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Generalized Razzaboni surfaces with the quasi frame in Minkowski 3-space

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Abstract. This study investigates the geometric properties of generalized Razzaboni surfaces in Minkowski 3-space utilizing the quasi-frame formalism. We derive the quasi-frame equations for these surfaces and employ them to analyze their characteristics. The conditions for surface developability and minimality are established. Furthermore, we determine the criteria under which the s-curve of the surface becomes an asymptotic, geodesic, or principal curve across three distinct cases. As the quasi-frame represents a generalization of the Frenet frame in Minkowski 3-space, our findings encompass and extend previous Frenet frame-based results. Finally, we provide an example of a curve and the corresponding generalized Razzaboni surface for this curve.

1. Introduction

The field of differential geometry is concerned with the properties and behaviors of geometric figures that exist in curved spaces. A pivotal area of research in differential geometry deals with the study of constant mean curvature surfaces, an incredibly significant field that has strong links with many other branches of mathematics, physics, and engineering.

Bertrand curves are a special class of curves in differential geometry characterized by the property that they share their principal normal vectors with another curve, called their Bertrand mate. The significance of Bertrand curves becomes apparent when studying Razzaboni surfaces, which are generated through the binormal motion of these special curves. Specifically, Razzaboni surfaces arise as the locus of points obtained by moving along the binormal direction of Bertrand curves, creating surfaces with remarkable geometric properties, including constant mean curvature and zero Gaussian curvature. Razzaboni surfaces represent an important class of constant mean curvature surfaces. These surfaces are defined as surfaces with constant mean curvature and zero Gaussian curvature. The name of Razzaboni surfaces comes from Francesco Razzaboni, an Italian mathematician who described them for the first time during the latter half of the 19^{th} century. The Razzaboni surfaces are generated by taking the surface of revolution acquired via the rotation of a curve around an axis in space. This surface is then parametrized using two parameters,

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leading to a set of surfaces with different shapes and sizes. Razzaboni surfaces have been the focus of extensive studies in Euclidean space over the years.

While Razzaboni surfaces have been extensively studied in Euclidean space, their investigation in Minkowski space using the quasi-frame approach represents a relatively new area of research. Previous works by Xu et al. (2015) in [20] and Erdogdu and Ozdemir (2019) in [6] have explored Razzaboni surfaces in Minkowski 3-space using the classical Frenet frame, but the application of the quasi-frame, which provides a more general and flexible framework, has not been thoroughly investigated. This frame has been introduced and studied in Euclidean space in [1, 3]. The quasi-frame approach allows for a unified treatment of different curve types (spacelike, timelike, and null) and provides deeper insights into the geometric properties of these surfaces.

Previous studies have investigated the integrability of Bertrand curves and Razzaboni surfaces in various spaces. For instance, Schief explored the integrability in Euclidean space in 2003 [16]. In 2015, Xu et al. analyzed these curves and surfaces in Minkowski 3−space [20]. More recently, Elzawy and Mosa studied Razzaboni surfaces in Galilean space G_3 in 2018 [5], while Erdogdu and Ozdemir investigated the Razzaboni transformation of surfaces in Minkowski 3−space in 2019 [6]. In addition to these studies, research on ruled surfaces in different spaces has been introduced in the literature [7, 10, 15, 17, 18]. These previous studies have contributed to our understanding of Bertrand curves and Razzaboni surfaces in various contexts, and have motivated our investigation of these surfaces in Minkowski 3-space with respect to the quasi frame. We can find more motivations for our work from several papers (see [9, 11–13]). Our findings provide further insights into the geometry of Razzaboni surfaces and their applications in other fields of mathematics and physics.

The purpose of this investigation is to delve into the intricacies of generalized Razzaboni surfaces in Minkowski 3-space with the aid of the quasi frame. The quasi frame is an orthogonal frame that is adapted to surfaces with constant mean curvature and serves as a valuable tool in the study of such surfaces. By analyzing the properties and characteristics of generalized Razzaboni surfaces in the context of the quasi frame, we aim to illuminate and provide new insights into the geometry of these surfaces, as well as gain an understanding of their behavior and properties within the Minkowski 3-space.

Our focus in this study is to investigate the geometric properties of generalized Razzaboni surfaces in Minkowski 3-space by utilizing the quasi frame. To achieve this, we derive the equations of the quasi frame from the first principles and utilize them to gain insight into the behavior of the principal curvatures and the asymptotic directions of the surfaces. By comparing these results to those obtained in Euclidean space, we reveal new insights into the geometry of generalized Razzaboni surfaces. These results provide the groundwork for future investigations of other classes of surfaces in Minkowski 3-space and may inform new mathematical and physical applications.

In order to provide a comprehensive analysis of generalized Razzaboni surfaces in Minkowski 3-space, this study is structured as follows: In Section 2, we introduce the fundamental tools and concepts that are utilized throughout the paper. In section 3, we undertake an in-depth investigation of generalized Razzaboni surfaces and their properties across three distinct cases. More specifically, we explore the conditions under which the surface can be classified as developable or minimal. Additionally, we examine the conditions required for the $s-$curve of the surface to be classified as either asymptotic, geodesic, or principal line across the three different cases under investigation. Our approach provides valuable insights into the geometric properties of generalized Razzaboni surfaces and advances the existing knowledge in the related fields of mathematics and physics.

2. Preliminaries

Minkowski 3-space is a mathematical construct representing a three-dimensional vector space that is equipped with the Minkowski metric, which is a fundamental concept in the study of this space. The Minkowski metric defines the geometry and distance measurements in the Minkowski 3-space. Specifically, in Minkowski 3-space, denoted by E_1^3, the Minkowski metric g is defined as:

$$g(m, n) = -m_1n_1 + m_2n_2 + m_3n_3,$$

where $m = (m_1, m_2, m_3)$ and $n = (n_1, n_2, n_3)$ are two vectors in E_1^3. The Minkowski metric has a specific signature that distinguishes it from other metrics [14, 19]. It is worth noting that the Minkowski metric plays an important role in various areas of physics and mathematics, including special relativity, differential geometry, and mathematical physics. In particular, it is used as a tool to study the properties of spacetime and gravitational fields.

