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Wave equations for the fractional Sturm-Liouville operator with singular coefficients

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Abstract. In this paper, we consider the wave equation for the fractional Sturm-Liouville operator with lower order terms and singular coefficients and data. We prove that the problem has a very weak solution. Furthermore, we prove the uniqueness in an appropriate sense and the consistency of the very weak solution concept with the classical theory.

1. Introduction

In the present paper, our investigation is devoted to the wave equation generated by the fractional Sturm-Liouville operator involving lower order terms and singularities in the coefficients and the data. That is, for $s \geq 0$ and $T > 0$, we study the equation

$$\partial_t^2 u(t,x) + \mathcal{L}^s u(t,x) + a(x)u(t,x) + b(x)u_t(t,x) = 0, \quad (t,x) \in [0,T] \times (0,1), \tag{1}$$

subject to the initial conditions

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in (0,1), \tag{2}$$

and boundary conditions

$$u(t,0) = u(t,1) = 0, \quad t \in [0,T], \tag{3}$$

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where a, b are assumed to be non-negative and $\mathcal{L}^s$ is the fractional differential operator associated to the Sturm-Liouville operator defined by

$$\mathcal{L}u(t,x) := -\partial_x^2 u(t,x) + q(x)u(t,x), \tag{4}$$

for a real valued function q.

The Sturm-Liouville operator with singular potential was studied by Savchuk and Shkalikov in [30]. In this work, asymptotic estimates for the eigenvalues and corresponding eigenfunctions, when the operator includes singular potential were obtained. We also cite [19], [28], [29] and [31] where the Sturm-Liouville operator with distributional potentials was explored.

So, our aim in the present work is to study the well-posedness of the initial/boundary problem (1)-(3), where the spatially dependent coefficients a, b and q and the initial data u_0 and u_1 are allowed to be non-regular functions, having in mind the Dirac delta function and its powers. We should mention here that powers of delta functions will be understood in the sense of powers of their regularisations, as it will be discussed later on. We do this study under the framework of the concept of very weak solutions. Our reasons to get into this framework lies in the fact that when the equation under consideration contains products of distributional terms, it is no longer possible to pose the problem in the distributional framework. This is related to the well known work of Schwartz [27] about the impossibility of multiplication of distributions.

In order to give a neat solution to this problem, the concept of very weak solutions was introduced in [15] for the analysis of second order hyperbolic equations with singular coefficients. Later on, this concept of solutions has been developed for a number of problems. We cite for instance [22], [23], [20], [4], [5], [6], [7], [9], [10], [11], [32] and [8] to mention only few. In [5], [6], [7], [9], [10] and [11], arguments were based on energy methods. In the recent works [21], [24], [25] and [26], the existence of solutions to initial/boundary value problems for the Sturm-Liouville operator including various types of time-dependent singular coefficients was considered. In these works, separation of variables techniques [16] were possible in order to obtain explicit formulas to the classical solutions. Our aim in the present paper is to combine separation of variables techniques with energy methods in order to extend the results obtained in [21] and [24], firstly by considering the fractional Sturm-Liouville operator instead of the classical one, and secondly, by including more terms in the equation under consideration. Most importantly, we allow coefficients to depend on space, so that the previous separation of variables methods do not readily apply.

The paper is organised as follows. After some preliminaries about the classical and the fractional Sturm-Liouville operator and about Duhamel's principle, we establish in Section 3, energy estimates in the regular case, which are key in proving existence and uniqueness of very weak solutions. We treat two cases. The general case when $s \geq 0$ and the case $s = 1$. In Section 4, we introduce the notion of very weak solutions adapted to our considered problem (1)-(3) and we prove that it is very weakly well-posed. Section 5 is devoted to showing the consistency of the concept of very weak solutions with the classical theory.

2. Preliminaries

The following notations and notions will be frequently used throughout this paper.

2.1. Notation

- By the notation $f \lesssim g$, we mean that there exists a positive constant C, such that $f \leq Cg$.
- We also define

$$\|u(t,\cdot)\|_s := \|u(t,\cdot)\|_{L^2} + \|\mathcal{L}^{\frac{s}{2}} u(t,\cdot)\|_{L^2} + \|u_t(t,\cdot)\|_{L^2}.$$

2.2. Classical Sturm-Liouville operator

Here we present some spectral properties of the Sturm-Liouville operator obtained in [30]. We consider the Sturm-Liouville operator $\mathcal{L}$ generated in the interval (0,1) by the differential expression

$$\mathcal{L}y := -\frac{d^2}{dx^2}y + q(x)y, \tag{5}$$

with boundary conditions

$$y(0) = y(1) = 0. \tag{6}$$

We first consider the real potential q satisfying

$$q(x) = \nu'(x) \geq 0, \text{ such that } \nu \in L^2(0,1). \tag{7}$$

The domain of the operator $\mathcal{L}$ is

$$\mathrm{Dom}(\mathcal{L}) = \left\{ y : y,\ y' - \nu y \in W_1^1(0,1),\ -y'' + qy \in L^2(0,1),\ y(0) = y(1) = 0 \right\}.$$

Let us introduce the quasi-derivative as follows

$$y^{[1]}(x) = y'(x) - \nu(x)y(x),$$

then the eigenvalue equation $\mathcal{L}y = \lambda y$ rewrites as a system

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}' = \begin{pmatrix} \nu & 1 \\ -\lambda - \nu^2 & -\nu \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad \phi_1(x) = y(x),\ \phi_2(x) = y^{[1]}(x).$$

Further on, we perform the so-called modified Prüfer transformation ([17])

$$\phi_1(x) = r(x) \sin\theta(x), \quad \phi_2(x) = \lambda^{\frac{1}{2}} r(x) \cos\theta(x),$$

where

$$\theta'(x,\lambda) = \lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}} \nu^2(x) \sin^2\theta(x,\lambda) + \nu(x) \sin 2\theta(x,\lambda), \tag{8}$$

and

$$r'(x,\lambda) = -r(x,\lambda) \left(\frac{1}{2} \nu^2(x) \lambda^{-\frac{1}{2}} \sin 2\theta(x,\lambda) + \nu(x) \cos 2\theta(x,\lambda) \right). \tag{9}$$

The solution to this equation has the form $\theta(x,\lambda) = \lambda^{\frac{1}{2}} x + \eta(x,\lambda)$, where

$$\eta(x,\lambda) = \lambda^{-\frac{1}{2}} \int_0^x \nu^2(\xi) \sin^2\theta(\xi,\lambda) d\xi + \int_0^x \nu(\xi) \sin\left(2\lambda^{\frac{1}{2}}\xi + 2\eta(\xi,\lambda)\right) d\xi.$$

By using the method of successive approximations, one can easily show that the last equation has a solution that is uniformly bounded for $0 \leq x \leq 1$ and $\lambda > 1$. Since $|\nu|^2 \in L^1(0,1)$, according to the Riemann-Lebesgue lemma, $\eta(x,\lambda) = o(1)$ as $\lambda \to \infty$. Hence,

$$\theta(x,\lambda) = \lambda^{\frac{1}{2}} x + o(1),$$

moreover $\theta(0,\lambda) = 0$.

Using the Riemann-Lebesgue lemma for the equation (9), we get

$$r(x,\lambda) = \exp\left(-\int_0^x \nu(\xi) \cos 2\theta(\xi,\lambda) d\xi - \frac{1}{2\sqrt{\lambda}} \int_0^x \nu^2(\xi) \sin 2\theta(\xi,\lambda) d\xi \right).$$

And finally, using the Dirichlet boundary conditions (6), we obtain

$$\phi_1(1,\lambda) = r(1,\lambda) \sin\theta(1,\lambda) = 0, \quad r(1,\lambda) \neq 0, \quad \theta(1,\lambda) = \pi n.$$

Then, the eigenvalues of the Sturm-Liouville operator $\mathcal{L}$ generated on the interval $(0,1)$ by the differential expression (5) with Dirichlet boundary conditions (6) are given by

$$\lambda_n = (\pi n)^2(1 + o(n^{-1})), \qquad n = 1, 2, ..., \tag{10}$$

and corresponding eigenfunctions

$$\tilde{\phi}_n(x) = r_n(x)\sin\theta_n(x) = r_n(x)\sin\Big(\sqrt{\lambda_n}x + \eta_n(x)\Big). \tag{11}$$

According to (7), ν is a real valued function. Then, the eigenfunctions $\tilde{\phi}_n$ are real. Here and below we will consider the positive operator $\langle\mathcal{L}y, y\rangle \geq 0$, which implies that all eigenvalues λ_n are real and non-negative. The first derivatives of $\tilde{\phi}_n$ have the following form

$$\tilde{\phi}_n'(x) = \sqrt{\lambda_n}r_n(x)\cos(\theta_n(x)) + \nu(x)\tilde{\phi}_n(x). \tag{12}$$

By Theorem 2 in [28] we have that

$$\tilde{\phi}_n(x) = \sin\sqrt{\lambda_n}x + \psi_n(x), \quad n = 1, 2, ..., \quad \sum_{n=1}^{\infty}\|\psi_n\|^2 \leq C\int_0^1 |\nu(x)|^2 dx. \tag{13}$$

The estimate for $\|\tilde{\phi}_n\|_{L^2}$ follows by taking the L^2 norm in (11) and by proceeding as follows

$$\begin{aligned}
\|\tilde{\phi}_n\|_{L^2}^2 &= \int_0^1 \left|r_n(x)\sin\left(\lambda_n^{\frac{1}{2}}x + \eta_n(x)\right)\right|^2 dx \lesssim \int_0^1 |r_n(x)|^2\, dx \\
&\lesssim \int_0^1 \left|\exp\left\{\left(-\int_0^x \nu(s)\cos 2\theta_n(s)ds - \frac{1}{2}\frac{1}{\sqrt{\lambda_n}}\int_0^x \nu^2(s)\sin 2\theta_n(s)ds\right)\right\}\right|^2 dx \\
&\lesssim \int_0^1 \exp\left\{\left(2\int_0^x |\nu(s)|ds + \frac{1}{\sqrt{\lambda_n}}\int_0^x |\nu^2(s)|ds\right)\right\}dx \\
&\lesssim \exp\left\{2\left(\|\nu\|_{L^2} + \lambda_n^{-\frac{1}{2}}\|\nu\|_{L^2}^2\right)\right\} < \infty.
\end{aligned} \tag{14}$$

In addition, according to Theorem 4 in [30], we get

$$\tilde{\phi}_n(x) = \sin(\pi n x) + o(1), \tag{15}$$

for sufficiently large n. Combining this with (11), we see that there exist a constant $C_0 > 0$, such that

$$0 < C_0 \leq \|\tilde{\phi}_n\|_{L^2} < \infty \quad \text{for all } n. \tag{16}$$

The family of eigenfunctions of the operator $\mathcal{L}$ form an orthogonal basis in $L^2(0,1)$. Moreover, we will normalize them and denote

$$\phi_n(x) = \frac{\tilde{\phi}_n(x)}{\sqrt{\langle\tilde{\phi}_n, \tilde{\phi}_n\rangle}} = \frac{\tilde{\phi}_n(x)}{\|\tilde{\phi}_n\|_{L^2}}. \tag{17}$$

2.3. Fractional Sturm-Liouville operator

Definition 2.1. *Let $\{\lambda_k, \phi_k\}_{k=1,\dots,\infty}$ be the family of eigenvalues and corresponding eigenfunctions to the classical Sturm-Liouville operator as defined above. Then, for $s \in \mathbb{R}$, $\mathcal{L}^s$ is defined in the sense that:*

$$\mathcal{L}^s \phi_k := \lambda^s \phi_k, \tag{18}$$

for all $k = 1, \dots$.

In other words, $\mathcal{L}^s$ is defined to be the operator having the family $\{\lambda_k^s, \phi_k\}_{k=1,\dots,}$ as family of eigenvalues and corresponding eigenfunctions.

