Filomat 39:35 (2025), 12611–12624
https://doi.org/10.2298/FIL2535611M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some characterizations for Kenmotsu manifolds admitting a general connection

Tuğba Mert[a,*], Mehmet Atçeken[b], Mohd Danish Siddiqi[c]

[a]*Department of Mathematics, Faculty of Science, Sivas Cumhuriyet University, 58140, Sivas, Turkey*
[b]*Department of Mathematics, Faculty of Art and Science, Aksaray University, 68100, Aksaray, Turkey*
[c]*Department of Mathematics College of Science, Jazan University, P.O. Box 277, Jazan 45142, Saudi Arabia*

Abstract. In this paper, we discuss η-Ricci solitons on Kenmotsu manifolds. While investigating η-Ricci solitons, we consider η-Ricci soliton in Kenmotsu manifold with respect to a general connection instead of the Levi-Civita connection. We present the geometry of Kenmotsu manifolds admitting a general connection under the conditions of Ricci pseudosymmetry and Ricci semisymmetry. Using the general connection, we obtain the characterizations of Ricci pseudosymmetric and Ricci semisymmetric Kenmotsu manifolds with respect to quarter-symmetric connection, Schouten-van Kampen connection, Tanaka-Webster connection, and Zamkovoy connection.

1. Introduction

Kenmotsu manifolds are a class of Riemannian manifolds characterized by their unique geometric properties. Kenmotsu manifolds have developed from a specific geometric study into a rich area of research with connections to various mathematical disciplines and applications. Their unique properties continue to inspire further exploration in both pure and applied mathematics.

A connection on a manifold provides a way to differentiate vector fields along curves. More formally, a connection allows the definition of a derivative of a vector field along another vector field, facilitating the study of how vectors change in a manifold's curved geometry. Levi-Civita connection is the most common type of connection, uniquely determined for a Riemannian manifold. It is compatible with the metric and is torsion-free, meaning the connection does not introduce any twisting in the vectors.

General connection, often referred to as a connection on a differentiable manifold, is a fundamental concept in differential geometry and plays a crucial role in the study of curved spaces. General connections are a powerful tool in understanding the geometric structure of manifolds. They provide the framework for defining differentiation in curved spaces and have significant implications in both mathematics and physics. The study of connections continues to be an active area of research, leading to deeper insights into the geometry and topology of manifolds.

2020 *Mathematics Subject Classification*. Primary 53C15; Secondary 53C44, 53C25, 53D10.
Keywords. Kenmotsu Manifold, η-Ricci Soliton, General Connection.
Received: 04 February 2025; Revised: 07 June 2025; Accepted: 23 June 2025
Communicated by Mića Stanković
* Corresponding author: Tuğba Mert
Email addresses: tmert@cumhuriyet.edu.tr (Tuğba Mert), mehmet.atceken382@gmail.com (Mehmet Atçeken), msiddiqi@jazanu.edu.sa (Mohd Danish Siddiqi)
ORCID iDs: https://orcid.org/0000-0001-8258-8298 (Tuğba Mert), https://orcid.org/0000-0002-1242-4359 (Mehmet Atçeken), https://orcid.org/0000-0002-1713-6831 (Mohd Danish Siddiqi)

Recently, Biswas and Baishya introduced and studied a new connection, named general connection in Sasakian geometry [1, 2]. The general connection $\mathcal{D}^G$ is defined as

$$\mathcal{D}^G_X Y = D^Y_X + \kappa_1 \left[(D_X \eta)(Y)\xi - \eta(Y) D_X \xi \right] + \kappa_2 \eta(X) \phi Y, \tag{1}$$

the pair (κ_1, κ_2) being real constants. The beauty of such connection $\mathcal{D}^G$ lies in the fact that it has the flavour of

· quarter symmetric metric connection for $(\kappa_1, \kappa_2) = (0, -1)$ in [3, 4],

· Schouten-van Kampen connection for $(\kappa_1, \kappa_2) = (1, 0)$ in [5],

· Tanaka Webster connection for $(\kappa_1, \kappa_2) = (1, -1)$ in [6],

· Zamkovoy connection for $(\kappa_1, \kappa_2) = (1, 1)$ in [7].

The notion of Ricci flow was introduced by Hamilton in 1982. With the help of this concept, Hamilton found the canonical metric on a smooth manifold. Then Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature. Perelman used Ricci flow and it surgery to prove Poncare conjecture in [8, 9]. The Ricci flow is an flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t} g(t) = -2S(g(t)).$$

A Ricci soliton emerges as the limit of the solitons of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group of diffeomorphism and scaling.

