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Common fixed point results via $C_{SA}IA$ using quasi non-expansive mappings in CAT(0) spaces

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Abstract. The paper presents a novel iterative algorithm named $C_{SA}IA$, which aims at finding common fixed points in CAT(0) space using quasi non-expansive mappings along with property (E). Also, we establish lemmas and theorems with the help of this algorithm and have demonstrated $\triangle$-convergence to a common fixed point for two operators D and C in CAT(0) space. Furthermore, the paper includes a numerical example to support the main results.

1. Introduction

In mathematical analysis and applied mathematics, fixed point theory, approximation, and iterative techniques are closely related fields. The idea behind fixed point theory is that, under specific circumstances, a function f will have at least one fixed point, or a point where $f(x) = x$. Furthermore, a common fixed point is defined as a point x where, given appropriate circumstances, $f(x) = g(x) = x$ when two functions f and g are taken into consideration. The idea is fundamental to both pure and practical contexts, particularly when developing numerical techniques for problem-solving. By producing a sequence that converges to the intended solution, iterative techniques, like the Banach contraction principle, are frequently used to approach fixed points. These sequences are particularly useful in computational mathematics and algorithm design because they are built so that the solution gets closer to the fixed point with each successive approximation (See [18–20]). Strong tools for resolving nonlinear equations, evaluating dynamical systems, and optimizing functions in both theoretical and real-world contexts are made possible by the combination of these ideas.

A metric space $(\mathcal{W}, d)$ is a CAT(0) space in which a geodesic path (shortest distance between points) will connect every pair of points, and if each geodesic triangle in $\mathcal{W}$ has a minimum thickness equal to that of its corresponding triangle in Euclidean space. This means for all a', b' points on the sides of the triangle in $\mathcal{W}$

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such that $d(a', b') \leq d(c', d')$ where c', d' are corresponding points on the comparison triangle in Euclidean space. Some examples of CAT(0) space are all pre-Hilbert spaces, hyperbolic spaces $\mathbb{H}^n$ or $\mathbb{R}$-trees.

After being first proposed by Alexandrov in the 1950s, CAT(0) spaces gained recognition when M. Gromov demonstrated that a significant portion of the theory of manifolds with non-positive sectional curvature could be built without employing far more than the condition of CAT(0) [2, 11]. There are various fields in which the CAT(0) space plays a role. Some of them are geometric group theory, non-linear analysis, and fixed point theory (uniformly convex), Banach spaces [3, 4, 12, 14, 15], and also provide a framework for defining $\triangle$-convergence.

2. Preliminaries

Some important features of CAT(0) spaces are included in the following lemma, which will be utilized later to support the main results of this study:

Lemma 2.1. *The characteristics listed below correspond to a given CAT(0) space* $(\mathcal{W}, d)$.

(a) *Consider* $\check{a}$ *and* $\check{b}$ *of* $\mathcal{W}$ *and* $l \in [0, 1]$, *where there exists a unique point* $\check{r} \in [\check{a}, \check{b}]$ *for every* l *such that* $d(\check{a}, \check{r}) = l d(\check{a}, \check{b})$ *and* $d(\check{b}, \check{r}) = (1 - l) d(\check{a}, \check{b})$ *([6]). We continue further by representing this unique point as* $(1 - l)\check{a} \oplus l\check{b}$.

(b) *For* $\check{z}, \check{n}, \check{r}$ *of* $\mathcal{W}$ *and* $l \in [0, 1]$ *one has convex inequality ([6]):*

$$d((1 - l)\check{z} \oplus l\check{n}, \check{r}) \leq (1 - l)d(\check{z}, \check{r}) + l d(\check{n}, \check{r}).$$

(c) *For* $\check{z}, \check{n}, \check{r} \in \mathcal{W}$ *and* $l \in [0, 1]$, *the successive inequality holds ([6])*

$$d((1 - l)\check{z} \oplus l(\check{n}, \check{r})^2 \leq (1 - l)d(\check{z}, \check{r})^2 + l d(\check{n}, \check{r})^2 - l(1 - l)d(\check{z}, \check{n})^2.$$

(d) *Consider a sequence* $\{\check{a}_k\}$ *which is bounded in* $\mathcal{W}$, *then the asymptotic center* $A(\{\check{a}_k\})$ *will consist of only a single point[5]. And in this context, the asymptotic center has already been defined as follows:*

$$A(\{\check{a}_k\}) = \{\check{a} \in \mathcal{W} : C(\check{a}, \{\check{a}_k\}) = C(\{\check{a}_k\})\},$$

where $C(\check{a}, \{\check{a}_k\}) = \lim_{k\to\infty} \sup d(\check{a}, \check{a}_k)$ *and* $C(\{\check{a}_k\}) = inf\{C(\check{a}, \{\check{a}_k\}) : \check{a} \in \mathcal{W}\}$ *is the asymptotic radius of* $\{\check{a}_k\}$.

(e) *Let* $\hat{B} \subseteq \mathcal{W}$ *where* $\hat{B}$ *is convex and closed, then the asymptotic center of* $\{\check{z}_k\} \in \hat{B}$, *which is bounded, will be in* $\hat{B}$ *itself [6, 14].*

Let $\{\check{f}_k\}$ be a subsequence of $\{\check{z}_k\}$ and $\check{z}$ be a unique asymptotic center of every $\{\check{f}_k\}$, then the sequence $\{\check{z}_k\}$ in $\mathcal{W}$ is said to be $\triangle$-convergent to $\check{z} \in \mathcal{W}$, which is expressed as $\triangle - \lim_{k\to\infty} \check{z}_k = \check{z}$[14–16].

The $\triangle$-convergence on a CAT(0) space $(\mathcal{W}, d)$ has the following properties:

Lemma 2.2. *[1] (a) Every sequence in* $\mathcal{W}$ *which are bounded contains of a* $\triangle$*-convergent subsequence.*
(b) The following property is satisfied by every CAT(0) space,

$$\lim_{k\to\infty} \sup d(\check{z}_k, \check{z}) < \lim_{k\to\infty} \sup d(\check{z}_k, q),$$

which is known as Opial property, for any sequence $\{\check{z}_k\} \subset \mathcal{W}$ $\triangle$*-converges to* $\check{z}$ *and* $q \neq \check{z}$.