Proposition 2.2. *Let $\mathcal{L}$ be the Sturm-Liouville operator generated in the interval (0,1) by the differential expression* (5) *with boundary conditions* (6)*. Assume that $(f, g) \in L^2(0,1) \times L^2(0,1)$ with $(\mathcal{L}^s f, \mathcal{L}^s g) \in L^2(0,1) \times L^2(0,1)$. Then*

$$\langle \mathcal{L}^s f, g\rangle_{L^2} = \langle f, \mathcal{L}^s g\rangle_{L^2} \text{ for any } s \in \mathbb{R}, \tag{19}$$

and

$$\mathcal{L}^{s+s'} f = \mathcal{L}^s \left(\mathcal{L}^{s'} f\right) \text{ for } s, s' \in \mathbb{R}. \tag{20}$$

Proof. Since for $n = 1, 2, \dots,$ the eigenfunctions ϕ_n of the Sturm-Liouville operator are orthonormal in $L^2(0,1)$ and using the fact that the operator $\mathcal{L}$ is self-adjoint ([18]) and using eigenfunction expansions for $f,\ g \in L^2(0,1)$, we obtain

$$\begin{aligned} \langle \mathcal{L}^s f, g\rangle_{L^2} &= \int_0^1 \sum_{n=1}^{\infty} \lambda_n^s f_n \phi_n(x) \sum_{m=1}^{\infty} g_m \phi_m(x) dx = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \int_0^1 \lambda_n^s f_n \phi_n(x) g_m \phi_m(x) dx \\ &= \sum_{n=1}^{\infty} \lambda_n^s f_n g_n \int_0^1 \phi_n^2(x) dx = \sum_{n=1}^{\infty} \lambda_n^s f_n g_n, \end{aligned} \tag{21}$$

where

$$f_n = \int_0^1 f(x)\phi_n(x)dx, \quad g_n = \int_0^1 g(x)\phi_n(x)dx.$$

On the other hand, we similarly get

$$\langle f, \mathcal{L}^s g\rangle_{L^2} = \sum_{n=1}^{\infty} f_n \lambda_n^s g_n = \sum_{n=1}^{\infty} \lambda_n^s f_n g_n. \tag{22}$$

This proves the first statement. For (20), the second statement of the proposition, we have

$$\begin{aligned} \mathcal{L}^{s+s'} f &= \sum_{n=1}^{\infty} \mathcal{L}^{s+s'} f_n \phi_n(x) = \sum_{n=1}^{\infty} \lambda_n^s \left(\lambda_n^{s'} f_n \phi_n(x)\right) \\ &= \sum_{n=1}^{\infty} \mathcal{L}^s \left(\mathcal{L}^{s'} f_n \phi_n(x)\right) = \mathcal{L}^s \left(\mathcal{L}^{s'} f\right), \end{aligned}$$

completing the proof. □

2.4. Sobolev spaces and embeddings

We define the Sobolev spaces $W^s_{\mathcal{L}}$ associated to $\mathcal{L}^s$, for any $s \in \mathbb{R}$, as the space

$$W^s_{\mathcal{L}}(0,1) := \left\{ f \in \mathcal{D}'_{\mathcal{L}}(0,1) : \ \mathcal{L}^{s/2} f \in L^2(0,1) \right\}, \tag{23}$$

with the norm $\|f\|_{W^s_{\mathcal{L}}} := \|\mathcal{L}^{s/2} f\|_{L^2}$. The global space of distributions $\mathcal{D}'_{\mathcal{L}}(0,1)$ is defined as follows.

The space $C^\infty_{\mathcal{L}}(0,1) := \mathrm{Dom}(\mathcal{L}^\infty)$ is called the space of test functions for $\mathcal{L}$, where we define

$$\mathrm{Dom}(\mathcal{L}^\infty) := \bigcap_{m=1}^{\infty} \mathrm{Dom}(\mathcal{L}^m),$$

where $\mathrm{Dom}(\mathcal{L}^m)$ is the domain of the operator $\mathcal{L}^m$, in turn defined as

$$\mathrm{Dom}(\mathcal{L}^m) := \left\{ f \in L^2(0,1) : \mathcal{L}^j f \in \mathrm{Dom}(\mathcal{L}), \ j = 0,1,2,...,m-1 \right\}.$$

The Fréchet topology of $C^\infty_{\mathcal{L}}(0,1)$ is given by the family of norms

$$\|\phi\|_{C^m_{\mathcal{L}}} := \max_{j \le m} \|\mathcal{L}^j \phi\|_{L^2(0,1)}, \quad m \in \mathbb{N}_0, \ \phi \in C^\infty_{\mathcal{L}}(0,1). \tag{24}$$

The space of $\mathcal{L}$-distributions

$$\mathcal{D}'_{\mathcal{L}}(0,1) := \mathbf{L}\left(C^\infty_{\mathcal{L}}(0,1), \mathbb{C} \right)$$

is the space of all linear continuous functionals on $C^\infty_{\mathcal{L}}(0,1)$. For $\omega \in \mathcal{D}'_{\mathcal{L}}(0,1)$ and $\phi \in C^\infty_{\mathcal{L}}(0,1)$, we shall write

$$\omega(\phi) = \langle \omega, \phi \rangle.$$

For any $\psi \in C^\infty_{\mathcal{L}}(0,1)$, the functional

$$C^\infty_{\mathcal{L}}(0,1) \ni \phi \mapsto \int_0^1 \psi(x)\phi(x) dx$$

is an $\mathcal{L}$-distribution, which gives an embedding $\psi \in C^\infty_{\mathcal{L}}(0,1) \hookrightarrow \mathcal{D}'_{\mathcal{L}}(0,1)$.

Proposition 2.3. *Let $0 < s \in \mathbb{R}$ and $f \in W^s_{\mathcal{L}}(0,1)$. Then, we have the continuous inclusions*

$$W^s_{\mathcal{L}}(0,1) \subset L^2(0,1) \subset W^{-s}_{\mathcal{L}}(0,1). \tag{25}$$

That is, for any $f \in W^s_{\mathcal{L}}(0,1)$, we have $f \in L^2(0,1)$ and accordingly $f \in W^{-s}_{\mathcal{L}}(0,1)$. Moreover, there exist positive constants C_1, C_2 independent of f such that

$$\|f\|_{W^{-s}_{\mathcal{L}}} \le C_1 \|f\|_{L^2}, \tag{26}$$

and

$$\|f\|_{L^2} \le C_2 \|f\|_{W^s_{\mathcal{L}}}. \tag{27}$$

Proof. The first embedding is a direct consequence of the definition of $W^s_{\mathcal{L}}(0,1)$ (see (23)). Let us prove the second statement. According to (10), the eigenvalues of the operator $\mathcal{L}$ are outside the unit ball, then

$$\lambda_n^{-\frac{s}{2}} \le 1$$

for all $n = 1, 2, \dots,$. This leads to the following estimate

$$\begin{aligned} \|f\|^2_{W^{-s}_{\mathcal{L}}} &= \|\mathcal{L}^{-\frac{s}{2}} f\|^2_{L^2} = \int_0^1 \left| \sum_{n=1}^{\infty} \mathcal{L}^{-\frac{s}{2}} f_n \phi_n(x) \right|^2 dx = \int_0^1 \left| \sum_{n=1}^{\infty} \lambda_n^{-\frac{s}{2}} f_n \phi_n(x) \right|^2 dx \\ &\lesssim \sum_{n=1}^{\infty} \int_0^1 \left| \lambda_n^{-\frac{s}{2}} f_n \phi_n(x) \right|^2 dx = \sum_{n=1}^{\infty} \left| \lambda_n^{-\frac{s}{2}} f_n \right|^2 \leq \sum_{n=1}^{\infty} |f_n|^2 = \|f\|^2_{L^2}, \end{aligned}$$

completing the proof. □

2.5. *Duhamel's principle*

Throughout this paper, we will often use the following version of Duhamel's principle for which the proof is given. For more details and applications about Duhamel's principle, we refer the reader to [12]. Let us consider the following initial/boundary problem,

$$\begin{cases} u_{tt}(t,x) + Lu(t,x) + \alpha(x)u_t(t,x) = f(t,x), & (t,x) \in (0,\infty) \times (0,1), \\ u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), & x \in (0,1), \\ u(t,0) = u(t,1) = 0, & t \in (0,\infty), \end{cases} \tag{28}$$

for a given function α and L is a linear partial differential operator acting over the spatial variable x.

Proposition 2.4. *The solution to the initial/boundary problem (28) is given by*

$$u(t,x) = w(t,x) + \int_0^t v(t,x;\tau)d\tau, \tag{29}$$

where $w(t,x)$ is the solution to the homogeneous problem

$$\begin{cases} w_{tt}(t,x) + Lw(t,x) + \alpha(x)w_t(t,x) = 0, & (t,x) \in (0,\infty) \times (0,1), \\ w(0,x) = u_0(x), \;\; w_t(0,x) = u_1(x), & x \in (0,1), \\ w(t,0) = w(t,1) = 0, & t \in (0,\infty), \end{cases} \tag{30}$$

and for fixed $\tau \in (0,\infty)$, $v(t,x;\tau)$ solves the auxiliary problem

$$\begin{cases} v_{tt}(t,x;\tau) + Lv(t,x;\tau) + a(x)v_t(t,x;\tau) = 0, & (t,x) \in (\tau,\infty) \times (0,1), \\ v(\tau,x;\tau) = 0, \;\; v_t(\tau,x;\tau) = f(\tau,x), & x \in (0,1), \\ v(t,0;\tau) = v(t,1;\tau) = 0, & t \in (\tau,\infty). \end{cases} \tag{31}$$

Proof. Firstly, we apply ∂_t to u in (29). We get

$$\partial_t u(t,x) = \partial_t w(t,x) + \int_0^t \partial_t v(t,x;\tau)d\tau, \tag{32}$$

where we used the fact that $v(t,x;t) = 0$ coming from the initial condition in (31). We differentiate again (32) with respect to t having in mind that $\partial_t v(t,x;t) = f(t,x)$, we get

$$\partial_{tt} u(t,x) = \partial_{tt} w(t,x) + f(t,x) + \int_0^t \partial_{tt} v(t,x;\tau)d\tau. \tag{33}$$

Now, the operator L when applied to u in (29) gives

$$Lu(t,x) = Lw(t,x) + \int_0^t Lv(t,x;\tau)d\tau. \tag{34}$$

Multiplying (32) by $a(x)$ yields

$$a(x)\partial_t u(t,x) = a(x)\partial_t w(t,x) + \int_0^t a(x)\partial_t v(t,x;\tau)d\tau. \tag{35}$$

Combining (33), (34) and (35) and taking into consideration that w and v are the solutions to (30) and (31), we arrive at

$$u_{tt}(t,x) + a(x)u_t(t,x) + Lu(t,x) = f(t,x).$$

Noting that $u(0,x) = w(0,x) = u_0(x)$ from (29) and that $u_t(0,x) = \partial_t w(0,x) = u_1(x)$ from (32) and that from (29), the boundary conditions $u(t,0) = u(t,1) = 0$ are satisfied concludes the proof. □

Remark 2.5. *We should highlight here that Duhamel's principle applies to weak solutions. This is due to the fact that the principle involves linearity and superposition, both of which hold in the weak formulation. Moreover, in our case we understand derivatives with respect to time as classical derivatives, and derivatives with respect to the spatial variable in the weak sense. Furthermore, in Sections 4 and 5, Duhamel's principle will often be applied at the level of regularisation, that is, for smooth solutions.*

3. Classical case: Energy estimates

In this section, we consider the case when the real potential q and the equation coefficients a and b are regular functions. We also assume that $s \geq 0$. In this case, we obtain the well-posedness in the Sobolev spaces $W^s_{\mathcal{L}}(0,1)$ associated to the operator $\mathcal{L}^s$. We start by proving the well-posedness of our initial/boudary problem (1)-(3) in the case when $a, b \equiv 0$. That is, for the equation

$$\partial_t^2 u(t,x) + \mathcal{L}^s u(t,x) = 0, \quad (t,x) \in [0,T] \times (0,1), \tag{36}$$

with initial conditions

$$u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), \quad x \in (0,1), \tag{37}$$

and Dirichlet boundary conditions

$$u(t,0) = u(t,1) = 0, \quad t \in [0,T], \tag{38}$$

where $\mathcal{L}^s$ is defined as in Definition 2.1.

Theorem 3.1. *Assume that $q \in L^\infty(0,1)$ is real. For any $s \geq 0$, if the initial data satisfy $(u_0,\, u_1) \in W^s_{\mathcal{L}}(0,1) \times L^2(0,1)$, then the equation* (36) *with the initial/boundary conditions* (37)-(38) *has a unique solution $u \in C([0,T], W^s_{\mathcal{L}}(0,1)) \cap C^1([0,T], L^2(0,1))$. It satisfies the estimates*

$$\|u(t,\cdot)\|_{L^2} \lesssim \|u_0\|_{L^2} + \|u_1\|_{W^{-s}_{\mathcal{L}}}, \tag{39}$$

$$\|u(t,\cdot)\|_{W^s_{\mathcal{L}}} \lesssim \|u_0\|_{W^s_{\mathcal{L}}} + \|u_1\|_{L^2}, \tag{40}$$

and

$$\|\partial_t u(t,\cdot)\|_{L^2} \lesssim \|u_0\|_{W^s_{\mathcal{L}}} + \|u_1\|_{L^2}, \tag{41}$$

where the constants are independent of u_0, u_1 and q.

