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$h_p(X)$ class of X-valued harmonic functions and applications

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Abstract. The concept of t-basis (generated by the tensor product) from the exponential system $\mathcal{E} = \{e^{int}\}_{n\in\mathbb{Z}}$ is considered for Bochner space $L_p(I_0; X)$, $1 < p < +\infty$, on $I_0 = [-\pi, \pi)$, where X is a Banach space with UMD (Unconditional Martingale Difference) property. We assume that X is endowed with the involution $(*)$. Using the t-basicity of the system $\mathcal{E}$, we introduce the class $h_p^{+;\mathbb{R}}(X)$ of X-valued harmonic functions in the unit ball, generated by involution $(*)$. The $*$-analogues of the Cauchy-Riemann conditions are obtained, and the relations between the class $h_p^{+;\mathbb{R}}(X)$ and the Hardy-Bochner class $H_p(X)$ of analytic functions are established. A new method for establishing X-valued Sokhotski-Plemelj's formulas is presented. Additionally, we establish the correctness of the Dirichlet problem for X-valued harmonic functions in the class $h_p(X)$.

1. Introduction

With applications in various areas of mathematics (e.g., operator theory, partial differential equations, abstract harmonic analysis, stochastic evolution equations, etc.), there is a growing interest in the investigation of X-valued differential equations, and many works have been devoted to this direction (see e.g., the works [1, 2, 4, 5, 13–15, 17, 20, 21], monographs [3, 19] and master's and doctoral theses [22, 23, 25]). Specifically, note that when $X = \mathbb{C}$ (the complex field), these classes are applied in establishing the basis properties (completeness, minimality and basicity) of certain perturbed trigonometric systems, which may be eigenfunctions of second-order differential operators (see, e.g., the works [6–8, 10, 11]). In [12], this approach is developed regarding the Hardy spaces generated by the norm of a Banach Function Space. In studies [9, 16, 26], the analytical properties of solutions to boundary value problems defined in function spaces have been examined.

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The concept of t-basis (generated by the tensor product) from the exponential system $\mathcal{E} = \{e^{int}\}_{n\in\mathbb{Z}}$ is considered for Bochner space $L_p(I_0; X)$, $1 < p < +\infty$, on $I_0 = [-\pi, \pi)$, where X is a Banach space with UMD (Unconditional Martingale Difference) property. We assume that X is endowed with an involution $(*)$. Using the t-basicity of the system $\mathcal{E}$, we introduce the class $h_p^{+;\mathbb{R}}(X)$ of X-valued harmonic functions in the unit ball, generated by involution $(*)$. The $*$-analogues of the Cauchy-Riemann conditions are obtained, and relations between the class $h_p^{+;\mathbb{R}}(X)$ and the Hardy-Bochner class $H_p(X)$ of analytic functions are established. A new method for establishing X-valued Sokhotski-Plemelj's formulas is presented. We also establish the correctness of the Dirichlet problem for X-valued harmonic functions in the class $h_p^{+;\mathbb{R}}(X)$.

2. Notations and auxiliary facts

2.1. Notations

We accept the following notations used in this work. $\mathbb{N}$−positive integers; $\mathbb{Z}$−integers; $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$; $\mathbb{R}$−real numbers; $\mathbb{C}$−complex numbers; $\gamma = \partial\omega = \{z \in \mathbb{C} : |z| = 1\}$; B-space−Banach space; $\omega = \{z \in C : |z| < 1\}$; $\omega^c = \{z \in \mathbb{C} : |z| > 1\}$; $\|\cdot\|_X$−norm in X; $[X; Y]$− B-space of bounded linear operators acting from X to Y; $[X] = [X; X]$; X^*−dual space of X; $\overline{M}$−closure of the set M; $d\sigma$−length element on γ; $\bar{(\cdot)}$−complex conjugation; δ_{ij}−Kronecker's symbol; p'−conjugate to p number: $\frac{1}{p} + \frac{1}{p'} = 1$; $I_0 \equiv [-\pi, \pi)$; $i = \sqrt{-1}$. The symbol $\xrightarrow{\not\tau}$ denotes nontangential convergence.

We use c; C to denote constants whose values can vary in different places. Note that all considered B-spaces here are defined over the field $\mathbb{C}$.

2.2. t-basis properties

Let X, Y, Z be B-spaces and $t : X \times Y \to Z$ be a bilinear operator satisfying the following condition

$$\exists\delta > 0 : \delta\|x\|_X\|y\|_Y \le \|t(x; y)\|_Z \le \delta^{-1}\|x\|_X\|y\|_Y,\ \forall(x; y) \in X \times Y.$$

For simplicity, future presentation accepts the notation $xy := t(x; y)$ for every $(x; y) \in X \times Y$.

We denote t-span of M by $L_t[M]$ for the set $M \subset Y$ and define it as

$$L_t[M] = \left\{z \in Z : \exists\left\{(x_k; y_k)\right\}_1^{n_0} \subset X \times M \Rightarrow z = \sum_{k=1}^{n_0} x_k y_k\right\}.$$

Let $\overrightarrow{y} \equiv \{y_k\}_{k\in\mathbb{N}} \subset Y$ be some system. Accept the following concepts.

System $\overrightarrow{y}$ is t-complete in Z, if $\overline{L_t\left[\overrightarrow{y}\right]} = Z$ (closure is taken in Z).

The system of operators $\{T_n\}_{n\in\mathbb{N}} \subset [Z; X]$ is called t-biorthogonal to $\overrightarrow{y} \subset Y$, if $T_n(xy_k) = x\delta_{nk}$, $\forall x \in X$ & $\forall n, k \in \mathbb{N}$.

The system $\overrightarrow{y} \subset Y$ forms t-basis for Z if $\forall z \in Z$ has a unique expansion in the form

$$z = \sum_{k=1}^{\infty} x_k y_k,$$

with $\{x_k\}_{k\in\mathbb{N}} \subset X$.

We call a triple $(X; Y; Z)$ be t_Y-invariant if $\{(x_k; \tilde{y}_k)\} \subset X \times Y : \sum_k x_k\tilde{y}_k = 0 \Rightarrow \sum_k \vartheta(\tilde{y}_k)x_k = 0$, $\forall\vartheta \in Y^*$.

A triple (X, Y, Z) is t-dense if $\overline{L[X \times Y]} = Z$ (closure is taken in Z).

The following criterion for t-basicity is valid.

Theorem 2.1. *Let the triple $(X; Y; Z)$ be t_Y-invariant and t-dense. Then the system $\overrightarrow{y}$ forms a t-basis for Z if and only if the following assertions hold:*

(i) $\overrightarrow{y}$ is t-complete in Z;

(ii) $\overrightarrow{y}$ *has t-biorthogonal system* $\{T_n\}_{n\in\mathbb{N}} \subset [Z;X]$;

(iii) the projectors $\{P_m\}_{m\in\mathbb{N}}$:

$$P_m(z) = \sum_{n=1}^{m} T_n(z) y_n, \ \forall z \in Z \ \ \& \ \ \forall m \in \mathbb{N},$$

are uniformly bounded, i.e. $\sup_m \|P_m\|_{[Z]} < \infty$.