Definition 2.3. *[1] Consider* $B' \subseteq (Y, d)$ *where* $B' \neq \emptyset$ *and* (Y, d) *is a CAT(0) space. For* $\mu \geq 1$, $C : B' \to \mathcal{W}$ *is said to have the enriched* (E_μ) *property on* Y *if there exists* $\mu \geq 1$ *such that for all* $\check{z}, \check{s} \in Y$,

$$d(\check{z}, C\check{s}) \leq \mu d(\check{z}, C\check{z}) + d(\check{z}, \check{s}).$$

for all $\check{s}, \check{z} \in Y$.

C has property (E) on B' *iff C satisfied the* (E_μ) *property with* $\mu \geq 1$.

In 1967, Diaz and Metcalf [7] provided a notion of quasi non-expansive mapping.

Definition 2.4. *[21] Consider $\bar{B} \subseteq (\mathcal{W}, d)$ where $\bar{B} \neq \emptyset$ and $(\mathcal{W}, d)$ is a CAT(0) space. A mapping $D : \bar{B} \to \bar{B}$ will be a non-expansive mapping iff $d(D\check{i}, D\check{j}) \leq d(\check{i}, \check{j})$, for all $\check{i}, \check{j} \in \bar{B}$. And D will be quasi-non-expansive if $\mathcal{K}(D) \neq \emptyset$, set of fixed points of D and $d(D\check{i}, q) \leq d(\check{i}, q)$, for all $\check{i} \in \bar{B}$ and $q \in \mathcal{K}(D)$.*

An example of not non-expansive mapping but quasi non-expansive was given by Dotson [8] in 1972.

Example 2.5. *Let $\bar{B} = \mathbb{R}^1$ and define a mapping $C : \bar{B} \to \bar{B}$ by:*

$$C\check{z} = \begin{cases} \dfrac{\check{z}}{2}, \ if \check{z} \neq 0 \\ 0, \ if \check{z} = 0 \end{cases}$$

Then C is not non-expansive but quasi non-expansive.

Proposition 2.6. *[9] Consider a mapping C having property (E) on $\bar{B}$, from $\bar{B}$ to $\mathcal{W}$. If there exists some fixed point in C, then C is quasi non-expansive.*

Very recently, Harmonchi[13] proposed a new iteration process by covering Thakur iteration [22], AK iteration [23] and a new iteration process introduced by Piri et *al.* [17], which is faster than Thakur iteration, as follows:

$$\begin{cases} \check{z}_1 \in \bar{B} \\ \check{z}_{k+1} = D(\bar{\xi}_k C(\check{s}_k) + (1 - \bar{\xi}_k) C(\check{r}_k)) \\ \check{s}_k = C(\check{r}_k) \\ \check{r}_k = C(\bar{\vartheta}_k C(\check{v}_k) + (1 - \bar{\vartheta}_k) C(\check{z}_k)) \\ \check{v}_k = (1 - \varphi_k)\check{z}_k + \varphi_k C(\check{z}_k) \end{cases}$$

In 2016, K. Ullah and M. Arshad introduced u_k iteration[23]. In 2022, GI Usurelu, T. Turcanu and M. Postolache further redefine the u_k iteration algorithm for the problem of approximating common fixed points of a pair of mappings $C\&D$ as follows:

(modified- u_k iteration[24]) *Consider a convex set $\bar{B} \neq \emptyset$ and $D, C : \bar{B} \to \bar{B}$ two given operators. For an arbitrary starting point $\check{z}_0 \in \bar{B}$, define the sequence $\{\check{z}_k\}$ repeatedly by*

$$\begin{cases} \check{r}_k = (1 - \xi_k)\check{z}_k + \xi_k C(\check{z}_k) \\ \check{s}_k = D((1 - \xi_k) C\check{z}_k + \xi_k D\check{r}_k) \\ \check{z}_{k+1} = (1 - \eta_k - \varphi_k) D\check{z}_k + \eta_k D\check{s}_k + \varphi_k D\check{r}_k, \end{cases}$$

where $\{\xi_k\}, \{\xi_k\}, \{\varphi_k\}, \{\eta_k\}$ and $\{\eta_k + \varphi_k\} \in \mathbb{R}^n$ in (0,1).

Motivated and inspired by the iterative process of Harmonchi[13] and Usurelu[24], we introduced a new iterative process which is known as $C_{SA}IA$ iteration in CAT(0) space $(\mathcal{W}, d)$ as below:

Two operators $D, C : \bar{B} \to \bar{B}$ where $\bar{B} \subseteq (\mathcal{W}, d)$ where $\bar{B} \neq \emptyset$ is closed convex,

$$\begin{cases} \check{z}_1 \in \bar{B} \\ \check{z}_{k+1} = C(\bar{\xi}_k C(\check{s}_k) \oplus (1 - \bar{\xi}_k) D(\check{r}_k)) \\ \check{s}_k = D((1 - \varphi_k) D(\check{v}_k) \oplus \varphi_k C(\check{r}_k)) \\ \check{r}_k = C(\bar{\vartheta}_k C(\check{v}_k) \oplus (1 - \bar{\vartheta}_k) D(\check{z}_k)) \\ \check{v}_k = (1 - \bar{\zeta}_k)\check{z}_k \oplus \bar{\zeta}_k D(\check{z}_k) \end{cases} \tag{1}$$

where $k \in \mathbb{N}$ where $\{\bar{\xi}_k\}, \{\varphi_k\}, \{\bar{\vartheta}_k\}$ and $\{\bar{\zeta}_k\} \in \mathbb{R}^n$ in (0,1).

3. Main results

In this paper, the $C_{SA}IA$ (1) iteration which was previously described in a CAT(0) space $(\mathcal{W}, d)$ will be studied in terms of its convergence and common fixed point for certain D and C operators which satisfy property (E). And let $\mathcal{K}$ be the set of common fixed points of C & D.