Within the context of Minkowski 3-space, the signature of the Minkowski metric is $(-, +, +)$, thereby indicating that the temporal component possesses a negative sign, while the spatial components exhibit positive signs [14, 19]. In this manifold, a vector m can be classified as either spacelike if $g(m, m) > 0$, timelike if $g(m, m) < 0$, or null or lightlike in the event that $g(m, m) = 0$ and $m \neq 0$. Similarly, for Minkowski 3-space, a curve is described as a function that maps a real number to a point in space, and it can be categorized as spacelike, timelike, or null contingent upon the configuration of the tangent vector at each point of the curve. Explicitly, a curve is classified as spacelike if its tangent vector is spacelike at all its points, timelike if its tangent vector is timelike at all its points, and null if its tangent vector is null at all its points [14, 19].

The Frenet frame, which is essential in the study of curves and their properties, is comprised of three mutually orthogonal unit vectors, namely the tangent vector (T), the principal normal vector (N_F), and the binormal vector (B_F). At each point along the curve, the tangent vector (T) is a unit vector that indicates the direction of the curve. Mathematically, T is obtained by taking the derivative of the curve with respect to arc length. The principal normal vector (N_F) is also a unit vector that is perpendicular to the tangent vector and points towards the center of curvature of the curve. It is derived by taking the derivative of the tangent vector with respect to arc length and normalizing it. Finally, the binormal vector (B_F) is another unit vector that is perpendicular to both the tangent vector and the principal normal vector. B_F can be obtained by taking the cross product of the tangent vector and the principal normal vector, therefore, forming a triplet of orthonormal vectors that describe the properties of curves in a precise and concise manner.

The Frenet frame satisfies the following differential equations:

$$\frac{dT}{ds} = \kappa N_F, \quad \frac{dN_F}{ds} = -\kappa T + \tau B_F, \quad \frac{dB_F}{ds} = -\tau N_F,$$

where κ is the curvature of the curve, τ is the torsion of the curve and ds is the differential of the arc length along the curve [14, 19].

The quasi frame in Euclidean 3−space is a useful tool for studying the theory of curves and surfaces. The utilization of the quasi frame finds its practical applications in the investigation of curves within Minkowski 3-space. Let $\alpha(s)$ be a regular parameterized curve in Minkowski 3-space E_1^3, where s represents the arc-length parameter. By employing a quasi frame that comprises three orthonormal vectors, which we name the unit tangent $T(s)$, the unit quasi-normal vector $N_q(s)$, and the unit quasi-binormal vector $B_q(s)$, one can effectively analyze the curve's behavior. This quasi frame, represented by $T(s), N_q(s), B_q(s)$, is established through the curve's Frenet-Serret frame and plays an indispensable role in a variety of geometrical computations:

$$T = \frac{\alpha'}{\|\alpha'\|}, \quad N_q = \frac{T \times u}{\|T \times u\|}, \quad B_q = T \times N_q. \tag{1}$$

The projection vector u is an arbitrary vector determined by either (1,0,0), (0,1,0), or (0,0,1). For our purposes, we have opted for $u = (1, 0, 0)$. In the context of the Frenet frame, which is represented by T, N_F, B_F, and given an angle $\varphi(s)$ between N_F and N_q, we may express N_q and B_q in relation to N_F and B_F, as follows [2].

$$N_q = cos\varphi N_F + sin\varphi B_F, \tag{2}$$

$$B_q = -sin\varphi N_F + cos\varphi B_F, \tag{3}$$

and we can write

$$N_F = cos\varphi N_q - sin\varphi B_q, \tag{4}$$

$$B_F = sin\varphi N_q + cos\varphi B_q. \tag{5}$$

When $\alpha(s)$ is a spacelike curve with a quasi spacelike normal vector field $N_q(s)$ and a quasi timelike binormal vector $B_q(s)$, then the quasi equations are given by

$$\begin{bmatrix} T' \\ N_q' \\ B_q' \end{bmatrix} = \begin{bmatrix} 0 & K_1 & -K_2 \\ -K_1 & 0 & K_3 \\ -K_2 & K_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_q \\ B_q \end{bmatrix},$$

where $K_1 = \kappa_1 \cosh\phi, \quad K_2 = \kappa_1 \sinh\phi, \quad K_3 = \kappa_2 + \phi'$, [2, 4, 8].

When $\alpha(s)$ is a spacelike curve with a quasi timelike normal vector field $N_q(s)$ and a quasi spacelike binormal vector $B_q(s)$, then the quasi equations are given by:

$$\begin{bmatrix} T' \\ N_q' \\ B_q' \end{bmatrix} = \begin{bmatrix} 0 & K_1 & -K_2 \\ K_1 & 0 & K_3 \\ K_2 & K_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_q \\ B_q \end{bmatrix},$$

where $K_1 = \kappa_1 \cosh\phi, \quad K_2 = \kappa_1 \sinh\phi, \quad K_3 = \kappa_2 + \phi'$.

When $\alpha(s)$ is a timelike curve with a quasi spacelike normal vector field $N_q(s)$ and a quasi spacelike binormal vector $B_q(s)$, then the quasi equations are given by

$$\begin{bmatrix} T' \\ N_q' \\ B_q' \end{bmatrix} = \begin{bmatrix} 0 & K_1 & K_2 \\ K_1 & 0 & K_3 \\ K_2 & -K_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_q \\ B_q \end{bmatrix},$$

where $K_1 = \kappa_1 \cosh\phi, K_2 = \kappa_1 \sinh\phi, \quad K_3 = \kappa_2 + \phi'$.

The Razzaboni surface is a surface that arises in Minkowski 3-space as a result of the binormal motion of Bertrand curves. Bertrand curves are a particular type of curve that share their principal normals with another curve, known as their Bertrand mate. When these Bertrand curves are embedded in a surface, the resulting surface is called a Razzaboni surface.

The characteristics of Razzaboni surfaces are analyzed based on the nature of the Bertrand geodesics in Minkowski 3-space. A Razzaboni transformation is defined for each generalized Razzaboni surface that is quasi-framed in Minkowski 3-space. This transformation maps the surface with constant curvature of Bertrand geodesics to a surface with similar Bertrand geodesics but with the opposite sign curvature. The study of Razzaboni surfaces and their associated Bertrand curves has many applications in physics and mathematics, including in the study of spacetime, relativity, and differential geometry.