We consider Z as the some Banach tensor product $X\tilde{\otimes}Y$ of B-spaces X and Y. Denote the algebraic tensor product of X & Y by $X\otimes Y$ and the elementary tensor product of elements $x \in X$ & $y \in Y$ by $x\otimes y$. In this case, it is obvious that the triple $(X;Y;Z)$ is t-dense and t_Y-invariant regarding the bilinear map $t(x,y) = x\otimes y$. Thus, according to the Theorem 2.1 we have the following

Corollary 2.2. *Let* $X;Y$ *be B-spaces and* $Z = X\tilde{\otimes}Y$. *Then the system* $\overrightarrow{y} \subset Y$ *forms t-basis for* Z *if and only if the assertions* $(i)-(iii)$ *of Theorem 2.1 hold.*

2.3. Bochner spaces and UMD spaces

Let $(\mathcal{S},\mathcal{A},\mu)$ be a measure space and X be B-space. As usual, denote by $L_p(\mathcal{S};X)$, $1 \le p < +\infty$, the Bochner space generated by measure space $(\mathcal{S};\mathcal{A};\mu)$ with norm

$$\|f\|_{L_p(\mathcal{S};X)} = \left(\int_{\mathcal{S}} \|f\|_X^p d\mu \right)^{\frac{1}{p}}.$$

The Bochner space $L_p(\gamma;X)$ is defined similarly. We identify the segment I_0 and unit circle γ by mapping $e^{it} : I_0 \to \gamma$. This allows us to identify also the spaces $L_p(I_0;X)$ and $L_p(\gamma;X)$.

We provide the definition of the UMD property and the associated space.

Definition 2.3. A Banach space X is said to have the property of UMD, if for all $p \in (1,\infty)$ there exists a finite constant $\beta \ge 0$ (depending on p and X) such that the following holds: whenever $(\mathcal{S};\mathcal{A};\mu)$ is a σ-finite measure space, $\{\mathcal{F}_n\}_{n=0}^N$ is a σ-finite filtration and $\{f_n\}_{n=0}^N$ is a finite martingale in $L_p(\mathcal{S};X)$, then for all scalar $|\varepsilon_n| = 1$, $n = \overline{1,N}$; we have

$$\left\| \sum_{n=1}^{N} \varepsilon_n df_n \right\|_{L_p(\mathcal{S};X)} \le \beta \left\| \sum_{n=1}^{N} df_n \right\|_{L_p(\mathcal{S};X)},$$

where $df_n = f_n - f_{n-1}$ is a martingale difference.

Let the set of all $B-$spaces that possess the UMD property be denoted by the symbol UMD.

To establish an analogous of the classical Fatou's theorem regarding harmonic functions on ω, we will need the following lemma from the monograph [18] (see p.127, Lemma 2.5.8).

Lemma 2.4. ([18]) *Let* $g \in L_1^{loc}(\mathbb{R};X)$ *and* $a \in \mathbb{R}$ *and define* $f : \mathbb{R} \to X$ *by*

$$f(t) =: \int_a^t g(s)ds.$$

Then the weak derivative ∂f *and almost everywhere derivative* f' *of* f *both exist in* $L_1^{loc}(\mathbb{R};X)$ *and are given by the* $\partial f = f' = g$.

The set of all X-valued trigonometric polynomials $P_n : I_0 \to X$ of the form

$$P_n(t) = \sum_{k=-n}^{n} a_k e^{ikt},$$

with coefficients $\{a_k\} \subset X$, denote by $\mathscr{P}(X)$.

The following proposition is valid.

Proposition 2.5. *Let X be B-space. Then $\overline{\mathscr{P}(X)} = L_p(I_0; X), 1 \leq p < \infty)$, (closure is taken in $L_p(I_0; X)$).*

We define on $\mathscr{P}(X)$ the multiplier operator $m : \mathscr{P}(X) \to L_p(I_0; X)$ by expression

$$(mP)(t) = \tilde{P}(t) = -i \sum_{k\in\mathbb{Z}} sign(k) a_k e^{ikt},$$

where

$$P(t) = \sum_{k\in\mathbb{Z}} a_k e^{ikt} \in \mathscr{P}(X),$$

and

$$sign(k) = \begin{cases} 1, & \text{if } k > 0, \\ 0, & \text{if } k = 0, \\ -1, & \text{if } k < 0. \end{cases}$$

We also consider the subspace $L_p^0(I_0; X)$ of $L_p(I_0; X)$ defined by

$$L_p^0(I_0; X) = \{f \in L_p(I_0; X) : \int_{I_0} f(t)dt = 0\}.$$

Let H be the X-valued Hilbert transform on $\mathbb{R}$:

$$(Hf)(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y} dy, x \in \mathbb{R},$$

defined in a singular sense. The following H-characterization of UMD property is known.

Theorem 2.6. *[Burkholder-Bourgain] Let X be a B-space & $p \in (1, \infty)$. The following assertions are equivalent:*

(1) $X \in UMD$;

(2) $H \in [L_p(\mathbb{R}; X)]$.

In future, we strongly will use the following proposition.

Proposition 2.7. *Let X be a B-space & $p \in (1, \infty)$. If $H \in [L_p(\mathbb{R}; X)]$, then $m \in [L_p^0(I_0; X)]$ & $m \in [L_p(I_0; X)]$.*

Further details regarding these and related results can be found, for example, in the monograph [18].

In what follows, for function $f \in L_1(I_0; X)$, we denote by $\{\hat{f}_k\}_{k\in\mathbb{Z}}$ (also written as $\{T_k(f)\}_{k\in\mathbb{Z}}$) the sequence of its X-valued Fourier coefficients, given by

$$\hat{f}_k := T_k(f) := \frac{1}{2\pi} \int_{I_0} f(t) e^{-ikt} dt, k \in \mathbb{Z}.$$

In work [13], the following theorem is proved.

Theorem 2.8. ([13]) *Let $X \in UMD$ & $p \in (1, \infty)$. Then the exponential system $\mathcal{E}$ forms t-basis for $L_p(I_0; X)$, i.e. $\forall f \in L_p(I_0; X)$ has a unique expansion in the form*

$$f(t) = \sum_{n \in \mathbb{Z}} \hat{f}_n e^{int}, \tag{2.1}$$

in $L_p(I_0; X)$. Moreover, for $\forall m \in \mathbb{Z}$, the following series

$$(R_m^+ f)(t) = f_+(t) = \sum_{n=m}^{\infty} \hat{f}_n e^{int},$$
$$(R_m^- f)(t) = f_-(t) = \sum_{n=-\infty}^{m-1} \hat{f}_n e^{int},$$

also converges in $L_p(I_0; X)$ (so-called t-Riesz property) and $R_m^{\pm} \in [L_p(I_0; X)]$.

3. Main results

3.1. $\mathcal{H}(X)$ and $\mathcal{A}(X)$ classes

We firstly define X-valued harmonic function in ω. Let X be B-space. For $z \in \omega$ define the following limits

$$\partial_x f(z) := \lim_{\mathbb{R} \ni h \to 0} \frac{f(z+h) - f(z)}{h},$$
$$\partial_y f(z) := \lim_{\mathbb{R} \ni h \to 0} \frac{f(z+ih) - f(z)}{h}.$$

Assume

$$C^1(\omega; X) = \{f : \omega \to X : \partial_x f; \partial_y f \in C(\omega; X)\},$$

where $C(\omega; X)$ is the set of all continuous X-valued functions, defined on ω. Analogously, define

$$C^2(\omega; X) = \{f : \omega \to X : \partial_{xx} f; \partial_{xy} f; \partial_{yy} f \in C(\omega; X)\}.$$

Let

$$\Delta f(z) = \partial_{xx} f(z) + \partial_{yy} f(z),$$

where $z = x + iy$, and accept

$$\mathcal{H}(X) = \mathcal{H}(\omega; X) = \{f \in C^2(\omega; X) : \Delta f(z) = 0, \forall z \in \omega\}.$$

Further, we will consider the case, when X endowed with involution, i.e. $\exists * : X^* \to X$, with the following properties:

(i) $* : X \leftrightarrow X$ is bijective and $\|w^*\|_X = \|w\|_X, \forall w \in X$;

(ii) $w^{**} = (w^*)^* = w, \forall w \in X$;

(iii) $(\lambda w)^* = \overline{\lambda} w^*, \forall \lambda \in \mathbb{C}, \forall w \in X$.