Lemma 3.1. *Consider $\bar{B} \subseteq (\mathcal{W}, d)$ where $\bar{B} \neq \emptyset$ is convex and $(\mathcal{W}, d)$ is complete. Let $D, C : \bar{B} \to \bar{B}$ have property (E) with $\mathcal{K} \neq \emptyset$. Consider a sequence $\{\check{z}_k\}$ defined by* (1). *Then $\lim_{k\to\infty} d(\check{z}_k, q)$ exists, for all $q \in \mathcal{K}$.*

Proof. Here, D and C are quasi non-expansive mappings as both D and C satisfy property (E) with $\mathcal{K}(C) \neq \emptyset$, $\mathcal{K}(D) \neq \emptyset$.

Now, by the convexity property, that is, by (b) of Lemma 2.1 and the quasi non-expansiveness of D, we get

$$\begin{aligned} d(\check{v}_k, q) &= d((1-\bar{\zeta}_k)\check{z}_k \oplus \bar{\zeta}_k D(\check{z}_k), q) \\ &\leq (1-\bar{\zeta}_k) d(\check{z}_k, q) + \bar{\zeta}_k d(D(\check{z}_k), q) \\ &\leq (1-\bar{\zeta}_k) d(\check{z}_k, q) + \bar{\zeta}_k d(\check{z}_k, q) \\ &= d(\check{z}_k, q) \end{aligned} \tag{2}$$

By (2), we continue

$$\begin{aligned} d(\check{r}_k, q) &= d(C(\bar{\vartheta}_k C(\check{v}_k) \oplus (1-\bar{\vartheta}_k) D(\check{z}_k)), q) \\ &\leq d(\bar{\vartheta}_k C(\check{v}_k) \oplus (1-\bar{\vartheta}_k) D(\check{z}_k), q) \\ &\leq \bar{\vartheta}_k d(C(\check{v}_k), q) + (1-\bar{\vartheta}_k) d(D(\check{z}_k), q) \\ &\leq \bar{\vartheta}_k d(\check{v}_k, q) + (1-\bar{\vartheta}_k) d(\check{z}_k, q) \\ &\leq \bar{\vartheta}_k d(\check{z}_k, q) + (1-\bar{\vartheta}_k) d(\check{z}_k, q) \\ &= d(\check{z}_k, q) \end{aligned} \tag{3}$$

By (2) and (3), we get

$$\begin{aligned} d(\check{s}_k, q) &= d(T((1-\varphi_k) D(\check{v}_k) \oplus \varphi_k C(\check{r}_k)), q) \\ &\leq d((1-\varphi_k) D(\check{v}_k) \oplus \varphi_k C(\check{r}_k)), q) \\ &\leq (1-\varphi_k) d(D(\check{v}_k), q) + \varphi_k d(C(\check{r}_k), q) \\ &\leq (1-\varphi_k) d(\check{v}_k, q) + \varphi_k d(\check{r}_k, q)) \\ &\leq (1-\varphi_k) d(\check{z}_k, q) + \varphi_k d(\check{z}_k, q) \\ &= d(\check{z}_k, q) \end{aligned} \tag{4}$$

From (3) and (4), we get

$$\begin{aligned} d(\check{z}_{k+1}, q) &= d(C(\bar{\xi}_k C(\check{s}_k) \oplus (1-\bar{\xi}_k) D(\check{r}_k)), q) \\ &\leq d(\bar{\xi}_k C(\check{s}_k) \oplus (1-\bar{\xi}_k) D(\check{r}_k), q) \\ &\leq \bar{\xi}_k d(C(\check{s}_k), q) + (1-\bar{\xi}_k) d(D(\check{r}_k), q) \\ &\leq \bar{\xi}_k d(\check{s}_k, q) + (1-\bar{\xi}_k) d(\check{r}_k, q) \\ &\leq \bar{\xi}_k d(\check{z}_k, q) + (1-\bar{\xi}_k) d(\check{z}_k, q) \\ &= d(\check{z}_k, q) \end{aligned}$$

According to (**??**), we can conclude

$$d(\check{z}_{k+1}, q) \leq d(\check{z}_k, q). \tag{5}$$

Therefore, $\{d(\check{z}_k, q)\}$ is a nonincreasing sequence that belongs to $\mathbb{R}^n$. Hence, $\{d(\check{z}_k, q)\}$ is convergent as it is nonincreasing and bounded. □

Lemma 3.2. *Consider $\bar{B} \subseteq (\mathcal{W}, d)$ where $\bar{B} \neq \emptyset$ is convex and $(\mathcal{W}, d)$ is complete. Let $D, C : \bar{B} \to \bar{B}$ have property (E) with $\mathcal{K} \neq \emptyset$. Consider a sequence $\{\check{z}_k\}$ defined by (1). Suppose $\{\bar{\xi}_k\}, \{\bar{\vartheta}_k\}, \{\bar{\xi}_k\}$ and $\{\varphi_k\}$ are in order that $0 < h \leq \bar{\xi}_k, \bar{\vartheta}_k, \bar{\xi}_k, \varphi_k \leq g < 1$, while $0 < h, g > 1$, then $\lim_{k\to\infty} d(\check{z}_k, C\check{z}_k) = 0$ and $\lim_{k\to\infty} d(\check{z}_k, D\check{z}_k) = 0$.*

Proof. By previous Lemma 3.1, we got, for each common fixed point s of D and C, there exists $\lim_{k\to\infty} d(\check{z}_k, q)$. Let

$$\lim_{k\to\infty} d(\check{z}_k, q) = e \tag{6}$$

By (2), (3), (4) and (5), we have

$$\begin{aligned} d(\check{z}_{k+1}, q) &\leq \bar{\xi}_k d(\check{s}_k, q) + (1 - \bar{\xi}_k) d(\check{r}_k, q) \\ &\leq \bar{\xi}_k[(1 - \varphi_k) d(\check{v}_k, q) + \varphi_k d(\check{r}_k, q)] + (1 - \bar{\xi}_k) d(\check{r}_k, q) \\ &\leq \bar{\xi}_k[d(\check{v}_k, q) - \varphi_k d(\check{v}_k, q) + \varphi_k d(\check{r}_k, q)] + (1 - \bar{\xi}_k) d(\check{r}_k, q) \\ &\leq \bar{\xi}_k[d(\check{v}_k, q) - \varphi_k d(\check{z}_k, q) + \varphi_k d(\check{z}_k, q)] + (1 - \bar{\xi}_k) d(\check{z}_k, q) \\ &\leq \bar{\xi}_k d(\check{v}_k, q) + (1 - \bar{\xi}_k) d(\check{z}_k, q) \end{aligned}$$