The Razzaboni transformation, first introduced by Schief [16] and later generalized by Erdogdu and Ozdemir [6], establishes a correspondence between two Razzaboni surfaces δ and $\delta^* = h(\delta)$ in E_1^3, where h denotes the transformation mapping.

If the Frenet Frame of Bertrand geodesics of surface δ^* is defined as T^*, N_F^*, and B_F^*, then the u^*-parameter curves constitute unit speed spacelike Bertrand geodesics while the v^*-parameter curves form orthogonal spacelike parallels. The transformation δ is referred to as a Razzaboni transformation if the following conditions hold true:

i) $|\delta - \delta^*|$=constant,

ii) $(\delta - \delta^*) \perp B_F$,

iii) $(\delta - \delta^*) \perp B_F^*$,

iv) $< B_F, B_F^* >=$ constant ,

After satisfying certain pre-determined conditions, the transformation δ is deemed a Razzaboni transformation, with the corresponding surface being identified as the dual Razzaboni surface of δ denoted by $\delta^* = h(\delta)$. The said transformation exhibits the characteristic of preserving the Euclidean distance between corresponding points on the surfaces, which remains constant throughout. The distance function across the dual Razzaboni surface can be described as $\delta^*(u, v) = h(\delta(u, v)) = \delta(u, v) + AN_F(u, v)$, wherein h represents a function that maps the original distance value to a modified value that takes into account the effect of the associated normal vector $N_F(u, v)$.

For a parametric surface $W(s,u)$ embedded in the given space, we define three essential quadratic forms that completely characterize its geometric structure. The primary, secondary, and tertiary fundamental forms are expressed as:

$$\mathrm{I} = E\,ds^2 + 2F\,ds\,du + G\,du^2, \tag{6}$$

$$\mathrm{II} = L\,ds^2 + 2M\,ds\,du + N\,du^2, \tag{7}$$

$$\mathrm{III} = e\,ds^2 + 2f\,ds\,du + r\,du^2, \tag{8}$$

where the metric coefficients and curvature-related coefficients are defined through the Minkowski inner product $g(\cdot,\cdot)$ as:

$$E = g(W_s, W_s), \quad F = g(W_s, W_u), \quad G = g(W_u, W_u), \tag{9}$$

$$L = g(W_{ss}, \mathbf{n}), \quad M = g(W_{su}, \mathbf{n}), \quad N = g(W_{uu}, \mathbf{n}), \tag{10}$$

$$e = g(\mathbf{n}_s, \mathbf{n}_s), \quad f = g(\mathbf{n}_s, \mathbf{n}_u), \quad r = g(\mathbf{n}_u, \mathbf{n}_u). \tag{11}$$

The intrinsic and extrinsic curvature measures of the surface are determined by the Gaussian curvature K and mean curvature H, respectively:

$$K = \frac{LN - M^2}{EG - F^2}, \qquad H = \frac{EN - 2MF + GL}{2(EG - F^2)}. \tag{12}$$

For a parameterized curve $\alpha(s)$ residing on the surface $W(s,u)$, we introduce three fundamental geometric invariants that describe its behavior relative to the ambient surface geometry:

$$\kappa_g = g(\mathbf{n}(s) \times \mathbf{T}(s), \mathbf{T}'(s)), \tag{13}$$

$$\kappa_n = g(\mathbf{n}(s), \alpha''(s)), \tag{14}$$

$$\tau_g = g(\mathbf{n} \times \mathbf{n}', \mathbf{T}'(s)), \tag{15}$$

where $\mathbf{n} = \frac{W_s \times W_u}{\|W_s \times W_u\|}$ denotes the unit normal vector field along the surface, and $\mathbf{T}$ represents the unit tangent vector to the curve $\alpha(s)$.

Definition 2.1. *[8] Consider a smooth curve $\alpha(s)$ embedded within a regular surface $W(s,u)$. The curve exhibits the following geometric properties:*

(i) $\alpha(s)$ constitutes a geodesic when its geodesic curvature vanishes: $\kappa_g = 0$.
(ii) $\alpha(s)$ forms an asymptotic line when its normal curvature vanishes: $\kappa_n = 0$.
(iii) $\alpha(s)$ represents a principal curvature line when its geodesic torsion vanishes: $\tau_g = 0$.

Definition 2.2. *[8] A regular surface $W(s,u)$ admits the following geometric characterizations:*

(i) The surface is developable (or locally flat) if and only if its Gaussian curvature vanishes identically: $K \equiv 0$.
(ii) The surface is minimal if and only if its mean curvature vanishes identically: $H \equiv 0$.

3. Main results

This section of our research delves into the investigation of the generalized Razzaboni surfaces in three distinct cases, while also presenting several characterizations of such surfaces in Minkowski 3-space. Our endeavor aims to expand the scientific understanding of these surfaces by exploring their properties and examining their behavior under varying conditions. Furthermore, our study highlights significant features of these surfaces that contribute to their distinctiveness within the realm of mathematical surfaces.

3.1. The spacelike geodesic Bertrand curves of the generalized Razzaboni surface have spacelike quasi-normal.

In this subsection, we study the generalized Razzaboni surfaces in which the spacelike geodesic Bertrand curves have spacelike quasi-normal.