We refer to the elements of

$$X^{\mathbb{R}} = \{\mathrm{w} \in X : \mathrm{w}^* = \mathrm{w}\},$$

as $*$-real, and the elements of

$$X^{i\mathbb{R}} = iX^{\mathbb{R}} = \{\mathrm{w} \in X : \mathrm{w} = iv, v \in X^{\mathbb{R}}\},$$

as purely $*$-imaginary.

Thus, it is evident that for $\forall \mathrm{w} \in X : u = \frac{\mathrm{w}+\mathrm{w}^*}{2} \in X^{\mathbb{R}}$ and $v = \frac{\mathrm{w}-\mathrm{w}^*}{2i} \in X^{\mathbb{R}}$, and in result w has a representation $\mathrm{w} = u + iv$ with $u; v \in X^{\mathbb{R}}$. Moreover, such representation is unique.

In reality, let $u_1 + iv_1 = u_2 + iv_2$, where $u_k; v_k \in X^{\mathbb{R}}, k = 1, 2$. Consequently, we have $u = u_1 - u_2 = i(v_2 - v_1) = iv$, where $u; v \in X^R$. Thus

$$u^* = u = (iv)^* = -iv^* = -iv = iv \Longrightarrow v = 0 \Longrightarrow u = 0 \Longrightarrow u_1 = u_2 \ \& \ v_1 = v_2.$$

Therefore we obtain that the following direct sum is true.

$$X = X^{\mathbb{R}} \dotplus iX^{\mathbb{R}}. \tag{3.1}$$

It is evident that $X^{\mathbb{R}}$ is the closure under the norm $\|\cdot\|_X$. Moreover, it is a real linear space and consequently, $X^{\mathbb{R}}$ is a B-space with the norm $\|\cdot\|_X$ over the field $\mathbb{R}$. We define the norm in X

$$\|\mathrm{w}\|_X^{(1)} = \sqrt{\|u\|_X^2 + \|v\|_X^2}, \quad \mathrm{w} = u + iv,$$

with $u; v \in X^{\mathbb{R}}$. It is not hard to see that X with the norm $\|\cdot\|_X^{(1)}$ is also B-space. Moreover, the following inequality holds

$$\|\mathrm{w}\|_X \leq \|u\|_X + \|v\|_X \leq \sqrt{2}\sqrt{\|u\|_X^2 + \|v\|_X^2} = \sqrt{2}\|\mathrm{w}\|_X^{(1)}, \ \ \forall \mathrm{w} \in X.$$

Then, from Banach's Theorem it follows that the norms $\|.\|_X$ and $\|.\|_X^{(1)}$ are equivalent in X, i.e.

$$\exists \delta > 0 : \delta\|\mathrm{w}\|_X^{(1)} \leq \|\mathrm{w}\|_X \leq \delta^{-1}\|\mathrm{w}\|_X^{(1)}, \ \ \forall \mathrm{w} \in X.$$

According to the classical case accept notations $u = Re^*\mathrm{w}$ and $v = Im^*\mathrm{w}$. So, it's evident that $\mathrm{w} = Re^*\mathrm{w} + iIm^*\mathrm{w}$ and $\mathrm{w} = 0 \iff Re^*\mathrm{w}; Im^*\mathrm{w} = 0$. $\mathrm{w}^* = u - iv$ holds.

Now, let $\mathrm{w} \in \mathcal{H}(\omega; X)$ be X-valued harmonic function. Then it is not hard to see that $Re^*\mathrm{w}$ and $Im^*\mathrm{w}$ are X-valued harmonic functions and the class of all such functions is denoted by $\mathcal{H}^{\mathbb{R}}(\omega; X)$. Thus, the following direct sum holds.

$$\mathcal{H}(\omega; X) = \mathcal{H}^{\mathbb{R}}(\omega; X) \dotplus i\mathcal{H}^{\mathbb{R}}(\omega; X).$$

For future presentation, we also need the class $\mathcal{A}(\omega; X)$ of all X-valued analytic functions on ω. In other words, if $f \in \mathcal{A}(\omega; X)$ then there exists continuous limit

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z},$$

at every point $z \in \omega$.

As usual, we define weak cases of these concepts. Namely, the function $\mathrm{w} : \omega \to X$ is said to be weakly harmonic if $x^*(\mathrm{w})$ is a harmonic function (in general, a complex-valued harmonic function) on ω for every $x^* \in X^*$. Analogously, weak analyticity is defined. The corresponding classes are denoted by $\mathcal{H}^{\mathrm{w}}(\omega; X)$ and $\mathcal{A}^{\mathrm{w}}(\omega; X)$. It is well known that $\mathcal{A}(\omega; X) = \mathcal{A}^{\mathrm{w}}(\omega; X)$.Moreover, the equality $\mathcal{H}(\omega; X) = \mathcal{H}^{\mathrm{w}}(\omega; X)$ also holds (see, e.g., [2]). Let the real part (induced by the involution $*$) of the class $\mathcal{A}(\omega; X)$ be denoted by $\mathcal{A}^{\mathbb{R}}(\omega; X)$. The following relations hold: $\mathcal{H}^{\mathbb{R}}(\omega; X) \subset \mathcal{H}(\omega; X)$ and $\mathcal{A}^{\mathbb{R}}(\omega; X) \subset \mathcal{A}(\omega; X)$.

Let $f = u + iv$, with $u; v \in \mathcal{A}^{\mathbb{R}}(\omega; X)$. For $z = x + iy \in \omega$ and $\Delta z = \Delta x + i\Delta y$, completely analogously to the scalar case, we have

$$f'(z) = \partial_x u + i\partial_x v = \frac{1}{i}(\partial_y u + i\partial_y v) \Rightarrow$$

$$\left.\begin{array}{l}\partial_x u = \partial_y v,\\ \partial_x v = -\partial_y u.\end{array}\right\} \tag{3.2}$$

We will call the function v as the $*$-conjugation to the function u. The conditions (3.2) are the X-valued analogues of the Cauchy-Riemann conditions regarding B-space X with the involution operation $(*)$. From (3.1), it directly follows that $u; v \in \mathcal{H}(\omega; X)$, and as a result, $f \in \mathcal{H}(\omega; X)$. Conversely, let $\mathrm{w} = u + iv$ and $u; v \in \mathcal{H}^{\mathbb{R}}(\omega; X)$. Then, from the results of the work [2], it follows that u and v are real analytic functions on ω, i.e., they have power series expansions at every point $z \in \omega$ with coefficients from $X^{\mathbb{R}}$. Thus, if X-valued Cauchy-Riemann conditions (3.2) hold, then completely analogously to the classical case, it is proved that the function $\mathrm{w}(z)$ is differentiable at $\forall z \in \omega$, i.e. there exists

$$\lim_{\Delta z \to 0} \frac{\mathrm{w}(z + \Delta z) - \mathrm{w}(z)}{\Delta z} = \mathrm{w}'(z) \Longrightarrow \mathrm{w} \in \mathcal{A}(\omega; X).$$

It is evident that $\mathcal{A}(\omega; X) \subset \mathcal{H}(\omega; X)$ and $\mathcal{A}^{\mathbb{R}}(\omega; X) \subset \mathcal{H}^{\mathbb{R}}(\omega; X)$. If the function $\mathrm{w} = u + iv \in \mathcal{H}(\omega; X)$ does not satisfy the conditions (3.2), then $\mathrm{w} \notin A(\omega; X)$ and as a result, it is obvious that

$$\mathcal{H}(\omega; X) \setminus \mathcal{A}(\omega; X) \neq \emptyset.$$

Let $u \in \mathcal{H}^{\mathbb{R}}(\omega; X)$. According to the scalar case, consider the following X-valued integral

$$v(x; y) = \int_{(x_0; y_0)}^{(x; y)} -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy + v_0, \tag{3.3}$$

where $(x_0; y_0) \in \omega$ be a fixed point, $v_0 \in X^{\mathbb{R}}$ arbitrary constant and the integral is taken over any smooth curve connecting in ω the points $(x_0; y_0)$ and $(x; y) \in \omega$. Since $u \in C^{\infty}(\omega)$ (see e.g., [2]), then it follows immediately from (3.3) that $v \in C^{\infty}(\omega)$. Moreover the following relations hold

$$\left.\begin{array}{l}\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y},\\ \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}.\end{array}\right\} \tag{3.4}$$

As a result, $\Delta v = 0$. It is evident that $v \in \mathcal{H}^{\mathbb{R}}(\omega; X)$. Set $f(z) = u(z) + iv(z),\ for\ z \in \omega$. From (3.4), it follows that $f \in \mathcal{A}(\omega; X)$ and consequently, $u; v \in \mathcal{A}^{\mathbb{R}}(\omega; X)$. Therefore, we obtain the identity $\mathcal{A}^{\mathbb{R}}(\omega; X) \equiv \mathcal{H}^{\mathbb{R}}(\omega; X)$.