Then,

$$\bar{\xi}_k d(\check{z}_k, q) \leq \bar{\xi}_k d(\check{v}_k, q) + d(\check{z}_k, q) - d(\check{z}_{k+1}, q)$$

which implies

$$\begin{aligned} d(\check{z}_k, q) &\leq d(\check{v}_k, q) + \frac{1}{\bar{\xi}_k}(d(\check{z}_k, q) - d(\check{z}_{k+1}, q)) \\ &\leq d(\check{v}_k, q) + \frac{1}{f}(d(\check{z}_k, q) - d(\check{z}_{k+1}, q)) \end{aligned}$$

Having limit inf in this previous inequality, we get

$$e \leq \liminf_{k\to\infty} d(\check{v}_k, q). \tag{7}$$

Also by (2)

$$\limsup_{k\to\infty} d(\check{v}_k, q) \leq \limsup_{k\to\infty} d(\check{z}_k, q) = e \tag{8}$$

By (7) and (8), we conclude that

$$\lim_{k\to\infty} d(\check{v}_k, q) = e \tag{9}$$

By (c) Lemma 2.1, we can define

$$\begin{aligned} d(\check{v}_k, q)^2 &= d((1 - \bar{\zeta}_k)\check{z}_k \oplus \bar{\zeta}_k D(\check{z}_k), q)^2 \\ &\leq (1 - \bar{\zeta}_k) d(\check{z}_k, q)^2 + \bar{\zeta}_k d(D(\check{z}_k), q)^2 - \bar{\zeta}_k(1 - \bar{\zeta}_k) d(\check{z}_k, D(\check{z}_k))^2 \\ &\leq d(\check{z}_k, q)^2 - \bar{\zeta}_k(1 - \bar{\zeta}_k) d(\check{z}_k, D(\check{z}_k))^2 \end{aligned}$$

This shows that

$$\bar{\zeta}_k(1 - \bar{\zeta}_k) d(\check{z}_k, D(\check{z}_k))^2 \leq d(\check{z}_k, q)^2 - d(\check{v}_k, q)^2.$$

For further, we have

$$\begin{aligned} d(\check{z}_k, D(\check{z}_k))^2 &\leq \frac{1}{\bar{\zeta}_k(1 - \bar{\zeta}_k)}(d(\check{z}_k, q)^2 - d(\check{v}_k, q)^2) \\ &\leq \frac{1}{h(1 - g)}(d(\check{z}_k, q)^2 - d(\check{v}_k, q)^2). \end{aligned}$$

Using (6) and (9), we get

$$\lim_{k\to\infty} d(\check{z}_k, D(\check{z}_k)) = 0.$$

By (1), we get

$$\begin{aligned} d(\check{v}_k, \check{z}_k) &= d((1-\bar{\zeta}_k)\check{z}_k \oplus \bar{\zeta}_k D(\check{z}_k), \check{z}_k) \\ &\leq (1-\bar{\zeta}_k) d(\check{z}_k, \check{z}_k) + \bar{\zeta}_k d(D(\check{z}_k), \check{z}_k) \\ &= \bar{\zeta}_k d(D(\check{z}_k), \check{z}_k) \end{aligned}$$

We also obtain

$$\lim_{k\to\infty} d(\check{z}_k, \check{v}_k) = 0. \tag{10}$$

By quasi non-expansiveness of C and (c) Lemma 2.1, we get

$$\begin{aligned} d(\check{z}_{k+1}, q)^2 &= d(C(\bar{\xi}_k C(\check{s}_k) \oplus (1-\bar{\xi}_k) D(\check{r}_k)), q)^2 \\ &\leq d(\bar{\xi}_k C(\check{s}_k) \oplus (1-\bar{\xi}_k) D(\check{r}_k), q)^2 \\ &\leq \bar{\xi}_k d(C(\check{s}_k), q)^2 + (1-\bar{\xi}_k) d(D(\check{r}_k), q)^2 - \bar{\xi}_k(1-\bar{\xi}_k) d(C(\check{s}_k), D(\check{r}_k))^2 \end{aligned}$$

Again, according to the quasi non-expansiveness of D and C, we say that

$$\bar{\xi}_k(1-\bar{\xi}_k) d(C(\check{s}_k), D(\check{r}_k))^2 \leq \bar{\xi}_k d(\check{s}_k, q)^2 + (1-\bar{\xi}_k) d(\check{r}_k, q)^2 - d(\check{z}_{k+1}, q)^2.$$

With the help of (2) and (3), we get

$$\begin{aligned} h(1-g) d(C(\check{s}_k), D(\check{r}_k))^2 &\leq \bar{\xi}_k d(\check{z}_k, q)^2 + (1-\bar{\xi}_k) d(\check{z}_k, q)^2 - d(\check{z}_{k+1}, q)^2 \\ &= d(\check{z}_k, q)^2 - d(\check{z}_{k+1}, q)^2. \end{aligned} \tag{11}$$

Taking limit in (11), we get

$$\lim_{k\to\infty} d(C(\check{s}_k), D(\check{r}_k)) = 0. \tag{12}$$

Since D has property (E), we have

$$d(\check{z}_k, D(\check{r}_k)) = \mu d(\check{z}_k, D(\check{z}_k)) + d(\check{z}_k, \check{r}_k). \tag{13}$$