Theorem 3.1. *Consider a parameterized family of spacelike geodesic Bertrand curves denoted by $\Gamma = \Gamma(s, v)$, featuring a quasi-spacelike principal normal within the three-dimensional Minkowski space E_1^3. If s and v are the geodesic coordinates of the generalized Razzaboni surface associated with Γ, then the variation in the quasi-frame $\{T, N_q, B_q\}$ of the Bertrand geodesics in the s and v directions is given by:*

$$\begin{bmatrix} T \\ N_q \\ B_q \end{bmatrix}_s = \begin{bmatrix} 0 & K_1 & -K_2 \\ -K_1 & 0 & K_3 \\ -K_2 & K_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_q \\ B_q \end{bmatrix}.$$

Further, the variation of T, N_q, B_q of the v-direction is

$$\begin{bmatrix} T \\ N_q \\ B_q \end{bmatrix}_v = \begin{bmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ \beta & \gamma & 0 \end{bmatrix} \begin{bmatrix} T \\ N_q \\ B_q \end{bmatrix},$$

and

$$\begin{aligned} K_{1v} - K_2\gamma &= 2\lambda_s K_3 + \lambda K_{3s}, \\ -K_1\lambda_s + K_{3v} &= \lambda K_2 K_3 + \gamma_s, \\ -K_{2v} + K_1\gamma &= \lambda K_3^2 + \lambda_s s. \end{aligned}$$

Proof. By using the compatibility $\Gamma_{uv} = \Gamma_{vu}$, we get $\alpha N_q + \beta B_q = (-\lambda K_2)T + (\lambda K_3)N_q + \lambda_s B_q$. Therefore, we can write

$$\begin{bmatrix} T \\ N_q \\ B_q \end{bmatrix}_v = \begin{bmatrix} 0 & \lambda K_3 & \lambda_s \\ -\lambda K_3 & 0 & \gamma \\ \lambda_s & \gamma & 0 \end{bmatrix} \begin{bmatrix} T \\ N_q \\ B_q \end{bmatrix}.$$

The compatibility conditions $T_{sv} = T_{vs}, (N_q)_{sv} = (N_q)_{vs}$ and $(B_q)_{sv} = (B_q)_{vs}$ gives the following undermined system

$$\begin{aligned} K_{1v} - K_2\gamma &= 2\lambda_s K_3 + \lambda K_{3s}, \\ -K_1\lambda_s + K_{3v} &= \lambda K_2 K_3 + \gamma_s, \\ -K_{2v} + K_1\gamma &= \lambda K_3^2 + \lambda_{ss}. \end{aligned}$$

The set of equations known as the Gauss-Minardi Codazzi equations can be regarded as applicable to a surface having geodesic coordinates. By imposing the constraint $A\sqrt{K_1^2 - K_2^2} + B(K_3 - \theta') = 1$, the system becomes well-defined and ensures that the surface Γ is a generalized Razzaboni surface. □

Theorem 3.2. *The 1^{st} form of the surface $\Gamma(s, v)$ is*

$$I = ds^2 - \lambda^2\, dv^2.$$

Proof. Here, the curves' s-parameter curves are unit speed spacelike with a quasi spacelike normal vector. Since, $\Gamma_s = T$ and $\Gamma_v = \lambda B_q$. Then, the 1^{st} form of Γ is given by

$$I = ds^2 - \lambda^2 dv^2. \tag{16}$$

□

Theorem 3.3. *The 2^{nd} form of the surface $\Gamma(s, v)$ is given by*

$$II = -K_1 ds^2 - 2\lambda K_3 ds dv - \gamma\lambda dv^2.$$

Proof. The normal vector n to the surface $\Gamma(s,v)$ is given by

$$n = \frac{\Gamma_s \times \Gamma_v}{|\Gamma_s \times \Gamma_v|} = -N_q. \tag{17}$$

The differentiation of the second order is

$$\Gamma_{ss} = -K_1 N_q - K_2 B_q,$$

$$\Gamma_{sv} = \lambda K_3 N_q + \lambda_s B_q,$$

$$\Gamma_{vv} = \lambda\lambda_s T + \lambda\gamma N_q + \Lambda_v B_q.$$

Then, the 2^{nd} form of Γ is given by

$$II = -K_1 ds^2 - 2\lambda K_3 dsdv - \gamma\lambda dv^2. \tag{18}$$

□

Theorem 3.4. *The 3^{rd} form of the surface $\Gamma(s,v)$ is of the form $III = (K_1^2 - K_2^2)ds^2 + 2(\lambda K_1 K_3 - \gamma K_3)dsdv + (\lambda^2 K_3^2 - \gamma^2)dv^2$.*

Proof. By partial differentiation Equation 17 with respect to s and v, we get:

$$n_s = K_1 T - K_3 B_q, \qquad n_v = \lambda K_3 T - \gamma B_q.$$

Therefore, the 3^{rd} form is given by

$$III = (K_1^2 - K_2^2)ds^2 + 2(\lambda K_1 K_3 - \gamma K_3)dsdv + (\lambda^2 K_3^2 - \gamma^2)dv^2. \tag{19}$$

□

Theorem 3.5. *The Gaussian and the mean curvatures of the surface Γ are given, respectively, by*

$$K = \frac{\lambda K_3^2 - K_1\gamma}{\lambda}, \qquad H = \frac{1}{2}(K_1\lambda + \gamma).$$

Proof. From the Equation 16 and Equation 18, we obtain the results. □

Theorem 3.6. *The κ_g, the κ_n and τ_g which associate $s-$ curve on the generalized Razzaboni surface are given, respectively, by*

$$\kappa_g = -K_1 K_3, \quad \kappa_n = -K_1, \quad \tau_g = K_1 K_2.$$

Corollary 3.7. *The generalized Razzaboni surfaces are developable if and only if $K_1\gamma = \lambda K_3^2$.*

Corollary 3.8. *The $s-$curve of the generalized Razzaboni surfaces is an asymptotic curve if and only if $K_1 = 0$.*

Corollary 3.9. *The $s-$curve of the generalized Razzaboni surfaces is geodesic if and only if $K_1 = 0$ or $K_3 = 0$.*

Corollary 3.10. *The $s-$curve of the generalized Razzaboni surfaces is the principal line if and only if $K_1 = 0$ or $K_2 = 0$.*

Corollary 3.11. *The generalized Razzaboni surface is a minimal surface if and only if $K_1 = -\frac{\gamma}{\lambda}$.*

Corollary 3.12. *The $s-$curve of the generalized Razzaboni surfaces is a line of curvature if and only if $K_3 = 0$.*

Now, let the dual generalized Razzaboni surface of Γ.