Before giving the proof, one observes that the assumption $q \in L^\infty(0,1)$ fits with (7) since $L^\infty(0,1)$ is embedded in $L^2(0,1)$.

Proof. Following the arguments in [21], we apply the technique of the separation of variables (see, e.g. [16]) to solve the equation (36) with the initial-boundary conditions (37)-(38). We look for a solution in the form

$$u(t,x) = T(t)X(x),$$

for functions $T(t)$ and $X(x)$ to be determined. Plugging $u(t,x) = T(t)X(x)$ into (36), we arrive at the equation

$$T''(t)X(x) + \mathcal{L}^s\left(T(t)X(x)\right) = 0,$$

since the operator does not depend on t, we obtain

$$T''(t)X(x) + T(t)\mathcal{L}^s X(x) = 0.$$

Dividing this equation by $T(t)X(x)$, we get

$$\frac{T''(t)}{T(t)} = \frac{-\mathcal{L}^s X(x)}{X(x)} = -\mu, \tag{42}$$

for some constant μ. Therefore, if there exists a solution $u(t,x) = T(t)X(x)$ of the wave equation, then $T(t)$ and $X(x)$ must satisfy the equations

$$\frac{T''(t)}{T(t)} = -\mu,$$

$$\frac{\mathcal{L}^s X(x)}{X(x)} = \mu,$$

for some constant μ. In addition, in order for u to satisfy the boundary conditions (38), we need our function X to satisfy the boundary conditions (6). That is, we need to find a function X and a scalar $\mu = \lambda^s$, such that

$$\mathcal{L}^s X(x) = \lambda^s X(x), \tag{43}$$

$$X(0) = X(1) = 0. \tag{44}$$

The equation (43) with the boundary condition (44) has eigenvalues of the form (10) with corresponding eigenfunctions as in (11) of the Sturm-Liouville operator $\mathcal{L}$ generated by the differential expression (5).

Further, we solve the left hand side of the equation (42) with respect to the independent variable t, that is,

$$T''(t) = -\lambda^s T(t), \qquad t \in [0,T]. \tag{45}$$

It is well known ([16]) that the solution of the equation (45) with the initial conditions (37) is

$$T_n(t) = A_n \cos\left(\sqrt{\lambda_n^s}t\right) + \frac{1}{\sqrt{\lambda_n^s}} B_n \sin\left(\sqrt{\lambda_n^s}t\right),$$

where

$$A_n = \int_0^1 u_0(x)\phi_n(x)dx, \qquad B_n = \int_0^1 u_1(x)\phi_n(x)dx.$$

Thus, the solution to (36) with the initial/boundary conditions (37)-(38) has the form

$$u(t,x) = \sum_{n=1}^{\infty}\left[A_n \cos\left(\sqrt{\lambda_n^s}t\right) + \frac{1}{\sqrt{\lambda_n^s}} B_n \sin \sqrt{\lambda_n^s}t\right]\phi_n(x). \tag{46}$$

Let us prove that $u \in C^1([0,T], L^2(0,1))$. By using the Cauchy-Schwarz inequality and fixed t, we can deduce that

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2}^2 &= \int_0^1 |u(t,x)|^2 dx \\
&= \int_0^1 \left| \sum_{n=1}^{\infty} \left[A_n \cos \sqrt{\lambda_n^s} t + \frac{1}{\sqrt{\lambda_n^s}} B_n \sin \sqrt{\lambda_n^s} t \right] \phi_n(x) \right|^2 dx \\
&\lesssim \int_0^1 \sum_{n=1}^{\infty} \left| A_n \cos \sqrt{\lambda_n^s} t + \frac{1}{\sqrt{\lambda_n^s}} B_n \sin \sqrt{\lambda_n^s} t \right|^2 |\phi_n(x)|^2 dx \\
&\leq \int_0^1 \sum_{n=1}^{\infty} \left(|A_n||\phi_n(x)| + \frac{1}{\sqrt{\lambda_n^s}} |B_n||\phi_n(x)| \right)^2 dx \\
&\lesssim \sum_{n=1}^{\infty} \left(\int_0^1 |A_n|^2 |\phi_n(x)|^2 dx + \int_0^1 \left| \frac{B_n}{\sqrt{\lambda_n^s}} \right|^2 |\phi_n(x)|^2 dx \right). \qquad (47)
\end{aligned}$$

By using Parseval's identity, we get

$$\sum_{n=1}^{\infty} \int_0^1 |A_n|^2 |\phi_n(x)|^2 dx = \sum_{n=1}^{\infty} |A_n|^2 = \int_0^1 |u_0(x)|^2 dx = \|u_0\|_{L^2}^2.$$

For the second term in (47), using the properties of the eigenvalues of the operator $\mathcal{L}$ and Parseval's identity again, we obtain the following estimate

$$\begin{aligned}
\sum_{n=1}^{\infty} \int_0^1 \left| \frac{B_n}{\sqrt{\lambda_n^s}} \right|^2 |\phi_n(x)|^2 dx &= \sum_{n=1}^{\infty} \left| \frac{B_n}{\sqrt{\lambda_n^s}} \right|^2 = \sum_{n=1}^{\infty} \left| \int_0^1 \frac{1}{\sqrt{\lambda_n^s}} u_1(x)\phi_n(x) dx \right|^2 \\
&= \sum_{n=1}^{\infty} \left| \int_0^1 \mathcal{L}^{-\frac{s}{2}} u_1(x)\phi_n(x) dx \right|^2 \\
&= \sum_{n=1}^{\infty} \left| \mathcal{L}^{-\frac{s}{2}} u_{1,n} \right|^2 = \|\mathcal{L}^{-\frac{s}{2}} u_1\|_{L_2}^2 = \|u_1\|_{W_{\mathcal{L}}^{-s}}^2. \qquad (48)
\end{aligned}$$

Therefore

$$\|u(t,\cdot)\|_{L^2}^2 \lesssim \|u_0\|_{L^2}^2 + \|u_1\|_{W_{\mathcal{L}}^{-s}}^2.$$

Now, let us estimate $\|\partial_t u(t,\cdot)\|_{L^2}$. We have

$$\begin{aligned}
\|\partial_t u(t,\cdot)\|_{L^2}^2 &= \int_0^1 |\partial_t u(t,x)|^2 dt \\
&= \int_0^1 \left| \sum_{n=1}^{\infty} \left[-\sqrt{\lambda_n^s} A_n \sin\left(\sqrt{\lambda_n^s} t\right) + \frac{1}{\sqrt{\lambda_n^s}} \sqrt{\lambda_n^s} B_n \cos \sqrt{\lambda_n^s} t \right] \phi_n(x) \right|^2 dx
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \int_0^1 \left(\sum_{n=1}^{\infty} |\sqrt{\lambda_n^s} A_n|^2 + \sum_{n=1}^{\infty} |B_n|^2 \right) |\phi_n(x)|^2 dx \\
&= \sum_{n=1}^{\infty} |\sqrt{\lambda_n^s} A_n|^2 + \sum_{n=1}^{\infty} |B_n|^2. \qquad (49)
\end{aligned}
$$

The second term in (49) gives the norm of $\|u_1\|^2_{L^2}$ by Parseval's identity. Now, since λ_n are eigenvalues and $\phi_n(x)$ are eigenfunctions of the operator $\mathcal{L}$, we have that

$$
\begin{aligned}
\sum_{n=1}^{\infty} |\sqrt{\lambda_n^s} A_n|^2 &= \sum_{n=1}^{\infty} \left| \sqrt{\lambda_n^s} \int_0^1 u_0(x)\phi_n(x)dx \right|^2 = \sum_{n=1}^{\infty} \left| \int_0^1 \sqrt{\lambda_n^s} u_0(x)\phi_n(x)dx \right|^2 \\
&= \sum_{n=1}^{\infty} \left| \int_0^1 \mathcal{L}^{\frac{s}{2}} u_0(x)\phi_n(x)dx \right|^2. \qquad (50)
\end{aligned}
$$

It follows from Parseval's identity that

$$
\sum_{n=1}^{\infty} \left| \int_0^1 \mathcal{L}^{\frac{s}{2}} u_0(x)\phi_n(x)dx \right|^2 = \|\mathcal{L}^{\frac{s}{2}} u_0\|^2_{L^2} = \|u_0\|^2_{W^s_{\mathcal{L}}}. \qquad (51)
$$

Thus,

$$
\|\partial_t u(t,\cdot)\|^2_{L^2} \lesssim \|u_0\|^2_{W^s_{\mathcal{L}}} + \|u_1\|^2_{L^2}.
$$

The proof of Theorem 3.1 is then complete. □

In the case when $s = 1$, the above estimates can be expressed in terms of all appearing coefficients. This will be needed later on, when the coefficients and data are singular. Let $s = 1$. Then the equation (36) with initial/boundary conditions (37)-(38) goes to the explicit form

$$
\begin{cases}
\partial_t^2 u(t,x) - \partial_x^2 u(t,x) + q(x)u(t,x) = 0, \quad (t,x) \in [0,T] \times (0,1), \\
u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), \quad x \in (0,1), \\
u(t,0) = u(t,1) = 0, \quad t \in [0,T].
\end{cases} \qquad (52)
$$

Corollary 3.2. *Let $q \in L^\infty(0,1)$ be real, and assume that $u_0 \in L^2(0,1)$ such that $u_0'' \in L^2(0,1)$ and that $u_1 \in L^2(0,1)$. Then the problem* (52) *has a unique solution $u \in C([0,T], L^2(0,1))$ which satisfies the estimates*

$$
\|u(t,\cdot)\|_{L^2} \lesssim \|u_0\|_{L^2} + \|u_1\|_{L^2}, \qquad (53)
$$

and

$$
\|\partial_t u(t,\cdot)\|_{L^2} \lesssim \|u_0''\|_{L^2} + \|q\|_{L^\infty}\|u_0\|_{L^2} + \|u_1\|_{L^2}, \qquad (54)
$$

uniformly in $t \in [0,T]$.

Proof. By using the inequality (47) for $s = 1$, we get

$$
\|u(t,\cdot)\|^2_{L^2} \lesssim \sum_{n=1}^{\infty} \left(\int_0^1 |A_n|^2 |\phi_n(x)|^2 dx + \int_0^1 \left| \frac{B_n}{\sqrt{\lambda_n}} \right|^2 |\phi_n(x)|^2 dx \right). \qquad (55)
$$

According to (10), we have that $\lambda_n > 1$ for $n = 1, \dots,$ thus

$$\int_0^1 \left| \frac{B_n}{\sqrt{\lambda_n}} \right|^2 |\phi_n(x)|^2 dx \leq \int_0^1 |B_n|^2 |\phi_n(x)|^2 dx. \tag{56}$$

By using Parseval's identity for (55) and taking into account the last inequality, we get

$$\|u(t,\cdot)\|_{L^2}^2 \lesssim \sum_{n=1}^{\infty} \left(|A_n|^2 + |B_n|^2 \right) = \|u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2,$$

implying (53). Now, from (49) we have that

$$\|\partial_t u(t,\cdot)\|_{L^2}^2 \lesssim \int_0^1 \left(\sum_{n=1}^{\infty} |\sqrt{\lambda_n} A_n|^2 + \sum_{n=1}^{\infty} |B_n|^2 \right) dx.$$

The second term of this sum gives the norm of $\|u_1\|_{L^2}^2$ by Parseval's identity. Since λ_n, for $n = 1, \dots,$ are eigenvalues of the operator $\mathcal{L}$, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} |\sqrt{\lambda_n} A_n|^2 &= \sum_{n=1}^{\infty} \left| \sqrt{\lambda_n} \int_0^1 u_0(x)\phi_n(x)dx \right|^2 \leq \sum_{n=1}^{\infty} \left| \int_0^1 \lambda_n u_0(x)\phi_n(x)dx \right|^2 \\ &= \sum_{n=1}^{\infty} \left| \int_0^1 \left(-u_0''(x) + q(x)u_0(x) \right) \phi_n(x)dx \right|^2 \\ &\lesssim \sum_{n=1}^{\infty} \left| \int_0^1 u_0''(x)\phi_n(x)dx \right|^2 + \sum_{n=1}^{\infty} \left| \int_0^1 q(x)u_0(x)\phi_n(x)dx \right|^2. \end{aligned}$$

