_{\{u_n>1\}\cap Q_T} (u_n - 1)^{r^+ -1} dxdt.\end{aligned}$$

Consequently, denoting $G_1(r) = r - T_1(r)$, we get the inequality

$$\int_{\{u_n>1\}\cap Q_T} u_n^{\theta p^- -1}|\nabla u_n|^{p^-} dxdt \le C_2 + C_1 \int_{\{u_n>1\}\cap Q_T} (G_1(u_n))^{r^+ -1} dxdt,$$

so, that

$$\int_{\{u_n>1\}\cap Q_T} |\nabla u_n|^{p^-} (G_1(u_n) + 1)^{\theta p^- -1} dxdt \le C_2 + C_1 \int_{\{u_n>1\}\cap Q_T} (G_1(u_n))^{r^+ -1} dxdt,$$

which yields

$$\begin{aligned}\left(1 + \frac{(\theta p^- - 1)}{p^-}\right)^{-p^-} \int_{\{u_n>1\}\cap Q_T} |\nabla (G_1(u_n) + 1)^{\left(1 - \frac{(1-\theta p^-)}{p^-}\right)}|^{p^-} dxdt \\ \le C_2 + C_1 \int_{\{u_n>1\}\cap Q_T} (G_1(u_n))^{r^+ -1} dxdt.\end{aligned}$$

Now Poincaré inequality implies

$$\int_{\{u_n>1\}\cap Q_T} (G_1(u_n) + 1)^{(\theta+1)p^- -1} dxdt \le C_3 + C_4 \int_{\{u_n>1\}\cap Q_T} (G_1(u_n))^{r^+ -1} dxdt.$$

Observe that $r^+ - 1 < (\theta + 1)p^- - 1$ (since (2)), by Young's inequality we obtain

$$\int_{\{u_n>1\}\cap Q_T} |G_1(u_n)|^{(\theta+1)p^- -1} dxdt \le C_5. \tag{35}$$

Therefore

$$\int_{\{u_n>1\}\cap Q_T} u_n^{r(x)-1}dxdt = \int_{\{u_n>1\}\cap Q_T} (G_1(u_n)+1)^{r^+-1}dxdt \le C_6.$$

From (18),(35), $r(x) < (\theta+1)p^-$ and Hölder inequality, we deduce that $(u_n^{r(x)-1})_n$ is equi-integrable in Q_T, then by Vitali's Theorem convergence , we have (33).

Now, we shall obtain local equi-integrability of $u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)}$ on Q_T. To this end. For every $\eta > 1$, we define the function $\psi_\eta : \mathbb{R}^+ \to \mathbb{R}$ by

$$\psi_\eta(\sigma) = (\eta-1)\int_0^\sigma \frac{dt}{(1+t)^\eta} = \left(1 - \frac{1}{(1+\sigma)^{\eta-1}}\right) \ge 0, \quad \sigma \in \mathbb{R}^+. \tag{36}$$

Let $k > 0$. We also define $\varphi_k : \mathbb{R}^+ \to \mathbb{R}$ by $\varphi_k(\sigma) = \psi_\eta(\sigma - k)$ if $\sigma \ge k$ and $\varphi_k(\sigma) = 0$ if $0 < \sigma < k$. We choose $\varphi_k(u_n)$ as test function in (P_n), we obtain

$$\begin{aligned}\int_\Omega dx\int_0^{u_n(T,x)} \varphi_k(\sigma)d\sigma + \int_{Q_T} d(t,x,u_n)|\nabla u_n|^{p(x)}\varphi_k'(u_n)dxdt + \int_{Q_T} u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)}\varphi_k(u_n)dxdt \\ \le \int_{Q_T}\varphi_k(u_n)u_n^{r(x)-1}dxdt + \int_\Omega dx\int_0^{u_n(0,x)}\varphi_k(\sigma)d\sigma.\end{aligned}$$