By the definition of $C_{SA}IA$ iteration (1) and quasi expansiveness of C and D, we conclude

$$\begin{aligned} d(\check{r}_k, \check{z}_k) &= d(C(\bar{\vartheta}_k C(\check{v}_k) \oplus (1-\bar{\vartheta}_k) D(\check{z}_k), \check{z}_k)) \\ &\leq d(\bar{\vartheta}_k C(\check{v}_k) \oplus (1-\bar{\vartheta}_k) D(\check{z}_k), \check{z}_k) \\ &\leq \bar{\vartheta}_k d(C(\check{v}_k), \check{z}_k) + (1-\bar{\vartheta}_k) d(D(\check{z}_k), \check{z}_k) \\ &\leq \bar{\vartheta}_k d(\check{v}_k, \check{z}_k) + (1-\bar{\vartheta}_k) d(\check{z}_k, \check{z}_k) \\ &\leq \bar{\vartheta}_k d(\check{v}_k, \check{z}_k) \\ &\leq h d(\check{v}_k, \check{z}_k). \end{aligned} \tag{14}$$

From (10), (13) and (14), we get

$$\lim_{k\to\infty} d(\check{z}_k, D(\check{r}_k)) = 0. \tag{15}$$

As

$$d(C(\check{s}_k), \check{z}_k) \leq d(C(\check{s}_k), D(\check{r}_k)) + d(D(\check{r}_k), \check{z}_k)$$

and using (12) and (15), we get

$$\lim_{k\to\infty} d(C(\check{s}_k), \check{z}_k) = 0. \tag{16}$$

Furthermore,

$$d(\check{s}_k, C(\check{s}_k)) \leq d(\check{s}_k, \check{z}_k) + d(\check{z}_k, C(\check{s}_k))$$

and again by $C_{SA}IA$ (1) and quasi non-expansiveness of C and D, we can say that

$$\begin{aligned} d(\check{s}_k, \check{z}_k) &\leq d(C(\varphi_k C(\check{v}_k) \oplus (1-\varphi_k) D(\check{z}_k)), \check{z}_k) \\ &\leq d(C(\varphi_k C(\check{v}_k) + (1-\varphi_k) D(\check{z}_k)), \check{z}_k) \\ &\leq \varphi_k d(\check{v}_k, \check{z}_k) + (1-\varphi_k) d(\check{z}_k, \check{z}_k) \\ &\leq g d(\check{v}_k, \check{z}_k) \end{aligned} \tag{17}$$

using (10), (16) and (17), we get

$$\lim_{k\to\infty} d(\check{s}_k, C(\check{s}_k)) = 0.$$

Again, as operator C has property (E)

$$d(C(\check{z}_k), \check{s}_k)) \leq \mu d(\check{s}_k, C(\check{s}_k)) + d(\check{z}_k, \check{s}_k) \tag{18}$$

Therefore, by using from (17) to (18), we can say

$$\lim_{k\to\infty} d(C(\check{z}_k), \check{s}_k) = 0.$$

At last, we use the following inequality

$$d(\check{z}_k, C(\check{z}_k)) \leq d(\check{z}_k, \check{s}_k)) + d(\check{s}_k, C(\check{z}_k))$$

to obtain that

$$\lim_{k\to\infty} d(\check{z}_k, C(\check{z}_k)) = 0$$

which proves the theorem. □

Theorem 3.3. *Consider $\bar{B} \subseteq (\mathcal{W}, d)$ where $\bar{B} \neq \emptyset$ is closed and convex and $(\mathcal{W}, d)$ is complete. Let $D, C : \bar{B} \to \bar{B}$ have property (E) with $\mathcal{K} \neq \emptyset$. Consider a sequence $\{\check{z}_k\}$ defined by (1). Suppose $\{\bar{\xi}_k\}, \{\bar{\vartheta}_k\}, \{\bar{\xi}_k\}$ and $\{\varphi_k\}$ are in order that $0 < h \leq \bar{\xi}_k, \bar{\vartheta}_k, \bar{\xi}_k, \varphi_k \leq g < 1$, as $0 < h, g > 1$, then $\{\check{z}_k\}$ $\triangle$-converges to a common fixed point s of C and D.*

Proof. By Lemma 3.1, there exists $\lim_{k\to\infty} d(\check{z}_k, q)$, for all $q \in F$. Therefore, $\{\check{z}_k\}$ is a bounded sequence and by Lemma 3.2, the successive equalities hold true:

$$\lim_{k\to\infty} d(\check{z}_k, C(\check{z}_k)) = \lim_{k\to\infty} d(\check{z}_k, D(\check{z}_k)) = 0.$$

Consider a set of all asymptotic centers of all the subsequences $\{\check{f}_k\}$ of $\{\check{z}_k\}$ which is named as $\hat{w}_\triangle(\{\check{z}_k\}) = \cup A(\{\check{f}_k\}$. By (e) Lemma 2.1, $\hat{w}_\triangle(\{\check{z}_k\}) \subseteq \bar{B}$. Initially, we want to prove that $\hat{w}_\triangle(\{\check{z}_k\}) \subseteq \mathcal{K}$. Let $\check{f} \in \hat{w}_\triangle(\{\check{z}_k\})$. Then, there exists a subsequence $\check{f}_k$ of $\check{z}_k$ such that $A(\{\check{f}_k\}) = \{\check{f}\}$.

As it is given, C has property (E), we get

$$d(\check{f}_k, C\check{f}) \leq \mu d(\check{f}_k, C(\check{f}_k)) + d(\check{f}_k, \check{f}).$$

Using the above inequality and since $\lim_{k\to\infty} d(\check{f}_k, C(\check{f}_k)) = 0$, taking limit sup on both sides, we get

$$\limsup_{k\to\infty} d(\check{f}_k, C\check{f}) \leq \limsup_{k\to\infty} d(\check{f}_k, \check{f})$$

Therefore, we get

$$u(C\check{f},\{\check{f}_k\}) \leq u(\check{f},\{\check{f}_k\}).$$

But as $\check{f}$ is the unique asymptotic center of $\{\check{f}_k\}$, which means $\check{f} = C\check{f}$, that is, $\check{f} \in \mathcal{K}(C)$. Thus, we can conclude that $\check{f} \in \mathcal{K}(D)$, hence $\check{f} \in \mathcal{K}$.