Again, using Parseval's identity for the second term and using that $q \in L^\infty$, we get

$$\sum_{n=1}^{\infty} \left| \int_0^1 q(x)u_0(x)\phi_n(x)dx \right|^2 = \sum_{n=1}^{\infty} \left| \langle (qu_0), \phi_n \rangle \right|^2 = \|qu_0\|_{L^2}^2 \leq \|q\|_{L^\infty}^2 \|u_0\|_{L^2}^2.$$

Similarly for the first term, we get

$$\sum_{n=1}^{\infty} \left| \int_0^1 u_0''(x)\phi_n(x)dx \right|^2 = \sum_{n=1}^{\infty} |u_{0,n}''|^2 = \|u_0''\|_{L^2}^2,$$

therefore

$$\sum_{n=1}^{\infty} |\sqrt{\lambda_n} A_n|^2 \lesssim \|u_0''\|_{L^2}^2 + \|q\|_{L^\infty}^2 \|u_0\|_{L^2}^2. \tag{57}$$

Thus,

$$\|\partial_t u(t,\cdot)\|_{L^2}^2 \lesssim \|u_0''\|_{L^2}^2 + \|q\|_{L^\infty}^2 \|u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2.$$

This completes the proof. □

Now we consider the case when $a, b \not\equiv 0$. That is, we consider the problem

$$\begin{cases} \partial_t^2 u(t,x) + \mathcal{L}^s u(t,x) + a(x)u(t,x) + b(x)u_t(t,x) = 0, \quad (t,x) \in [0,T] \times (0,1), \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \\ u(t,0) = u(t,1) = 0, \quad t \in [0,T]. \end{cases} \tag{58}$$

Theorem 3.3. *Let $T > 0$ and $s \geq 0$. Assume $a, b \in L^\infty(0,1)$ to be non-negative, and $q \in L^\infty(0,1)$ is real, and let $u_0 \in W^s_{\mathcal{L}}(0,1)$ and $u_1 \in L^2(0,1)$. Then, there exists a unique solution $u \in C([0,T]; W^s_{\mathcal{L}}(0,1)) \cap C^1([0,T]; L^2(0,1))$ to the problem* (58) *and it satisfies the estimates*

$$\begin{aligned} &\left\{ \|u(t,\cdot)\|_{L^2}, \|\mathcal{L}^{\frac{s}{2}} u(t,\cdot)\|_{L^2}, \|u_t(t,\cdot)\|_{L^2} \right\} \lesssim \\ &\qquad (1 + \|a\|_{L^\infty} + \|b\|_{L^\infty}) \Big[\|u_0\|_{W^s_{\mathcal{L}}} + \|u_1\|_{L^2} \Big]. \end{aligned} \tag{59}$$

Proof. By multiplying the equation in (58) by u_t and integrating with respect to the variable x over $[0,1]$, we get

$$\langle u_{tt}(t,\cdot), u_t(t,\cdot)\rangle_{L^2} + \langle \mathcal{L}^s u(t,\cdot), u_t(t,\cdot)\rangle_{L^2} + \langle a(\cdot)u(t,\cdot), u_t(t,\cdot)\rangle_{L^2} + \langle b(\cdot)u_t(t,\cdot), u_t(t,\cdot)\rangle_{L^2} = 0. \tag{60}$$

It is easy to see that

$$\langle u_{tt}(t,\cdot), u_t(t,\cdot)\rangle_{L^2} = \frac{1}{2}\partial_t \langle u_t(t,\cdot), u_t(t,\cdot)\rangle_{L^2} = \frac{1}{2}\partial_t \|u_t(t,\cdot)\|^2_{L^2}, \tag{61}$$

and since the fractional Sturm-Liouville operator is self-adjoint (see Proposition 2.2) and by the use of the semi-group property (20), we get

$$\begin{aligned} \langle \mathcal{L}^s u(t,\cdot), u_t(t,\cdot)\rangle_{L^2} &= \frac{1}{2}\partial_t \langle \mathcal{L}^{\frac{s}{2}} u(t,\cdot), \mathcal{L}^{\frac{s}{2}} u_t(t,\cdot)\rangle_{L^2} \\ &= \frac{1}{2}\partial_t \|\mathcal{L}^{\frac{s}{2}} u(t,\cdot)\|^2_{L^2}. \end{aligned} \tag{62}$$

Moreover, we have

$$\langle a(\cdot)u(t,\cdot), u_t(t,\cdot)\rangle_{L^2} = \frac{1}{2}\partial_t \|a^{\frac{1}{2}}(\cdot)u(t,\cdot)\|^2_{L^2}, \tag{63}$$

and

$$\langle b(\cdot)u_t(t,\cdot), u_t(t,\cdot)\rangle_{L^2} = \|b^{\frac{1}{2}}(\cdot)u_t(t,\cdot)\|^2_{L^2}. \tag{64}$$

By substituting all these terms in (60) we arrive at

$$\partial_t \Big[\|u_t(t,\cdot)\|^2_{L^2} + \|\mathcal{L}^{\frac{s}{2}} u(t,\cdot)\|^2_{L^2} + \|a^{\frac{1}{2}}(\cdot)u(t,\cdot)\|^2_{L^2} \Big] = -2\|b^{\frac{1}{2}}(\cdot)u_t(t,\cdot)\|^2_{L^2}. \tag{65}$$

By denoting

$$E(t) := \|u_t(t,\cdot)\|^2_{L^2} + \|\mathcal{L}^{\frac{s}{2}} u(t,\cdot)\|^2_{L^2} + \|a^{\frac{1}{2}}(\cdot)u(t,\cdot)\|^2_{L^2}, \tag{66}$$

it follows that the functional $E(t)$ is decreasing over $[0,1]$ and thus we have $E(t) \leq E(0)$, for all $t \in [0,T]$. We get the estimates

$$\begin{aligned} \|u_t(t,\cdot)\|^2_{L^2} &\lesssim \|u_1\|^2_{L^2} + \|\mathcal{L}^{\frac{s}{2}} u_0\|^2_{L^2} + \|a^{\frac{1}{2}} u_0\|^2_{L^2} \\ &\lesssim \|u_1\|^2_{L^2} + \|u_0\|^2_{W^s_{\mathcal{L}}} + \|a\|_{L^\infty} \|u_0\|^2_{L^2}, \end{aligned} \tag{67}$$

and similarly

$$\|\mathcal{L}^{\frac{s}{2}}u(t,\cdot)\|_{L^2}^2 \lesssim \|u_1\|_{L^2}^2 + \|u_0\|_{W^s_{\mathcal{L}}}^2 + \|a\|_{L^\infty}\|u_0\|_{L^2}^2, \tag{68}$$

and

$$\|a^{\frac{1}{2}}(\cdot)u(t,\cdot)\|_{L^2}^2 \lesssim \|u_1\|_{L^2}^2 + \|u_0\|_{W^s_{\mathcal{L}}}^2 + \|a\|_{L^\infty}\|u_0\|_{L^2}^2. \tag{69}$$

To estimate the solution u, we proceed as follows. We rewrite the equation in (58) as

$$\partial_t^2 u(t,x) + \mathcal{L}^s u(t,x) = f(t,x), \quad (t,x) \in [0,T] \times (0,1), \tag{70}$$

where

$$f(t,x) := -a(x)u(t,x) - b(x)u_t(t,x),$$

and we apply Duhamel's principle. According to Proposition 2.4, the solution to (70) has the representation

$$u(t,x) = w(t,x) + \int_0^t v(t,x;\tau)d\tau, \tag{71}$$

where $w(t,x)$ is the solution to the homogeneous problem

$$\begin{cases} w_{tt}(t,x) + \mathcal{L}^s w(t,x) = 0, \quad (t,x) \in [0,T] \times (0,1), \\ w(0,x) = u_0(x), \quad w_t(0,x) = u_1(x), \quad x \in (0,1), \\ w(t,0) = w(t,1) = 0, \quad t \in [0,T]. \end{cases} \tag{72}$$

and $v(t,x;s)$ solves

$$\begin{cases} v_{tt}(t,x;\tau) + \mathcal{L}^s v(t,x;\tau) = 0, \quad (t,x) \in (\tau,T) \times (0,1), \\ v(\tau,x;\tau) = 0, \; v_t(\tau,x;\tau) = f(\tau,x), \quad x \in (0,1), \\ v(t,0;\tau) = v(t,1;\tau) = 0, \quad t \in [0,T]. \end{cases} \tag{73}$$

Taking the L^2 norm in (71) gives

$$\|u(t,\cdot)\|_{L^2} \leq \|w(t,\cdot)\|_{L^2} + \int_0^t \|v(t,\cdot;\tau)\|_{L^2}d\tau, \tag{74}$$

where we used Minkowski's integral inequality. The terms on the right hand side can be estimated as follows:

$$\|w(t,\cdot)\|_{L^2} \lesssim \|u_0\|_{L^2} + \|u_1\|_{W^{-s}_{\mathcal{L}}}, \tag{75}$$

which follows from (39) since w is the solution to the homogeneous problem and still by (39), the second term is estimated by

$$\begin{aligned} \|v(t,\cdot;\tau)\|_{L^2} &\lesssim \|f(\tau,\cdot)\|_{W^{-s}_{\mathcal{L}}} \\ &\leq \|a(\cdot)u(\tau,\cdot)\|_{W^{-s}_{\mathcal{L}}} + \|b(\cdot)u_t(\tau,\cdot)\|_{W^{-s}_{\mathcal{L}}} \\ &\lesssim \|a(\cdot)u(\tau,\cdot)\|_{L^2} + \|b(\cdot)u_t(\tau,\cdot)\|_{L^2}. \end{aligned} \tag{76}$$

The last inequality follows from the fact that from (67) we have that $u_t \in L^2(0,1)$, and since $\{\phi_n\}_{n=1,\ldots,}$, the family of eigenfunctions is an orthonormal basis in $L^2(0,1)$, then, u_t can be expanded in terms of this basis as

$$u_t(t,x) = \sum_{n=1}^{\infty} \tilde{u}_{t,n}(t)\phi_n(x), \text{ for } (t,x) \in [0,T] \times (0,1), \tag{77}$$

where $\tilde{u}_{t,n} = \langle u_t, \phi_n \rangle_{L^2}$ for $n = 1, \ldots,$. It follows from (77) that $u_t(t, 0) = u_t(t, 1) = 0$ for all $t \in [0, T]$, which allows us to use Proposition 2.3 as $s \geq 0$.
On the one hand we have

$$\begin{aligned} \|a(\cdot)u(\tau,\cdot)\|_{L^2} &\leq \|a\|_{L^\infty}^{\frac{1}{2}} \|a^{\frac{1}{2}}(\cdot)u(\tau,\cdot)\|_{L^2} \\ &\lesssim \|a\|_{L^\infty}^{\frac{1}{2}} \Big[\|u_1\|_{L^2}^2 + \|u_0\|_{W^s_{\mathcal{L}}}^2 + \|a\|_{L^\infty}\|u_0\|_{L^2}^2\Big]^{\frac{1}{2}}, \end{aligned} \tag{78}$$

which comes from (69). On the other hand

$$\begin{aligned} \|b(\cdot)u_t(\tau,\cdot)\|_{L^2} &\leq \|b\|_{L^\infty} \|u_t(\tau,\cdot)\|_{L^2} \\ &\lesssim \|b\|_{L^\infty} \Big[\|u_1\|_{L^2}^2 + \|u_0\|_{W^s_{\mathcal{L}}}^2 + \|a\|_{L^\infty}\|u_0\|_{L^2}^2\Big]^{\frac{1}{2}}. \end{aligned} \tag{79}$$

The latter results from (67). We obtain then

$$\|v(t,\cdot;\tau)\|_{L^2} \lesssim (1 + \|a\|_{L^\infty} + \|b\|_{L^\infty})\Big[\|u_0\|_{L^2} + \|u_0\|_{W^s_{\mathcal{L}}} + \|u_1\|_{L^2}\Big]. \tag{80}$$

We substitute (75) and (80) into (71), we get our estimate for u,

$$\begin{aligned} \|u(t,\cdot)\|_{L^2} &\lesssim (1 + \|a\|_{L^\infty} + \|b\|_{L^\infty})\Big[\|u_0\|_{L^2} + \|u_0\|_{W^s_{\mathcal{L}}} + \|u_1\|_{L^2} + \|u_1\|_{W^{-s}_{\mathcal{L}}}\Big] \\ &\lesssim (1 + \|a\|_{L^\infty} + \|b\|_{L^\infty})\Big[\|u_0\|_{W^s_{\mathcal{L}}} + \|u_1\|_{L^2}\Big]. \end{aligned} \tag{81}$$

The last estimate follows by Proposition 2.3. This completes the proof of the theorem. □

In the case when $s = 1$, the estimates in Theorem 3.3 can be expressed in terms of all coefficients appearing in (58), including the potential q. This removes the dependence of the estimates on $\mathcal{L}$.

