Filomat 39:35 (2025), 12537–12554
https://doi.org/10.2298/FIL2535537P

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Point-wise Semi-Slant Riemannian maps from Riemannian manifolds into Kaehler Manifolds

Rajendra Prasad[a], **Saroj Kumar**[a,*]

[a]*Department of mathematics and Astronomy, University of Lucknow, Lucknow, India*

Abstract. In this paper, we introduce $\mathcal{PSSRM}$ from Riemannian manifold to Kaehler manifolds. We provide examples and characterizations of these maps, along with an investigation into their harmonicity. Additionally, we derive a Chen-Ricci inequality for $\mathcal{PSSRM}$, and explore curvature relations in complex space forms, particularly involving the Casorati curvatures for $\mathcal{PSSRM}$.

1. Introduction

We will use some abbreviations in this article as follows:

$\mathcal{RM}$: Riemannian Map
$\mathcal{PRM}$: Pointwise Riemannian Map:
$\mathcal{PSSRM}$: Pointwise Semi-Slant Riemannian Map:

In mathematical physics, complex techniques have proven to be highly effective tools for understanding spacetime geometry. complex manifolds have two interesting classes of Kaehler manifolds. One is Calabi-Yau manifolds, which have their applications in superstring theory [8] and the other one is Teichmuller spaces applicable to relativity [44]. The theory of submanifolds of complex manifolds plays a central role in submanifold geometry. Submanifolds are classified based on how their tangent spaces behaves in relation to the complex structure of the manifold. Some important families of complex submanifolds: holomorphic (invariant) submanifolds, totally real submanifolds, semi-invariant submanifolds, CR-submanifolds, slant submanifolds. Semi-slant submanifolds were introduced as a generalization of slant and CR-submanifolds [27]. In [9], Casorati introduced Casorati curvature which is a very natural concept of regular surfaces in the three-dimensional Euclidean space. For the general framework of manifolds, the Casorati curvature in the context of submanifold withen the Riemannian manifold is defined as the normalized square of the length of the second fundamental form, and it is well known that is an extrinsic invariant. Such a curvature is applied to the visual perception of shapes and appearances [18]. Recently, many geometers published some optimal inequalities involving Casorati curvatures in different spaces: [5, 22, 43, 45, 47]. On the other

2020 *Mathematics Subject Classification*. Primary 53C15.
Keywords. Riemannain map, pointwise semi-slant Riemannian map, pointwise semi-slant submanifold, kaehler manifold.
Received: 25 September 2024; Revised: 23 December 2024; Accepted: 21 February 2025
Communicated by Ljubica Velimirović
* Corresponding author: Saroj Kumar
Email addresses: rp.lucknow@rediffmail.com (Rajendra Prasad), sk67bhu@gmail.com (Saroj Kumar)
ORCID iDs: https://orcid.org/0000-0002-7502-0239 (Rajendra Prasad), https://orcid.org/0009-0008-0790-3049 (Saroj Kumar)

hand, in [12], F. Etayo defined pointwise slant submanifolds (as a generalization of holomorphic (invariant), totally real and slant submanifolds,) under the name of quasi-slant submanifolds and in [11], B. Y. Chen and A. Gray studied this kind of manifolds and obtained simple characterizations. See also: [20, 29]. In this case, the angle between tangent space and non-zero vector field under the almost complex structure depends on the point. The angle was called the slant function. As a generalization of pointwise slant submanifolds, B. Sahin defined the notion of pointwise semi-slant submanifolds in [39]. Riemannian submersions were independently introduced by B. O. Neill in 1960s [25] and A. Gray [16] (see also: [38]). In [37], B. Sahin introduced slant submersions from almost Hermitian submersions onto Riemannian manifolds as a natural generalization of some important families of Riemannian submersions: invariant, anti-invariant and semi-invariant submersions. As a generalization of these submersions, semi-slant submersions have been introduced by K. S. Park and R. Prasad in [30], who explored their properties, including the integrability of distributions, foliation geometry, harmonic conditions, and totally geodesic maps. For further developments on semi-slant submersions in different spaces, see [30, 41]. As a natural generalization of slant submersions, J. W. Lee and B. Sahin defined the notion of pointwise slant submersions. C. Sayar et al.defined the notion of pointwise semi-slant submersions as a generalization of pointwise slant submersions and obtained many new results for the submersions in [41, 42]. Riemannian maps, which generalize both Riemannian submersions and isometric immersions, were introduced by A. Fischer [14]. Given two Riemannian manifolds (N_1, g_1) and (N_2, g_2) and π is smooth map between them Then the tangent bundle of N_1 has the following decompostion

$$TN_1 = ker\pi_* \oplus (ker\pi_*)^\perp$$

where $ker\pi_*$ denotes the kernel space of π_* and $(ker\pi_*)^\perp$ is the orthogonal complementary space to $ker\pi_*$. In a similar way, the tangent bundle of N_2 has the following decompostion

$$TN_2 = range\pi_* \oplus (range\pi_*)^\perp$$

where $range\pi_*$ denotes range of π_* and $(range\pi_*)^\perp$ is the orthogonal complementary space to $range\pi_*$. Now, if the horizontal restriction

$$\pi^h_{*,p} : (ker\pi_*)^\perp \to (range\pi_{*,p})$$

is a linear isometry between the inner product spaces $(ker\pi^\perp_{*,p}, g_1(p))$ and $(range\pi_{*,p}, g_2(q))$, $\pi(p) = q$ then a smooth map

$$\pi : (N_1, g_1) \to (N_2, g_2)$$

is called $\mathcal{RM}$ at $p \in N_1$. One can see that Riemannian submersions and isometric immersions are the particular Riemannian maps with
$(range\pi_*)^\perp = \{0\}$ and $ker\pi_* = \{0\}$, respectively. Taking into account of by Fischer's article, B. Sahin introduced new kinds of $\mathcal{RM}$: holomorphic Riemannian maps and anti invariant Riemannian maps [40], semi-invariant Riemannian maps [38] and slant Riemannian maps [36]. These concepts have opened new avenues in Riemannian map theory. In [31], K. S. Park and B. Sahin introduced the concept of semi-slant Riemannian maps as a natural extension of semi-slant submanifolds and semi-slant submersions. Since then, many geometers have studied Riemannian maps in various spaces: [1, 3, 4, 15, 18, 19, 21, 28, 32, 33, 35, 42]. In the Figure 1, one can see the progress of the theory of RM. In 2022, the authors of [18] introduced the notion of pointwise slant Riemannian maps, as a natural generalization many notions: holomorphic (invariant) submanifold, holomorphic submersions, anti-invariant Riemannian submersions, anti invariant submanifolds, slant submanifolds, slant Riemannian submersions etc. The purpose of the present paper is to introduce and study a new class of $\mathcal{RM}$ which are called $\mathcal{PSSRM}$ as a generalization of many concepts mentioned in the figure 2. The structure of the paper is as follows: Section 2 outlines essential preliminary concepts needed for the subsequent sections. In Section 3, the concept of pointwise semi-slant Riemannian maps $\mathcal{PSSRM}$ from Riemannian manifolds to almost Hermitian manifolds is introduced, and key properties of these maps are explored. This section also provides examples of this new category of Riemannian maps and examines the geometry of the foliations related to the distributions involved. Sections 4 and 5 focus on presenting the Chen-Ricci and Casorati inequalities relevant to pointwise semi-slant Riemannian maps.

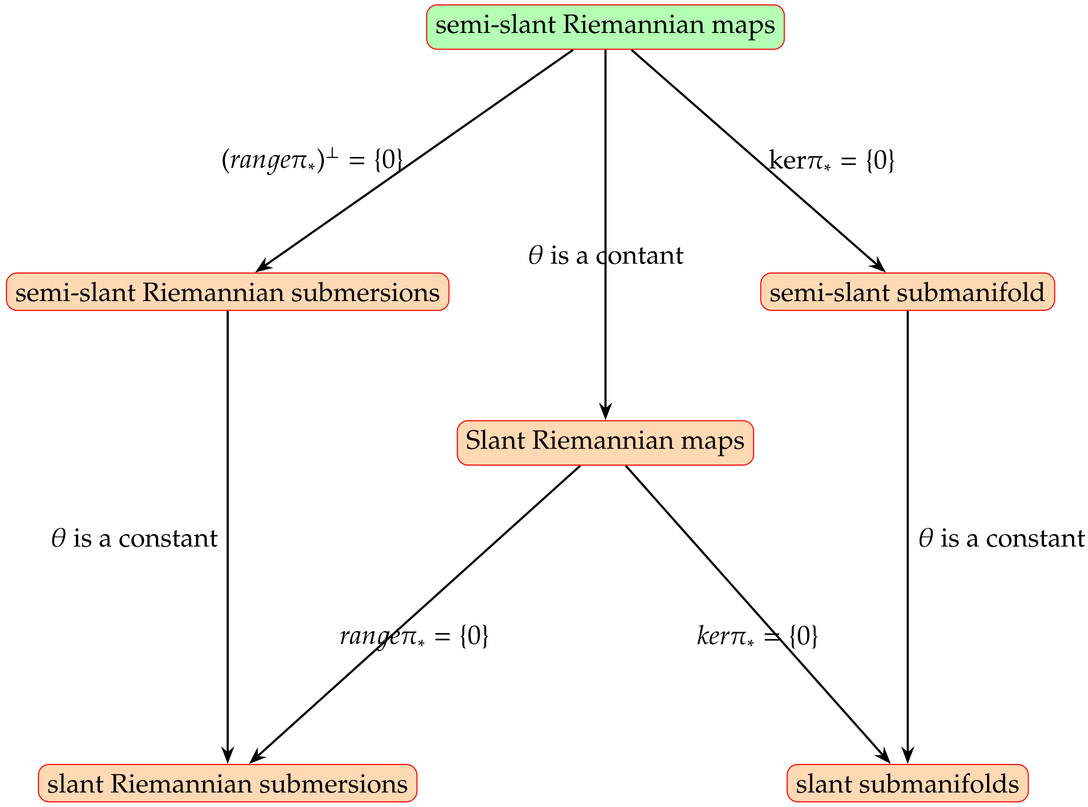

Figure 1. The class of $\mathcal{RM}$

We also give a method to obtain examples for these maps.