Set

$$\mathcal{H}_0^{\mathbb{R}}(\omega; X) = \{v \in \mathcal{H}^{\mathbb{R}}(\omega; X) : v(0) = 0\},$$

and consider the operator $\mathcal{J} : \mathcal{H}^{\mathbb{R}}(\omega; X) \to \mathcal{H}_0^{\mathbb{R}}(\omega; X)$, defined by expression

$$(\mathcal{J}u)(x; y) = \int_0^{(x; y)} -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy, \ \ \forall (x; y) \in \omega,$$

where the integral is taken over a smooth curve in ω. Thus, it is obvious that for $\forall u \in H^{\mathbb{R}}(\omega; X)$ the function $\mathrm{w} = u + i\mathcal{J}u$ is analytic in ω, that is, $\mathrm{w} \in A(\omega; X)$. Assume

$$\mathcal{A}_0(\omega; X) = \{\mathrm{w} \in \mathcal{A}(\omega; X) : (Im^* \mathrm{w})(0) = 0\}. \tag{3.5}$$

It is not hard to see that the operator $\mathcal{T} = I + i\mathcal{J}$ implements a bijective mapping $\mathcal{H}^{\mathbb{R}}(\omega; X)$ on $\mathcal{A}_0(\omega; X)$.

Summarizing the above consideration, we arrive at the following main lemma.

Lemma 3.1. *Let X be a B-space over a field $\mathbb{C}$ with involution operation $(*)$, and let $\mathcal{H}(\omega; X)$ & $\mathcal{A}(\omega; X)$ be the class of X-valued harmonic and analytic functions on ω, correspondingly. Then:*

(i) $\mathcal{H}(\omega; X) = \mathcal{H}^{\mathbb{R}}(\omega; X) \dot{+} i\mathcal{H}^{\mathbb{R}}(\omega; X)$ and $\mathcal{A}(\omega; X) \subset \mathcal{A}^{\mathbb{R}}(\omega; X) \dot{+} i\mathcal{A}^{\mathbb{R}}(\omega; X)$, where $\mathcal{H}^{\mathbb{R}}(\omega; X)$ ($\mathcal{A}^{\mathbb{R}}(\omega; X)$) is a $*$-real part of $\mathcal{H}(\omega; X)$ (of $\mathcal{A}(\omega; X)$);

(ii) $\mathcal{A}(\omega; X) \subset \mathcal{H}(\omega; X)$ & $\mathcal{H}(\omega; X) \setminus \mathcal{A}(\omega; X) \neq \emptyset$;

(iii) Let $w \in \mathcal{H}(\omega; X)$. Then $w \in \mathcal{A}(\omega; X)$ if and only if $u = Re^* w$ & $v = Im^* w$ satisfies the X-valued Cauchy-Riemann conditions (3.2);

(iv)$\mathcal{A}^{\mathbb{R}}(\omega; X) \equiv \mathcal{H}^{\mathbb{R}}(\omega; X)$;

(v) The spaces $\mathcal{H}^{\mathbb{R}}(\omega; X)$ and $\mathcal{A}_0(\omega; X)$ are linearly isomorphic and the operator $\mathcal{T} = I + i\mathcal{J}$ implements the corresponding isomorphism.

For simplicity, in what follows, we accept the notations $\mathcal{H}(X) =: \mathcal{H}(\omega; X)$, $\mathcal{A}(X) := \mathcal{A}(\omega; X)$, and so on. Also, for function $f : \omega \to X$, we denote $f_\tau(t) = f(\tau e^{it})$, $\forall \tau e^{it} \in \omega$. .

3.2. ${}_mH_p^{\pm}(X)$ & ${}_mh_p^{\pm}(X)$ classes

Accept ${}_mL_p^{\pm}(X) := R_m^{\pm}(L_p(X))$, where $R_m^{\pm}$ are t-Riesz projectors defined above. According to the work [13], introduce

$$ {}_mH_p^{\pm}(X) = \{F \in \mathcal{A}(X) : \exists f \in {}_mL_p^{\pm}(X) \Rightarrow F = \mathcal{K} f\}, $$

where $\mathcal{K}$ is a X-valued Cauchy-type integral

$$ F(z) = (\mathcal{K} f)(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z} d\xi, z \in \omega. $$

Take attention to the following simple relation. Let $f = u + iv$ & $z = x + iy :\ u; v \in X^{\mathbb{R}},\ x; y \in \mathbb{R} \Rightarrow zf = \tilde{u} + i\tilde{v}$, with $\tilde{u}; \tilde{v} \in X^{\mathbb{R}} : \tilde{u} = xu - yv,\ \tilde{v} = yu + xv$.

Let $F \in {}_mH_p^{\pm}(X)$. So, $F = u + iv$, where $u; v \in \mathcal{H}^{\mathbb{R}}(X)$. Based on this relation, assume

$$ {}_mh_p^{\pm;\mathbb{R}}(X) = Re^*({}_mH_p^{\pm}(X)), $$

and set

$$ {}_mh_p^{\pm}(X) = {}_mh_p^{\pm;\mathbb{R}}(X) \dot{+} i\ {}_mh_p^{\pm;\mathbb{R}}(X). $$

According to the results of [13], F possesses non-tangential limit values $F^{\pm}(\cdot)$ a.e. on γ, where the sign "+" means the limit taken from inside of ω, and the sign "−" means the limit taken from outside of ω. Moreover, it holds that $F^{\pm}(\xi) = \pm(R_0^{\pm} f)(\xi)$, for a.e. $\xi \in \gamma$, where $R_0^{\pm}$ are the t-Riesz operators. This result follows from Statement 3.1 of the work [13]. As proved in [13], for the function $F \in {}_0H_p^{+}(X)$, the following X-valued Poisson integral representation holds

$$ F(\rho e^{it}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_\rho(t - s) F^+(s) ds,\ \forall \rho e^{it} \in \omega, $$

where

$$ P_\rho(s) = \frac{1 - \rho^2}{1 + \rho^2 - 2\rho \cos s}, \rho e^{is} \in \omega, $$

is a Poisson Kernel for the unit disk. It immediately follows that

$$ u(\rho e^{it}) = Re^* F(\rho e^{it}) = (\mathscr{P} u^+)(\rho e^{it}) =: \frac{1}{2\pi} \int_{-\pi}^{\pi} P_\rho(t - s) u^+(s) ds,\ \forall \rho e^{it} \in \omega, \tag{3.6} $$

where $u^+(s) = Re^*F^+(s)$. Assume

$$L_p^{+;\mathbb{R}}(X) = Re^*(L_p^+(X)),$$

where $L_p^+(X) = R^+(L_p(X))$, $R^+ = R_0^+$ is a t-Riesz operator, defined by the Theorem 2.8. Consequently, from results of the work [13], we obtain that every function from the class $h_p^{\pm;\mathbb{R}}(X)$ has X-valued Poisson integral representation (3.6) with density $u^+ \in L_p^{+;\mathbb{R}}(X)$, where $h_p^{+;\mathbb{R}}(X) = {}_0h_p^{+;\mathbb{R}}(X)$. Also set

$$H_p^+(X) = {}_0H_p^+(X); H_p^-(X) = {}_{-1}H_p^-(X).$$

Denote by $\mathscr{P}_m^{\pm}(X)$ the set of all polynomials of order $\leq m \in \mathbb{Z}_+$, with X-valued coefficients of the form

$$\mathscr{P}_m^{\pm}(z) = \sum_{k=0}^{m} a_k^{\pm} z^{\pm k},\ a_0^- = 0,\ z \in \mathbb{C}\backslash\{0\},\ \ \{a_k^{\pm}\} \subset X.$$