Now, we have to show that there exists only one common fixed point in $\hat{w}_\triangle(\{\check{z}_k\})$. Let $\check{f}, \ddot{c} \in \hat{w}_\triangle(\{\check{z}_k\})$ and let $\{\check{f}_k\}$ and $\{\ddot{c}_k\}$ be subsequences of $\{\check{z}_k\}$ such that $A(\{\check{f}_k\}) = \{\check{f}\}$. Similarly $A(\{\ddot{c}_k\}) = \{\ddot{c}\}$. Let us assume that $\check{f} \neq \ddot{c}$. Due to (a) Lemma 2.1, there exists a subsequence $\{\check{f}'_k\}$ which $\triangle$-converges to $\check{f}'_k$ of $\{\check{f}_k\}$. Definitely, $\check{f}' \in \hat{w}_\triangle(\{\check{z}_k\}) \subset \mathcal{K}$. Thus, $\lim_{k\to\infty} d(\check{z}_k,\check{f})$, $\lim_{k\to\infty} d(\check{z}_k,\check{f}')$ will exist definitely, and for further $\lim_{k\to\infty} d(\check{z}_k,\check{f}) = \lim_{k\to\infty} d(\check{f}_k,\check{f}) = \lim_{k\to\infty} d(\check{f}'_k,\check{f})$ and in this same way $\lim_{k\to\infty} d(\check{z}_k,\check{f}') = \lim_{k\to\infty} d(\check{f}_k,\check{f}') = \lim_{k\to\infty} d(\check{f}'_k,\check{f}')$. Considering that $\check{f}' \neq \check{f}$, then the inequality (b) of Lemma 2.2 implies that

$$\limsup_{k\to\infty} d(\check{f}_k,\check{f}') = \limsup_{k\to\infty} d(\check{f}'_k,\check{f}') < \limsup_{k\to\infty} d(\check{f}'_k,\check{f}) = \limsup_{k\to\infty} d(\check{f}_k,\check{f}),$$

that is,

$$u(\check{f}',\{\check{f}_k\}) < u(\check{f},\{\check{f}_k\}),$$

which is not possible. Hence, $\triangle - \lim_{k\to\infty} \check{f}'_k = \check{f}$. Likewise, $\{\ddot{c}_k\}$ contains a subsequence $\{\ddot{c}'_k\}$, that is, $\triangle - \lim_{k\to\infty} \ddot{c}'_k = \ddot{c}$. Again with (b) of Lemma 2.2, we get

$$\begin{aligned}\lim_{k\to\infty} d(\check{z}_k,\ddot{c}) &= \lim_{k\to\infty} d(\ddot{c}'_k,\ddot{c}) < \lim_{k\to\infty} d(\ddot{c}'_k,\check{f}) = \lim_{k\to\infty} d(\check{z}_k,\check{f}) = \lim_{k\to\infty} d(\check{f}'_k,\check{f}) \\ &< \lim_{k\to\infty} d(\check{f}'_k,\ddot{c}) = \lim_{k\to\infty} d(\check{z}_k,\ddot{c}),\end{aligned}$$

which is not a possible inequality. This shows that our assumption $\check{f} \neq \ddot{c}$ is wrong. Therefore, $\check{f} = \ddot{c}$. In the end, $\hat{w}_\triangle(\{\check{z}_k\})$ is a singleton set, and $\check{f}$ is a common fixed point of C and D. This proves the $\triangle$-convergence of $\{\check{z}_k\}$. □

Corollary 3.4. *Consider $\bar{B} \subseteq (\mathcal{W}, d)$ where $\bar{B} \neq \emptyset$ is convex and closed and $(\mathcal{W}, d)$ is complete. Let $C : \bar{B} \to \bar{B}$ has property (E) with $\mathcal{K}(C) \neq \emptyset$. Consider a sequence $\{\check{z}_k\}$ defined by* (1). *Suppose $\{\bar{\xi}_k\}, \{\bar{\vartheta}_k\}, \{\bar{\xi}_k\}$ and $\{\varphi_k\}$ are in order that $0 < h \leq \bar{\xi}_k, \bar{\vartheta}_k, \bar{\xi}_k, \varphi_k \leq g < 1$, as $0 < h, g > 1$, then $\{\check{z}_k\}$ $\triangle$-converges to a fixed point of C.*

Proof. If $D = C$ in Theorem 2.1, then the exact conclusion will be obtained. □

Next, we will use a further condition, that is, (A') condition[1, 10] to prove a strong convergence result.

Definition 3.5. *[1] Let $\hat{B} \subseteq (\mathcal{W}, d)$ where $\hat{B} \neq \emptyset$ and $(\mathcal{W}, d)$ is a CAT(0) space. Let $D, C : \hat{B} \to \hat{B}$ along with $\mathcal{K} \neq \emptyset$ be said to satisfy the condition (A') if there exists a function $l : [0,\infty) \to [0,\infty)$ which is increasing along with $l(0) = 0, l(u) > 0$, for all $u \in (0,\infty)$, that is, either*

$$d(\check{y}, C\check{y}) \geq l(d(\check{y},\mathcal{K}))$$

or

$$d(\check{z}, D\check{y}) \geq l(d(\check{y},\mathcal{K})),$$

for all $\check{y} \in \hat{B}$. If $D = C$, then the above definition will be define with mappings which satisfy (A) condition.

Theorem 3.6. *Consider $\bar{B} \subseteq (\mathcal{W}, d)$ where $\bar{B} \neq \emptyset$ is convex, closed, and bounded and $(\mathcal{W}, d)$ is complete. Let $D, C : \bar{B} \to \bar{B}$, which have the property (E) along with $\mathcal{K} \neq \emptyset$. And consider a sequence $\{\check{z}_k\}$ defined by* (1). *Suppose $\{\bar{\xi}_k\}, \{\bar{\vartheta}_k\}, \{\bar{\xi}_k\}$ and $\{\varphi_k\}$ are in order that $0 < h \leq \bar{\xi}_k, \bar{\vartheta}_k, \bar{\xi}_k, \varphi_k \leq g < 1$, while $0 < h, g > 1$ and, moreover, C&D satisfy the (A') condition, then $\{\check{z}_k\}$ converges strongly to a common fixed point of C and D.*

Proof. By Lemma 3.1, there exists $\lim_{k\to\infty} d(\check{z}_k, \ddot{z})$, for all $\ddot{z} \in \mathcal{K}$. Let $e \geq 0$ be its value. If $e = 0$, the end is clear. Hence, moreover we consider that $e > 0$.