As $|\varphi_k| \le 1$ and (1), we have

$$\begin{aligned}\int_\Omega dx\int_0^{u_n(T,x)} \varphi_k(\sigma)d\sigma + \alpha\int_{Q_T} \frac{|\nabla u_n|^{p(x)}}{(1+u_n)^{\rho(x)+\eta}}dxdt + \int_{Q_T} u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)}\varphi_k(u_n)dxdt \\ \le \int_{Q_T\cap\{u_n\ge k\}} u_n^{r(x)-1}dxdt + \int_{\Omega\cap\{u_n\ge k\}} u_0 dx.\end{aligned}$$

Dropping the non negative terms and using (18), we deduce from the above inequality that

$$\int_0^T\int_\Omega u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)}\varphi_k(u_n)dxdt \to 0 \quad \text{as} \quad k \to +\infty \text{ uniformly with respect to } n.$$

Using the properties of the function φ_k, we get as $k \to +\infty$

$$\int_{\Omega\cap\{u_n\ge 2k\}} u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)}dxdt \le \frac{1}{\varphi_k(2k)}\int_0^T\int_\Omega u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)}\varphi_k(u_n)dxdt \to 0. \tag{37}$$

Consequently, we have

$$\begin{aligned}\int_0^T\int_\Omega u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)}dxdt \le \int_0^T\int_{\Omega\cap\{u_n\ge 2k\}} u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)}dxdt \\ +((2k)^{\theta p^+-1}+1)\int_0^T\int_{\Omega\cap\{u_n\le 2k\}} |\nabla T_{2k}(u_n)|^p dxdt.\end{aligned} \tag{38}$$

From (37) and (19) give equi-integrability of $u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)}$ on Q_T. By this, and (22),(24) and Vitali's Theorem, we obtain (34).

The second condition is a consequence of proving that the sequence (u_n) is a Cauchy sequence in $C([0,T];L^1(\Omega))$.To do this fix $t \in [0,T]$ and we define the vector-valued function $\widehat{a}(t,x,s,\xi) : Q_t \times \mathbb{R} \times \mathbb{R}^N \longrightarrow$

$\mathbb{R}^N$, where $\widehat{a}(t,x,s,\xi) = d(t,x,s)|\xi|^{p(x)-2}\xi$. Taking $T_\gamma(u_n - u_\epsilon)$ as test function in (P_n) for u_n and u_ϵ, subtracting up both identities, we deduce that

$$\int_\Omega \Theta_\gamma(u_n(t) - u_\epsilon(t))dx + \int_0^T \int_\Omega \left[\widehat{a}(t,x,u_n,\nabla u_n) - \widehat{a}(t,x,u_n,\nabla u_\epsilon)\right] \nabla T_\gamma(u_n - u_\epsilon)$$
$$+ \int_0^T \int_\Omega \left[u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)} - u_\epsilon^{\theta p(x)-1}|\nabla u_\epsilon|^{p(x)}\right] T_\gamma(u_n - u_\epsilon)dxdt$$
$$\leq \int_0^T \int_\Omega |u_n^{r(x)-1} - u_\epsilon^{r(x)-1}||T_\gamma(u_n - u_\epsilon)|dxdt + \int_\Omega |\Theta_\gamma(u_n(0) - u_\epsilon(0))|dx.$$

Note that I_n^ϵ is well defined and $I_n^\epsilon \geq 0$, where

$$I_n^\epsilon = \int_0^T \int_\Omega [\widehat{a}(t,x,u_n,\nabla u_n) - \widehat{a}(t,x,u_n,\nabla u_\epsilon)]\nabla(u_n - u_\epsilon) \geq 0.$$