During the last two decades, the geometry of Ricci solitons has been the focus of attention of many mathematicians. In particular, it has become more important after Perelman applied Ricci solitons to solve the long standing Poincare conjecture posed in 1904. In [10], Sharma studied the Ricci solitons in contact geometry. Thereafter Ricci solitons in contact metric manifolds have been studied by various authors such as Bagewadi et al. in [11–14], Bejan and Crasmareanu in [15], Blaga in [16], Chandra et al. in [17], Chen and Deshmukh in [18], Deshmukh et al. in [19], He and Zhu in [20], Atçeken et al. in [21], Nagaraja and Premalatta in [22], Tripathi in [23] and many others.

Motivated by all these studies, we discuss η-Ricci solitons on Kenmotsu manifolds. While investigating η-Ricci solitons, we consider η-Ricci soliton in Kenmotsu manifold with respect to a general connection instead of the Levi-Civita connection. We present the geometry of Kenmotsu manifolds admitting a general connection under the conditions of Ricci pseudosymmetry and Ricci semisymmetry. Using the general connection, we obtain the characterizations of Ricci pseudosymmetric and Ricci semisymmetric Kenmotsu manifolds with respect to quarter-symmetric connection, Schouten-van Kampen connection, Tanaka-Webster connection, and Zamkovoy connection.

2. Preliminary

Let M be an $n = (2m+1)$-dimensional differentiable manifold, it said to be an almost contact Riemannian manifold if either its structural group can be reduced to $U(n) \times \{I\}$ or there is an almost contact metric structure (ϕ, ξ, η, g) consisting of a vector field ξ, $(1,1)$ tensor field ϕ, 1-form η and Riemannian metric g satisfying

$$\phi^2 X = -X + \eta(X)\xi, \tag{2}$$

$$\eta(\xi) = 1, \eta(\phi X) = 0, \phi\xi = 0. \tag{3}$$

In contact manifolds (M^n, g) the following relations hold:

$$g(X, \phi Y) = -g(\phi X, Y), g(X, \xi) = \eta(X), \tag{4}$$

$$g\left(\phi X,\phi Y\right)=g\left(X,Y\right)-\eta\left(X\right)\eta\left(Y\right),g\left(QX,Y\right)=S\left(X,Y\right), \tag{5}$$

$$S\left(\phi X,\phi Y\right)=S\left(X,Y\right)+\left(n-1\right)\eta\left(X\right)\eta\left(Y\right), \tag{6}$$

$$\left(D_X\phi\right)Y=-g\left(X,\phi Y\right)\xi-\eta\left(Y\right)\phi\left(X\right), \tag{7}$$

$$D_X\xi=X-\eta\left(X\right)\xi, \tag{8}$$

$$\left(D_X\eta\right)Y=g\left(X,Y\right)-\eta\left(X\right)\eta\left(Y\right), \tag{9}$$

for all $X,Y\in TM$, D is Levi-Civita connection ([24]-[27]).

Further, for Kenmotsu manifold with structure $\left(\phi,\xi,\eta,g\right)$ following relations holds:

$$R\left(X,Y\right)\xi=\eta\left(X\right)Y-\eta\left(Y\right)X, \tag{10}$$

$$S\left(X,\xi\right)=-\left(n-1\right)\eta\left(X\right), \tag{11}$$

$$R\left(X,\xi\right)Y=g\left(X,Y\right)\xi-\eta\left(Y\right)X, \tag{12}$$

$$R\left(\xi,X\right)Y=\eta\left(Y\right)X-g\left(X,Y\right)\xi, \tag{13}$$

$$Q\xi=-\left(n-1\right)\xi, \tag{14}$$

where S and Q are Ricci tensor and Ricci operator.

By the help of $(2),(8),(9)$ the relation (1) reduces to

$$\mathcal{D}^G_XY=D_XY+\kappa_1\left[g\left(X,Y\right)\xi-\eta\left(Y\right)X\right]+\kappa_2\eta\left(X\right)\phi Y. \tag{15}$$

Substituting Y by ξ in (15) and using $(2),(8)$, we get

$$\mathcal{D}^G_X\xi=\left(1-\kappa_1\right)\left[X-\eta\left(X\right)\xi\right]. \tag{16}$$

In Kenmotsu manifolds (M^n,g) admitting general connection D^G the following relations hold:

$$\mathcal{D}^G_X\eta\left(Y\right)=\eta\left(D_XY\right)+g\left(X,Y\right)-\eta\left(X\right)\eta\left(Y\right), \tag{17}$$

$$\mathcal{D}^G_X\left(\phi Y\right)=D_X\left(\phi Y\right)+\kappa_1g\left(X,\phi Y\right)\xi-\kappa_2\left[\eta\left(X\right)Y-\eta\left(X\right)\eta\left(Y\right)\xi\right], \tag{18}$$