Corollary 3.4. *Let $T > 0$. Assume $a, b \in L^\infty(0, 1)$ to be non-negative, and $q \in L^\infty(0, 1)$ is real. Let $u_0 \in L^2(0, 1)$ be such that $u_0'' \in L^2(0, 1)$, and let $u_1 \in L^2(0, 1)$. Then, the problem*

$$\begin{cases} \partial_t^2 u(t,x) + \mathcal{L}u(t,x) + a(x)u(t,x) + b(x)u_t(t,x) = 0, & (t,x) \in [0,T]\times(0,1), \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \\ u(t,0) = u(t,1) = 0, \quad t \in [0,T], \end{cases} \tag{82}$$

has a unique solution $u \in C([0, T]; W^1_{\mathcal{L}}(0, 1)) \cap C^1([0, T]; L^2(0, 1))$ and it satisfies

$$\begin{aligned} &\Big\{\|u(t,\cdot)\|_{L^2}, \|\mathcal{L}^{\frac{1}{2}}u(t,\cdot)\|_{L^2}, \|u_t(t,\cdot)\|_{L^2}\Big\} \lesssim \\ &\qquad (1 + \|q\|_{L^\infty})(1 + \|a\|_{L^\infty})(1 + \|b\|_{L^\infty})\Big[\|u_0\|_{L^2} + \|u_0''\|_{L^2} + \|u_1\|_{L^2}\Big]. \end{aligned} \tag{83}$$

Proof. We consider $s = 1$ and argue similarly as in the proof of Theorem 3.3. Firstly, we get the estimates

$$\begin{aligned} &\Big\{\|u_t(t,\cdot)\|_{L^2}, \|\mathcal{L}^{\frac{1}{2}}u(t,\cdot)\|_{L^2}, \|a^{\frac{1}{2}}(\cdot)u(t,\cdot)\|_{L^2}\Big\} \\ &\qquad \lesssim \|u_1\|_{L^2} + \|u_0\|_{W^1_{\mathcal{L}}} + \|a\|_{L^\infty}^{\frac{1}{2}}\|u_0\|_{L^2} \\ &\qquad \lesssim \|u_1\|_{L^2} + \|u_0''\|_{L^2} + \|q\|_{L^\infty}\|u_0\|_{L^2} + \|a\|_{L^\infty}^{\frac{1}{2}}\|u_0\|_{L^2}, \end{aligned} \tag{84}$$

where we used that for $s = 1$ and by arguing as in (50) and (51), the term $\|u_0\|_{W^1_{\mathcal{L}}}$ can be estimated by

$$\|u_0\|_{W^1_{\mathcal{L}}}^2 = \|\mathcal{L}^{\frac{1}{2}}u_0\|_{L^2}^2 = \sum_{n=1}^{n}\Big|\sqrt{\lambda_n}A_n\Big|^2 \leq \sum_{n=1}^{n}|\lambda_n A_n|^2 = \|\mathcal{L}u_0\|_{L^2}^2$$

$$\leq \left(\|u_0''\|_{L^2} + \|q\|_{L^\infty}\|u_0\|_{L^2}\right)^2,$$

which is valid since $\lambda_n > 1$ for $n = 1, 2, \ldots$, resulting from (10). Once again, to estimate u, we rewrite the equation in (82) as

$$\partial_t^2 u(t,x) + \mathcal{L}u(t,x) = f(t,x), \quad (t,x) \in [0,T] \times (0,1), \tag{85}$$

where

$$f(t,x) := -a(x)u(t,x) - b(x)u_t(t,x),$$

and we apply Duhamel's principle to get the following representation for the solution

$$u(t,x) = w(t,x) + \int_0^t v(t,x;\tau)d\tau, \tag{86}$$

where $w(t,x)$ is the solution to the homogeneous problem

$$\begin{cases} w_{tt}(t,x) + \mathcal{L}w(t,x) = 0, \quad (t,x) \in [0,T] \times (0,1), \\ w(0,x) = u_0(x), \quad w_t(0,x) = u_1(x), \quad x \in (0,1), \\ w(t,0) = w(t,1) = 0, \quad t \in [0,T]. \end{cases} \tag{87}$$

and $v(t,x;s)$ solves

$$\begin{cases} v_{tt}(t,x;\tau) + \mathcal{L}v(t,x;\tau) = 0, \quad (t,x) \in (\tau,T) \times (0,1), \\ v(\tau,x;\tau) = 0, \;\; v_t(\tau,x;\tau) = f(\tau,x), \quad x \in (0,1), \\ v(t,0;\tau) = v(t,1;\tau) = 0, \quad t \in [0,T]. \end{cases} \tag{88}$$

By using the estimate (53) from Corollary 3.2 to estimate w and v in combination with (84) and proceeding as in Theorem 3.3 we easily get

$$\|u(t,\cdot)\|_{L^2} \lesssim (1 + \|q\|_{L^\infty})(1 + \|a\|_{L^\infty})(1 + \|b\|_{L^\infty})\Big[\|u_0\|_{L^2} + \|u_0''\|_{L^2} + \|u_1\|_{L^2}\Big],$$

$$\|\mathcal{L}^{\frac{1}{2}}u(t,\cdot)\|_{L^2} \lesssim (1 + \|q\|_{L^\infty})(1 + \|a\|_{L^\infty})(1 + \|b\|_{L^\infty})\Big[\|u_0\|_{L^2} + \|u_0''\|_{L^2} + \|u_1\|_{L^2}\Big],$$

and

$$\|u_t(t,\cdot)\|_{L^2} \lesssim (1 + \|q\|_{L^\infty})(1 + \|a\|_{L^\infty})(1 + \|b\|_{L^\infty})\Big[\|u_0\|_{L^2} + \|u_0''\|_{L^2} + \|u_1\|_{L^2}\Big],$$

ending the proof. □

4. Very weak well-posedness

For $s \geq 0$ and $T > 0$, we consider the initial/boundary problem

$$\begin{cases} \partial_t^2 u(t,x) + \mathcal{L}^s u(t,x) + a(x)u(t,x) + b(x)u_t(t,x) = 0, \quad (t,x) \in [0,T] \times (0,1), \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \\ u(t,0) = u(t,1) = 0, \quad t \in [0,T]. \end{cases} \tag{89}$$

Now, we want to analyse solutions to (89) for less regular coefficients a, b, q and initial data u_0, u_1 having in mind distributions. To obtain the well-posedness in such cases, we will be using the concept of very weak solutions. To start with, for $\varepsilon \in (0,1]$ we consider families of regularised problems to (89) arising from the regularising nets

$$(a_\varepsilon)_\varepsilon = (a * \psi_\varepsilon)_\varepsilon, \; (b_\varepsilon)_\varepsilon = (b * \psi_\varepsilon)_\varepsilon, \; (q_\varepsilon)_\varepsilon = (q * \psi_\varepsilon)_\varepsilon, \tag{90}$$

and

$$(u_{0,\varepsilon})_\varepsilon = (u_0 * \psi_\varepsilon)_\varepsilon,\ (u_{1,\varepsilon})_\varepsilon = (u_1 * \psi_\varepsilon)_\varepsilon, \tag{91}$$

where $\psi_\varepsilon(x) = \varepsilon^{-1}\psi(x/\varepsilon)$. The function ψ is a Friedrichs-mollifier, i.e. $\psi \in C_0^\infty(\mathbb{R}^d)$, $\psi \geq 0$ and $\int \psi = 1$. We introduce the following definitions.

Definition 4.1 (Moderateness). *Let X be a normed space of functions on $\mathbb{R}$ endowed with the norm $\|\cdot\|_X$.*

1. *A net of functions $(f_\varepsilon)_{\varepsilon\in(0,1]}$ from X is said to be X-moderate, if there exist $N \in \mathbb{N}_0$ such that*

$$\|f_\varepsilon\|_X \lesssim \varepsilon^{-N}, \tag{92}$$

and in particular

2. *a net of functions $(f_\varepsilon)_{\varepsilon\in(0,1]}$ from $L^2(0,1)$ is said to be H^2-moderate, if there exist $N \in \mathbb{N}_0$ such that*

$$\|f_\varepsilon\|_{L^2} + \|f_\varepsilon''\|_{L^2} \lesssim \varepsilon^{-N}. \tag{93}$$

3. *For $T > 0$, $s \geq 0$ and q smooth enough, a net of functions $(u_\varepsilon(\cdot,\cdot))_{\varepsilon\in(0,1]}$ from $C([0,T]; W^s_{\mathcal{L}}(0,1)) \cap C^1([0,T]; L^2(0,1))$ is said to be $C([0,T]; W^s_{\mathcal{L}}(0,1)) \cap C^1([0,T]; L^2(0,1))$-moderate and we shortly write uniformly s-moderate, if there exist $N \in \mathbb{N}_0$ such that*

$$\sup_{t\in[0,T]} \|u_\varepsilon(t,\cdot)\|_s \lesssim \varepsilon^{-N}. \tag{94}$$

4. *For $T > 0$ and $s = 1$. A net of functions $(u_\varepsilon(\cdot,\cdot))_{\varepsilon\in(0,1]}$ from $C([0,T]; W^1_{\mathcal{L}}(0,1)) \cap C^1([0,T]; L^2(0,1))$ is said to be $C([0,T]; W^1_{\mathcal{L}}(0,1)) \cap C^1([0,T]; L^2(0,1))$-moderate and we shortly write uniformly 1-moderate, if there exist $N \in \mathbb{N}_0$ such that*

$$\sup_{t\in[0,T]} \|u_\varepsilon(t,\cdot)\|_1 \lesssim \varepsilon^{-N}. \tag{95}$$

Remark 4.2. *We note that such assumptions are natural for distributional coefficients in the sense that regularisations of distributions are moderate. Precisely, by the structure theorems for distributions (see, e.g. [13], [14]), we know that*

$$\mathcal{D}'(0,1) \subset \{L^p(0,1) - \text{moderate families}\}, \tag{96}$$

for any $p \in [1,\infty)$, which means that a solution to an initial/boundary problem may not exist in the sense of distributions, while it may exist in the set of L^p-moderate functions.

For instance, if we consider $f \in L^2(0,1)$, $f : (0,1) \to \mathbb{C}$. We define

$$\tilde{f} = \begin{cases} f, & \text{on } (0,1), \\ 0, & \text{on } \mathbb{R}\setminus(0,1). \end{cases}$$

We have then $\tilde{f} : \mathbb{R} \to \mathbb{C}$, and $\tilde{f} \in \mathcal{E}'(\mathbb{R})$.

Let $\tilde{f}_\varepsilon = \tilde{f} * \psi_\varepsilon$ be obtained via convolution of $\tilde{f}$ with a mollifying net ψ_ε, where

$$\psi_\varepsilon(x) = \frac{1}{\varepsilon}\psi\left(\frac{x}{\varepsilon}\right), \quad \text{for } \psi \in C_0^\infty(\mathbb{R}),\ \int \psi = 1.$$

Then the regularising net $(\tilde{f}_\varepsilon)$ is L^p-moderate for any $p \in [1,\infty)$, and it approximates f on $(0,1)$:

$$0 \leftarrow \|\tilde{f}_\varepsilon - \tilde{f}\|^p_{L^p(\mathbb{R})} \approx \|\tilde{f}_\varepsilon - f\|^p_{L^p(0,1)} + \|\tilde{f}_\varepsilon\|^p_{L^p(\mathbb{R}\setminus(0,1))}.$$

In order to prove uniqueness of very weak solutions to (89), we will need the following definition.

Definition 4.3 (Negligibility). *Let X be a normed space with the norm $\|\cdot\|_X$. A net of functions $(f_\varepsilon)_{\varepsilon\in(0,1]}$ from X is said to be X-negligible, if the estimate*

$$\|f_\varepsilon\|_X \leq C_k\varepsilon^k, \tag{97}$$

is valid for all $k > 0$, where C_k may depend on k. We shortly write $\|f_\varepsilon\|_X \lesssim \varepsilon^k$. In particular, a net a functions $(f_\varepsilon)_{\varepsilon\in(0,1]}$ from L^2 such that $(f''_\varepsilon)_{\varepsilon\in(0,1]} \in L^2(0,1)$ is said to be H^2-negligible, if the estimate

$$\|f_\varepsilon\|_{L^2} + \|f''_\varepsilon\|_{L^2} \leq C_k\varepsilon^k, \tag{98}$$

is valid for all $k > 0$.