Dropping non-negative terms, we get

$$\int_\Omega \Theta_\gamma(u_n(t) - u_\epsilon(t))dx \leq \gamma \int_0^T \int_\Omega \left|u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)} - u_\epsilon^{\theta p(x)-1}|\nabla u_\epsilon|^{p(x)}\right| dxdt$$
$$+\gamma \int_0^T \int_\Omega |u_n^{r(x)-1} - u_\epsilon^{r(x)-1}|dxdt + \gamma \int_\Omega |(u_n(0) - u_\epsilon(0))|dx.$$

Next, we divide this inequality by γ and let $\gamma \to 0$. Since (8) and by applying the monotone convergence theorem, we obtain

$$\int_\Omega |u_n(t) - u_\epsilon(t)|dx \leq \int_0^T \int_\Omega \left|u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)} - u_\epsilon^{\theta p(x)-1}|\nabla u_\epsilon|^{p(x)}\right| dxdt$$
$$+ \int_0^T \int_\Omega |u_n^{r(x)-1} - u_\epsilon^{r(x)-1}|dxdt + \int_\Omega |(u_n(0) - u_\epsilon(0))|dx.$$

Hence

$$\sup_{t\in[0,T]} \int_\Omega |u_n(t) - u_\epsilon(t)|dx \leq \int_0^T \int_\Omega \left|u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)} - u_\epsilon^{\theta p(x)-1}|\nabla u_\epsilon|^{p(x)}\right| dxdt$$
$$+ \int_0^T \int_\Omega |u_n^{r(x)-1} - u_\epsilon^{r(x)-1}|dxdt + \int_\Omega |(u_n(0) - u_\epsilon(0)|dx.$$

Taking into account $u_0 \in L^\infty(\Omega)$,(33) and (34), we deduce that

$$\int_\Omega |u_n(t) - u_\epsilon(t)|dx \longrightarrow 0 \quad \text{as} \quad n, \epsilon \longrightarrow \infty.$$

Hence, u_n is a Cauchy sequence in $u \in C([0,T]; L^1(\Omega))$, thus $u \in C([0,T]; L^1(\Omega))$.

To finish the proof, we choose $T_\gamma(u_n - \varphi)$ with $\varphi \in L^{p^-}(0,T; W_0^{1,p(.)}(\Omega)) \cap L^\infty(Q_T)$ as a test function in (P_n), we obtain

$$\int_0^T \langle \partial_t u_n, T_\gamma(u_n - \varphi)\rangle dt + \int_{Q_T} d(t,x,,u_n)|\nabla u_n|^{p(x)-2}\nabla u_n \nabla T_\gamma(u_n - \varphi)$$
$$+ \int_{Q_T} u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)} T_\gamma(u_n - \varphi)dxdt = \int_{Q_T} u_n^{r(x)-1} T_\gamma(u_n - \varphi). \quad (39)$$

For the first term on the left-hand side of (39), we have

$$\int_0^T \langle \partial_t u_n, T_\gamma(u_n-\varphi)\rangle dt = \int_\Omega \Theta_\gamma(u_n-\varphi)(T)dx - \int_\Omega \Theta_\gamma(u_n-\varphi)(0)dx + \int_{Q_T} \partial_t\varphi T_\gamma(u_n-\varphi)dxdt.$$

Because that $u \in C([0,T];L^1(\Omega))$, we have $\forall t \in [0,T]$, $u_n(t) \longrightarrow u(t)$ in $L^1(\Omega)$, and since Θ_γ is Lipschitz continuous, we obtain

$$\int_\Omega \Theta_\gamma(u_n-\varphi)(T)dx \longrightarrow \int_\Omega \Theta_\gamma(u-\varphi)(T)dx,$$

and

$$\int_\Omega \Theta_\gamma(u_n-\varphi)(0)dx \longrightarrow \int_\Omega \Theta_\gamma(u_0-\varphi(0))dx.$$