2. Preliminaries

In this section we provide an overview of fundamental concepts and results related to geometric structures for Riemannian maps. Let(N_1, g_1) represent a Riemannian manifold and (N_2, g_2, J) denote an almost Hermitian manifold, meaning N_2 supports a tensor field J of type $(1,1)$ on N_2 such that

$$J^2 = -I, \quad g_2(JX, JY) = g_2(X, Y), \quad X, Y \in \Gamma(TN_2). \tag{1}$$

An almost Hermitian manifold N_2 is called Kähler manifold [46] if

$$(\nabla_X J)Y = 0, \quad X, Y \in \Gamma(TN_2) \tag{2}$$

where ∇ denotes the Riemannian connection of the metric g_2 on N_2. Let (N_1, g_1) and (N_2, g_2) be Riemannian manifolds and π is a differentiable map between them. Then the differential π_* of π can be viewed a section of the bundle $Hom(TN_1, \pi^{-1}TN_2) \to N_1$ is the pullback bundle which has fibres $(\pi^{-1}TN_2)_p = T_{\pi(p)}N_2$, $p \in N_2$. $Hom(TN_1, \pi^{-1}TN_2)$ has a connection ∇ induced from the Levi-civita connection ∇^{N_1} and the pullback connection. The second fundamental form of π is given by [6, 26]

$$(\nabla \pi_*)(X_1, X_2) = \nabla^{\pi}_{X_1} \pi_* X_2 - \pi_*(\nabla_{X_1} X_2) \tag{3}$$

for $X_1, X_2 \in \Gamma(TN_1)$, where ∇^{π} is the pullback connection. the second fundamental form is symmetric if ∇ and ∇^{π} is torsion free.

The map π is hormonic if we get the tension field $\tau(\pi) = trace(\nabla\pi_*) = 0$ and we call the map is totally geodesic if $(\nabla\pi_*)(X_1, X_2) = 0$.
meanwhile, it is shown in [40] that $\nabla\pi_*(X_1, X_2)$ has no components in $\Gamma(range\pi_*)$, provided that $X_1, X_2 \in \Gamma(ker\pi_*)^\perp$. That is,

$$(\nabla\pi_*)(X_1, X_2) \in \Gamma(range\pi_*)^\perp, \forall X_1, X_2 \in \Gamma(ker\pi_*)^\perp \tag{4}$$

here $\Gamma(range\pi_*)^\perp$ is the subbundle of $\pi^{-1}(TN_2)$ with fiber $\Gamma(\pi_*(T_pN_1)^\perp)$, $p \in N_1$.
Now we define $\mathcal{S}_V$ as

$$\overset{2}{\nabla}{}^{\pi}{}_X V = -\mathcal{S}_V \pi_* X + \nabla_X^{\pi\perp} V \tag{5}$$

where $\overset{2}{\nabla}$ denotes both levi-civita and its pullback connection of (N_2, g_2) along π. Where $\mathcal{S}_V\pi_*X$ is tangential component of $\overset{2}{\nabla}_{\pi_*X}V$ and $\nabla_X^{\pi\perp}V$ is an orthogonal projection of $\overset{2}{\nabla}_{\pi_*X}V$ on $(\pi_*\,((T_pN_1))^\perp$- such that $\nabla^{\pi\perp}g_2 = 0$. $\mathcal{S}_V\pi_*X$ is bilinear in V and π_*X and $\mathcal{S}_V\pi_*X$ at p depends only on V_p and $\pi_{*p}X_p$. Hence, for $X_1, X_2 \in \Gamma(ker\pi_*)^\perp$ and $V \in \Gamma(range\pi_*)^\perp$, we get

$$g_2(\mathcal{S}_V\pi_*X_1, \pi_*X_2) = g_2(V, (\nabla\pi_*)(X_1, X_2)) \tag{6}$$

since $(\nabla\pi_*)$ is symmetric, Consequently $\mathcal{S}_V$ is a symmetric linear transformation of $range\pi_*$.

3. $\mathcal{PSSRM}$ to Kaehler Manifolds

Let $\pi : (N_1, g_1) \to (N_2, g_2, J)$ represent a Riemannian map. Where (N_1, g_1) is a Riemannian manifold and (N_2, g_2, J) is an almost Hermitian manifold. If, for every point $q \in N_2$, the angle $\theta(X)$ between $J\pi_*(X)$ and the subspace $range\pi_*$ called Wirtinger angle does not depend on the particular choice of the $0 \neq \pi_*(X)$ within $range\pi_*$, then π is referred to as a pointwise slant Riemannian map $\mathcal{PSSRM}$. The angle θ is defined as a function on N_2 and is called the slant function of the $\mathcal{PSRM}$

Definition 3.1. Let (N_1, g_1) is a Riemannian manifold and (N_2, g_2, J) be an almost Hermitian manifold. Then we say that a $\mathcal{RM}$ $\pi : (N_1, g_1) \to (N_2, g_2, J)$ is a $\mathcal{PSSRM}$ if $\exists$ a pair of orthogonal distributions $\mathcal{D}^{\mathcal{T}}$ and $\mathcal{D}^\theta$ on $range\pi_*$ such that

i. The space $range\pi_* = \mathcal{D}^{\mathcal{T}} \oplus \mathcal{D}^\theta$.
ii. The distribution $\mathcal{D}^{\mathcal{T}}$ is invarient under J.
iii. The distribution $\mathcal{D}^\theta$ is pointwise slant with semi-slant function θ.In this case, the angle θ can be regarded as a function on N_2, which is known as semi-slant function of the $\mathcal{PSSRM}$.

Now we say that the $\mathcal{PSSR}$ map π is proper if $\mathcal{D}^\theta \neq \{0\}$ and $\theta \neq 0, \frac{\pi}{2}$.

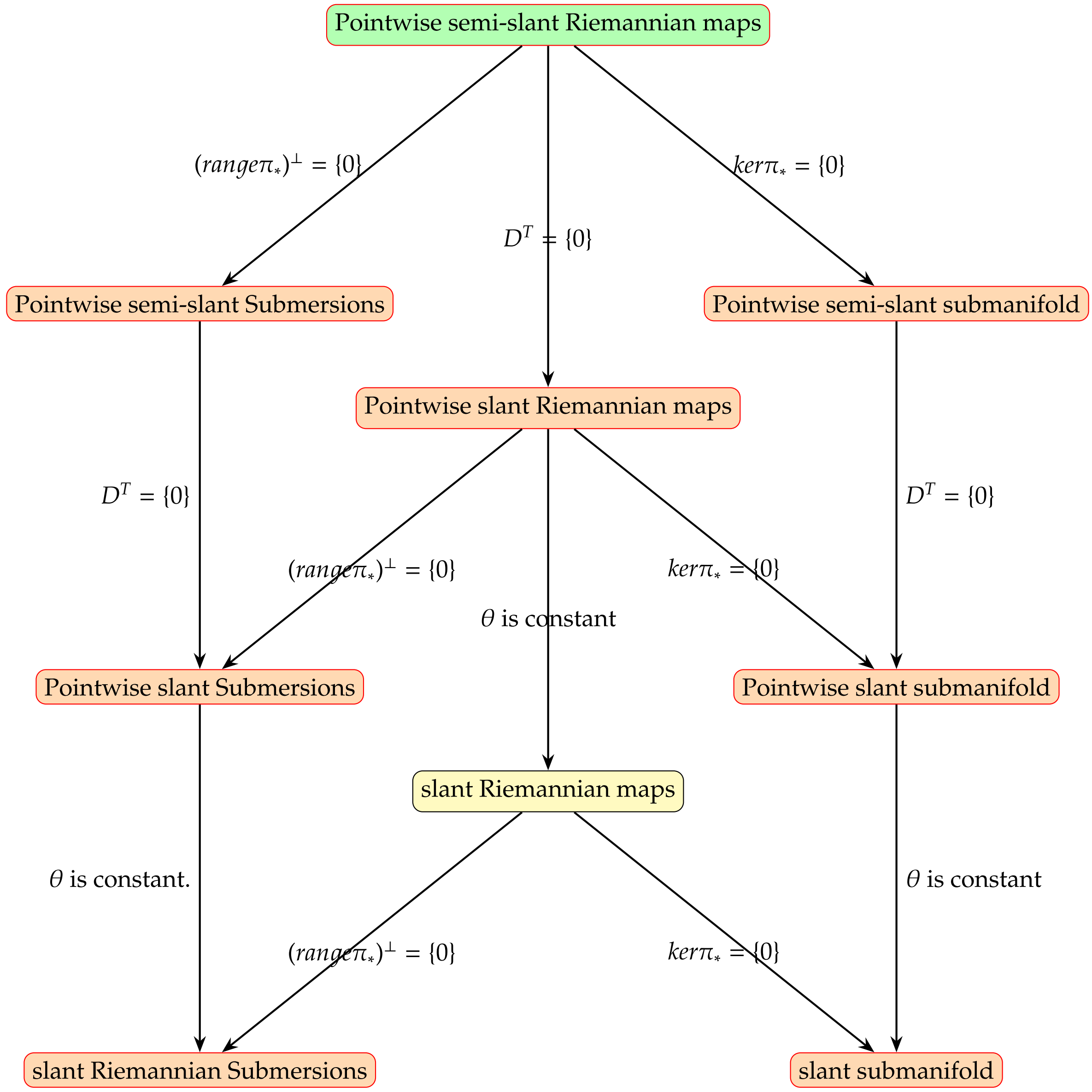

FIGURE 2. Class of $\mathcal{PSSRM}$s.

Let $\pi : (N_1, g_1) \to (N_2, g_2, J)$ be a $\mathcal{PSSRM}$ Then for $\pi_*(X) \in \Gamma(range\pi_*), X \in \Gamma(ker\pi_*)^\perp$, we write

$$J\pi_*(X) = \phi\pi_*(X) + \omega\pi_*(X) \tag{7}$$

where $\pi_*(X) \in \Gamma(\mathcal{D}^{\mathcal{T}})$ and $\omega\pi_*(X) \in \Gamma(\mathcal{D}^\theta)$. On the other hand, for $V \in (range\pi_*)^\perp$, we have

$$JV = \mathcal{B}V + \mathcal{C}V, \tag{8}$$

where $\mathcal{B}V \in \Gamma(range\pi_*)$ and $\mathcal{C}V \in \Gamma(range\pi_*)^\perp$.