As $d(\check{z}_{k+1}, \ddot{z}) \leq d(\check{z}_k, \ddot{z})$, we get

$$\inf_{\ddot{z}\in\mathcal{K}} d(\check{z}_{k+1}, \ddot{z}) \leq \inf_{\ddot{z}\in\mathcal{K}} d(\check{z}_k, \ddot{z}).$$

Therefore, $d(\check{z}_{k+1}, \mathcal{K}) \leq d(\check{z}_k, \mathcal{K})$ and it is simple to declare that $\lim_{k\to\infty} d(\check{z}_k, \mathcal{K})$ exists. By the (A') condition, whether

$$\lim_{k\to\infty} h(d(\check{z}_k, \mathcal{K})) \leq \lim_{k\to\infty} d(\check{z}_k, C(\check{z}_k)) = 0$$

or

$$\lim_{k\to\infty} h(d(\check{z}_k, \mathcal{K})) \leq \lim_{k\to\infty} d(\check{z}_k, D(\check{z}_k)) = 0.$$

In these two cases, we get $\lim_{k\to\infty} h(d(\check{z}_k, \mathcal{K})) = 0$. Further, since h is a nondecreasing function with $h(0) = 0$, it pursue that $\lim_{k\to\infty} d(\check{z}_k, \mathcal{K}) = 0$. According to this, we able to conclude that for each $\xi > 0$, there exists a $p \in \mathbb{Z}$ such that

$$d(\check{z}_k, \mathcal{K}) < \frac{\xi}{4}, \text{ for all } k \geq p.$$

Starting with, $\inf\{d(\check{z}_p, \hat{v}) : \hat{v} \in \mathcal{K}\} < \frac{\xi}{4}$, so there exists $\ddot{v} \in \mathcal{K}$, that is, $d(\check{z}_s, \ddot{v}) < \frac{\xi}{2}$. Further, for all $a, b \geq p$, we have

$$d(\check{z}_a, \check{v}_b) < d(\check{z}_a, \ddot{v}) + d(\ddot{v}, \check{z}_b) \leq 2d(\check{z}_p, \ddot{v}) < 2.\frac{\xi}{2} = \xi.$$

Therefore, $\{\check{z}_k\}$ is a Cauchy in $\bar{B} \subseteq (\mathcal{W}, d)$, hence it converges to $\check{z} \in \bar{B}$. We get $d(\check{z}, \mathcal{K}) = 0$ since $\lim_{k\to\infty} d(\check{z}_k, \mathcal{K}) = 0$. As $\mathcal{K}$ is closed, that means $\check{z} \in \mathcal{K}$, which concludes the proof. □

Corollary 3.7. *Consider $\bar{B} \subseteq (\mathcal{W}, d)$ where $\bar{B} \neq \emptyset$ is convex, closed and bounded and $(\mathcal{W}, d)$ is complete. Let $C : \bar{B} \to \bar{B}$ which has property (E) with $\mathcal{K}(C) \neq \emptyset$. Consider a sequence $\{\check{z}_k\}$ defined by (1). Suppose $\{\bar{\xi}_k\}, \{\bar{\vartheta}_k\}, \{\bar{\xi}_k\}$ and $\{\varphi_k\}$ are in order that $0 < h \leq \bar{\xi}_k, \bar{\vartheta}_k, \bar{\xi}_k, \varphi_k \leq g < 1$, while $0 < h, g > 1$ and, moreover, C satisfies the (A) condition, then $\{\check{z}_k\}$ converges strongly to a fixed point of C.*

Proof. If $D = C$ in Theorem 2.2, then one can obtain the exact conclusion. □

Example 3.8. *Let $\mathcal{W} = \mathbb{R}^2$ be a CAT(0) space and $\bar{B} = [0, 1]^2$ be a convex bounded and closed subset of $\mathcal{W}$. We define $D, C : [0, 1]^2 \to [0, 1]^2$ as follows (say, $\check{z} = (x, y)$):*

$$C((x, y)) = \begin{cases} \left(\frac{x}{2}, \frac{y}{2}\right), \ if \ (x, y) \in \left[0, \frac{1}{2}\right]^2 \\ (0, 0), \quad otherwise \end{cases}$$

and

$$D((x, y)) = \begin{cases} (x, y), \ if \ (x, y) \in \left[0, \frac{1}{2}\right]^2 \\ (0, 0), \quad otherwise \end{cases}.$$

Here, it is very easy to verify that both C and D are quasi non-expansive (but not non-expansive), and that their common fixed point is $(0, 0)$. They also satisfy property (E). Determine a sequence $\{\check{z}_k\}$ with respect to (1) by initiating from arbitrary $\check{z}_1 = (1, 1) \in \bar{B}$,

$$\begin{aligned}
&\check{z}_1 \in \bar{B} \\
&\check{z}_{k+1} = C(\bar{\xi}_k C(\check{s}_k) + (1 - \bar{\xi}_k) D(\check{r}_k)) \\
&\check{s}_k = C((1 - \varphi_k) D(\check{v}_k) + \varphi_k C(\check{r}_k)) \\
&\check{r}_k = C(\bar{\vartheta}_k C(\check{v}_k) + (1 - \bar{\vartheta}_k) D(\check{z}_k)) \\
&\check{v}_k = (1 - \bar{\zeta}_k)\check{z}_k + \bar{\zeta}_k D(\check{z}_k)
\end{aligned}$$