We have $\partial_t\varphi \in L^{p'^-}(0,T;W^{-1,p'(.)}(\Omega)) + L^1(Q_T)$, and since

$$T_\gamma(u_n-\varphi) \rightharpoonup T_\gamma(u-\varphi) \text{ weakly* in } L^\infty(Q_T),$$

then,

$$\int_{Q_T} \partial_t\varphi T_\gamma(u_n-\varphi)dxdt \longrightarrow \int_{Q_T} \partial_t\varphi T_\gamma(u-\varphi)dxdt.$$

Also, by the dominated convergence and observing the strong convergence of $\{u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)}\}_n$ and $\{u_n^{r(x)-1}\}_n$ in $L^1(Q_T)$, hence

$$\int_{Q_T} u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)} T_\gamma(u_n-\varphi)dxdt \longrightarrow \int_{Q_T} u^{\theta p(x)-1}|\nabla u|^{p(x)} T_\gamma(u-\varphi)dxdt,$$

and

$$\int_{Q_T} u_n^{r(x)-1} T_\gamma(u_n-\varphi)dxdt \longrightarrow \int_{Q_T} u^{r(x)-1} T_\gamma(u-\varphi)dxdt.$$

Concerning the second term on the left-hand side of (39) and let $n > M = \gamma + \|\varphi\|_{L^\infty(Q_T)}$, we can be rewritten as

$$\int_{Q_T} d(t,x,u_n)|\nabla T_M(u_n)|^{p(x)-2}\nabla T_M(u_n)\nabla T_\gamma(u_n-\varphi)dxdt,$$

according to (22), (24) and Fatou's Lemma, we obtain

$$\begin{aligned}
&\liminf_{n\longrightarrow+\infty} \int_{Q_T} d(t,x,u_n)|\nabla T_M(u_n)|^{p(x)-2}\nabla T_M(u_n)\nabla T_\gamma(u_n-\varphi)dxdt \\
&\geq \int_{Q_T} d(t,x,u)|\nabla T_M(u)|^{p(x)-2}\nabla T_M(u)\nabla T_\gamma(u-\varphi)dxdt \\
&= \int_{Q_T} d(t,x,u)|\nabla u|^{p(x)-2}\nabla u\nabla T_\gamma(u-\varphi)dxdt.
\end{aligned}$$

Thus, we get (9) by putting all terms together, This completes the proof of Lemma 4.4 and so the proof of Theorem 3.2 is concluded. □

Now, we are going to prove Theorem 3.3.

Proof of Theorem 3.3

We start by defining the function $S_\gamma(.)$ in $C^2(\mathbb{R})$, such that $S_\gamma(x) = x$ for $|x| \leq \gamma$ and supp$S'_\gamma \subset [-\gamma-1, \gamma+1]$, then supp$S''_\gamma \subset [-\gamma-1, -\gamma] \cup [\gamma, \gamma+1]$.

The weak convergence of $\partial_t S_\gamma(u_n)$ in $L^{p'^-}(0,T;W^{-1,p'(.)}(\Omega))+L^1(Q_T)$

Taking $S'_\gamma(u_n)\varphi$ as a test function in (P_n) with $\varphi\in L^{p'^-}(0,T;W^{-1,p'(.)}(\Omega))+L^\infty(Q_T)$, we obtain

$$\int_0^T\langle\partial_t u_n,S'_\gamma(u_n)\varphi\rangle dt+\int_{Q_T}d(t,x,u_n)|\nabla u_n|^{p(x)-2}\nabla u_n(S''_\gamma(u_n)\varphi\nabla u_n+S'_\gamma(u_n)\nabla\varphi)dxdt$$
$$+\int_{Q_T}u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)}S'_\gamma(u_n)\varphi dxdt=\int_{Q_T}u_n^{r(x)-1}S'_\gamma(u_n)\varphi dxdt.$$