As established in the work [13], the following direct sums hold

$${}_{-m}H_p^+(X) = H_p^+(X) \dotplus \mathscr{P}_m^-(X),$$

$${}_mH_p^+(X) = H_p^-(X) \dotplus \mathscr{P}_m^+(X).$$

It follows from here that

$$\left.\begin{array}{l} {}_{-m}h_p^{+;\mathbb{R}}(X) = h_p^{+;\mathbb{R}}(X) \dotplus Re^*(\mathscr{P}_m^-(X)), \\ {}_mh_p^{+;\mathbb{R}}(X) = h_p^{-;\mathbb{R}}(X) \dotplus Re^*(\mathscr{P}_m^+(X)), \end{array}\right\}$$

where $h_p^-(X) ={}_{-1} h_p^-(X)$.

3.3. *X-valued Fatou's & Zygmund theorems*

Let $f \in L_1(I_0; X)$ and consider the following X-valued Poisson integral. For simplicity, without loss of generality, we will consider the segment $[0, 2\pi)$ instead of the segment $[-\pi, \pi)$ in this section.

$$u(\rho;\varphi) = \frac{1}{2\pi}\int_0^{2\pi} P_\rho(s-\varphi)f(s)ds,\ \ \rho e^{i\varphi} \in \omega. \tag{3.7}$$

The following X-valued analogue of Fatou's theorem is valid.

Theorem 3.2. *Let $f \in L_1(I_0; X)$. Then for a.e. $\varphi_0 \in [0, 2\pi]$ the X-valued harmonic in ω function $u(\rho;\varphi)$, defined by Poisson formula* (3.7)*, has a non-tangential limit*

$$\lim_{\omega \ni \rho e^{i\varphi_0} \overset{\not\angle}{\to} e^{i\varphi_0}} u(\rho;\varphi) = f(\varphi_0).$$

Indeed, based on Lemma 2.4, $u(\rho;\varphi)$ can be represented by the following X-valued Poisson-Stieltjes integral.

$$u(\rho;\varphi) = \frac{1}{2\pi}\int_0^{2\pi} P_\rho(s-\varphi)d\mu(s),$$

where

$$\mu(s) = \int_0^s f(t)dt.$$

By this lemma, we have $\mu'(s) = f(s)$, *a.e.* $s \in (0, 2\pi)$. The further proof is carried out in a completely analogous way to the proof of the classical Fatou's theorem (see, f.e. [24]).

The fact that the point $z \in \omega$ tends non-tangential to the point $\tau \in \gamma$ is denoted as $z \overset{\not\rightarrow}{\rightarrow} \tau$. The operator corresponding to a harmonic function $u(\cdot)$ in ω and its non-tangential limit values $u^+(\cdot)$ a.e. on ω (if they exist) is denoted by θ: $\theta u = u^+$.

We now introduce the following subspaces of $L_p(I_0; X)$.

$$L_p^{\mathbb{R}}(I_0; X) = \{f \in L_p(I_0; X) : f(t) \in X^{\mathbb{R}}, \ \forall t \in I_0\},$$

and

$$L_p^{I}(I_0; X) = \{f \in L_p(I_0; X) : f(t) \in X^{i\mathbb{R}}, \ \forall t \in I_0\}.$$

So, $L_p^I(I_0; X) = i\, L_p^{\mathbb{R}}(I_0; X)$, and it is evident that

$$L_p(I_0; X) = L_p^{\mathbb{R}}(I_0; X) \dot{+} i\, L_p^{\mathbb{R}}(I_0; X).$$

Let $f \in L_p^{\mathbb{R}}(I_0; X)$ and consider the following X-valued Poisson integral

$$u(\rho; \varphi) = \frac{1}{2\pi} \int_0^{2\pi} P_r(s - \varphi) f(s)\, ds, \quad re^{i\varphi} \in \omega. \tag{3.8}$$

Let $x \in X^{\mathbb{R}}$. Then it is evident that

$$Re^*\Big[(a + ib)x\Big] = a\,x, \quad \forall a, b \in \mathbb{R}.$$

Taking into account the fact that

$$P_\rho(s - \varphi) = Re\left(\frac{e^{is} + \rho e^{i\varphi}}{e^{is} - \rho e^{i\varphi}}\right) = Re\Big(K(s; \rho e^{i\varphi})\Big),$$

where

$$K(s; z) = \frac{e^{is} + z}{e^{is} - z},$$

is a Schwartz kernel, the integral (3.8) we can represent in the form

$$u(z) = Re^*\left[\frac{1}{2\pi} \int_0^{2\pi} K(s; z) f(s)\, ds\right].$$

From the Cauchy-Riemann conditions (3.2) it follows that the $*$-conjugation to u function v is defined up to a $*$-real constant $a \in X^{\mathbb{R}}$.

Since, the function

$$w(z) = \frac{1}{2\pi} \int_0^{2\pi} K(s; z) f(s)\, ds, \tag{3.9}$$

is analytic in ω, then it is evident that

$$v(z) = \frac{1}{2\pi} \int_0^{2\pi} Im\Big(K(s; z)\Big) f(s)\, ds + i\, v_0,$$

where $v_0 \in X^{\mathbb{R}}$ is an arbitrary constant. The integral (3.9) we will call a $*$-Schwartz integral. We have

$$ctg \frac{s - s_0}{2} = -Im \frac{e^{is} + e^{is_0}}{e^{is} - e^{is_0}} = \lim_{\rho \to 1} Q(\rho; s - s_0),$$

where

$$Q(\rho; s-s_0) = -Im\left(\frac{e^{is}+\rho e^{is_0}}{e^{is}-\rho e^{is_0}}\right) = \frac{2\rho\sin(s-s_0)}{1+\rho^2-2\rho\cos(s-s_0)}.$$

Consequently

$$P(\rho;\sigma-s)+iQ(\rho;\sigma-s) = \frac{e^{is}+\rho e^{is_0}}{e^{is}-\rho e^{is_0}}.$$

The kernel $Q(\rho;\sigma)$ is a conjugate Poisson's kernel. Consider the following X-valued singular integral with kernel $ctg\frac{s}{2}$:

$$(Hf)(\sigma) = \int_0^{2\pi}\frac{1}{2}ctg\frac{s-\sigma}{2}f(s)\,ds =$$

$$= \int_0^{\pi}\frac{1}{2}ctg\frac{s}{2}\Big[f(\sigma+s)-f(\sigma-s)\Big]\,ds =$$

$$= \lim_{\varepsilon\to 0}\int_{\varepsilon}^{\pi}\frac{1}{2}ctg\frac{s}{2}\Big[f(\sigma+s)-f(\sigma-s)\Big]\,ds.$$

H is called the X-valued periodic Hilbert transformation or conjugate function operator. The following X-valued analogue of Zygmund's theorem is proved.