where $k \in \mathbb{N}$, $\{\bar{\xi}_k\}, \{\varphi_k\}, \{\bar{\vartheta}_k\}$ *and* $\{\bar{\zeta}_k\} \in \mathbb{R}^2$ *such that* $0 < \bar{\xi}_k, \bar{\vartheta}_k, \bar{\xi}_k, \varphi_k, \bar{\zeta}_k < 1$. *Considering* $\bar{\xi}_k = \bar{\vartheta}_k = \bar{\xi}_k = \varphi_k = \bar{\zeta}_k = \frac{1}{2}$. *Next, we create a sequence* $\{\check{z}_k\}$. *Initiating from* $\check{z}_1 = 1$, *we continue*

$$v_1 = \frac{1}{2}z_1 + \frac{1}{2}D(z_1) = \frac{1}{2}(1,1) + \frac{1}{2}\left(\frac{1}{2}, \frac{1}{2}\right) = \left(\frac{3}{4}, \frac{3}{4}\right),$$

$$r_1 = C\left(\frac{1}{2}C(v_1) + \frac{1}{2}D(z_1)\right) = C\left(\frac{1}{2}(0,0) + \frac{1}{2}\left(\frac{1}{2}, \frac{1}{2}\right)\right) = C\left(\frac{1}{4}, \frac{1}{4}\right) = \left(\frac{1}{8}, \frac{1}{8}\right),$$

$$s_1 = D\left(\frac{1}{2}D(v_1) + \frac{1}{2}C(r_1)\right) = D\left(\frac{1}{2}\left(\frac{1}{2}, \frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{16}, \frac{1}{16}\right)\right) = D\left(\left(\frac{9}{32}, \frac{9}{32}\right)\right) = \left(\frac{9}{32}, \frac{9}{32}\right),$$

we get

$$z_2 = C\left(\frac{1}{2}C(s_1) + \frac{1}{2}D(r_1)\right) = C\left(\frac{1}{2}\left(\frac{9}{64}, \frac{9}{64}\right) + \frac{1}{2}\left(\frac{1}{8}, \frac{1}{8}\right)\right) = \left(\frac{41}{1024}, \frac{41}{1024}\right) \approx (0.04, 0.04),$$

$$v_2 = \frac{1}{2}z_2 + \frac{1}{2}D(z_2) = \frac{1}{2} \cdot \left(\frac{41}{1024}, \frac{41}{1024}\right) + \frac{1}{2} \cdot \left(\frac{41}{1024}, \frac{41}{1024}\right) = \left(\frac{41}{1024}, \frac{41}{1024}\right),$$

$$r_2 = C\left(\frac{1}{2}C(v_2) + \frac{1}{2}D(z_2)\right) = C\left(\frac{1}{2} \cdot \left(\frac{41}{2048}, \frac{41}{2048}\right) + \frac{1}{2} \cdot \left(\frac{41}{1024}, \frac{41}{1024}\right)\right) = \left(\frac{123}{8192}, \frac{123}{8192}\right),$$

$$s_2 = D\left(\frac{1}{2}D(v_2) + \frac{1}{2}C(r_2)\right) = D\left(\frac{1}{2}\left(\frac{41}{1024}, \frac{41}{1024}\right) + \frac{1}{2}\left(\frac{123}{16384}, \frac{123}{16384}\right)\right) = \left(\frac{943}{16384}, \frac{943}{16384}\right),$$

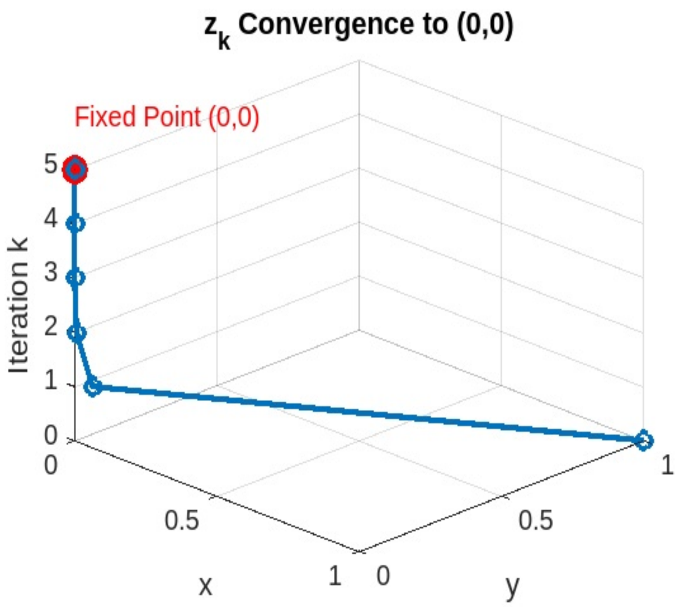
we get

$$z_3 = C\left(\frac{1}{2}C(s_2) + \frac{1}{2}D(r_2)\right) = C\left(\left(\frac{943}{32768} + \frac{123}{8192}\right)\right) = \left(\frac{3619}{65536}, \frac{3619}{65536}\right) \approx (0.055, 0.055).$$

Ongoing in this way, we will obtain $\{\check{z}_k\} \subset \bar{B}$ *which converges to the common fixed point*

$$\check{z}_k \to \check{0} = (0,0) \in \bar{B} \text{ as } k \to \infty.$$

This confirms the convergence of the CS AIA iterative algorithm in R^2 *with piecewise-defined quasi non-expansive mappings C and D, which converges to* $\check{0} \in \bar{B}$ *(as shown in Figure), the common fixed point of C and D.*



4. Conclusion

This paper introduces a new iterative algorithm called $C_{SA}IA$, which is designed for mappings T and S that satisfy property (E) in a CAT(0) space. The algorithm is specifically intended for quasi non-expansive mappings. It is used to find common fixed points of two operators, which is a key result in the field of operator theory. The $C_{SA}IA$ iteration is used for two mappings, and the paper presents two lemmas and two theorems that are obtained by using this iteration. Additionally, the paper provides two corollaries that apply the $C_{SA}IA$ iteration for a single mapping. Furthermore, to support the findings related to the $C_{SA}IA$ iteration, a numerical example in a CAT(0) space is provided.

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