Define the O'Neill tensors $\mathcal{A}$ and $\mathcal{T}$ by

$$\mathcal{A}_E F = h\nabla_{hE} vF + v\nabla_{hE} hF \tag{9}$$

$$\mathcal{T}_E F = v\nabla_{vE} hF + h\nabla_{vE} vF \tag{10}$$

for every $E, F \in \Gamma(TN_1)$, where ∇ is the Levi-Civita connection of g_1. Here h and v are the orthogonal projections on horizontal and vertical distributions, respectively. It is known that tensor fields $\mathcal{T}$ is symetric and $\mathcal{A}$ is anti-symetric tensors.
By using (9) and (10), we get

$$\nabla_{X_1} X_2 = \mathcal{T}_{X_1} X_2 + v\nabla_{X_1} X_2; \tag{11}$$

$$\nabla_{X_1} Y_1 = \mathcal{T}_{X_1} Y_1 + h\nabla_{X_1} Y_1; \tag{12}$$

$$\nabla_{Y_1} X_1 = \mathcal{A}_{Y_1} X_1 + v\nabla_{Y_1} X_1; \tag{13}$$

$$\nabla_{Y_1} Y_2 = \mathcal{A}_{Y_1} Y_2 + h\nabla_{Y_1} Y_2; \tag{14}$$

for any $Y_1, Y_2 \in \Gamma((ker\pi_*)^\perp)$, $X_1, X_2 \in \Gamma(ker\pi_*)$

Theorem 3.1. *Let $\pi : (N_1, g_1) \to (N_2, g_2, J)$ be a $\mathcal{PSSRM}$ with semi-slant function θ. Then we have*

$$\phi^2 \pi_* X = -cos^2(\theta)\pi_* X \tag{15}$$

for any $\pi_* X \in \Gamma(\mathcal{D}^\theta)$

Proof. Since,

$$cos\theta = \frac{g_2(J_2\pi_* X, \phi\pi_* X)}{|J_2\pi_* X||\phi\pi_* X|} = -\frac{g_2(\pi_* X, \phi^2\pi_* X)}{|\pi_* X||\phi\pi_* X|}$$

Hence,

$$\phi^2\pi_* X = -cos^2\theta\pi_* X$$

also convers of the above theorem, it can be directly verified. □

Moreover, for any $\pi_* X, \pi_* Y \in \Gamma(\mathcal{D}^\theta)$ we have

$$g_2(\phi\pi_* X, \phi\pi_* Y) = cos^2\theta g_2(\pi_* X, \pi_* Y) \tag{16}$$

$$g_2(\omega\pi_* X, \omega\pi_* Y) = sin^2\theta g_2(\pi_* X, \pi_* Y) \tag{17}$$

Example 3.1. *Let (R^8, g_{R^8}) is an Euclidean space. Consider $\{J, J'\}$ an almost complex structures on $\mathbb{R}^8$ satisfying $JJ' = J'J$, here*

$$J(x_1, ..., x_8) = (-x_2, x_1, -x_4, x_3, -x_6, x_5, -x_8, x_7)$$

and

$$J'(x_1, ...,_8) = (-x_3, x_4, x_1, -x_2, -x_7, -x_8, x_5, x_6)$$

for any $f : \mathbb{R}^8 \to \mathbb{R}$, a real-valued function we introduce a new almost complex structure J_f on $\mathbb{R}^8$ by $J_f = (cos(f))J + (sin(f))J'$.
Then, $\mathbb{R}^8_f = (\mathbb{R}^8, J_f, g_{\mathbb{R}^8})$ is an almost Hermitian manifold. Consider a Riemannian map $\pi : \mathbb{R}^8 \to \mathbb{R}^8_f$ by

$$\pi : (x_1, ..., x_8) = (x_1, x_2, x_3, x_4, e, x_6, a, x_8)$$

Then, the map π is a proper $\mathcal{PSSRM}$ with the semi-slnat function f such that

$$\mathcal{D}^{\theta} = span\{\partial_6, \partial_8\},$$

and

$$\mathcal{D}^{\mathcal{T}} = \{\partial_1, \partial_2, \partial_3, \partial_4\}$$

Also, we obtain

$$(range\pi_*)^{\perp} = span\{\partial_5, \partial_7\}$$

here $\{\partial_i = \frac{\partial}{\partial y_i}\}$ and $\{y_i\}$ are the local coordinates on $\mathbb{R}^8$.

Theorem 3.2. *[34] Let $\pi_1 : (N_1, g_1) \to (N_2, g_2, J_1)$ be a Riemannian submersion and $\pi_2 : (N_2, g_2, J_1) \to (N_3, g_3, J_2)$ a pointwise semi-slant immersion. Then $\pi_2 \circ \pi_1$ is a $\mathcal{PSSRM}$.*

As an application of the above theorem, we give the fallowing example of proper $\mathcal{PSSRM}$

Example 3.2. *Let $(\mathbb{R}^8, g_{\mathbb{R}^8})$ be the Euclid space. Consider $\{J, J'\}$ be two almost complex structures on $\mathbb{R}^8$ satisfying $JJ' = -J'J$, here*

$$J(x_1, ..., x_8) = (-x_2, x_1, -x_4, x_3, -x_6, x_5, -x_8, x_7)$$

.

$$J'(x_1, ..., x_8) = (-x_3, x_4, x_1, -x_2, -x_7, -x_8, x_5, x_6)$$

For any real valued function $\mu : \mathbb{R}^8 \to \mathbb{R}^8$, we construct new almost complex structure J_μ on $\mathbb{R}^8$ by

$$J_\mu = (cos\mu)J + (sin\mu)J'$$

Then, $\mathbb{R}^8_\mu = (\mathbb{R}^8, J_\mu, g_{\mathbb{R}^8})$ is an almost Hermitian manifold. suppose the map

$$\pi : (\mathbb{R}^8, g) \to (\mathbb{R}^8, J_\mu, g_{\mathbb{R}^8})$$

by

$$\pi(x_1, ..., x_8) = (x_1, x_2, x_3, x_4, a, x_6, 0, x_8)$$

which is the composition of the Riemannian submersion

$$\pi_1 : (\mathbb{R}^8, g) \to \mathbb{R}^6$$

by

$$\pi_1(x_1, ..., x_8) = (x_1, x_2, x_3, x_4, x_6, x_8)$$

followed by the pointwise semi-slant immersion

$$\pi_2 : \mathbb{R}^6 \to (\mathbb{R}^8, J_\mu, g_{\mathbb{R}^8})$$

by

$$\pi_2(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1, x_2, x_3, x_4, 0, x_5, 0, x_6)$$

It is easy to verify that $\pi = \pi_2 \circ \pi_1$ is a $\mathcal{PSSRM}$ with the semi-slant function $\theta = \mu$ such that

$$\mathcal{D}^{\theta} = span\{\frac{\partial}{\partial y_6}, \frac{\partial}{\partial y_8}\},$$

and

$$\mathcal{D}^{\mathcal{T}} = \{\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3}, \frac{\partial}{\partial y_4}\}.$$

Also, we obtain

$$(range\pi_*)^\perp = span\{\frac{\partial}{\partial y_5}, \frac{\partial}{\partial y_7}\},$$

here $\{y_1, ..., y_8\}$ *are the local coordinates on* $\mathcal{R}^8$.

We note that for $\pi_* w_1 \in \mathcal{D}^\theta$ *and* $\pi_* w_2 \in \mathcal{D}^\mathcal{T}$, *we get* $g_2(\pi_* w_1, \pi_* w_2) = 0$. *Then, Riemannian map* π *implies that* $g_1(w_1, w_2) = 0$. *So we obtain two orthogonal distributions* $\tilde{\mathcal{D}}^\theta$ *and* $\tilde{\mathcal{D}}^\mathcal{T}$ *such that*

$$(ker\pi_*)^\perp = \tilde{\mathcal{D}}^\theta \oplus \tilde{\mathcal{D}}^\mathcal{T}$$

Let π be a $C^\infty-$ map from (N_1, g_1) to Riemannian manifold (N_2, g_2). Then, the adjoint map ${}^*(\pi_*)_p$ of the differential $(\pi_*)_p$, $p \in N_1$, is given by

$$g_2((\pi_*)_p X, Y) = g_1(X, {}^*(\pi_*)_p Y)$$

for any $X \in T_p N_1$ and $Y \in T_{\pi(p)} N_2$. Furthermore if the map π is a Riemannian map, then for $X \in \Gamma(range\pi_*)_{\pi(p)}$ and $Y \in (ker(\pi_*)_p)^\perp$, We obtain

$${}^*(\pi_*)_p(\pi_*)_p X = X, \quad {}^*(\pi_*)_p(\pi_*)_p Y = Y,$$

thus the linear map ${}^*(\pi_*)_p : (range\pi_*)_{\pi(p)} \to (ker(\pi_*)_p)^\perp$ is an isomorphism. Define $C = {}^*(\pi_*)_p \phi(\pi_*)$.

Corollary 3.1. *Let* $\pi : (N_1, g_1) \to (N_2, g_2, J_2)$ *be a* $\mathcal{PSSRM}$. *Where* (N_1, g_1) *Riemannian manifold and* (N_2, g_2, J_2) *is an almost Hermitian manifold with the semi-slant function* θ. *Then,* $X \in \Gamma(D^\theta)$ *we have*

$$C^2 X = -cos^2\theta X \tag{18}$$

For $Y_1, Y_2, Y_3 \in \Gamma((ker\pi_*)_p)^\perp$ with $\pi_* Y_3 = \phi\pi_* Y_2$, we define

$$(\nabla^\pi_{Y_1}\omega)\pi_* Y_2 = C(\nabla\pi_*)(Y_1, Y_2) - (\nabla\pi_*)(Y_1, Y_3) \tag{19}$$

Proposition 3.1. *Let* π *be a* $\mathcal{PSSRM}$ *from a Riemannian manifold* (N_1, g_1) *to a Käehler manifold* (N_2, g_2, J_2) *with semi-slant function* θ. *If the tensor* ω *is parallel, then for* $U_1, U_2 \in \Gamma(\mathcal{D}^\theta)$, *we obtain*

$$(\nabla\pi_*)(CU_1, CU_2) = -cos^2\theta(U_1, U_2) \tag{20}$$

Proof. *Given that* ω *is parallel that is* $\nabla\omega = 0$. *Then using (19), for* $U_1, U_2 \in \Gamma(\mathcal{D}^\theta)$ *we get*

$$C(\nabla\pi_*)(U_1, U_2) = (\nabla\pi_*)(U_1, CU_2)$$

by interchanging U_1, U_2, *we have*

$$C(\nabla\pi_*)(U_2, U_1) = (\nabla\pi_*)(U_2, CU_1)$$

Since the tensor $(\nabla\pi_*)$ *is symmetric, we obtain*

$$(\nabla\pi_*)(U_1, CU_2) = (\nabla\pi_*)(U_2, CU_1)$$

Thus, we have

$$(\nabla\pi_*)(CU_1, CU_2) = (\nabla\pi_*)(U_1, C^2 U_2) = -cos^2\theta(\nabla\pi_*)(U_1, U_2)$$

□

Now, we are going to investigate the geometry of the leaves of the invarient distribution $\mathcal{D}^\mathcal{T}$ and the slant distribution $\mathcal{D}^\theta$.