$$\Gamma^*(u,v) = \phi(\Gamma(u,v)) = \Gamma(u,v) + AN_q(u,v). \tag{20}$$

Theorem 3.13. *The quasi frame of* Γ^* *is given*

$$T^* = \frac{1}{D}[(1 - AK_1)T + AK_3B_q],$$

$$N_q^* = N_q,$$

$$B_q^* = \frac{1}{D}[-AK_3T - (1 - AK_1)B_q],$$

where $D = \sqrt{|(1 - AK_1)^2 - A^2K_3^2|}$

Proof. By differentiation Equation 20 with respect to s, we get

$$\Gamma_s^* = (1 - AK_1)T + AK_3B_q,$$

then the norm of Γ_s^* is denoted by

$$||\Gamma_s^*|| = \sqrt{|(1 - AK_1)^2 - A^2K_3^2|} = D.$$

Therefore,

$$T^* = \frac{\Gamma_s^*}{||\Gamma_s^*||} = \frac{1}{D}[(1 - AK_1)T + AK_3B_q],$$

$$N_q^* = N_q,$$

$$B_q^* = -T^* \times N_q^* = \frac{1}{D}[-AK_3T - (1 - AK_1)B_q].$$

□

Theorem 3.14. *The quasi-curvatures of the dual generalized Razzaboni surface are given by*

$$K_1^* = \frac{1}{D^3}[(1 - AK_1)(K_1(1 - AK_1) + AK_3^2) + AK_3(AK_1K_3 + K_3(1 - AK_1))],$$

$$K_2^* = \frac{1}{D^3}[(1 - AK_1)(AK_1K_3 + K_3(1 - AK_1)) + AK_3(K_1(1 - AK_1) + AK_3^2)],$$

$$K_3^* = \frac{1}{D}[K_3 + \frac{A}{D^2}(K_1(1 - AK_1) + AK_3^2)].$$

Proof. By differentiation T^* with respect to s, we get

$$T_s^* = \frac{1}{D^3}[(K_1(1 - AK_1) + AK_3^2)T + (AK_1K_3 + K_3(1 - AK_1))B_q].$$

Then, we obtain the results. □

3.2. The spacelike geodesic Bertrand curves of the generalized Razzaboni surface have timelike quasi-normal.

In this subsection, we study the generalized Razzaboni surfaces in which the spacelike geodesic Bertrand curves have timelike quasi-normal.

Theorem 3.15. *Consider a parameterized family of spacelike geodesic Bertrand curves denoted by $\Gamma = \Gamma(s, v)$, featuring a quasi-timelike principal normal within the three-dimensional Minkowski space E_1^3. If s and v are the geodesic coordinates of the generalized Razzaboni surface associated with Γ, then the variation in the quasi-frame $\{T, N_q, B_q\}$ of the Bertrand geodesics in the s and v directions is given by:*

$$\begin{bmatrix} T \\ N_q \\ B_q \end{bmatrix}_s = \begin{bmatrix} 0 & K_1 & -K_2 \\ K_1 & 0 & K_3 \\ K_2 & K_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_q \\ B_q \end{bmatrix}.$$

Further, the variation of T, N_q, B_q of the v-direction is

$$\begin{bmatrix} T \\ N_q \\ B_q \end{bmatrix}_v = \begin{bmatrix} 0 & \alpha & \beta \\ \alpha & 0 & \gamma \\ \beta & \gamma & 0 \end{bmatrix} \begin{bmatrix} T \\ N_q \\ B_q \end{bmatrix},$$

and

$$K_{1v} - K_2\gamma = 2\lambda_s K_3 + \lambda K_{3s},$$
$$-K_1\lambda_s + K_{3v} = \lambda K_2 K_3 + \gamma_s,$$
$$-K_{2v} + K_1\gamma = \lambda K_3^2 + \lambda_s s.$$

Proof. By using the compatibility $\Gamma_{uv} = \Gamma_{vu}$, we get $\alpha N_q + \beta B_q = (-\lambda K_2)T + (\lambda K_3)N_q + \lambda_s B_q$. Therefore, we can write

$$\begin{bmatrix} T \\ N_q \\ B_q \end{bmatrix}_v = \begin{bmatrix} 0 & \lambda K_3 & \lambda_s \\ \lambda K_3 & 0 & \gamma \\ \lambda_s & \gamma & 0 \end{bmatrix} \begin{bmatrix} T \\ N_q \\ B_q \end{bmatrix}.$$

The compatibility conditions $T_{sv} = T_{vs}, (N_q)_{sv} = (N_q)_{vs}$ and $(B_q)_{sv} = (B_q)_{vs}$ gives the following undermined system

$$K_{1v} - K_2\gamma = 2\lambda_s K_3 + \lambda K_{3s},$$
$$-K_1\lambda_s + K_{3v} = \lambda K_2 K_3 + \gamma_s,$$
$$-K_{2v} + K_1\gamma = \lambda K_3^2 + \lambda_{ss}.$$

The set of equations known as the Gauss-Minardi Codazzi equations can be regarded as applicable to a surface having geodesic coordinates. By imposing the constraint $A\sqrt{K_1^2 - K_2^2} + B(K_3 - \theta') = 1$, the system becomes well-defined and ensures that the surface Γ is a generalized Razzaboni surface. □

Theorem 3.16. *The 1^{st} form of the surface $\Gamma(s, v)$ is*

$$I = ds^2 + \lambda^2\, dv^2.$$

Proof. Here, the curves' s-parameter curves are unit speed spacelike with a quasi timelike normal vector. Since, $\Gamma_s = T$ and $\Gamma_v = \lambda B_q$. Then, the 1^{st} form of Γ is given by

$$I = ds^2 + \lambda^2 dv^2. \tag{21}$$

□

Theorem 3.17. *The 2^{nd} form of the surface $\Gamma(s, v)$ is given by*

$$II = K_1 ds^2 + 2\lambda K_3 ds dv + \gamma\lambda dv^2.$$

Proof. The normal vector n to the surface $\Gamma(s, v)$ is given by

$$n = \frac{\Gamma_s \times \Gamma_v}{|\Gamma_s \times \Gamma_v|} = N_q. \tag{22}$$

The differentiation of the second order is

$$\Gamma_{ss} = K_1 N_q + K_2 B_q,$$
$$\Gamma_{sv} = \lambda K_3 N_q + \lambda_s B_q,$$
$$\Gamma_{vv} = \lambda \lambda_s T + \lambda \gamma N_q + \Lambda_v B_q.$$