Now, we are ready to introduce the notion of very weak solutions adapted to our problem. We treat two cases. Firstly, we treat the case when $s \geq 0$ and the potential q is smooth enough. We also treat the case when $s = 1$ where q is allowed to be singular.

4.1. Case 1: $s \geq 0$

We should mention that the reason we consider here regular potential q, lies in the fact that the estimates obtained in Theorem 3.3 depend on $\mathcal{L}$.

Definition 4.4 (Very weak solution). *Let $a, b, u_0, u_1 \in \mathcal{D}'(0,1)$ be such that a, b are non-negative (in the sense of their representatives) and assume $q \in L^\infty(0,1)$ is real. A net of functions $(u_\varepsilon)_{\varepsilon\in(0,1]}$ is said to be a very weak solution to the initial/boundary problem* (89) *if there exist non-negative L^∞-moderate regularisations $(a_\varepsilon)_\varepsilon$ and $(b_\varepsilon)_\varepsilon$ of a and b, a $W^s_{\mathcal{L}}(0,1)$-moderate regularisation $(u_{0,\varepsilon})_\varepsilon$ of u_0 and an $L^2(0,1)$-moderate regularisation $(u_{1,\varepsilon})_\varepsilon$ of u_1 such that the family $(u_\varepsilon)_\varepsilon$ solves the ε-dependent problems*

$$\begin{cases} \partial_t^2 u_\varepsilon(t,x) + \mathcal{L}^s u_\varepsilon(t,x) + a_\varepsilon(x)u_\varepsilon(t,x) + b_\varepsilon(x)\partial_t u_\varepsilon(t,x) = 0, \quad (t,x) \in [0,T]\times(0,1), \\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x), \quad \partial_t u_\varepsilon(0,x) = u_{1,\varepsilon}(x), \quad x \in (0,1), \\ u_\varepsilon(t,0) = 0 = u_\varepsilon(t,1), \quad t \in [0,T], \end{cases} \tag{99}$$

for any $\varepsilon \in (0,1]$, and $(u_\varepsilon)_\varepsilon$ is uniformly s-moderate.

Theorem 4.5 (Existence). *Let a, b, u_0, u_1 and q as in Definition 4.4. Then the initial/boundary problem* (89) *has a very weak solution.*

Proof. Since a, b, u_0, u_1 are moderate, then, there exists $N_1, N_2, N_3, N_4 \in \mathbb{N}$, such that

$$\|a_\varepsilon\|_{L^\infty} \lesssim \varepsilon^{-N_1}, \quad \|b_\varepsilon\|_{L^\infty} \lesssim \varepsilon^{-N_2},$$

and

$$\|u_{0,\varepsilon}\|_{W^s_{\mathcal{L}}} \lesssim \varepsilon^{-N_3}, \quad \|u_{1,\varepsilon}\|_{L^2} \lesssim \varepsilon^{-N_4}.$$

Using the estimate (59), we get

$$\|u_\varepsilon(t,\cdot)\|_1 \lesssim \varepsilon^{-\max\{N_1,N_2\}-\max\{N_3,N_4\}},$$

for all $t \in [0,T]$. Thus, $(u_\varepsilon)_\varepsilon$ is uniformly s-moderate and the existence of a very weak solution follows, ending the proof. □

The uniqueness of the very weak solution is proved in the sense of the following definition.

Definition 4.6 (Uniqueness of very weak solutions). *We say that the initial/boundary problem* (89) *has a unique very weak solution, if for any non-negative L^∞-moderate nets $(a_\varepsilon)_\varepsilon$, $(\tilde{a}_\varepsilon)_\varepsilon$, $(b_\varepsilon)_\varepsilon$, $(\tilde{b}_\varepsilon)_\varepsilon$, such that $(a_\varepsilon - \tilde{a}_\varepsilon)_\varepsilon$ and $(b_\varepsilon - \tilde{b}_\varepsilon)_\varepsilon$ are L^∞-negligible; for any $W^s_{\mathcal{L}}$-moderate regularisations $(u_{0,\varepsilon}, \tilde{u}_{0,\varepsilon})_\varepsilon$, such that $(u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})_\varepsilon$ is $W^s_{\mathcal{L}}$-negligible and for any L^2-moderate regularisations $(u_{1,\varepsilon}, \tilde{u}_{1,\varepsilon})_\varepsilon$, such that $(u_{1,\varepsilon} - \tilde{u}_{1,\varepsilon})_\varepsilon$ is L^2-negligible, we have that $(u_\varepsilon - \tilde{u}_\varepsilon)_\varepsilon$ is L^2-negligible for all $t \in [0,T]$, where $(u_\varepsilon)_\varepsilon$ and $(\tilde{u}_\varepsilon)_\varepsilon$ are the families of solutions to the corresponding regularised problems*

$$\begin{cases} \partial_t^2 u_\varepsilon(t,x) + \mathcal{L}^s u_\varepsilon(t,x) + a_\varepsilon(x)u_\varepsilon(t,x) + b_\varepsilon(x)\partial_t u_\varepsilon(t,x) = 0, & (t,x) \in [0,T]\times(0,1), \\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x), \quad \partial_t u_\varepsilon(0,x) = u_{1,\varepsilon}(x), & x \in (0,1), \\ u_\varepsilon(t,0) = 0 = u_\varepsilon(t,1), \quad t \in [0,T], \end{cases} \tag{100}$$

and

$$\begin{cases} \partial_t^2 \tilde{u}_\varepsilon(t,x) + \mathcal{L}^s \tilde{u}_\varepsilon(t,x) + \tilde{a}_\varepsilon(x)\tilde{u}_\varepsilon(t,x) + \tilde{b}_\varepsilon(x)\partial_t \tilde{u}_\varepsilon(t,x) = 0, & (t,x) \in [0,T]\times(0,1), \\ \tilde{u}_\varepsilon(0,x) = \tilde{u}_{0,\varepsilon}(x), \quad \partial_t \tilde{u}_\varepsilon(0,x) = \tilde{u}_{1,\varepsilon}(x), & x \in (0,1), \\ \tilde{u}_\varepsilon(t,0) = 0 = \tilde{u}_\varepsilon(t,1), \quad t \in [0,T], \end{cases} \tag{101}$$

respectively.

Theorem 4.7 (Uniqueness). *Let $a, b, u_0, u_1 \in \mathcal{D}'(0,1)$ be such that a, b are non-negative, and assume $q \in L^\infty(0,1)$ is real. In the background of Theorem 4.5, the very weak solution to the initial/boundary problem* (89) *is unique.*

Proof. Let $(u_\varepsilon)_\varepsilon$ and $(\tilde{u}_\varepsilon)_\varepsilon$ be the nets of solutions to (100) and (101) corresponding to the families of regularised coefficients and initial data $\left(a_\varepsilon, b_\varepsilon, u_{0,\varepsilon}, u_{1,\varepsilon}\right)_\varepsilon$ and $\left(\tilde{a}_\varepsilon, \tilde{b}_\varepsilon, \tilde{u}_{0,\varepsilon}, \tilde{u}_{1,\varepsilon}\right)_\varepsilon$ respectively. Assume that the nets $(a_\varepsilon - \tilde{a}_\varepsilon)_\varepsilon$ and $(b_\varepsilon - \tilde{b}_\varepsilon)_\varepsilon$ are L^∞-negligible; $(u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})_\varepsilon$ is $W^s_{\mathcal{L}}$-negligible and $(u_{1,\varepsilon} - \tilde{u}_{1,\varepsilon})_\varepsilon$ is L^2-negligible. Let us introduce

$$U_\varepsilon(t,x) := u_\varepsilon(t,x) - \tilde{u}_\varepsilon(t,x),$$

then, $U_\varepsilon(t,x)$ is solution to

$$\begin{cases} \partial_t^2 U_\varepsilon(t,x) + \mathcal{L}^s U_\varepsilon(t,x) + a_\varepsilon(x)U_\varepsilon(t,x) + b_\varepsilon(x)\partial_t U_\varepsilon(t,x) = f_\varepsilon(t,x), \\ U_\varepsilon(0,x) = (u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})(x), \quad \partial_t U_\varepsilon(0,x) = (u_{1,\varepsilon} - \tilde{u}_{1,\varepsilon})(x), \\ U_\varepsilon(t,0) = U_\varepsilon(t,1) = 0, \end{cases} \tag{102}$$

for $(t,x) \in [0,T]\times(0,1)$, where

$$f_\varepsilon(t,x) := \left(\tilde{a}_\varepsilon(x) - a_\varepsilon(x)\right)\tilde{u}_\varepsilon(t,x) + \left(\tilde{b}_\varepsilon(x) - b_\varepsilon(x)\right)\partial_t \tilde{u}_\varepsilon(t,x).$$

By using Duhamel's principle, $U_\varepsilon(t,x)$ is given by

$$U_\varepsilon(t,x) = W_\varepsilon(t,x) + \int_0^t V_\varepsilon(t,x;\tau)d\tau, \tag{103}$$

where $W_\varepsilon(t,x)$ is the solution to the problem

$$\begin{cases} \partial_t^2 W_\varepsilon(t,x) + \mathcal{L}^s W_\varepsilon(t,x) + a_\varepsilon(x)W_\varepsilon(t,x) + b_\varepsilon(x)\partial_t W_\varepsilon(t,x) = 0, \\ W_\varepsilon(0,x) = (u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})(x), \quad \partial_t U_\varepsilon(0,x) = (u_{1,\varepsilon} - \tilde{u}_{1,\varepsilon})(x), \\ W_\varepsilon(t,0) = W_\varepsilon(t,1) = 0, \end{cases} \tag{104}$$

for $(t,x) \in [0,T]\times(0,1)$, and $V_\varepsilon(t,x;s)$ solves

$$\begin{cases} \partial_t^2 V_\varepsilon(t,x;\tau) + \mathcal{L}^s V_\varepsilon(t,x;\tau) + a_\varepsilon(x)V_\varepsilon(t,x;\tau) + b_\varepsilon(x)\partial_t V_\varepsilon(t,x;\tau) = 0, \\ V_\varepsilon(\tau,x;\tau) = 0, \quad \partial_t V_\varepsilon(\tau,x;\tau) = f_\varepsilon(\tau,x), \\ V_\varepsilon(t,0;\tau) = V_\varepsilon(t,1;\tau) = 0, \end{cases} \tag{105}$$

for $(t,x) \in [\tau, T] \times (0,1)$ and $s \in [0,T]$. By taking the L^2-norm in both sides in (103) we get

$$\|U_\varepsilon(t,\cdot)\|_{L^2} \leq \|W_\varepsilon(t,\cdot)\|_{L^2} + \int_0^t \|V_\varepsilon(t,\cdot;\tau)\|_{L^2} d\tau. \tag{106}$$

Using (59) to estimate $\|W_\varepsilon(t,\cdot)\|_{L^2}$ and $\|V_\varepsilon(t,\cdot;\tau)\|_{L^2}$, we get

$$\|W_\varepsilon(t,\cdot)\|_{L^2} \lesssim \Big(1 + \|a_\varepsilon\|_{L^\infty} + \|b_\varepsilon\|_{L^\infty}\Big)\Big[\|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{W^s_{\mathcal{L}}} + \|u_{1,\varepsilon} - \tilde{u}_{1,\varepsilon}\|_{L^2}\Big],$$

and

$$\|V_\varepsilon(t,\cdot;\tau)\|_{L^2} \lesssim \Big(1 + \|a_\varepsilon\|_{L^\infty} + \|b_\varepsilon\|_{L^\infty}\Big)\Big[\|f_\varepsilon(\tau,\cdot)\|_{L^2}\Big].$$