Theorem 3.3. *Let π be a $\mathcal{PSSRM}$ from a Riemannian manifold (N_1, g_1) to a Käehler manifold (N_2, g_2, J_2). Then the invarient distribution $\mathcal{D}^{\mathcal{T}}$ defines totally geodesic foliation on N_2 iff*

i. $\mathcal{S}_{CV}\pi_*(X_1) - \pi_*(\nabla_{X_1}Z)$ *has no component in* $\Gamma(\mathcal{D}^{\mathcal{T}})$
ii. $\phi(\pi_*(\nabla_{X_1}W_1') - \mathcal{S}_{\omega\pi_*W_1}X_1)$. *has no component in* $\Gamma(\mathcal{D}^{\mathcal{T}})$.

for any $X_1, X_2, Z, W_1 \in \Gamma(ker\pi_*)^\perp$ *such that* $\pi_*X_1, \pi_*X_2 \in \Gamma(\mathcal{D}^{\mathcal{T}}), \pi_*W_1 \in \Gamma(D^\theta)$ *and* $V \in \Gamma(range\pi_*)^\perp$*:* $\pi_*Z = \mathcal{B}V$

Proof. For $\pi_*X_1, \pi_*X_2 \in \Gamma(\mathcal{D}^{\mathcal{T}})$ and $V \in \Gamma(range\pi_*)^\perp$, since π is a $\mathcal{PSSRM}$, using (1) and (8) we have

$$g_2(\nabla^\pi_{X_1}\pi_*X_2, V) = -g_2(\nabla^\pi_{X_1}\mathcal{B}V, J\pi_*X_2) - g_2(\nabla^\pi_{X_1}CV, J\pi_*X_2)$$

from (2.3) and (2.5) we get

$$\begin{aligned} g_2(\nabla^\pi_{X_1}\pi_*X_2, V) &= -g_2((\nabla\pi_*)(X_1, Z) + \pi_*\nabla_{X_1}Z, J\pi_*X_2) - g_2(-\mathcal{S}_{CV}\pi_*X_1 + \nabla^{\pi\perp}_{X_1}CV, J\pi_*X_2) \\ &= g_2(-\mathcal{S}_{CV}\pi_*X_1 - \pi_*\nabla_{X_1}Z, J\pi_*X_2) \end{aligned}$$

where $\pi_*Z = \mathcal{B}V \in \Gamma(\mathcal{D}^\theta)$ for $Z \in \Gamma(ker\pi_*)^\perp$. Since $J\pi_*X_2 \in \Gamma(\mathcal{D}^{\mathcal{T}})$, we obtain (*i*) On the other hand, for $\pi_*W_1 \in \Gamma(\mathcal{D}^\theta)$, by using (7), we get

$$g_2(\nabla^\pi_{X_1}\pi_*X_2, \pi_*W_1) = -g_2(\nabla^\pi_{X_1}\phi\pi_*W_1, J\pi_*X_2) - g_2(\nabla^\pi_{X_1}\omega\pi_*W_1, J\pi_*X_2)$$

from (2.3) and (2.5) we obtain

$$\begin{aligned} g_2(\nabla^\pi_{X_1}\pi_*X_2, \pi_*W_1) = &-g_2((\nabla\pi_*)(X_1, W') + \pi_*(\nabla_{X_1}W'), J\pi_*X_2) \\ &-g_2(-\mathcal{S}_{\omega\pi_*W_1}\pi_*(X_1) + \nabla^{\pi\perp}_{X_1}\omega\pi_*W_1, J\pi_*X_2) \end{aligned}$$

where $\phi\pi_*W_1 = \pi_*W'$ for $W_1' \in \Gamma(ker\pi_*)^\perp$. Then by using (1) and (7), we get

$$g_2(\nabla^\pi_{X_1}\pi_*X_2, \pi_*W_1) = g_2(\phi(\pi_*(\nabla_{X_1}W') - \mathcal{S}_{\omega\pi_*W_1}\pi_*X_1), \pi_*X_2)$$

This completes the proof. □

Theorem 3.4. *Let π be a $\mathcal{PSSRM}$ from a Riemannian manifold (N_1, g_1) to a Kaehler manifold (N_2, g_2, J_2). Then the slant distribution $\mathcal{D}^\theta$ defines totally geodesic foliation on N_2 iff*

i. $-sin^2\theta[W_1, V] + sin(2\theta)V(\theta)W_1 + \nabla^\pi_V\omega\phi\pi_*W_1 + \phi\nabla^\pi_V\omega\pi_*W_1$ *has no components in* $\Gamma(range\pi_*)$.
ii. $\phi(\mathcal{S}_{\omega\pi_*W_2}\pi_*W_1 - \nabla^\pi_{W_1}\phi\pi_*W_2)$ *has no components in* $\Gamma(\mathcal{D}^{\mathcal{T}})$.

for any $W_1, W_2 \in \Gamma(ker\pi_*)^\perp$ *such that* $\pi_*W_1, \pi_*W_2 \in \Gamma(\mathcal{D}^\theta)$ *and* $V \in \Gamma(range\pi_*)^\perp$.

Proof. Given that for any $W_1, W_2 \in \Gamma(ker\pi_*)^\perp$ *such that* $\pi_*W_1, \pi_*W_2 \in \Gamma(\mathcal{D}^\theta)$ *and* $V \in \Gamma(range\pi_*)^\perp$*, by using (1) and (7) we have*

$$\begin{aligned} g_2(\nabla^\pi_{W_1}\pi_*W_2, V) &= -g_2([W_1, V], \pi_*W_2) + g_2(\nabla^\pi_V\phi^2\pi_*W_1, \pi_*W_2) \\ &+ g_2(\nabla^\pi_V\omega\phi\pi_*W_1, \pi_*W_2) + g_2(\phi\nabla^\pi_V\omega\pi_*W_1, \pi_*W_2) \end{aligned}$$

using (3.9)*, we obtain*

$$\begin{aligned} g_2(\nabla^\pi_{W_1}\pi_*W_2, V) &= -g_2([W_1, V], \pi_*W_2) + sin(2\theta)V(\theta)g_2(\pi_*W_1, \pi_*W_2) \\ &- cos^2\theta g_2(\nabla^\pi_V\pi_*W_1, \pi_*W_2) + g_2(\nabla^\pi_V\omega\phi\pi_*W_1, \pi_*W_2) \\ &+ g_2(\phi\nabla^\pi_V\omega\pi_*W_1, \pi_*W_2) \end{aligned}$$

obviously, we have

$$\begin{aligned} g_2(\nabla^\pi_{W_1}\pi_*W_2, V) &= -sin^2\theta g_2([W_1, V], \pi_*W_2) + cos^2\theta g_2(\nabla^\pi_{W_1}\pi_*W_2, V) \\ &+ sin(2\theta)V(\theta)g_2(\pi_*W_1, \pi_*W_2) + g_2(\nabla^\pi_V\omega\phi\pi_*W_1, \pi_*W_2) \\ &+ g_2(\phi\nabla^\pi_V\omega\pi_*W_1, \pi_*W_2) \end{aligned}$$

$$\begin{aligned} sin^2\theta g_2(\nabla^{\pi}_{W_1}\pi_*W_2, V) &= g_2(-sin^2\theta[W_1, V] + sin(2\theta)V(\theta)\pi_*W_1 + \nabla^{\pi}_V\omega\phi\pi_*W_1 \\ &+ \phi\nabla^{\pi}_V\omega\pi_*W_1, \pi_*W_2) \end{aligned}$$

which gives (i) Now, by using (1), (7) and (5) we obtain

$$\begin{aligned} g_2(\nabla^{\pi}_{W_1}\pi_*W_2, \pi_*X_1) &= g_2(\nabla^{\pi}_{W_1}\phi\pi_*W_2, J\pi_*X_1) + g_2(\nabla^{\pi}_{W_1}\omega\pi_*W_2, J\pi_*X_1) \\ &= -g_2(J\nabla^{\pi}_{W_1}\phi\pi_*W_2, \pi_*X_1) + g_2(-\mathcal{S}_{\omega\pi_*W_2}\pi_*W_1 + \nabla^{\pi\perp}_{W_1}\omega\pi_*W_2, J\pi_*X_1) \\ &= g_2(\phi(\mathcal{S}_{\omega\pi_*W_2}\pi_*W_1 - \nabla^{\pi}_{W_1}\phi\pi_*W_2), \pi_*X_1) \end{aligned}$$

which gives (ii). This completes the proof. □

Now, we investigate the geometry of the leaves of the distribution $(range\pi_*)^{\perp}$.

Theorem 3.5. *Let π be a $\mathcal{PSSRM}$ from a Riemannian manifold (N_1, g_1) to a Kaehler manifold (N_2, g_2, J_2). Then the distribution $(range\pi_*)^{\perp}$ defines totally geodesic foliation on N_2 if and only if*

i. $[\pi_*X_1, V] - \nabla^{\pi\perp}_{X_1}V$ *has no components in* $\Gamma(range\pi_*)^{\perp}$
ii. $\nabla^{\pi}_V\omega\phi\pi_*X_2 + \omega\nabla^{\pi}_V\omega\pi_*X_2$ *has no components in* $\Gamma(range\pi_*)^{\perp}$

for any $X_1, X_2 \in \Gamma(ker\pi_*)^{\perp}$, $\pi_*X_1 \in \Gamma(\mathcal{D}^{\mathcal{T}})$, $\pi_*X_2 \in \Gamma(\mathcal{D}^{\theta})$ and $V, W \in \Gamma(range\pi_*)^{\perp}$,

Proof. *Given for any $X_1 \in \Gamma(ker\pi_*)^{\perp}, \pi_*X_1 \in \Gamma(\mathcal{D}^{\mathcal{T}})$ and $V, W \in \Gamma(range\pi_*)^{\perp}$, since the connection is metric and using (5), we have*

$$\begin{aligned} g_2(\nabla^{\pi}_V W, \pi_*X_1) &= -g_2(-[\pi_*X_1, V] + \nabla^{\pi}_{X_1}V, W) \\ &= g_2([\pi_*X_1, V], W) - g_2(\mathcal{S}_V\pi_*X_1 + \nabla^{\pi\perp}_{X_1}V, W) \end{aligned}$$

obviously, we get

$$g_2(\nabla^{\pi}_V W, \pi_*X_1) = g_2([\pi_*X_1, V] - \nabla^{\pi\perp}_{X_1}V, W)$$

which gives (i). Now for any $\pi_*X_2 \in \Gamma(\mathcal{D}^{\theta})$ by using (1), (7) and (11), we get

$$\begin{aligned} g_2(\nabla^{\pi}_V W, \pi_*X_2) &= -g_2(\nabla^{\pi}_V\phi\pi_*X_2, JW) - g_2(\nabla^{\pi}_V\omega\pi_*X_2, JW) \\ &= sin2\theta V\theta g_2(\pi_*X_2, W) - cos^2\theta g_2(\nabla_V\pi_*X_2, W) + g_2(\nabla^{\pi}_V\omega\phi\pi_*X_2, W) + g_2(\omega\nabla^{\pi}_V\omega\pi_*X_2, W) \end{aligned}$$

By straight computations, we obtain

$$sin^2\theta g_2(\nabla^{\pi}_V W, \pi_*X_2) = g_2(\nabla^{\pi}_V\omega\phi\pi_*X_2 + \omega\nabla^{\pi}_V\omega\pi_*X_2, W)$$

which gives (ii). This completes the proof. □

As a consequence of the Theorem (3.3), (3.4)and (3.5), we derive the following.