Then, the 2^{nd} form of Γ is given by

$$II = K_1 ds^2 + 2\lambda K_3 ds dv + \gamma \lambda dv^2. \tag{23}$$

□

Theorem 3.18. *The 3^{rd} form of the surface $\Gamma(s, v)$ is of the form $III = (K_1^2 - K_2^2)ds^2 + 2(\lambda K_1 K_3 - \gamma K_3)dsdv + (\lambda^2 K_3^2 - \gamma^2)dv^2$.*

Proof. By partial differentiation Equation 22 with respect to s and v, we get:

$$n_s = -K_1 T - K_3 B_q, \qquad n_v = -\lambda K_3 T - \gamma B_q.$$

Therefore, the 3^{rd} form is given by

$$III = (K_1^2 - K_2^2)ds^2 + 2(\lambda K_1 K_3 - \gamma K_3)dsdv + (\lambda^2 K_3^2 - \gamma^2)dv^2. \tag{24}$$

□

Theorem 3.19. *The Gaussian and the mean curvatures of the surface Γ are given, respectively, by*

$$K = \frac{\lambda K_3^2 - K_1 \gamma}{\lambda}, \qquad H = \frac{1}{2}(K_1 \lambda + \gamma).$$

Proof. From the Equation 21 and Equation 23, we obtain the results. □

Theorem 3.20. *The κ_g, the κ_n and τ_g which associate $s-$ curve on the generalized Razzaboni surface are given, respectively, by*

$$\kappa_g = K_1 K_3, \quad \kappa_n = K_1, \quad \tau_g = K_1 K_2.$$

Corollary 3.21. *The generalized Razzaboni surfaces are developable if and only if $K_1 \gamma = \lambda K_3^2$.*

Corollary 3.22. *The $s-$curve of the generalized Razzaboni surfaces is an asymptotic curve if and only if $K_1 = 0$.*

Corollary 3.23. *The $s-$curve of the generalized Razzaboni surfaces is geodesic if and only if $K_1 = 0$ or $K_3 = 0$.*

Corollary 3.24. *The $s-$curve of the generalized Razzaboni surfaces is the principal line if and only if $K_1 = 0$ or $K_2 = 0$.*

Corollary 3.25. *The generalized Razzaboni surface is a minimal surface if and only if $K_1 = -\frac{\gamma}{\lambda}$.*

Corollary 3.26. *The $s-$curve of the generalized Razzaboni surfaces is a line of curvature if and only if $K_3 = 0$.*

Now, let the dual generalized Razzaboni surface of Γ.

$$\Gamma^*(u,v) = \phi(\Gamma(u,v)) = \Gamma(u,v) + AN_q(u,v). \tag{25}$$

Theorem 3.27. *The quasi frame of* Γ^* *is given*

$$T^* = \frac{1}{D}[(1-AK_1)T + AK_3B_q],$$

$$N_q^* = N_q,$$

$$B_q^* = \frac{1}{D}[-AK_3T - (1-AK_1)B_q],$$

where $D = \sqrt{|(1-AK_1)^2 + A^2K_3^2|}$

Proof. By differentiation Equation 25 with respect to s, we get

$$\Gamma_s^* = (1-AK_1)T + AK_3B_q,$$

then the norm of Γ_s^* is denoted by

$$||\Gamma_s^*|| = \sqrt{|(1-AK_1)^2 + A^2K_3^2|} = D.$$

Therefore,

$$T^* = \frac{\Gamma_s^*}{||\Gamma_s^*||} = \frac{1}{D}[(1-AK_1)T + AK_3B_q],$$

$$N_q^* = N_q,$$

$$B_q^* = -T^* \times N_q^* = \frac{1}{D}[-AK_3T - (1-AK_1)B_q].$$

□

Theorem 3.28. *The quasi-curvatures of the dual generalized Razzaboni surface are given by*

$$K_1^* = \frac{1}{D^3}[(1-AK_1)(K_1(1-AK_1) - AK_3^2) + AK_3(AK_1K_3 + K_3(1-AK_1))],$$

$$K_2^* = \frac{1}{D^3}[(1-AK_1)(AK_1K_3 + K_3(1-AK_1)) + AK_3(K_1(1-AK_1) - AK_3^2)],$$

$$K_3^* = \frac{1}{D}[K_3 + \frac{A}{D^2}(K_1(1-AK_1) - AK_3^2)].$$

Proof. By differentiation T^* with respect to s, we get

$$T_s^* = \frac{1}{D^3}[(K_1(1-AK_1) - AK_3^2)T + (AK_1K_3 + K_3(1-AK_1))B_q].$$

Then, we obtain the results. □

3.3. The timelike geodesic Bertrand curves of the generalized Razzaboni surface have spacelike quasi-normal.

In this subsection, we study the generalized Razzaboni surfaces in which the timelike geodesic Bertrand curves have spacelike quasi-normal.

Theorem 3.29. *Consider a parameterized family of timelike geodesic Bertrand curves denoted by $\Gamma = \Gamma(s, v)$, featuring a quasi-spacelike principal normal within the three-dimensional Minkowski space E_1^3. If s and v are the geodesic coordinates of the generalized Razzaboni surface associated with Γ, then the variation in the quasi-frame $\{T, N_q, B_q\}$ of the Bertrand geodesics in the s and v directions is given by:*

$$\begin{bmatrix} T \\ N_q \\ B_q \end{bmatrix}_s = \begin{bmatrix} 0 & K_1 & K_2 \\ K_1 & 0 & K_3 \\ K_2 & -K_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_q \\ B_q \end{bmatrix}.$$

Further, the variation of T, N_q, B_q of the v-direction is

$$\begin{bmatrix} T \\ N_q \\ B_q \end{bmatrix}_v = \begin{bmatrix} 0 & \alpha & \beta \\ \alpha & 0 & \gamma \\ \beta & -\gamma & 0 \end{bmatrix} \begin{bmatrix} T \\ N_q \\ B_q \end{bmatrix},$$

and

$$K_{1v} - K_2\gamma = 2\lambda_s K_3 + \lambda K_{3s},$$
$$-K_1\lambda_s + K_{3v} = \lambda K_2 K_3 + \gamma_s,$$
$$-K_{2v} + K_1\gamma = \lambda K_3^2 + \lambda_s s.$$