It follows from (106) that

$$\|U_\varepsilon(t,\cdot)\|_{L^2} \lesssim \Big(1 + \|a_\varepsilon\|_{L^\infty} + \|b_\varepsilon\|_{L^\infty}\Big)\Big[\|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{W^s_{\mathcal{L}}} + \|u_{1,\varepsilon} - \tilde{u}_{1,\varepsilon}\|_{L^2} + \int_0^T \|f_\varepsilon(\tau,\cdot)\|_{L^2} d\tau\Big], \tag{107}$$

since $t \in [0,T]$. Let us estimate $\|f_\varepsilon(\tau,\cdot)\|_{L^2}$. We have,

$$\begin{aligned}\|f_\varepsilon(\tau,\cdot)\|_{L^2} &\leq \|(\tilde{a}_\varepsilon(\cdot) - a_\varepsilon(\cdot))\tilde{u}_\varepsilon(\tau,\cdot)\|_{L^2} + \|(\tilde{b}_\varepsilon(\cdot) - b_\varepsilon(\cdot))\partial_t\tilde{u}_\varepsilon(\tau,\cdot)\|_{L^2} \\ &\lesssim \|\tilde{a}_\varepsilon - a_\varepsilon\|_{L^\infty}\|\tilde{u}_\varepsilon(\tau,\cdot)\|_{L^2} + \|\tilde{b}_\varepsilon - b_\varepsilon\|_{L^\infty}\|\partial_t\tilde{u}_\varepsilon(\tau,\cdot)\|_{L^2}.\end{aligned} \tag{108}$$

Thus, we get

$$\begin{aligned}\|U_\varepsilon(t,\cdot)\|_{L^2} \lesssim &\Big(1 + \|a_\varepsilon\|_{L^\infty} + \|b_\varepsilon\|_{L^\infty}\Big)\Big[\|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{W^s_{\mathcal{L}}} + \|u_{1,\varepsilon} - \tilde{u}_{1,\varepsilon}\|_{L^2} \\ &+ \|\tilde{a}_\varepsilon - a_\varepsilon\|_{L^\infty}\int_0^T \|\tilde{u}_\varepsilon(\tau,\cdot)\|_{L^2} d\tau + \|\tilde{b}_\varepsilon - b_\varepsilon\|_{L^\infty}\int_0^T \|\partial_t\tilde{u}_\varepsilon(\tau,\cdot)\|_{L^2} d\tau\Big].\end{aligned} \tag{109}$$

Now, using the fact that $(a_\varepsilon)_\varepsilon$ and $(b_\varepsilon)_\varepsilon$ are L^∞-moderate by assumption, and that the net $(\tilde{u}_\varepsilon)_\varepsilon$ is uniformly s-moderate being a very weak solution to (89) this on one hand and from the other hand that the nets $(a_\varepsilon - \tilde{a}_\varepsilon)_\varepsilon$ and $(b_\varepsilon - \tilde{b}_\varepsilon)_\varepsilon$ are L^∞-negligible; $(u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})_\varepsilon$ is $W^s_{\mathcal{L}}$-negligible and $(u_{1,\varepsilon} - \tilde{u}_{1,\varepsilon})_\varepsilon$ is L^2-negligible, it follows from (109) that

$$\|U_\varepsilon(t,\cdot)\|_{L^2} \lesssim \varepsilon^k,$$

for all $k > 0$. This completes the proof. □

4.2. *Case 2: $s = 1$*

In this case, the energy estimates obtained in Theorem 3.3 are expressed in terms of all appearing coefficients and initial data including the potential q as it was shown in Corollary 3.4. this allows us to consider singular potentials. So, the problem to be analysed here is the initial/boundary problem

$$\begin{cases} \partial_t^2 u(t,x) + \mathcal{L}_q u(t,x) + +a(x)u(t,x) + b(x)u_t(t,x) = 0, & (t,x) \in [0,T] \times (0,1), \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in (0,1), \\ u(t,0) = u(t,1) = 0, \quad t \in [0,T], \end{cases} \tag{110}$$

where

$$\mathcal{L}_q u(t,x) := -\partial_x^2 u(t,x) + q(x)u(t,x). \tag{111}$$

Here, the coefficients a, b and the initial data u_0, u_1 together with the potential q are assumed to be distributions on $(0,1)$. Let us first adapt our previous definitions to this case.

Definition 4.8 (Very weak solution). *Let $a, b, q, u_0, u_1 \in \mathcal{D}'(0,1)$ be such that a, b are non-negative and assume that $q \in L^\infty(0,1)$ is real. A net of functions $(u_\varepsilon)_{\varepsilon\in(0,1]}$ is said to be a very weak solution to the initial/boundary problem* (110) *if there exist non-negative L^∞-moderate regularisations $(a_\varepsilon)_\varepsilon$, $(b_\varepsilon)_\varepsilon$ of a, b, and $(q_\varepsilon)_\varepsilon$ of q, an H^2-moderate regularisation $(u_{0,\varepsilon})_\varepsilon$ of u_0 and an L^2-moderate regularisation $(u_{1,\varepsilon})_\varepsilon$ of u_1 such that the family $(u_\varepsilon)_\varepsilon$ solves the ε-dependent problems*

$$\begin{cases} \partial_t^2 u_\varepsilon(t,x) + \mathcal{L}_{q_\varepsilon} u_\varepsilon(t,x) + a_\varepsilon(x) u_\varepsilon(t,x) + b_\varepsilon(x)\partial_t u_\varepsilon(t,x) = 0, & (t,x) \in [0,T]\times(0,1), \\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x), \quad \partial_t u_\varepsilon(0,x) = u_{1,\varepsilon}(x), & x \in (0,1), \\ u_\varepsilon(t,0) = 0 = u_\varepsilon(t,1), & t \in [0,T], \end{cases} \tag{112}$$

for any $\varepsilon \in (0,1]$ and $(u_\varepsilon)_\varepsilon$ is 1-moderate.

Theorem 4.9 (Existence). *Let a, b, q, u_0, u_1 be as in Definition 4.8. Assume that there exist non-negative L^∞-moderate regularisations $(a_\varepsilon)_\varepsilon$, $(b_\varepsilon)_\varepsilon$ of a, b, and $(q_\varepsilon)_\varepsilon$ of q, an H^2-moderate regularisation $(u_{0,\varepsilon})_\varepsilon$ of u_0 and an L^2-moderate regularisation $(u_{1,\varepsilon})_\varepsilon$ of u_1. Then, the initial/boundary problem* (110) *has a very weak solution.*

Proof. Since a, b, q, u_0, u_1 are moderate, this means that there exists $N_1, N_2, N_3, N_4, N_5 \in \mathbb{N}$, such that

$$\|a_\varepsilon\|_{L^\infty} \lesssim \varepsilon^{-N_1}, \quad \|b_\varepsilon\|_{L^\infty} \lesssim \varepsilon^{-N_2}, \quad \|q_\varepsilon\|_{L^\infty} \lesssim \varepsilon^{-N_3},$$

and

$$\|u_{0,\varepsilon}\|_{L^2} + \|u''_{0,\varepsilon}\|_{L^2} \lesssim \varepsilon^{-N_4}, \quad \|u_{1,\varepsilon}\|_{L^2} \lesssim \varepsilon^{-N_5}.$$

From (83), we have

$$\begin{aligned} \|u_\varepsilon(t,\cdot)\|_1 &= \|u_\varepsilon(t,\cdot)\|_{L^2} + \|\mathcal{L}^{\frac{1}{2}} u_\varepsilon(t,\cdot)\|_{L^2} + \|u_{t,\varepsilon}(t,\cdot)\|_{L^2} \\ &\lesssim (1 + \|q_\varepsilon\|_{L^\infty})(1 + \|a_\varepsilon\|_{L^\infty})(1 + \|b_\varepsilon\|_{L^\infty})\Big[\|u_{0,\varepsilon}\|_{L^2} + \|u''_{0,\varepsilon}\|_{L^2} + \|u_{1,\varepsilon}\|_{L^2}\Big] \\ &\lesssim (1 + \varepsilon^{-N_3})(1 + \varepsilon^{-N_1})(1 + \varepsilon^{-N_2})\Big[\varepsilon^{-N_4} + \varepsilon^{-N_5}\Big] \\ &\lesssim \varepsilon^{-\max\{N_1,N_2,N_3\}-\max\{N_4,N_5\}}, \end{aligned}$$

for all $t \in [0,T]$. Thus, $(u_\varepsilon)_\varepsilon$ is C^1-moderate and the existence of a very weak solution follows. □

In order to prove the uniqueness of the very weak solution in the case when $s = 1$, we need to adapt Definition 4.6 to this case. The definition reads,

Definition 4.10 (Uniqueness of very weak solutions). *We say that the initial/boundary problem* (110) *has a unique very weak solution, if for any non-negative L^∞-moderate nets $(a_\varepsilon)_\varepsilon$, $(\tilde{a}_\varepsilon)_\varepsilon$, $(b_\varepsilon)_\varepsilon$, $(\tilde{b}_\varepsilon)_\varepsilon$, and real $(q_\varepsilon)_\varepsilon$, $(\tilde{q}_\varepsilon)_\varepsilon$, such that $(a_\varepsilon - \tilde{a}_\varepsilon)_\varepsilon$, $(b_\varepsilon - \tilde{b}_\varepsilon)_\varepsilon$ and $(q_\varepsilon - \tilde{q}_\varepsilon)_\varepsilon$ are L^∞-negligible; for any H^2-moderate regularisations $(u_{0,\varepsilon}, \tilde{u}_{0,\varepsilon})_\varepsilon$ such that $(u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})_\varepsilon$ is H^2-negligible and for any L^2-moderate regularisations $(u_{1,\varepsilon}, \tilde{u}_{1,\varepsilon})_\varepsilon$, such that $(u_{1,\varepsilon} - \tilde{u}_{1,\varepsilon})_\varepsilon$ is L^2-negligible, we have that $(u_\varepsilon - \tilde{u}_\varepsilon)_\varepsilon$ is L^2-negligible for all $t \in [0,T]$, where $(u_\varepsilon)_\varepsilon$ and $(\tilde{u}_\varepsilon)_\varepsilon$ are the families of solutions to the corresponding regularised problems*

$$\begin{cases} \partial_t^2 u_\varepsilon(t,x) + \mathcal{L}_{q_\varepsilon} u_\varepsilon(t,x) + a_\varepsilon(x) u_\varepsilon(t,x) + b_\varepsilon(x)\partial_t u_\varepsilon(t,x) = 0, & (t,x) \in [0,T]\times(0,1), \\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x), \quad \partial_t u_\varepsilon(0,x) = u_{1,\varepsilon}(x), & x \in (0,1), \\ u_\varepsilon(t,0) = 0 = u_\varepsilon(t,1), & t \in [0,T], \end{cases} \tag{113}$$

and

$$\begin{cases} \partial_t^2 \tilde{u}_\varepsilon(t,x) + \mathcal{L}_{\tilde{q}_\varepsilon} \tilde{u}_\varepsilon(t,x) + \tilde{a}_\varepsilon(x) \tilde{u}_\varepsilon(t,x) + \tilde{b}_\varepsilon(x)\partial_t \tilde{u}_\varepsilon(t,x) = 0, & (t,x) \in [0,T]\times(0,1), \\ \tilde{u}_\varepsilon(0,x) = \tilde{u}_{0,\varepsilon}(x), \quad \partial_t \tilde{u}_\varepsilon(0,x) = \tilde{u}_{1,\varepsilon}(x), & x \in (0,1), \\ \tilde{u}_\varepsilon(t,0) = 0 = \tilde{u}_\varepsilon(t,1), & t \in [0,T], \end{cases} \tag{114}$$

respectively.