Corollary 3.2. *Let π be a $\mathcal{PSSRM}$ from a Riemannian manifold (N_1, g_1) to a Kaehler manifold (N_2, g_2, J_2). Then the total space N_2 is a locally product manifold of the leaves of $\mathcal{D}^{\mathcal{T}}$, $\mathcal{D}^{\theta}$ and $(range\pi_*)^{\perp}$, i.e.*
$N_2 = N_{2\mathcal{D}^{\mathcal{T}}} \times N_{2\mathcal{D}^{\theta}} \times N_{2(range\pi_*)^{\perp}}$, if and only if

I. $\mathcal{S}_{CV}\pi_*(X_1) - \pi_*(\nabla_{X_1}Z)$ *has no component in* $\Gamma(\mathcal{D}^{\mathcal{T}})$,
II. $\phi(\pi_*(\nabla_{X_1}W_1') - \mathcal{S}_{\omega\pi_*W_1}X_1)$. *has no component in* $\Gamma(\mathcal{D}^{\mathcal{T}})$,
III. $-sin^2\theta[W_1, V] + sin(2\theta)V(\theta)W_1 + \nabla^{\pi}_V\omega\phi\pi_*W_1 + \phi\nabla^{\pi}_V\omega\pi_*W_1$ *has no components in* $\Gamma(range\pi_*)$,
IV. $\phi(\mathcal{S}_{\omega\pi_*W_2}\pi_*W_1 - \nabla^{\pi}_{W_1}\phi\pi_*W_2)$ *has no components in* $\Gamma(\mathcal{D}^{\mathcal{T}})$,
V. $[\pi_*X_1, V] - \nabla^{\pi\perp}_{X_1}V$ *has no components in* $\Gamma(range\pi_*)^{\perp}$,
VI. $\nabla^{\pi}_V\omega\phi\pi_*X_2 + \omega\nabla^{\pi}_V\omega\pi_*X_2$ *has no components in* $\Gamma(range\pi_*)^{\perp}$.

for any $X_1, X_2, Z, W_1, W_2 \in \Gamma(ker\pi_*)^{\perp}$ and $V \in \Gamma(range\pi_*)^{\perp}$.

Now, we give necessary and sufficient conditions for a $\mathcal{PSSRM}$ π to be totally geodesic.

Theorem 3.6. *Let π be a $\mathcal{PSSRM}$ from a Riemannian manifold (N_1, g_1) to a Kaehler manifold (N_2, g_2, J_2) with the semi-slant function θ. Then, π is totally geodesic if and only if following condition are satisfied:*

a. *all the fibers $\pi^{-1}(p)$ are totally geodesic for $p \in N_1$,*
b. *$(ker\pi_*)^\perp$ defines a totally geodesic foliation on N_1,*
c. *$C(\nabla\pi_*)(X, Y') - \omega\pi_*(\nabla_X Y')$ has no components in $\Gamma(range\pi_*)^\perp$,*
d. *for $\pi_*X, \pi_*Y \in \Gamma(\mathcal{D}^\theta)$ for any $X, Y \in \Gamma(ker\pi_*)^\perp$ and $Z \in \Gamma(range\pi_*)^\perp$, the following equality is satisfied.*

$$\begin{aligned} sin(2\theta)Z(\theta)g_2(\pi_*X.\pi_*Y) &= -g_2(\pi_*Y, [\pi_*X, Z]) - cos^2\theta g_2(\pi_*Y.\nabla^\pi_Z\pi_*X) \\ &+ g_2(\pi_*Y, \mathcal{B}\nabla^\pi_Z\omega\pi_*X) + g_2(\pi_*Y, \nabla^\pi_Z\omega\phi\pi_*X) \end{aligned}$$

Theorem 3.7. *Let π be a $\mathcal{PSSRM}$ from a Riemannian manifold (N_1, g_1) to a Kaehler manifold (N_2, g_2, J) with the semi-slant function θ. Then, π is harmonic if and only if the following conditions are satisfiedd:*

a. the fiberes are minimal.
b. trace$\{\nabla^{\pi\perp}_{(.)}\omega\phi\pi_*(.) + \omega\mathcal{S}_{\omega\pi_*(.)}(.) - C\nabla^{\pi\perp}_{(.)}\omega\pi_*(.)\} = 0$

4. Chen-Ricci inequality

Let (N_2, g_2, J) be a Kaekler manifold. The Riemannian- cristoffel curvature tensor of a complex space form $N_2(\nu)$ of constant holomorphic sectional curvature ν satisfies

$$\begin{aligned} R_2(Y_1, Y_2, Y_3, Y_4) &= \frac{\nu}{4}\{g_2(Y_1, Y_3)g_2(Y_2, Y_4) - g_2(Y_2, Y_3)g_2(Y_1, Y_4) + g_2(Y_1, JY_3)g_2(Y_2, JY_4) \\ &- g_2(Y_2, JY_3)g_2(Y_1, JY_4) + 2g_2(Y_1, JY_2)g_2(Y_3, JY_4)\} \end{aligned} \quad (21)$$

$\forall\ Y_1, Y_2, Y_3, Y_4 \in \Gamma(TN_2)$[46].

Let π be a $\mathcal{RM}$ from a Riemannian manifold (N_1, g_1) to a Riemannian manifold (N_2, g_2). Let R_1 and R_2 be the curvature tensor fields of ∇^{N_1} and ∇^{N_2}, respectively. Then, $\forall\ Y_1, Y_2, Y_3, Y_4 \in \Gamma(ker\pi_*)^\perp$, we obtain the Gauss formula given by [35]

$$\begin{aligned} R_2(\pi_*Y_1, \pi_*Y_2, \pi_*Y_3, \pi_*Y_4) &= R_1(Y_1, Y_2, Y_3, Y_4) + g_2((\nabla\pi_*)(Y_1, Y_3), (\nabla\pi_*)(Y_2, Y_4)) \\ &- g_2((\nabla\pi_*)(Y_1, Y_4), (\nabla\pi_*)(Y_2, Y_3)) \end{aligned} \quad (22)$$

Now, we suppose that π is a $\mathcal{PSSRM}$ from a Riemannian manifold $(N_1^{b_1}, g_1)$ to the complex space form $(N_2^{2m}(\nu), g_2)$ such that $3 \leq r = rank\pi \leq (b_1, 2m)$. Using (21) in (22) we get for all $Y_1, Y_2, Y_3, Y_4 \in \Gamma(ker\pi_*)^\perp$.

$$\begin{aligned} R_1(Y_1, Y_2, Y_3, Y_4) &= \frac{\nu}{4}\{g_1(Y_1, Y_4)g_1(Y_2, Y_3) - g_1(Y_1, Y_3)g_1(Y_2, Y_4) \\ &+ g_2(\pi_*Y_1, J\pi_*Y_3)g_2(J\pi_*Y_2, \pi_*Y_4) \\ &- g_2(\pi_*Y_2, J\pi_*Y_3)g_2(J\pi_*Y_1, \pi_*Y_4) \\ &+ 2g_2(\pi_*Y_1, J\pi_*Y_2)g_2(J\pi_*Y_3, \pi_*Y_4)\} \\ &- g_2((\nabla\pi_*)(Y_1, Y_3), (\nabla\pi_*)(Y_2, Y_4)) \\ &+ g_2((\nabla\pi_*)(Y_1, Y_4), (\nabla\pi_*)(Y_2, Y_3)) \end{aligned} \quad (23)$$

Let $p \in N_1$ and consider

$\{\pi_*e_1, \pi_*e_2 = J\pi_*e_1, ..., \pi_*e_{2r_1-1}, \pi_*e_{2r_1} = J\pi_*e_{2r_1-1}, \pi_*e_{2r_1+1},$
$\pi_*e_{2r_1+2} = sec\theta\phi\pi_*e_{2r_1+1}, ..., \pi_*e_{2r_1+2r_2-1}, \pi_*e_r = J\pi_*e_{2r_1+2r_2-1}\}$
and $\{e_{r+1}, e_{r+2}, ..., e_{2m}\}$ be an orthonormal bases for $range\pi_*$ and $(range\pi_*)^\perp$, respectively. Then, the dimension of range of π_* is $r = 2r_1 + 2r_2$.

$$g_2^2(J\pi_*e_i, \pi_*e_{i+1}) = \begin{cases} 1 & i \in \{1, ..., 2r_1 - 1\}; \\ cos^2\theta & i \in \{2r_1 + 1, ..., 2r_1 + 2r_2 - 1\} \end{cases}$$

Then

$$\sum_{1\le k<s\le r} g_2^2(J\pi_* e_i, \pi_* e_s) = 2r_1 + 2r_2 cos^2\theta$$

Let's denote

$$\psi^{\alpha}_{ks} = g_2((\nabla\pi_*)(e_k, e_s), e_\alpha), k, s = 1, ..., r, \alpha = r+1, ..., 2m \tag{24}$$

$$\|\psi\|^2 = g_2((\nabla\pi_*)(e_k, e_s), (\nabla\pi_*)(e_k, e_s)) \tag{25}$$

$$trace\psi = \nabla\pi_*(e_k, e_k) \tag{26}$$

$$\|tace\psi\|^2 = g_2(trace\psi, trace\psi) \tag{27}$$

Now, for $(ker\pi_*)^\perp$ using (23), since π is $\mathcal{PSSRM}$ then, for every unit vector field $E_1 \in \Gamma(ker\pi_*)^\perp$ we arrive at

$$Ric^{(ker\pi_*)^\perp}(E_1) = \frac{\nu}{4}[r + 2 + 3cos^2\theta] - \|\nabla\pi_*(e_i, E_1)\|^2 + rg_2(\mathcal{H}, \nabla\pi_*(E_1, E_1))$$

$\mathcal{H}$ denotes the *mean curvature* vector field of the fiber.

Theorem 4.1. *Let $\pi : N_1, g_1 \to (N_2^{2m}(\nu), g_2, J)$ be a $\mathcal{PSSRM}$ then, we have*

$$Ric^{(ker\pi_*)^\perp}(E_1) \ge \frac{\nu}{4}[r + 2 + 3cos^2\theta] - \|\nabla\pi_*(e_i, E_1)\|^2$$

For a unit horizontal vector field $E_1 \in \Gamma(ker\pi_*)^\perp$, the equality status of the inequality satisfies if and only if every fiber is totally geodesic.