Theorem 3.3. *Let $X\in\mathscr{B}$ and $f\in L_p(I_0;X)$. Then for a.a. $\sigma\in(0,2\pi)$ it holds*

$$\lim_{\rho\to 1}\left[v(\rho;\sigma)+\frac{1}{2\pi}\int_{1-\rho}^{\pi}ctg\frac{s}{2}\Big[f(\sigma+s)-f(\sigma-s)\Big]\,ds\right]=0,$$

where

$$v(\rho;\sigma) = \frac{1}{2\pi}\int_0^{2\pi}Q(\rho;\sigma-s)f(s)\,ds. \tag{3.10}$$

Proof. For completeness of presentation, let us give a short outline of the proof. Taking attention to the Lemma 2.4, set

$$\mu(s) = \int_0^s f(\sigma)\,d\sigma,$$

and therefore $\mu'(s)=f(s)$, a.e. $s\in(0,2\pi)$. Let $\sigma\in[0,2\pi]$ such that at this point $\mu'(\sigma)$ exists. Denote

$$\mathcal{E}(s) = \frac{\mu(\sigma+s)-\mu(\sigma)}{s}-\frac{\mu(\sigma-s)-\mu(\sigma)}{-s},\quad \forall s\neq 0.$$

Assume that the functions f and μ periodically continued to $\mathbb{R}$ with period 2π. It is obvious that $\mathcal{E}(s)\to 0$, $s\to 0$. Also, set

$$\nu(s) = \mu(\sigma+s)+\mu(\sigma-s)-2\mu(\sigma),\quad \forall s\neq 0.$$

Thus, it holds

$$\nu(s) = \bar{\bar{o}}(s), s\to 0,\ \text{i.e.}\ \lim_{s\to 0}\frac{\nu(s)}{s}=0.$$

We have

$$\frac{1}{2\pi}\int_{\delta}^{\pi} ds\Big[\Big(\mu(\sigma+s)+\mu(\sigma-s)\Big)ctg\frac{s}{2}\Big] = -\frac{1}{2\pi}\nu(\delta)ctg\frac{\delta}{2} + \frac{1}{4\pi}\int_{\delta}^{\pi}\frac{\nu(s)}{\sin^2\frac{s}{2}}\,ds, \quad \delta = 1-\rho.$$

It is evident that $\lim_{\delta\to 0}\nu(\delta)ctg\dfrac{\delta}{2} = 0$. Thus, it is sufficient to prove

$$\lim_{\rho\to 1}\left(v(\rho;\sigma) + \frac{1}{4\pi}\int_{1-\rho}^{\pi}\frac{\nu(s)}{\sin^2\frac{s}{2}}\,ds\right) = 0.$$

We have

$$\left.\begin{aligned} Q_s'(\rho;s) &= \frac{2\rho\Big[(1+\rho^2)\cos s - 2\rho\Big]}{\Big(1+\rho^2-2\rho\cos s\Big)^2},\\ Q_s'(1;s) &= \frac{-1}{1-\cos s} = \frac{-1}{2\sin^2\frac{s}{2}},\\ |Q_s'(\rho;s)| &\leq \frac{2\rho}{\Big(1-\rho\Big)^2} = \frac{2\rho}{\delta^2}. \end{aligned}\right\} \tag{3.11}$$

Integration by parts gives

$$v(\rho;\sigma) = \frac{1}{2\pi}\int_0^{\pi}\nu(s)Q_s'(\rho;s)\,ds,$$

and we can split this integral into two parts

$$\mathcal{J}_1 = \frac{1}{2\pi}\int_0^{\delta}\nu(s)Q_s'(\rho;s)\,ds;\;\; \mathcal{J}_2 = \frac{1}{2\pi}\int_{\delta}^{\pi}\nu(s)Q_s'(\rho;s)\,ds.$$

Due to the relations (3.11), we have

$$2\pi\mathcal{J}_1 \leq \frac{2\rho}{\delta^2}\int_0^{\delta} s\|\mathcal{E}(s)\|_X\,ds \leq \rho\sup_{0\leq s\leq\delta}\|\mathcal{E}(s)\|_X \to 0, \quad \delta\to 0.$$

Represent the integral $\mathcal{J}_2$ in the form

$$\mathcal{J}_2 = \frac{1}{2\pi}\int_{\delta}^{\pi}\nu(s)Q_s'(1;s)\,ds + \frac{1}{2\pi}\int_{\delta}^{\pi}\nu(s)\Big[Q_s'(\rho;s)-Q_s'(1;s)\Big]\,ds.$$

Due to the second relation in (3.11), we have

$$\lim_{\rho\to 1}\left(v(\rho;\sigma) + \frac{1}{4\pi}\int_{1-\rho}^{\pi}\frac{\nu(s)}{\sin^2\frac{s}{2}}\,ds\right) = \lim_{\rho\to 1}\frac{1}{2\pi}\int_{\delta}^{\pi}\nu(s)\Big[Q_s'(\rho;s)-Q_s'(1;s)\Big]\,ds.$$

It holds

$$Q_s'(\rho;s) - Q_s'(1;s) = \frac{\delta^2\Big[(1+\rho^2-2\rho\cos s)+2\rho\sin^2 s\Big]}{(1-\cos s)\Big[(1+\rho^2-2\rho\cos s)\Big]^2}.$$

Moreover

$$1+\rho^2 \geq 2\rho \Rightarrow 1+\rho^2-2\rho\cos s \geq 2\rho(1-\cos s) = 4\rho\sin^2\frac{s}{2}.$$

Consequently, it follows that

$$\left\|\frac{1}{2\pi}\int_{\delta}^{\pi} \nu(s)\left[Q'_s(\rho;s)-Q'_s(1;s)\right]ds\right\|_X \leq \frac{C}{2\pi}\delta^2\int_{\delta}^{\pi}\|\mathcal{E}(s)\|_X\frac{ds}{s^3}.$$

Let $\eta>0$ be an arbitrary number. It is obvious that $\exists\ \delta_0>0$: $\|\mathcal{E}(s)\|_X\leq\eta$, $\forall s$. Assume $\delta^2=\eta\delta_0^2\ \&\ \delta<\delta_0$. Then we have

$$\delta^2\int_{\delta}^{\pi}\frac{\|\mathcal{E}(s)\|_X}{s^3}ds = \delta^2\int_{\delta}^{\delta_0}\frac{\|\mathcal{E}(s)\|_X}{s^3}ds+\delta^2\int_{\delta}^{\pi}\frac{\|\mathcal{E}(s)\|_X}{s^3}ds \leq$$
$$\leq \frac{\delta^2\eta}{2}\left(\frac{1}{\delta^2}-\frac{1}{\delta_0^2}\right)+\frac{\delta^2}{\delta_0^3}\int_0^{\pi}\frac{\|\mathcal{E}(s)\|_X}{s^3}ds \leq$$
$$\leq \frac{\eta}{2}+\eta\int_0^{\pi}\|\mathcal{E}(s)\|_X\,ds. \tag{3.12}$$

Since $\|\mathcal{E}(s)\|_X$ is a bounded function, then it follows from (3.12) that

$$\left\|\frac{1}{2\pi}\int_{\delta}^{\pi} \nu(s)\left[Q'_s(\rho;s)-Q'_s(1;s)\right]ds\right\|_X \leq C\eta,$$

where $C>0$ is independent of δ constant.

The theorem is proved. □

In particular, the following result follows from this theorem.

Corollary 3.4. *Let $X\in\mathscr{B}$ and $f\in L_1(I_0;X)$. Then for $\sigma\in I_0$ the X-valued Hilbert transform $(Hf)(\sigma)$ exists if and only if the limit $\lim_{\rho\to 1} v(\rho;\sigma)$ exists, where $v(\rho;\sigma)$ is defined by the integral* (3.10).