Then, we have

$$\left|\int_0^T\langle\partial_t S_\gamma(u_n),\varphi\rangle dt\right|\le\int_{Q_T}|d(t,x,u_n)|\left(|S''_\gamma(u_n)||\nabla u_n|^{p(x)}|\varphi|+|S'_\gamma(u_n)||\nabla u_n|^{p(x)-1}|\nabla\varphi|\right)$$
$$+\int_{Q_T}u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)}|S'_\gamma(u_n)\varphi|+\int_{Q_T}u_n^{r(x)-1}|S'_\gamma(u_n)\varphi|. \tag{40}$$

For the first term on the right-hand side of (40), using Hölder's inequality and the fact that $(\nabla T_\gamma(u_n))_n$ is bounded in $L^{p(.)}(Q_T)$ for all $\gamma>0$, we find

$$\int_{Q_T}|d(t,x,u_n)|\left(|S''_\gamma(u_n)||\nabla u_n|^{p(x)}|\varphi|+|S'_\gamma(u_n)||\nabla u_n|^{p(x)-1}|\nabla\varphi|\right)dxdt$$
$$\le\beta\int_0^T\int_{\{|u_n|\le\gamma+1\}}\left(|S''_\gamma(u_n)||\nabla u_n|^{p(x)}|\varphi|+|S'_\gamma(u_n)||\nabla u_n|^{p(x)-1}|\nabla\varphi|\right)dxdt$$
$$\le 2\beta\|S''_\gamma(.)\|_{L^\infty(\mathbb{R})}\|\varphi\|_{L^\infty(Q_T)}\|\nabla T_{\gamma+1}(u_n)\|_{L^{p(x)}(Q_T)}\|1\|_{L^{p(x)}(Q_T)}$$
$$+2\beta\|S'_\gamma(.)\|_{L^\infty(\mathbb{R})}\|\nabla\varphi\|_{L^{p(x)}(Q_T)}\|\nabla T^{p(x)-1}_{\gamma+1}(u_n)\|_{L^{p'(x)}(Q_T)}$$
$$\le C_1(\|\nabla\varphi\|_{L^{p(x)}(Q_T)}+\|\varphi\|_{L^\infty(Q_T)}).$$

So, that

$$\int_{Q_T}|d(t,x,u_n)|\left(|S''_\gamma(u_n)||\nabla u_n|^{p(x)}|\varphi|+|S'_\gamma(u_n)||\nabla u_n|^{p(x)-1}|\nabla\varphi|\right)$$
$$\le C_1\left(\|\varphi\|_{L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))}+\|\varphi\|_{L^\infty(Q_T)}\right). \tag{41}$$

Concerning the last two terms on the right-hand side of (40), we get

$$\int_{Q_T}u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)}|S'_\gamma(u_n)\varphi|dxdt+\int_{Q_T}u_n^{r(x)-1}|S'_\gamma(u_n)\varphi|dxdt$$
$$\le\|S'_\gamma(.)\|_{L^\infty(\mathbb{R})}\|\varphi\|_{L^\infty(Q_T)}(\|u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)}\|_{L^1(Q_T)}+\|u_n^{r(x)-1}\|_{L^1(Q_T)})$$
$$\le C_2\|\varphi\|_{L^\infty(Q_T)}. \tag{42}$$

Using (40),(41) and (42), we obtain

$$\left|\int_0^T\langle\partial_t S_\gamma(u_n),\varphi\rangle dt\right|\le C_3\left(\|\varphi\|_{L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))}+\|\varphi\|_{L^\infty(Q_T)}\right)+C_2\|\varphi\|_{L^\infty(Q_T)}.$$

Hence

$$\partial_t S_\gamma(u_n)\rightharpoonup\partial_t S_\gamma(u)\quad\text{in}\quad L^{p'^-}(0,T;W^{-1,p'(.)}(\Omega))+L^1(Q_T). \tag{43}$$

Existence of renormalized solutions.