Theorem 4.2. *Let $\pi : N_1, g_1 \to (N_2^{2m}(\nu), g_2, J)$ be a $\mathcal{PSSRM}$ then, the Ricci tensor $S^{ker\pi_*^\perp}$ on $ker\pi_*^\perp$ satisfies*

$$S^{ker\pi_*^\perp}(E_1, E_2) \ge \frac{\nu}{4}[r + 2 + 3cos^2\theta]g_2(E_1, E_2) - g_2((\nabla\pi_*)(e_i, E_2), (\nabla\pi_*)(E_1, e_i))$$

For a unit horizontal vector field $E_1, E_2 \in \Gamma(ker\pi_*)^\perp$, the equality status of the inequality satisfies if and only if every fiber is totally geodesic.

similarly, for $ker\pi_*^\perp$ using (23) we obtain

$$2\tau^{ker\pi_*^\perp} = \frac{\nu}{4}[r(r-1) + 3(2r_1 + 2r_2cos^2\theta)] + r^2\|\mathcal{H}\|^2 - g_2((\nabla\pi_*)(e_k, e_s), (\nabla\pi_*)(e_k, e_s)) \tag{28}$$

$\tau^{ker\pi_*^\perp}$ is scalar curvature of horizontal space.

Theorem 4.3. *Let $\pi : N_1, g_1 \to (N_2^{2m}(\nu), g_2, J)$ be a $\mathcal{PSSRM}$ then, we have*

$$2\tau^{ker\pi_*^\perp} \ge \frac{\nu}{4}[r(r-1) + 3(2r_1 + 2r_2cos^2\theta)] - g_2((\nabla\pi_*)(e_k, e_s), (\nabla\pi_*)(e_k, e_s))$$

the equality status of the inequality satisfies if and only if every fiber is totally geodesic.

Now, we give Chen-Ricci inequality on $ker\pi_*^\perp$
from (24) and (28) we arrive at

$$2\tau^{ker\pi_*^\perp} = \frac{\nu}{4}[r(r-1)+3(2r_1+2r_2cos^2\theta)] + r^2\|\mathcal{H}\|^2 - \sum_{\alpha=r+1}^{2m}\sum_{k,s=1}^{r}\psi_{ks}^\alpha \tag{29}$$

From [17], we know that

$$\begin{aligned}\sum_{\alpha=r+1}^{2m}\sum_{k,s=1}^{r}(\psi_{ks}^\alpha)^2 &= \frac{1}{2}r^2\|\mathcal{H}\|^2 + \frac{1}{2}\sum_{\alpha=r+1}^{2m}[\psi_{11}^\alpha - \psi_{22}^\alpha - \ldots - \psi_{rr}^\alpha]^2 \\ &+ 2\sum_{\alpha=r+1}^{2m}\sum_{s=2}^{r}(\psi_{1s})^2 - 2\sum_{\alpha=r+1}^{2m}\sum_{2\le k<s\le r}[\psi_{kk}^{alpha}\psi_{ss}^\alpha - (\psi_{ks}^\alpha)^2]\end{aligned}$$

if we put this value in (29), we get

$$\begin{aligned}2\tau^{ker\pi_*^\perp} &= \frac{\nu}{4}[r(r-1)+3(2r_1+2r_2cos^2\theta)] + \frac{1}{2}r^2\|\mathcal{H}\|^2 - \frac{1}{2}\sum_{\alpha=r+1}^{2m}[\psi_{11}^\alpha - \psi_{22}^\alpha - \ldots - \psi_{rr}^\alpha]^2 \\ &- 2\sum_{\alpha=r+1}^{2m}\sum_{s=2}^{r}(\psi_{1s})^2 + 2\sum_{\alpha=r+1}^{2m}\sum_{2\le k<s\le r}[\psi_{kk}^\alpha\psi_{ss}^\alpha - (\psi_{ks}^\alpha)^2]\end{aligned}$$

From here we get

$$2\tau^{ker\pi_*^\perp} \le \frac{\nu}{4}[r(r-1)+3(2r_1+2r_2cos^2\theta)] + \frac{1}{2}r^2\|\mathcal{H}\|^2 + 2\sum_{\alpha=r+1}^{2m}\sum_{2\le k<s\le r}[\psi_{kk}^\alpha\psi_{ss}^\alpha - (\psi_{ks}^\alpha)^2]$$

Now from (22)

$$2\sum_{2\le k<s\le r}R_2(\pi_*e_k,\pi_*e_s,\pi_*e_s,\pi_*e_k) = 2\sum_{2\le k<s\le r}R^{ker\pi_*^\perp}(e_k,e_s,e_s,e_k) + 2\sum_{\alpha=r+1}^{2m}\sum_{2\le k<s\le r}[\psi_{kk}^\alpha\psi_{ss}^\alpha - (\psi_{ks}^\alpha)^2]$$

from the last inequality, we can write

$$\begin{aligned}2\tau^{ker\pi_*^\perp} &\le \frac{\nu}{4}[r(r-1)+3(2r_1+2r_2cos^2\theta)] + \frac{1}{2}r^2\|\mathcal{H}\|^2 + 2\sum_{2\le k<s\le r}R_2(\pi_*e_k,\pi_*e_s,\pi_*e_s,\pi_*e_k) \\ &- 2\sum_{2\le k<s\le r}R^{ker\pi_*^\perp}(e_k,e_s,e_s,e_k)\end{aligned}$$

Also, using

$$2\tau^{ker\pi_*^\perp} = 2\sum_{2\le k<s\le r}R^{ker\pi_*^\perp}(e_k,e_s,e_s,e_k) + 2\sum_{s=1}^{r}R^{\ker\pi_*^\perp}(e_1,e_s,e_s,e_1).$$

we get,

$$\begin{aligned}2Ric^{ker\pi_*^\perp}(e_1) &\le \frac{\nu}{4}[r(r-1)+3(2r_1+2r_2cos^2\theta)] \\ &+ \frac{1}{2}r^2\|\mathcal{H}\|^2 + 2\sum_{2\le k<s\le r}R_2(\pi_*e_k,\pi_*e_s,\pi_*e_s,\pi_*e_k) \\ &- 4\sum_{2\le k<s\le r}R^{ker\pi_*^\perp}(e_k,e_s,e_s,e_k)\end{aligned}$$

if we put value of R_2 from (21) we get

$$Ric^{ker\pi_*^\perp}(e_1) \leq \frac{\nu}{4}[(r-1)(r-2) + \frac{3\nu}{4}\{4r_1 - 3 + (4r_2-3)cos^2\theta\} + \frac{1}{4}r^2\|\mathcal{H}\|^2$$

Thus we can give the following result:

Theorem 4.4. *Let $\pi : N_1, g_1 \to (N_2^{2m}(\nu), g_2, J)$ be a $\mathcal{PSSRM}$ then, we have*

$$Ric^{ker\pi_*^\perp}(e_1) \leq \frac{\nu}{4}[(r-1)(r-2) + \frac{3\nu}{4}\{4r_1 - 3 + (4r_2-3)cos^2\theta\} + \frac{1}{4}r^2\|\mathcal{H}\|^2$$

the equality of the inequality satisfies if and only if

$$\psi_{11}^\alpha = \psi_{22} + ... + \psi_{rr}^\alpha$$

$$\psi_{1s}^\alpha = 0, s = 2, ..., r.$$

Corollary 4.1. *Let $\pi : N_1, g_1 \to (N_2^{2m}(\nu), g_2, J)$ be a $\mathcal{PSSRM}$ and the semi-slant function $\theta = \frac{\pi}{2}$ then, we get*

$$Ric^{ker\pi_*^\perp}(e_1) \leq \frac{\nu}{4}[(r-1)(r-2) + \frac{3\nu}{4}\{4r_1 - 3\} + \frac{1}{4}r^2\|\mathcal{H}\|^2$$

the equality of the inequality satisfies if and only if

$$\psi_{11}^\alpha = \psi_{22} + ... + \psi_{rr}^\alpha$$

$$\psi_{1s}^\alpha = 0, s = 2, ..., r.$$

Corollary 4.2. *Let $\pi : N_1, g_1 \to (N_2^{2m}(\nu), g_2, J)$ be a $\mathcal{PSSRM}$ and the semi-slant function $\theta = \frac{\pi}{2}$ then, we get*

$$Ric^{ker\pi_*^\perp}(e_1) \leq \frac{\nu}{4}[(r-1)(r-2) + \frac{3\nu}{2}\{r - 3\} + \frac{1}{4}r^2\|\mathcal{H}\|^2$$

the equality of the inequality satisfies if and only if

$$\psi_{11}^\alpha = \psi_{22} + ... + \psi_{rr}^\alpha$$

$$\psi_{1s}^\alpha = 0, s = 2, ..., r.$$

5. Casorati curvatures

The lemma below is crucial for the proof of our theorem.

Lemma 5.1. *Let $\mathcal{W} = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_1 + x_2 +, ..., x_n = k\}$ be a hyperplane of $\mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}$ is a quadratic form*

$$f(x_1, ..., x_n) = a\sum_{i=1}^{n-1}(x_i)^2 + b(x_n)^2 - 2\sum_{1\leq i<j\leq n} x_i x_j, a > 0, b > 0.$$

Thus, the constrained optimization problem

$$\min_{(x_1,...,x_n)\in\mathcal{W}} f$$

has a global solution given by

$$x_1 = x_2 = ... = x_{n-1} = \frac{k}{a+1}, x_n = \frac{k}{b+1} = (a-n+2)\frac{k}{a+1}$$

provided that $b = \frac{n-1}{a-n+2}$[43]

Let π be a $\mathcal{PSSRM}$ from a Riemannian manifold (N_1, g_1) to a complex space form $(N_2^{2m}(\nu), J, g_2)$ suppose $\{\pi_* e_1, ..., \pi_* e_r\}$ is an orthonormal basis of the vertical space $(range\pi_*)_{\pi(p)}$, for $p \in N_1$, and $\{e_{r+1}, e_{r+2}, .., e_{2m}\}$ be an orthonormal basis the horizontal space $(range\pi_*)^{\perp}$. We define scalar curvature $\tau^{(range\pi_*)}$ on the horizontal space $(ker\pi_*)^{\perp}$ by

$$\tau^{(ker\pi_*)^{\perp}} = \sum_{1\leq k<s\leq r} R_1(e_k, e_s, e_s, e_k) \tag{30}$$

and the normalized scalar curvature $\tau_{Nor}^{ker\pi_*^{\perp}}$ of $ker\pi_*^{\perp}$ as

$$\tau_{Nor}^{ker\pi_*^{\perp}} = \frac{2\tau^{ker\pi_*^{\perp}}}{r(r-1)} \tag{31}$$

Casorati curvature of the horizontal space $ker\pi_*^{\perp}$ is the normalised squared length of second fundamental form $\nabla\pi_*$ of the horizontal space $ker\pi_*^{\perp}$ over the manifold $(N_2^{2m}(\nu), J, g_2)$ and it is denoted by C Thus, from (25) we get

$$C = \frac{1}{r}\|\psi\|^2 = \frac{1}{r}\sum_{\alpha=r+1}^{2m}\sum_{k,s=1}^{r}(\psi_{ks}^{\alpha})^2 \tag{32}$$