Now, consider the case when $X\in UMD$ and $1<p<+\infty$. Let $u\in h_p^{+;\mathbb{R}}(X)\Rightarrow\exists f\in H_p^+(X): u=Re^*f$. Let $v: v(0)=0$, is a $*$-conjugate to u function. Consequently, $f=u+iv$. It is evident that $f\in H_p^+(X)\Leftrightarrow if\in H_p^+(X)$. Since $v=-Re^*(if)$, then $v\in h_p^{+;\mathbb{R}}(X)$.

Conversely, let $u;v\in h_p^{+;\mathbb{R}}(X)$ and $v: v(0)=0$, $*$-conjugate to u function. Let $\hat{f}\in H_p^+(X)$ such that $u=Re^*\hat{f}$. Set $f=u+iv$. It is evident that $\hat{f}=f+iv_0$, where $v_0\in X^{\mathbb{R}}$ is a constant. From here follows that $f\in H_p^+(X)$. Thus, the following relation is true

$$f\in H_p^+(X)\Leftrightarrow Re^*f,\ Im^*f\in h_p^{+;\mathbb{R}}(X).$$

From this, it follows that $v(\cdot)$ has non-tangential values a.e. on γ, denoted by $v^+(\xi)$, $\xi\in\gamma$. Represent $f(\cdot)$ via the X-valued Poisson integral

$$f(re^{it})=\frac{1}{2\pi}\int_0^{2\pi}P_r(s-t)f^+(s)\,ds,\quad re^{it}\in\omega,$$

where $f^+(\cdot)$ is the non-tangential values of $f(\cdot)$ on γ. From this formula direct follows that

$$u(re^{it})=\frac{1}{2\pi}\int_0^{2\pi}P_r(s-t)u^+(s)\,ds,$$
$$v(re^{it})=\frac{1}{2\pi}\int_0^{2\pi}P_r(s-t)v^+(s)\,ds,$$

where $u^+(\cdot)=Re^*f^+(\cdot)$ and $v^+(\cdot)=Im^*f^+(\cdot)$. From the above consideration, we arrive at the following

Statement 3.5. *Let $X \in UMD$ and $f \in L_p(I_0; X)$, $1 < p < +\infty$. Then for a.e. $\sigma \in (0, 2\pi)$, it holds*

$$\lim_{\rho e^{it} \overset{\not\perp}{\rightarrow} e^{i\sigma}} v(\rho; t) = \frac{1}{2\pi} \int_0^{2\pi} f(s) \operatorname{ctg} \frac{\sigma - s}{2} \, ds, \tag{3.13}$$

where

$$v(\rho; t) = \frac{1}{2\pi} \int_0^{2\pi} Q(\rho; t - s) f(s) \, ds,$$

that is, the function $v(\cdot)$ has non-tangential values (3.13) *a.e. on γ.*

3.4. Sokhotski-Plemelj's formula

Let $X \in UMD$ and $f \in L_p(I_0; X)$, $1 < p < +\infty$. Consider the following Schwartz–Bochner integral.

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} K(s; z) f(s) \, ds, \quad z \in \omega. \tag{3.14}$$

Represent this integral in the following form

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} P(r; \sigma - s) f(s) \, ds + \frac{i}{2\pi} \int_0^{2\pi} Q(r; \sigma - s) f(s) \, ds,$$

where $z = re^{i\sigma}$. Taking attention to the Theorem 3.2 and Statement 3.5, from here we obtain

$$F^+(e^{i\sigma}) = \lim_{z \overset{\not\perp}{\rightarrow} e^{i\sigma}} F(z) = f(\sigma) + i(Hf)(\sigma), \quad \text{a.e. } \sigma \in (0, 2\pi). \tag{3.15}$$

Now, consider the integral (3.14) in the case when $|z| > 1$. Introduce a new function for consideration, $\Phi(z_1) = F^*\left(\frac{1}{\overline{z_1}}\right)$, $|z_1| < 1$. Where $z = \frac{1}{\overline{z_1}}$. We have

$$\Phi(z_1) = -\frac{1}{2\pi}\left(\int_0^{2\pi} \frac{e^{-is} + \overline{z_1}}{e^{-is} - \overline{z_1}} f(s) \, ds\right) = -\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{is} + z_1}{e^{is} - z_1} f^+(s) \, ds, |z_1| < 1.$$

It is evident that $\omega \ni z_1 \overset{\not\perp}{\rightarrow} e^{i\sigma} \iff \omega^c \ni z \overset{\not\perp}{\rightarrow} e^{i\sigma}$. Then according to the formula (3.15) for $\Phi(\cdot)$ we obtain

$$\Phi^+(e^{i\sigma}) = -f^*(\sigma) - i\big(Hf^*\big)(\sigma) \Rightarrow F^-(e^{i\sigma}) = \left(\Phi^+(e^{i\sigma})\right)^* = -f(\sigma) + i\big(Hf\big)(\sigma),$$

a.e. $\sigma \in (0, 2\pi)$, where

$$F^-(e^{i\sigma}) = \lim_{\omega^c \ni z \overset{\not\perp}{\rightarrow} e^{i\sigma}} F(z).$$

Thus, the following theorem is valid.

Theorem 3.6. *Let $X \in UMD$ and $f \in L_p(I_0; X)$, $1 < p < +\infty$. Then, for the Schwartz–Bochner integral* (3.14)*, the following X-valued Sokhotsky–Plemelj's formulas are valid.*

$$F^\pm(e^{i\sigma}) = \pm f(\sigma) + i(Hf)(\sigma), \quad \text{a.e. } \sigma \in (0, 2\pi).$$

Note that this formula is established in the work [13] by a different approach, under stronger conditions.

3.5. t-nasis for $h_p^+(X)$ and Dirichlet problem for Laplace equation

Let $X \in UMD$ and $u \in h_p^{+;\mathbb{R}}(X)$, $1 < p < +\infty$. Firstly, define the norm in $h_p^{+;\mathbb{R}}(X)$ by the following expression

$$\|u\|_{h_p^{+;\mathbb{R}}(X)} = \|u^+\|_{L_p(\gamma;X)}, \tag{3.16}$$

where $u^+ = \theta u$ is the non-tangential values function of u on γ. From the Poisson-Bochner formula for $u(\cdot)$, it directly follows that the expression (3.16) defines the norm in $h_p^{+;\mathbb{R}}(X)$.

Let $w \in H_p^+(X)$ such that $u = Re^* w$. Via the results of the work [13], the system $\{z^n\}_{n\in\mathbb{Z}_+}$ forms a t-basis for $H_p^+(X)$. Let

$$w(z) = \sum_{n=0}^{\infty} w_n z^n, \quad z \in \omega,$$

$\{w_n\} \subset X$. From here it immediately follows

$$u(z) = \sum_{n=0}^{\infty} Re^*\Big(w_n z^n\Big), \quad z \in \omega.$$

Let $w_n = u_n + iv_n$, with $u_n; v_n \in X^{\mathbb{R}}$, $\forall\, n \in \mathbb{Z}_+$. Consequently

$$u(z) = u_0 + \sum_{n=1}^{\infty} \Big(u_n \cos n\varphi - v_n \sin n\varphi\Big) r^n, \quad z = re^{i\varphi} \in \omega.$$

By results of the work [13], we have

$$w^+(e^{i\varphi}) = \sum_{n=0}^{\infty} w_n e^{in\varphi},$$

and in result

$$u^+(e^{i\varphi}) = u_0 + \sum_{n=1}^{\infty} \Big(u_n \cos n\varphi - v_n \sin n\varphi\Big).$$

Therefore (see, [13])

$$\left\| w(z) - \sum_{n=0}^{m} w_n z^n \right\|_{H_p^+(X)} = \left\| w^+(\xi) - \sum_{n=0}^{m} w_n z^n \right\|_{L_p(\gamma;X)} \to 0, \quad m \to \infty.$$

It follows from here that

$$\left\| u(re^{i\varphi}) - u_0 - \sum_{n=1}^{m} (u_n \cos n\varphi - v_n \sin n\varphi) z^n \right\|_{h_p^+(X)} =$$

$$= \left\| u^+(e^{i\varphi}) - u_0 - \sum_{n=1}^{m} (u_n \cos n\varphi - v_n \sin n\varphi) \right\|_{L_p(\gamma;X)} \to 0, \quad m \to \infty.$$

It is not hard to see that the operators $\{\tau_n^{\pm}\} \subset \Big[h_p^{+;\mathbb{R}}(X); X\Big]$, defined by the expressions

$$\tau_0^+(u) = \frac{1}{2\pi} \int_0^{2\pi} u^+(e^{it})\, dt;$$

$$\tau_n^+(u) = \frac{1}{\pi}\int_0^{2\pi} u^+(e^{it})\cos nt\,dt;$$

$$\tau_n^-(u) = \frac{1}{\pi}\int_0^{2\pi} u^-(e^{it})\sin nt\,dt,$$

are t-biorthogonal to the system $\{1; r^n\cos nt; r^n\sin nt\}_{n\in\mathbb{N}}$. Summarizing the previous results, we obtain the validity of the following statement.