Choosing $\phi_\gamma(u_n) = T_{\gamma+1}(u_n) - T_\gamma(u_n)$ as a test function in (P_n), we obtain

$$\int_0^T \langle \partial_t u_n, \phi_\gamma(u_n)\rangle dt + \int_{Q_T} d(t,x,u_n)|\nabla u_n|^{p(x)-2}\nabla u_n \nabla(\phi_\gamma(u_n)) + \int_{Q_T} u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)}\phi_\gamma(u_n)dxdt = \int_{Q_T} u_n^{r(x)-1}\phi_\gamma(u_n)dxdt.$$

Using (7) and the definition of Θ, we can write the first term as follows

$$\int_0^T \langle \partial_t u_n, \phi_\gamma(u_n)\rangle dt = \int_\Omega \Theta_{\gamma+1}(u_n(T))dx - \int_\Omega \Theta_\gamma(u_n(T))dx + \int_\Omega \Theta_\gamma(u_n(0))dx - \int_\Omega \Theta_{\gamma+1}(u_n(0))dx, \tag{44}$$

and since

$$\int_\Omega \Theta_{\gamma+1}(u_n(T))dx - \int_\Omega \Theta_\gamma(u_n(T))dx = \int_{\{\gamma\le|u_n(T)|\le\gamma+1\}}\left(\frac{|u_n(T)|^2}{2} - \gamma|u_n(T)| + \frac{\gamma^2}{2}\right) - \int_{\{|u_n(T)|>\gamma+1\}}\left(|u_n(T)| - \gamma - \frac{1}{2}\right) \ge 0, \tag{45}$$

we have

$$\int_{B_\gamma} d(t,x,u_n)|\nabla u_n|^{p(x)} + \int_{\{u_n\ge\gamma\}} u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)}(T_{\gamma+1}(u_n) - T_\gamma(u_n)) \le \int_{\{u_n\ge\gamma\}} u_n^{r(x)-1} + \int_{\{\gamma\le u_{0n}\le\gamma+1\}}\left(\frac{u_{0n}^2}{2} - \gamma u_{0n} + \frac{\gamma^2}{2}\right) - \int_{\{u_{0n}>\gamma+1\}}\left(u_{0n} - \gamma - \frac{1}{2}\right). \tag{46}$$

Since $u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)} \ge 0$, we can write (46) as

$$\int_{B_\gamma} d(t,x,u_n)|\nabla u_n|^{p(x)}dxdt \le \int_{\{u_n\ge\gamma\}} u_n^{r(x)-1}dxdt + \int_{\{\gamma\le u_0\le\gamma+1\}}\frac{1}{2} - \int_{\{u_0>\gamma+1\}}\left(u_0 - \gamma - \frac{1}{2}\right).$$

Thanks to Lemma 4.2, we have

$$\int_{\{u_n\ge\gamma\}} u_n^{r(x)-1}dxdt \le (\|u_n^{r^+-1}\|_{L^\infty} + 1)|(\{u_n \ge \gamma\})| \to 0 \quad \text{as} \quad \gamma \longrightarrow +\infty,$$

and

$$\int_{\{\gamma\le u_0\le\gamma+1\}}\frac{1}{2} - \int_{\{u_0>\gamma+1\}}\left(u_0 - \gamma - \frac{1}{2}\right) \longrightarrow 0, \quad \text{as} \quad \gamma \longrightarrow +\infty.$$

so, that

$$\int_{B_\gamma} d(t,x,u_n)|\nabla u_n|^{p(x)}dxdt \longrightarrow 0 \quad \text{as} \quad \gamma \longrightarrow \infty. \tag{47}$$

In view of Fatou's Lemma, we deduce that

$$\int_{B_\gamma} d(t,x,u)|\nabla u|^{p(x)}dxdt$$
$$\leq \liminf_{n\longrightarrow 0}\int_{B_\gamma} d(t,x,u_n)|\nabla u_n|^{p(x)}dxdt \longrightarrow 0 \quad \text{as} \quad \gamma \longrightarrow \infty.$$