Now, assume that $L^{ker\pi_*^{\perp}}$ is a $t-$dimensional subspace $(ker\pi_*)_p^{\perp}$, $2 \leq t$ and let $\{e_1, ..., e_t\}$ be an orthonormal basis of $L^{ker\pi_*^{\perp}}$. Then the casorati curvature $C^{ker\pi_*^{\perp}}(L^{ker\pi_*^{\perp}})$ of $L^{ker\pi_*^{\perp}}$ defined as

$$C^{ker\pi_*^{\perp}}(L^{ker\pi_*^{\perp}}) = \frac{1}{t}\|T\|^2 = \frac{1}{t}\sum_{\alpha=r+1}^{2m}\sum_{k,s=1}^{t}(T_{ks}^{\alpha})^2.$$

The normalized $\delta-$ casorati curvature $\hat{\delta}_c^{ker\pi_*^{\perp}}(r-1)$, $\delta_c^{ker\pi_*^{\perp}}(r-1)$ of $ker\pi_*^{\perp}$ are given by
$[\hat{\delta}_c^{ker\pi_*^{\perp}}(r-1)]_p = 2C_p^{ker\pi_*^{\perp}} - \frac{2r-1}{2r}Sup\{C^{ker\pi_*^{\perp}}(L^{ker\pi_*^{\perp}}) : L^{ker\pi_*^{\perp}}$ is a hyperplane of $(ker\pi_*^{\perp})_p$ }

$[\delta_c^{ker\pi_*^{\perp}}(r-1)]_p = \frac{1}{2}C_p^{ker\pi_*^{\perp}} + \frac{r+1}{2r}inf\{C^{ker\pi_*^{\perp}}(L^{ker\pi_*^{\perp}}) : L^{ker\pi_*^{\perp}}$ is a hyperplane of $(ker\pi_*^{\perp})_p\}$
Using (23) and (32) we get

$$2\tau^{ker\pi_*^{\perp}} = \frac{\nu}{4}[r^2 - r + 6(r_1 + r_2cos^2\theta)] - rC^{ker\pi_*^{\perp}} + \|trace\psi\|^2 \tag{33}$$

Now we define a function $Q^{\ker\pi_*^{\perp}}$ associated with the following quadratic polynomial with respect to the components of ψ:

$$Q^{ker\pi_*^{\perp}} = \frac{1}{2}[(r^2-r)C^{ker\pi_*^{\perp}} + (r^2-1)C^{ker\pi_*}(L^{ker\pi_*^{\perp}})] - 2\tau^{ker\pi_*^{\perp}} + \frac{\nu}{4}(r^2-r) + \frac{3\nu}{2}(r_1 + r_2cos^2\theta).$$

Without loss of genrality, by supposing that the hyperplane $L^{ker\pi_*^{\perp}}$ is spanned by $\{e_1, ..., r-1\}$, using (33) one can produce

$$\begin{aligned} Q^{ker\pi_*^{\perp}} &= \sum_{\alpha=r+1}^{2m}\sum_{r=1}^{r-1}[r(\psi_{kk}^{\alpha})^2 + (r+1)(\psi_{kr}^{\alpha})^2] - \sum_{\alpha=r+1}^{2m}[2(r+1)\sum_{1=k<s}^{r-1}(\psi_{ks}^{\alpha})^2 + \frac{r-1}{2}(\psi_{rr}^{\alpha})^2] \\ &- 2\sum_{1=k<s}^{r}\psi_{kk}^{\alpha}\psi_{ss}^{\alpha} + \frac{r-1}{2}(\psi_{rr}^{\alpha})^2 \\ &\geq \sum_{\alpha=r+1}^{2m}[\sum_{k=1}^{r-1}r(\psi_{kk}^{\alpha})^2 + \frac{r-1}{2}(\psi_{rr}^{\alpha})^2 - 2\sum_{1=k<s}^{r}\psi_{kk}^{\alpha}\psi_{ss}^{\alpha}] \end{aligned}$$

For $\alpha = r+1, ..., 2m$, let us consider the quadratic form $g_\alpha : \mathbb{R}^{2m} \to \mathbb{R}$ defined by

$$g_\alpha(\psi_{11}^\alpha, ..., \psi_{rr}^\alpha) = \sum_{k=1}^{r-1} r(\psi_{kk}^\alpha)^2 + \frac{r-1}{2}(\psi_{rr}^\alpha)^2 - 2\sum_{1=k<s}^{r} \psi_{kk}^\alpha \psi_{ss}^\alpha$$

and the constrained extrimum problem, $min g_\alpha$, subject to

$$\mathcal{W}^\alpha : \psi_{11}^\alpha +, ..., \psi_{rr}^\alpha = k^\alpha$$

here k^α is a real constant. From Lemma (5.1) we obtain $a = r$, $b = \frac{r-1}{2}$. Thus, by Lemma (5.1) we get the critical point $(\psi_{11}^\alpha, ..., \psi_{rr}^\alpha)$, given by

$$\psi_{11}^\alpha = \psi_{22}^\alpha =, .., \psi_{r-1r-1}^\alpha = \frac{k^\alpha}{r+1}, \psi_{rr}^\alpha = \frac{2k^\alpha}{r+1}$$

is a global minimum point. Also, $g_\alpha(\psi_{11}^\alpha, ..., \psi_{rr}^\alpha) = 0$. Moreover we obtain

$$\mathcal{Q}^{ker\pi_*^\perp} \geq 0 \tag{34}$$

Which gives

$$2\tau^{ker\pi_*^\perp} \leq \frac{1}{2}[(r^2-r)\mathcal{C}^{ker\pi_*^\perp} + (r^2-1)\mathcal{C}^{ker\pi_*^\perp}(L^{ker\pi_*^\perp})] + \frac{\nu}{4}[r^2 - r + 6(r_1 + r_2 cos^2\theta)]$$

using (31)

$$\tau_{Nor}^{ker\pi_*{}^\perp} \leq [\frac{1}{2}\mathcal{C}^{ker\pi_*^\perp} + \frac{r+1}{2r}\mathcal{C}^{ker\pi_*^\perp}(L^{ker\pi_*^\perp})] + \frac{\nu}{4} + \frac{3\nu(r_1 + r_2cos^2\theta)}{2r(r-1)} \tag{35}$$

for all hyperplane $L^{ker\pi_*^\perp}$ of $(ker\pi_*^\perp)$. Similarly, we can write

$$\mathcal{F}^{ker\pi_*^\perp} = 2(r^2-r)\mathcal{C}^{(ker\pi_*)^\perp} - \frac{1}{2}(2r^2 - 3r + 1)\mathcal{C}^{(ker\pi_*)^\perp}(L^{(ker\pi_*)^\perp}) - 2\tau^{(ker\pi_*)^\perp} + \frac{\nu}{4}[r^2 - r + 6(r_1 + r_2cos^2\theta)],$$

here hyperplane $L^{(ker\pi_*)^\perp}$ is a hyperplane of $ker\pi_*^\perp$. From here,

$$\mathcal{F}^{(ker\pi_*)^\perp} \geq 0 \tag{36}$$

which implies

$$\tau_{Nor}^{(\ker\pi_*)^\perp} \leq 2\mathcal{C}^{(\ker\pi_*)^\perp} - \frac{2r-1}{2r}\mathcal{C}^{(\ker\pi_*)^\perp}(L^{(\ker\pi_*)^\perp}) + \frac{\nu}{4} + \frac{3\nu(r_1 + r_2\cos^2\theta)}{2r(r-1)} \tag{37}$$

Now taking the infimum in (35) and the supremum in (37) over all hyperplanes $L^{ker\pi_*^\perp}$ of $ker\pi_*^\perp$ we get

Theorem 5.1. *Let π be a $\mathcal{PSSRM}$ a Riemannian manifold (N^{b_1,g_1}) to a complex space form $(N_2^{2m}(\nu), J, g_2)$ with semi-slant function θ, $3 \leq r = rank\pi < min\{b_1, 2m\}$. Then the normalized δ casorati curvatures $\hat{\delta}_c^{ker\pi_*^\perp}(r-1)$ $\delta_c^{ker\pi_*^\perp}(r-1)$ on $ker\pi_*^\perp$ satisfy*

1. $\tau_{Nor}^{(\ker\pi_*)^\perp} \leq \delta_c^{ker\pi_*^\perp}(r-1) + \frac{\nu}{4} + \frac{3\nu(r_1+r_2cos^2\theta)}{2r(r-1)}$
2. $\tau_{Nor}^{(\ker\pi_*)^\perp} \leq \hat{\delta}_c^{ker\pi_*^\perp}(r-1) + \frac{\nu}{4} + \frac{3\nu(r_1+r_2cos^2\theta)}{2r(r-1)}$

Furthermore, equality case holds in any inequalities at a point $p \in N_1$ iff with respect to suitable orthonormal basis $\{e_1, ..., e_r\}$ on $ker\pi_*^\perp$ and $\{e_{r+1}, ..., e_{2m}\}$ on $range\pi_*^\perp$, the components of ψ satisfy

$$\psi_{11}^\alpha = \psi_{22}^\alpha =, ..., = \psi_{r-1r-1}^\alpha = \frac{1}{2}\psi_{rr}^\alpha, \alpha \in \{r+1, ..., 2m\}$$

$$\psi_{ks}^\alpha = 0, k, s \in \{1, ..., r\}(k \neq s), \alpha \in \{r+1, ..., 2m\}$$

Corollary 5.1. *Let π be a $\mathcal{PSSRM}$ a Riemannian manifold (N^{b_1,g_1}) to a complex space form $(N_2^{2m}(\nu), J, g_2)$ with semi-slant function $\theta = \frac{\pi}{2}$, $3 \leq r = rank\pi < min\{b_1, 2m\}$. Then the normalized δ casorati curvatures $\hat{\delta}_c^{ker\pi_*^{\perp}}(r-1)$ $\delta_c^{ker\pi_*^{\perp}}(r-1)$ on $ker\pi_*^{\perp}$ satisfy*

1. $\tau_{Nor}^{(\ker\pi_*)^{\perp}} \leq \delta_c^{ker\pi_*^{\perp}}(r-1) + \frac{\nu}{4} + \frac{3\nu(r_1}{2r(r-1)}$
2. $\tau_{Nor}^{(\ker\pi_*)^{\perp}} \leq \hat{\delta}_c^{ker\pi_*^{\perp}}(r-1) + \frac{\nu}{4} + \frac{3\nu(r_1}{2r(r-1)}$

Furthermore, equality holds in any inequalities at $p \in N_1$ iff with respect to appropriate orthonormal basis $\{e_1, ..., e_r\}$ on $ker\pi_*^{\perp}$ and $\{e_{r+1}, ..., e_{2m}\}$ on $range\pi_*^{\perp}$, the components of ψ satisfy

$$\psi_{11}^{\alpha} = \psi_{22}^{\alpha} =, ..., = \psi_{r-1r-1}^{\alpha} = \frac{1}{2}\psi_{rr}^{\alpha}, \alpha \in \{r+1, ..., 2m\}$$

$$\psi_{ks}^{\alpha} = 0, k, s \in \{1, ..., r\}(k \neq s), \alpha \in \{r+1, ..., 2m\}$$