Proof. By using the compatibility $\Gamma_{uv} = \Gamma_{vu}$, we get $\alpha N_q + \beta B_q = (-\lambda K_2)T + (\lambda K_3)N_q + \lambda_s B_q$. Therefore, we can write

$$\begin{bmatrix} T \\ N_q \\ B_q \end{bmatrix}_v = \begin{bmatrix} 0 & \lambda K_3 & \lambda_s \\ \lambda K_3 & 0 & \gamma \\ \lambda_s & -\gamma & 0 \end{bmatrix} \begin{bmatrix} T \\ N_q \\ B_q \end{bmatrix}.$$

The compatibility conditions $T_{sv} = T_{vs}, (N_q)_{sv} = (N_q)_{vs}$ and $(B_q)_{sv} = (B_q)_{vs}$ gives the following undermined system

$$K_{1v} - K_2\gamma = 2\lambda_s K_3 + \lambda K_{3s},$$
$$-K_1\lambda_s + K_{3v} = \lambda K_2 K_3 + \gamma_s,$$
$$-K_{2v} + K_1\gamma = \lambda K_3^2 + \lambda_{ss}.$$

The set of equations known as the Gauss-Minardi Codazzi equations can be regarded as applicable to a surface having geodesic coordinates. By imposing the constraint $A\sqrt{K_1^2 - K_2^2} + B(K_3 - \theta') = 1$, the system becomes well-defined and ensures that the surface Γ is a generalized Razzaboni surface. □

Theorem 3.30. *The 1^{st} form of the surface $\Gamma(s, v)$ is*

$$I = -ds^2 + \lambda^2\, dv^2.$$

Proof. Here, the curves' s-parameter curves are unit speed timelike with a quasi spacelike normal vector. Since, $\Gamma_s = T$ and $\Gamma_v = \lambda B_q$. Then, the 1^{st} form of Γ is given by

$$I = -ds^2 + \lambda^2 dv^2. \tag{26}$$

□

Theorem 3.31. *The 2^{nd} form of the surface $\Gamma(s, v)$ is given by*

$$II = -K_1 ds^2 - 2\lambda K_3 dsdv - \gamma\lambda dv^2.$$

Proof. The normal vector n to the surface $\Gamma(s, v)$ is given by

$$n = \frac{\Gamma_s \times \Gamma_v}{|\Gamma_s \times \Gamma_v|} = N_q. \tag{27}$$

The differentiation of the second order is

$$\Gamma_{ss} = K_1 N_q + K_2 B_q,$$

$$\Gamma_{sv} = \lambda K_3 N_q + \lambda_s B_q,$$

$$\Gamma_{vv} = \lambda \lambda_s T + \lambda \gamma N_q + \Lambda_v B_q.$$

Then, the 2^{nd} form of Γ is given by

$$II = -K_1 ds^2 - 2\lambda K_3 ds dv - \gamma \lambda dv^2. \tag{28}$$

□

Theorem 3.32. *The 3^{rd} form of the surface $\Gamma(s, v)$ is of the form $III = (K_1^2 - K_2^2)ds^2 + 2(\lambda K_1 K_3 - \gamma K_3)dsdv + (\lambda^2 K_3^2 - \gamma^2)dv^2$.*

Proof. By partial differentiation Equation 27 with respect to s and v, we get:

$$n_s = -K_1 T - K_3 B_q, \qquad n_v = -\lambda K_3 T - \gamma B_q.$$

Therefore, the 3^{rd} form is given by

$$III = (K_1^2 - K_2^2)ds^2 + 2(\lambda K_1 K_3 - \gamma K_3)dsdv + (\lambda^2 K_3^2 - \gamma^2)dv^2. \tag{29}$$

□

Theorem 3.33. *The Gaussian and the mean curvatures of the surface Γ are given, respectively, by*

$$K = \frac{\lambda K_3^2 - K_1 \gamma}{\lambda}, \qquad H = \frac{1}{2}(K_1 \lambda + \gamma).$$

Proof. From the Equation 26 and Equation 28, we obtain the results. □

Theorem 3.34. *The κ_g, the κ_n and τ_g which associate $s-$ curve on the generalized Razzaboni surface are given, respectively, by*

$$\kappa_g = -K_1 K_3, \quad \kappa_n = -K_1, \quad \tau_g = K_1 K_2.$$

Corollary 3.35. *The generalized Razzaboni surfaces are developable if and only if $K_1 \gamma = \lambda K_3^2$.*

Corollary 3.36. *The $s-$curve of the generalized Razzaboni surfaces is an asymptotic curve if and only if $K_1 = 0$.*

Corollary 3.37. *The $s-$curve of the generalized Razzaboni surfaces is geodesic if and only if $K_1 = 0$ or $K_3 = 0$.*

Corollary 3.38. *The $s-$curve of the generalized Razzaboni surfaces is the principal line if and only if $K_1 = 0$ or $K_2 = 0$.*

Corollary 3.39. *The generalized Razzaboni surface is a minimal surface if and only if $K_1 = -\frac{\gamma}{\lambda}$.*

Corollary 3.40. *The $s-$curve of the generalized Razzaboni surfaces is a line of curvature if and only if $K_3 = 0$.*

Now, let the dual generalized Razzaboni surface of Γ.