We will now focus on proving the equality (10). Let $\varphi \in L^{p^-}(0,T;W_0^{1,p(.)}(\Omega)) \cap L^\infty(Q_T)$ and $S(.) \in C^\infty(\mathbb{R})$ such that supp$S'(.) \in [-M,M]$ for some constant $M > 0$. By Taking $S'(u_n)\varphi$ as a test function in (P_n), we get

$$\int_0^T \langle \partial_t u_n, S'(u_n)\varphi\rangle dt + \int_{Q_T} d(t,x,u_n)|\nabla u_n|^{p(x)-2}\nabla u_n(S''(u_n)\varphi\nabla u_n + S'(u_n)\nabla\varphi)dxdt$$
$$+ \int_{Q_T} u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)}S'(u_n)\varphi dxdt = \int_{Q_T} u_n^{r(x)-1}S'(u_n)\varphi dxdt. \tag{48}$$

According to (43), we have

$$\partial_t S(u_n) \rightharpoonup \partial_t S(u) \quad \text{in} \quad L^{p'^-}(0,T;W^{-1,p'(.)}(\Omega)) + L^1(Q_T),$$

and

$$\lim_{n\longrightarrow+\infty}\int_0^T \langle \partial_t u_n, S'(u_n)\varphi\rangle dt = \lim_{n\longrightarrow+\infty}\int_0^T \langle \partial_t S(u_n),\varphi\rangle dt = \int_0^T \langle \partial_t S(u),\varphi\rangle dt.$$

By looking at the following :

$$\int_{Q_T} d(t,x,u_n)|\nabla u_n|^{p(x)-2}\nabla u_n(S''(u_n)\varphi\nabla u_n + S'(u_n)\nabla\varphi)$$
$$= \int_{Q_T} d(t,x,u_n)|\nabla T_M(u_n)|^{p(x)-2}\nabla T_M(u_n)(S''(u_n)\varphi\nabla T_M(u_n) + S'(u_n)\nabla\varphi),$$

and since (24), it follows that

$$d(t,x,u_n)|\nabla T_M(u_n)|^{p(x)-2}\nabla T_M(u_n) \rightharpoonup d(t,x,u)|\nabla T_M(u)|^{p(x)-2}\nabla T_M(u),$$

in $L^{p'(.)}(Q_T)$, and the covergence strongly of $S''(u_n)\varphi\nabla T_M(u_n) + S'(u_n)\nabla\varphi$ in $L^{p(.)}(Q_T)$, (32), we conclude that

$$\lim_{n\longrightarrow+\infty}\int_{Q_T} d(t,x,u_n)|\nabla u_n|^{p(x)-2}\nabla u_n(S''(u_n)\varphi\nabla u_n + S'(u_n)\nabla\varphi)dxdt$$
$$= \int_{Q_T} d(t,x,u)|\nabla u|^{p(x)-2}\nabla u(S''(u)\varphi\nabla u + S'(u)\nabla\varphi)dxdt.$$

Moreover, since the convergence weakly* of $S'(u_n)\varphi$ in $L^\infty(Q_T)$ and (33),(34), then

$$\lim_{n\longrightarrow+\infty}\int_{Q_T} u_n^{\theta p(x)-1}|\nabla u_n|^{p(x)}S'(u_n)\varphi dxdt = \int_{Q_T} u^{\theta p(x)-1}|\nabla u|^{p(x)}S'(u)\varphi dxdt$$

and

$$\lim_{n\longrightarrow+\infty}\int_{Q_T} u_n^{r(x)-1}S'(u_n)\varphi dxdt = \int_{Q_T} u^{r(x)-1}S'(u)\varphi dxdt$$

Passing to the limite in (48) as $n \to +\infty$ and based on the previous endings, we deduce that (10). Therefore, u is a renormalized solution to problem (P), this completes the proof of Theorem 3.3

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Declarations

Conflicts of Interest

The author declare that have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability statement

The paper does not contain a data that can be consulted.

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