Theorem 4.11 (Uniqueness). *Let $a, b, q, u_0, u_1 \in \mathcal{D}'(0,1)$. Under the assumptions of of Theorem 4.9, the very weak solution to the initial/boundary problem* (110) *is unique.*

Proof. Let $(u_\varepsilon)_\varepsilon$ and $(\tilde{u}_\varepsilon)_\varepsilon$ be the nets of solutions to (113) and (114) corresponding to the families of regularised coefficients and initial data $\left(a_\varepsilon, b_\varepsilon, q_\varepsilon, u_{0,\varepsilon}, u_{1,\varepsilon}\right)_\varepsilon$ and $\left(\tilde{a}_\varepsilon, \tilde{b}_\varepsilon, \tilde{q}_\varepsilon, \tilde{u}_{0,\varepsilon}, \tilde{u}_{1,\varepsilon}\right)_\varepsilon$ respectively. Assume that the nets $(a_\varepsilon - \tilde{a}_\varepsilon)_\varepsilon$, $(b_\varepsilon - \tilde{b}_\varepsilon)_\varepsilon$ and $(q_\varepsilon - \tilde{q}_\varepsilon)_\varepsilon$ are L^∞-negligible; $(u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})_\varepsilon$ is H^2-negligible and $(u_{1,\varepsilon} - \tilde{u}_{1,\varepsilon})_\varepsilon$ is L^2-negligible. Then, $(U_\varepsilon(t,x))_\varepsilon := (u_\varepsilon(t,x) - \tilde{u}_\varepsilon(t,x))_\varepsilon$ is solution to

$$\begin{cases} \partial_t^2 U_\varepsilon(t,x) + \mathcal{L}_{q_\varepsilon} U_\varepsilon(t,x) + a_\varepsilon(x) U_\varepsilon(t,x) + b_\varepsilon(x)\partial_t U_\varepsilon(t,x) = f_\varepsilon(t,x), \\ U_\varepsilon(0,x) = (u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})(x), \quad \partial_t U_\varepsilon(0,x) = (u_{1,\varepsilon} - \tilde{u}_{1,\varepsilon})(x), \\ U_\varepsilon(t,0) = U_\varepsilon(t,1) = 0, \end{cases} \tag{115}$$

for $(t,x) \in [0,T] \times (0,1))$, where,

$$f_\varepsilon(t,x) := \left[\left(\tilde{a}_\varepsilon(x) - a_\varepsilon(x)\right) + \left(\tilde{q}_\varepsilon(x) - q_\varepsilon(x)\right)\right]\tilde{u}_\varepsilon(t,x) + \left(\tilde{b}_\varepsilon(x) - b_\varepsilon(x)\right)\partial_t \tilde{u}_\varepsilon(t,x).$$

Thanks to Duhamel's principle, $U_\varepsilon(t,x)$ can be represented as

$$U_\varepsilon(t,x) = W_\varepsilon(t,x) + \int_0^t V_\varepsilon(t,x;\tau)d\tau, \tag{116}$$

where $W_\varepsilon(t,x)$ is the solution to the problem

$$\begin{cases} \partial_t^2 W_\varepsilon(t,x) + \mathcal{L}_{q_\varepsilon} W_\varepsilon(t,x) + a_\varepsilon(x) W_\varepsilon(t,x) + b_\varepsilon(x)\partial_t W_\varepsilon(t,x) = 0, \\ W_\varepsilon(0,x) = (u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})(x), \quad \partial_t U_\varepsilon(0,x) = (u_{1,\varepsilon} - \tilde{u}_{1,\varepsilon})(x), \\ W_\varepsilon(t,0) = W_\varepsilon(t,1) = 0, \end{cases} \tag{117}$$

for $(t,x) \in [0,T] \times (0,1)$, and $V_\varepsilon(t,x;\tau)$ solves

$$\begin{cases} \partial_t^2 V_\varepsilon(t,x;\tau) + \mathcal{L}_{q_\varepsilon} V_\varepsilon(t,x;\tau) + a_\varepsilon(x) V_\varepsilon(t,x;\tau) + b_\varepsilon(x)\partial_t V_\varepsilon(t,x;\tau) = 0, \\ V_\varepsilon(\tau,x;\tau) = 0, \quad \partial_t V_\varepsilon(\tau,x;\tau) = f_\varepsilon(\tau,x), \\ V_\varepsilon(t,0;\tau) = V_\varepsilon(t,1;\tau) = 0, \end{cases} \tag{118}$$

for $(t,x) \in [\tau,T] \times (0,1)$ and $\tau \in [0,T]$. Using the estimate (83) and reasoning similarly as in the proof of Theorem 4.7, we arrive at

$$\|U_\varepsilon(t,\cdot)\|_{L^2} \lesssim \left(1 + \|a_\varepsilon\|_{L^\infty}\right)\left(1 + \|b_\varepsilon\|_{L^\infty}\right)\left(1 + \|b_\varepsilon\|_{L^\infty}\right)\Big[\|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} \tag{119}$$

$$+ \|u''_{0,\varepsilon} - \tilde{u}''_{0,\varepsilon}\|_{L^2} + \|u_{1,\varepsilon} - \tilde{u}_{1,\varepsilon}\|_{L^2} + \int_0^T \|f_\varepsilon(\tau,\cdot)\|_{L^2} d\tau\Big], \tag{120}$$

and we easily show that $\|f_\varepsilon(\tau,\cdot)\|_{L^2}$ can be estimated by

$$\|f_\varepsilon(\tau,\cdot)\|_{L^2} \leq \left(\|\tilde{a}_\varepsilon - a_\varepsilon\|_{L^\infty} + \|\tilde{q}_\varepsilon - q_\varepsilon\|_{L^\infty}\right)\|\tilde{u}_\varepsilon(\tau,\cdot)\|_{L^2} + \|\tilde{b}_\varepsilon - b_\varepsilon\|_{L^\infty}\|\partial_t \tilde{u}_\varepsilon(\tau,\cdot)\|_{L^2}. \tag{121}$$

Combining (119) and (121) and using the moderateness and negligibility assumptions, one can easily see that

$$\|U_\varepsilon(t,\cdot)\|_{L^2} \lesssim \varepsilon^k,$$

for all $k > 0$ and $t \in [0,T]$, showing the uniqueness of the very weak solution. □

5. Consistency with classical theory

We conclude this article with the important question of proving that the classical solutions to the initial/boundary problem (1), as given in Theorem 3.3 and Corollary 3.4, can be recaptured by the very weak solutions as $\varepsilon \to 0$. We prove the following theorems in both cases: the case when $s \geq 0$ and the case when $s = 1$.

Theorem 5.1 (Consistency, case: $s \geq 0$). *Let $T > 0$ and $s \geq 0$. Assume $a, b \in L^\infty(0,1)$ to be non-negative and $q \in L^\infty(0,1)$ is real and let $u_0 \in W^s_{\mathcal{L}}(0,1)$ and $u_1 \in L^2(0,1)$, in such way that a classical solution to (1) exists. Then, for any regularising families $(a_\varepsilon)_\varepsilon$, $(b_\varepsilon)_\varepsilon$ for the equation coefficients, satisfying*

$$\|a_\varepsilon - a\|_{L^\infty} \to 0 \quad \text{and} \quad \|b_\varepsilon - b\|_{L^\infty} \to 0, \text{ as } \varepsilon \to 0, \tag{122}$$

and any regularising families $(u_{0,\varepsilon})_\varepsilon$ and $(u_{1,\varepsilon})_\varepsilon$ for the initial data, satisfying

$$\|u_{0,\varepsilon} - u_0\|_{W^s_{\mathcal{L}}} \to 0 \quad \text{and} \quad \|u_{1,\varepsilon} - u_1\|_{L^2} \to 0 \text{ as } \varepsilon \to 0, \tag{123}$$

the net $(u_\varepsilon)_\varepsilon$ converges to the classical solution of the initial/boundary problem (1) in L^2 as $\varepsilon \to 0$.

Proof. Let u be the classical solution to (1) and $(u_\varepsilon)_\varepsilon$ its very weak solution. Then, for $\varepsilon \in (0,1]$, $U_\varepsilon(t,x) := u_\varepsilon(t,x) - u(t,x)$ is solution to

$$\begin{cases} \partial_t^2 U_\varepsilon(t,x) + \mathcal{L}^s U_\varepsilon(t,x) + a_\varepsilon(x) U_\varepsilon(t,x) + b_\varepsilon(x)\partial_t U_\varepsilon(t,x) = f_\varepsilon(t,x), \\ U_\varepsilon(0,x) = (u_{0,\varepsilon} - u_0)(x), \quad \partial_t U_\varepsilon(0,x) = (u_{1,\varepsilon} - u_1)(x), \\ U_\varepsilon(t,0) = U_\varepsilon(t,1) = 0, \end{cases} \tag{124}$$

where $(t,x) \in [0,T] \times (0,1)$ and

$$f_\varepsilon(t,x) := -\big(a_\varepsilon(x) - a(x)\big)u(t,x) - \big(b_\varepsilon(x) - b(x)\big)\partial_t u(t,x).$$

By arguing as we did in Theorem 4.7, we obtain

$$\begin{aligned} \|U_\varepsilon(t,\cdot)\|_{L^2} \lesssim & \big(1 + \|a_\varepsilon\|_{L^\infty} + \|b_\varepsilon\|_{L^\infty}\big)\Big[\|u_{0,\varepsilon} - u_0\|_{W^s_{\mathcal{L}}} + \|u_{1,\varepsilon} - u_1\|_{L^2} \\ & + \|a_\varepsilon - a\|_{L^\infty} \int_0^T \|u(\tau,\cdot)\|_{L^2} d\tau + \|b_\varepsilon - b\|_{L^\infty} \int_0^T \|\partial_t u(\tau,\cdot)\|_{L^2} d\tau\Big], \end{aligned} \tag{125}$$

uniformly in $t \in [0,T]$. Since

$$\|a_\varepsilon - a\|_{L^\infty} \to 0,\ \|b_\varepsilon - b\|_{L^\infty} \to 0,\ \|u_{0,\varepsilon} - u_0\|_{W^s_{\mathcal{L}}} \to 0 \text{ and } \|u_{1,\varepsilon} - u_1\|_{L^2} \to 0$$

as $\varepsilon \to 0$ by assumption and the terms $\|a_\varepsilon\|_{L^\infty}$, $\|b_\varepsilon\|_{L^\infty}$, $\|u(\tau,\cdot)\|_{L^2}$ and $\|\partial_t u(\tau,\cdot)\|_{L^2}$ are bounded, it follows that

$$\|U_\varepsilon(t,\cdot)\|_{L^2} \to 0, \quad \text{as } \varepsilon \to 0,$$

uniformly in $t \in [0,T]$. This completes the proof of Theorem 5.1. □

In the case when $s = 1$, the consistency theorem reads as following.

Theorem 5.2 (Consistency, case: $s = 1$). *Let $T > 0$. Assume $a, b \in L^\infty(0,1)$ to be non-negative and that $q \in L^\infty(0,1)$ is real. Let $u_0 \in L^2(0,1)$ such that $u_0'' \in L^2(0,1)$ and $u_1 \in L^2(0,1)$, in such way that a classical solution to (1) exists. Then, for any regularising families $(a_\varepsilon)_\varepsilon$, $(b_\varepsilon)_\varepsilon$ and $(q_\varepsilon)_\varepsilon$ for the equation coefficients, satisfying*

$$\|a_\varepsilon - a\|_{L^\infty} \to 0, \quad \|b_\varepsilon - b\|_{L^\infty} \to 0 \quad \text{and } \|q_\varepsilon - q\|_{L^\infty} \to 0, \text{ as } \varepsilon \to 0, \tag{126}$$

and any regularising families $(u_{0,\varepsilon})_\varepsilon$ and $(u_{1,\varepsilon})_\varepsilon$ for the initial data, the net $(u_\varepsilon)_\varepsilon$ converges to the classical solution of the initial/boundary problem (1) in L^2 as $\varepsilon \to 0$.

Proof. For u being the classical solution to (1) and $(u_\varepsilon)_\varepsilon$ the very weak solution, by reasoning as in Theorem 5.1, we obtain

$$\begin{aligned}\|U_\varepsilon(t,\cdot)\|_{L^2} \lesssim& \Big(1+\|a_\varepsilon\|_{L^\infty}\Big)\Big(1+\|b_\varepsilon\|_{L^\infty}\Big)\Big(1+\|b_\varepsilon\|_{L^\infty}\Big)\Big[\|u_{0,\varepsilon}-u_0\|_{L^2}+\|u''_{0,\varepsilon}-u''_0\|_{L^2}\\ &+\|u_{1,\varepsilon}-u_1\|_{L^2}+\Big(\|a_\varepsilon-a\|_{L^\infty}+\|q_\varepsilon-q\|_{L^\infty}\Big)\int_0^T\|u(\tau,\cdot)\|_{L^2}d\tau\\ &+\|b_\varepsilon-b\|_{L^\infty}\int_0^T\|\partial_t u(\tau,\cdot)\|_{L^2}d\tau\Big].\end{aligned}$$

It follows from assumptions and that

$$\|u_{0,\varepsilon}-u_0\|_{L^2}\to 0,\quad \|u''_{0,\varepsilon}-u''_0\|_{L^2}\to 0 \quad\text{and}\quad \|u_{1,\varepsilon}-u_1\|_{L^2}\to 0 \text{ as } \varepsilon\to 0,$$

and u being the classical solution to (1) that

$$\|U_\varepsilon(t,\cdot)\|_{L^2}\to 0,\quad \text{as } \varepsilon\to 0,$$

uniformly in $t\in[0,T]$, ending the proof. □

Declarations

Ethical Approval

Not applicable.

Competing interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Author's contributions

Equal.

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Availability of data and materials

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All of the material is owned by the authors, and no permissions are required.

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