$$\Gamma^*(u,v) = \phi(\Gamma(u,v)) = \Gamma(u,v) + AN_q(u,v). \tag{30}$$

Theorem 3.41. *The quasi frame of* Γ^* *is given*

$$T^* = \frac{1}{D}[(1 - AK_1)T + AK_3B_q],$$

$$N_q^* = N_q,$$

$$B_q^* = \frac{1}{D}[-AK_3T - (1 - AK_1)B_q],$$

where $D = \sqrt{|(1 - AK_1)^2 + A^2K_3^2|}$

Proof. By differentiation Equation 30 with respect to s, we get

$$\Gamma_s^* = (1 - AK_1)T + AK_3B_q,$$

then the norm of Γ_s^* is denoted by

$$||\Gamma_s^*|| = \sqrt{|(1 - AK_1)^2 + A^2K_3^2|} = D.$$

Therefore,

$$T^* = \frac{\Gamma_s^*}{||\Gamma_s^*||} = \frac{1}{D}[(1 - AK_1)T + AK_3B_q],$$

$$N_q^* = N_q,$$

$$B_q^* = -T^* \times N_q^* = \frac{1}{D}[-AK_3T - (1 - AK_1)B_q].$$

□

Theorem 3.42. *The quasi-curvatures of the dual generalized Razzaboni surface are given by*

$$K_1^* = \frac{1}{D^3}[(1 - AK_1)(K_1(1 - AK_1) - AK_3^2) + AK_3(AK_1K_3 + K_3(1 - AK_1))],$$

$$K_2^* = \frac{1}{D^3}[(1 - AK_1)(AK_1K_3 + K_3(1 - AK_1)) + AK_3(K_1(1 - AK_1) - AK_3^2)],$$

$$K_3^* = \frac{1}{D}[K_3 + \frac{A}{D^2}(K_1(1 - AK_1) - AK_3^2)].$$

Proof. By differentiation T^* with respect to s, we get

$$T_s^* = \frac{1}{D^3}[(K_1(1 - AK_1) - AK_3^2)T + (AK_1K_3 + K_3(1 - AK_1))B_q].$$

Then, we obtain the results. □

4. Example

Consider the curve $\alpha(s)$ in Minkowski 3-space E_1^3 given by

$$\alpha(s) = \left(\frac{1}{2}\cosh(2s), \frac{1}{2}\sinh(2s), s\right).$$

This curve is a spacelike curve with a quasi-spacelike normal vector field. The quasi-frame for this curve is given by:

$$T(s) = (-\sinh(2s), \cosh(2s), 0)\,,$$
$$N_q(s) = (0, 0, -1)\,,$$
$$B_q(s) = (-\cosh(2s), \sinh(2s), 0)\,.$$

The quasi-curvatures are:

$$K_1 = 2, \quad K_2 = 0, \quad K_3 = 0.$$

Now, consider the generalized Razzaboni surface generated by this curve:

$$\Gamma(s, v) = \alpha(s) + vB_q(s) = \left(\frac{1}{2}\cosh(2s) - v\cosh(2s), \frac{1}{2}\sinh(2s) + v\sinh(2s), s\right).$$

The first fundamental form coefficients are:

$$E = 1, \quad F = 0, \quad G = -1.$$

The second fundamental form coefficients are:

$$L = -2, \quad M = 0, \quad N = 0.$$

The Gaussian and mean curvatures are:

$$K = 0, \quad H = -1.$$

This surface is developable but not minimal. The s-curve is both geodesic and principal line, but not asymptotic.

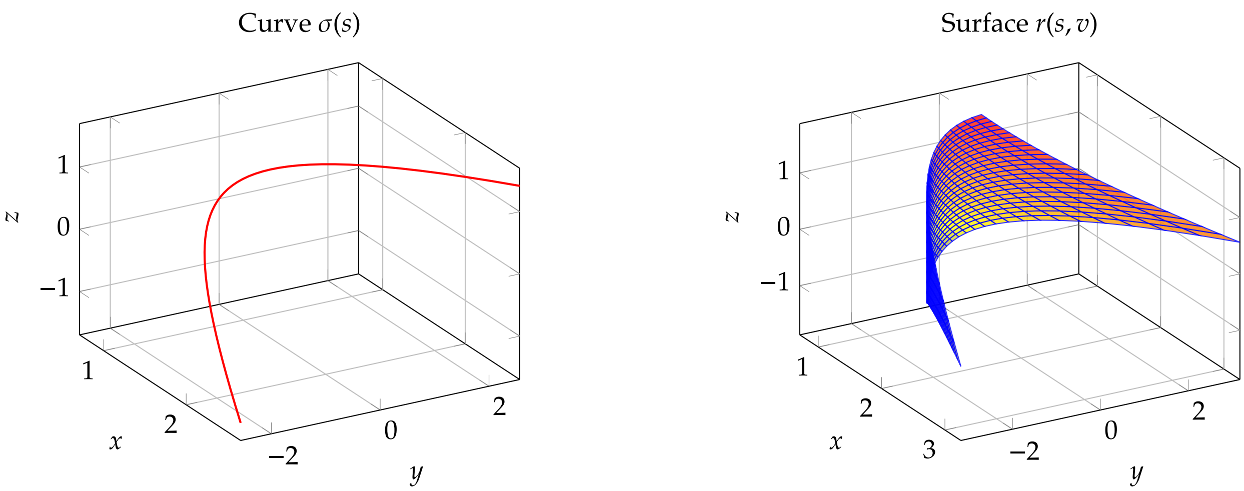

Figure 1: Curve $\sigma(s)$ and Generalized Razzaboni Surface $r(s, v)$

5. Conclusion

In this paper, we have investigated the geometric properties of generalized Razzaboni surfaces in Minkowski 3-space using the quasi-frame formalism. Our main contributions can be summarized as follows:

1. We derived the quasi-frame equations for generalized Razzaboni surfaces and established the compatibility conditions through the Gauss-Minardi-Codazzi equations.

2. We computed the fundamental forms (first, second, and third) for these surfaces across three different cases, depending on the nature of the geodesic Bertrand curves and their quasi-normal vectors.

3. We obtained explicit formulas for the Gaussian curvature, mean curvature, and the geometric invariants (κ_g, κ_n, τ_g) associated with the s-curves on these surfaces.

4. We established necessary and sufficient conditions for these surfaces to be developable or minimal, and for the s-curves to be asymptotic, geodesic, or principal lines.

5. We constructed the dual generalized Razzaboni surfaces and computed their quasi-frames and quasi-curvatures.

6. We provided a concrete example illustrating our theoretical results.

Our results generalize previous work on Razzaboni surfaces in Minkowski space and provide a unified framework for studying these surfaces using the quasi-frame approach. The quasi-frame formalism offers advantages over the classical Frenet frame as it provides a more flexible and comprehensive treatment of curves and surfaces in Minkowski space.

Future research directions include studying Razzaboni surfaces in higher-dimensional Minkowski spaces, investigating their applications in physics (particularly in relativity theory), and exploring their connections with other special surfaces in differential geometry.

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