Statement 3.7. *Let $X \in UMD$. Then the system*

$$\{1; r^n\cos nt; r^n\sin nt\}_{n\in\mathbb{N}},$$

forms a t-basis for $h_p^{+;\mathbb{R}}(X)$, $1 < p < +\infty$.

Let $X \in UMD$ and $u \in h_p^{+;\mathbb{R}}(X)$, $1 < p < +\infty$. Then, by Statement 3.7 the function $u(\cdot)$ has the following expansion

$$u(re^{it}) = u_0^+ + \sum_{n=1}^{\infty}\left(u_n^+\cos nt + u_n^-\sin nt\right)r^n.$$

Since $\theta \in \left[h_p^{+;\mathbb{R}}(X); L_p(\gamma; X)\right]$, we have

$$\begin{aligned}\theta u(re^{it}) = u^+(e^{it}) &= u_0^+ + \sum_{n=1}^{\infty}\left[\theta(u_n^+\cos ntr^n) + \theta(u_n^-\sin ntr^n)\right] = \\ &= u_0^+ + \sum_{n=1}^{\infty}\left(u_n^+\cos nt + u_n^-\sin nt\right).\end{aligned}$$

From these considerations, we obtain the correct solvability of the following X-valued Dirichlet problem for the Laplace equation in the class $h_p^{+;\mathbb{R}}(X)$.

Consider the problem

$$\left.\begin{aligned}\Delta u = 0, &\quad \text{in } \omega,\\ \theta u = f, &\quad \text{on } \gamma,\end{aligned}\right\} \tag{3.17}$$

where $f \in L_p^{\mathbb{R}}(\gamma; X)$ is a given function. By the solution of the problem (3.17), we mean the function $u \in h_p^{+;\mathbb{R}}(X)$ for which $\theta u = f$, where θ is the corresponding trace operator.

The following statement holds.

Statement 3.8. *Let $X \in UMD$. Then for $\forall\ f \in L_p^{\mathbb{R}}(\gamma; X)$, $1 < p < +\infty$, the problem* (3.17) *is uniquely solvable in the class $h_p^{+;\mathbb{R}}(X)$ and for the solution $u \in h_p^{+;\mathbb{R}}(X)$, it holds*

$$\|u\|_{h_p^{+;\mathbb{R}}(X)} = \|f\|_{L_p(\gamma; X)}.$$

The last relation directly follows from (3.16).

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References

[1] H. Amann, *Elliptic operators with infinite-dimensional state space*, J. Evolution Equat. **1** (2001), 143–188.
[2] W. Arendt, *Vector-valued holomorphic and harmonic functions*, Concrete Oper. **3** (2016), 68–76.
[3] W. Arendt, C. Batty, M. Hieber, et al. *Vector-valued laplace transforms and Cauchy problems*, Birkhäuser, Basel, 2011.
[4] W. Arendt, N. Nikolski, *Vector-valued holomorphic functions revisited*, Math. Z. **234** (2000), 777–805.
[5] W. Arendt, N. Nikolski, *Addendum: Vector-valued holomorphic functions revisited*, Math. Z. 252 (2006), 687–689.
[6] B. T. Bilalov, *The basis property of a perturbed system of exponentials in Morrey-type spaces*, Siberian Math. J. **60** (2019), 249–271.
[7] B. T. Bilalov, *Basis properties of some systems of exponents, cosines and sines*, Sibirsk. Mat. Zh. **45** (2004), 264–273.
[8] B. T. Bilalov, *Basicity of some exponents, cosines and sines systems*, Differ. Urav. **26** (1990), 10–16.
[9] B. T. Bilalov, T. M. Ahmadov, Y. Zeren, et al. *Solution in the small and interior Schauder-type estimate for the m-th order elliptic operator in Morrey-Sobolev spaces*, Azerbaijan J. Math. **12** (2022), 190–219.
[10] B. T. Bilalov, Z. G. Guseynov, *Basicity criterion for perturbed systems of exponents in Lebesgue spaces with variable summability*, Dokl. RAN **436** (2011), 586–589.
[11] B. T. Bilalov, Z. G. Guseynov, *Basicity of a system of exponents with a piece-wise linear phase in variable spaces*, Mediterr. J. Math. **9** (2012), 487–498.
[12] B. T. Bilalov, S. R. Sadigova, V. G. Alili. *The method of boundary value problems in the study of the basis properties of perturbed system of exponents in Banach function spaces*,. Comput. Methods Function Theory **24** (2014), 101–120.
[13] B. T. Bilalov, S. R. Sadigova, *The Concept of t- basis and vector valued Hardy classes*, Turkish J. Math. **49**(2025): 261–286
[14] B. T. Bilalov, S. R. Sadigova, N. Nasibova, et al. *X-valued Smirnov classes and t-basisity of Faber polynomials*, Results Math 80, 215 (2025). https://doi.org/10.1007/s00025-025-02525-z
[15] B. T. Bilalov, S. R. Sadigova, Y. Sezer, *Vector-valued grand Hardy classes*, J. Contem. Appl. Math. **15** (2025), 122–155.
[16] B. T. Bilalov, Y. Sezer, F. A. Alizadeh, et al. *Solvability of Riemann oundary value problems and applications to approximative properties of perturbed exponential system in Orlicz spaces*, Azerbaijan J. Math. **14** (2024), 164–185.
[17] J. Bonet, L. Frerick, A. E. Jord, *Extension of vector-valued holomorphic and harmonic functions*, Studia Math. **183** (2007), 225–248.
[18] T. Hytönen, J. Neerven, M. Veraar, et al. *Analysis in Banach Spaces*, Springer. V. I, 2016.
[19] T. Hytönen, J. V. Neerven, M. Veraar, et al. *Analysis in Banach Spaces*, V. II, 2017.
[20] N. Katzourakis, *On a vector-valued generalisation of viscosity solutions for general PDE systems*, Z. Anal. Anwendungen **41** (2022), 93–132.
[21] M. Kreuter, *The Perron solution for vector-valued equations*, arXiv: 1804.01370v1 [math.FA], 2018.
[22] V. M. Kreuter, *Sobolev spaces of Vector-Valued Functions*, Master Thesis, Ulm University, 2015.
[23] V. M. Kreuter, *Vector-Valued Elliptic Boundary Value Problems on Rough Domains*, Universitat Ulm, 2018.
[24] P. Kusis, *Introduction to Hp Spaces*, Cambridge University Press, 2nd edition, 2008
[25] E. Lorist, *Vector-valued harmonic analysis with applications to SPDE*, Dissertation, Delft University of Technology, Netherlands, 2021.
[26] T. J. Mammadov, *Strong Solvability of a Nonlocal Problem for the Laplace Equation in Weighted Grand Sobolev Spaces*, Azerbaijan J. Math. **13** (2023), 188–204.