Corollary 5.2. *Let π be a $\mathcal{PSSRM}$ a Riemannian manifold (N^{b_1,g_1}) to a complex space form $(N_2^{2m}(\nu), J, g_2)$ with semi-slant function $\theta = 0$, $3 \leq r = rank\pi < min\{b_1, 2m\}$. Then the normalized δ casorati curvatures $\hat{\delta}_c^{ker\pi_*^{\perp}}(r-1)$ $\delta_c^{ker\pi_*^{\perp}}(r-1)$ on $ker\pi_*^{\perp}$ satisfy*

1. $\tau_{Nor}^{(\ker\pi_*)^{\perp}} \leq \delta_c^{ker\pi_*^{\perp}}(r-1) + \frac{\nu(r+2)}{4(r-1)}$
2. $\tau_{Nor}^{(\ker\pi_*)^{\perp}} \leq \hat{\delta}_c^{ker\pi_*^{\perp}}(r-1) + \frac{\nu(r+2)}{4(r-1)}$

Furthermore, equality holds in any inequalities at $p \in N_1$ iff with respect to appropriate orthonormal basis $\{e_1, ..., e_r\}$ on $ker\pi_*^{\perp}$ and $\{e_{r+1}, ..., e_{2m}\}$ on $range\pi_*^{\perp}$, the components of ψ satisfy

$$\psi_{11}^{\alpha} = \psi_{22}^{\alpha} =, ..., = \psi_{r-1r-1}^{\alpha} = \frac{1}{2}\psi_{rr}^{\alpha}, \alpha \in \{r+1, ..., 2m\}$$

$$\psi_{ks}^{\alpha} = 0, k, s \in \{1, ..., r\}(k \neq s), \alpha \in \{r+1, ..., 2m\}$$

Corollary 5.3. *Let $\pi : (N^{b_1}, g_1) \to (N_2^{2m}(\nu), J, g_2)$ be a $\mathcal{PSSRM}$, where $(N_2^{2m}(\nu), J, g_2)$ is complex space form with semi-slant function θ, $3 \leq r = rank\pi < min\{b_1, 2m\}$ then we get*

1. $\tau_{Nor}^{(\ker\pi_*)^{\perp}} \leq \delta_c^{ker\pi_*^{\perp}}(r-1)$
2. $\tau_{Nor}^{(\ker\pi_*)^{\perp}} \leq \hat{\delta}_c^{ker\pi_*^{\perp}}(r-1)$

References

[1] R. Abraham, J.E. Marsden and T. Ratiu, *Manifolds, Tensor Analysis and Applications*, Applied Mathematical Sciences, Vol. 75. Springer, New York, 1988.
[2] M. A. Akyol and B. Sahin, *Conformal anti-invariant Riemannian maps to Kahler manifolds*, U.P.B. Sci. Bull., Series A, Vol. 80, Iss. 4, 2018.
[3] M. A. Akyol and B. Sahin, *Conformal semi-invariant Riemannian maps to Kahler manifolds*, Revista de la Union Matematica Argentina, 60(2), 2019, 459-468.
[4] M. A. Akyol and B. Sahin, *Conformal slant Riemannian maps to Kahler manifolds*, Tokyo J. Math. 42(1), 225-237, (2019).
[5] M. Aquib and M. H. Shahid, *Generalized normalized δ-Casorati curvature for statistical submanifolds in quaternion Kaehler-like statistical space forms*, J. Geom. 109(1), Art. 13 (2018).
[6] P. Baird, J. C. Wood, *Harmonic Morphisms Between Riemannian Manifolds*, Clarendon Press, Oxford, 2003.
[7] J. L. Cabrerizo, A. Carriazo, L. M. Fernandez and M. Fernandez, *Slant submanifolds in Sasakian manifolds*, Glasgow Math. J. 42(1), (2000) 125-138.
[8] P. Candelas, G. T. Horowitz, A. Strominger, E.Witten: Vacuum configurations for superstrings. Nuclear Phys. B (electronic only) 258 (1985), 46-74.
[9] F. Casorati, *Nuova defnizione della curvatura delle superfcie e suo confronto con quella di Gauss. (New defnition of the curvature of the surface and its comparison with that of Gauss)*. Rend. Inst. Matem. Accad. Lomb. Ser. II 22(8), 335-346 (1889).
[10] B. Y. Chen, *Slant immersions*, Bull. Australian Math. Soc. 41(1), (1990), 135-147.
[11] B-Y. Chen and O. J. Garay, *Pointwise slant submanifolds in almost Hermitian manifolds*, Turk J Math 36 (2012), 630-640.
[12] F. Etayo, *On quasi-slant submanifolds of an almost Hermitian manifold*, Publ. Math. Debrecen 53, (1998), 217-223.

[13] M. Falcitelli, S. Ianus, A. M. Pastore, *Riemannian Submersions and Related Topics*, World Scientific, 2004.
[14] A. E. Fischer, *Riemannian maps between Riemannian manifolds*, Contemp. Math. 132 (1992), 331- 366.
[15] E. Garcia-Rio and D. N. Kupeli, *Semi-Riemannian Maps and Their Applications* (Kluwer Academic, Dordrecht, 1999).
[16] A. Gray, *Pseudo-Riemannian almost product manifolds and submersions*, J. Math. Mech. 1967; 16: 715-737.
[17] M. Gulbahar, S.E. Meric and E. Kılıc, *Sharp inequalities involving the Ricci curvature for Riemannian submersions*, Kragujevac Journal of Mathematics, 41(2) (2017), 279-293.
[18] Y. Gunduzalp and M. A. Akyol, *Pointwise slant Riemannian maps from Kaehler manifolds*, Journal of Geometry and Physcis, Volume 179, 104589, (2022).
[19] Y. Gunduzalp and M. A. Akyol, *Remarks on conformal anti-invariant Riemannian maps to cosymplectic manifolds*, Hacet. J. Math. Stat. 50(4) (2021), 1131-1139.
[20] S. Kumar, S. Kumar, S. Pandey, R. Prasad *Conformal hemi-slant submersions from almost Hermitian manifolds* - Communications of the Korean Mathematical Society, 2020
[21] S. Kumar, R. Prasad, P.K. Singh, *Conformal semi-slant submersions from Lorentzian para Sasakian manifolds*, Communications of the Korean Mathematical Society 34 (2), 2019, 637-655.
[22] C. W. Lee, J. W. Lee and G. E. Vılcu, *Optimal inequalities for the normalized* $\delta-$*Casorati curvatures of submanifolds in Kenmotsu space forms*, Adv. Geom. 17(3), 355-362 (2017).
[23] J. W. Lee and B. Sahin, *Pointwise slant submersions*, Bulletin of the Korean Mathematical Sosiety, 51(4),(2014), 115-1126.
[24] M. T. Mustafa, *Applications of harmonic morphisms to gravity*, J. Math. Phys. 41, 6918-6929 (2000).
[25] B. O. Neill, *The fundamental equations of a submersion*, Mich Math J, 1966; 13: 458-469
[26] T. Nore, *Second fundamental form of a map*, Ann. Mat. Pure Appl. 146(1987), 281-310.
[27] N. Papaghiuc, *Semi-slant submanifolds of a Kaehlerian manifold*. An. S¸tiint. Univ. Al. I. Cuza Ia¸si, Ser. Noua, Mat. 40 (1994), 55-61.
[28] K-S. Park, *Almost h-semi-slant Riemannian maps*, Taiwanese J. Math. 17(3), (2013), 937-956.
[29] K-S. Park, *On the Pointwise Slant Submanifolds*, In: Suh, Y., Ohnita, Y., Zhou, J., Kim, B., Lee, H. (eds) Hermitian–Grassmannian Submanifolds. Springer Proceedings in Mathematics and Statistics, vol 203. Springer, Singapore.
[30] K-S. Park and R. Prasad, *Semi-slant submersions*. Bull. Korean Math. Soc. 50 (2013), No. 3, pp. 951962.
[31] K-S. Park and B. Sahin, *Semi-slant Riemannian maps into almost Hermitian manifolds* Czechoslovak Mathematical Journal, 64(4), 2014, 1045-1061.
[32] R. Prasad, S. Kumar, *Conformal semi-invariant submersions from almost contact metric manifolds onto Riemannian manifolds*, Khayyam Journal of Mathematics 5 (2), 77-95.
[33] R. Prasad and S. Pandey, *Slant Riemannian maps from an almost contact manifold*, Filomat, 31(13), (2017), 3999-4007.
[34] B. Sahin, *Conformal Riemannian maps between Riemannian manifolds, their harmonicity and decomposition theorems*, Acta Appl. Math. 109(3), (2010), 829-847.
[35] B. Sahin, *Slant Riemannian maps from almost Hermitian manifolds*, Questiones Mathematicae, 36(3), 449-461, (2013).
[36] B. Sahin, *Slant Riemannian maps to Kaehler manifolds*, Int. J. Geom. Methods Mod. Phys. 10 (2013), 1250080.
[37] B. Sahin, *Slant submersions from almost Hermitian manifolds*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) 54(102) (2011) 93–105.
[38] B. Sahin, *Riemannian Submersions, Riemannian Maps in Hermitian Geometry, and their Applications*, Elsevier, Academic Press, (2017).
[39] B. Sahin, *Warped product pointwise semi-slant submanifolds of Kaehler manifolds*. Port. Math. 2013, 70, 252-268.
[40] B. Sahin, *Invariant and anti-invariant Riemannian maps to Kaehler manifolds*, Int. J. Geom. Meth. Mod. Phys. 7(3), (2010), 1-19.
[41] C. Sayar, F. Ozdemir and H. M. Ta¸stan, *Pointwise semi-slant submersions whose total manifolds are locally product Riemannian manifolds*. International Journal of Maps in Mathematics, Volume:1, Issue:1, (2018), 91-115.
[42] S. A. Sepet and M. Ergut, *Pointwise slant submersions from cosymplectic manifolds*. Turkish J Math 2016, 40,2 582-593.
[43] M. M. Tripathi, *Inequalities for algebraic Casorati curvatures and their applications*, Note Mat. 37(1), 161-186 (2017).
[44] A. J. Tromba, *Teichmuller Theory in Riemannian Geometry: Based on Lecture Notes by Jochen Denzler*. Lectures in Mathematics ETH Zurich, Birkhauser, Basel, 1992.
[45] G. E. Vilcu, *An optimal inequality for Lagrangian submanifolds in complex space forms involving Casorati curvatures*, J. Math. Anal. Appl. 465(2), 1209-1222 (2018).
[46] K. Yano and M. Kon, *structures on manifolds*, World scientific, 1985.
[47] P. Zhang and L. Zhang, *Inequalities for Casorati curvatures of submanifolds in real space forms*, Adv. Geom. 16(3), 329–335 (2016).