$$\begin{aligned}R^G\left(X,Y\right)Z&=R\left(X,Y\right)Z+\left(\kappa_1\kappa_2-\kappa_2\right)\left[g\left(X,\phi Z\right)\eta\left(Y\right)\xi-g\left(Y,\phi Z\right)\eta\left(X\right)\xi\right]\\&\quad+\left(\kappa_1\kappa_2-\kappa_2\right)\left[\eta\left(Y\right)\eta\left(Z\right)\phi X-\eta\left(X\right)\eta\left(Z\right)\phi Y\right]\\&\quad+\kappa_1\left(1-\kappa_1\right)\left[g\left(X,Z\right)\eta\left(Y\right)\xi-g\left(Y,Z\right)\eta\left(X\right)\xi\right]\\&\quad+\kappa_1\left(2-\kappa_1\right)\left[g\left(Y,Z\right)X-g\left(X,Z\right)Y\right],\end{aligned} \tag{19}$$

$$\begin{aligned}Q^GY&=QY-\kappa_2\left(1-\kappa_1\right)\phi Y\\&\quad+\left[2n\kappa_1-n\kappa_1^2-3\kappa_1+2\kappa_1^2\right]Y\\&\quad+\kappa_1\left(1-\kappa_1\right)\eta\left(Y\right)\xi,\end{aligned} \tag{20}$$

$$\begin{aligned} S^G(Y,Z) &= S(Y,Z) + \kappa_2(1-\kappa_1)g\left(Y,\phi Z\right) \\ &+ \left[2n\kappa_1 - n\kappa_1^2 - 3\kappa_1 + 2\kappa_1^2\right]g(Y,Z) \\ &+\kappa_1(1-\kappa_1)\eta(Y)\eta(Z), \end{aligned} \tag{21}$$

$$r^G = r + \kappa_1(1-\kappa_1) + n\left(2n\kappa_1 - n\kappa_1^2 - 3\kappa_1 + 2\kappa_1^2\right), \tag{22}$$

$$S^G(Y,\xi) = (1-n)(1-\kappa_1)^2\eta(Y), \tag{23}$$

$$\begin{aligned} R^G(Y,Z)\xi &= (1-\kappa_1)^2[\eta(Y)Z - \eta(Z)Y] \\ &+\kappa_2(\kappa_1-1)\left[\eta(Z)\phi Y - \eta(Y)\phi Z\right], \end{aligned} \tag{24}$$

$$\begin{aligned} \eta\left(R^G(X,Y)Z\right) &= (1-\kappa_1)^2 g(U,\eta(Y)X - \eta(X)Y) \\ &+\kappa_2(\kappa_1-1)g\left(U,\eta(X)\phi Y - \eta(Y)\phi X\right), \end{aligned} \tag{25}$$

A Ricci soliton emerges as the limit of the solutions of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group of diffeomorphism and scaling. A Ricci soliton (g, V, λ) on a Riemannian manifold (M, g) is a generalization of an Einstein metric such that

$$(L_V g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0, \tag{26}$$

where S is the Ricci tensor, L_V is the Lie derivative operator along the vector field V on M and λ is a real number. The Ricci soliton is said to be shrinking, steady or expanding according to λ being negative, zero or positive, respectively.

As a generalization of Ricci soliton, the notion of η-Ricci soliton was introduced by Cho and Kimura. They have studied Ricci soliton of real hypersurfaces in a non-flat complex space form and defined η-Ricci soliton, which satisfies the equation

$$(L_V g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0, \tag{27}$$

where λ and μ are real numbers. In particular, if $\mu = 0$, then the notion of η-Ricci soliton (g, V, λ, μ) reduces to the notion of Ricci soliton (g, V, λ).

3. Almost η-Ricci Solitons on Kenmotsu Manifolds Admitting General Connection

We consider a Kenmotsu manifold with respect to general connection admitting an η-Ricci soliton (g, ξ, λ, μ). Then from (27), it obvious that

$$\left(L_\xi^G g\right)(X,Y) + 2S^G(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0, \tag{28}$$

where L_ξ is the Lie derivative operator along the vector field ξ on M. Now, we express the Lie derivative along ξ on M with respect to general connection as follows:

$$\begin{aligned} \left(L_\xi^G g\right)(X,Y) &= L_\xi^G g(X,Y) - g\left(L_\xi^G X, Y\right) - g\left(X, L_\xi^G Y\right) \\ &= L_\xi^G g(X,Y) - g([\xi,X],Y) - g(X,[\xi,Y]). \end{aligned} \tag{29}$$

By the help of (1) and (29) and using (15) , (16) , the (29) reduces to

$$\left(L_{\xi}^{G}g\right)(X,Y)=2\left(1-\kappa_{1}\right)\left[g\left(X,Y\right)-\eta\left(X\right)\eta\left(Y\right)\right]. \tag{30}$$

By virtue of (30), the equation (28) takes the following form

$$S^{G}\left(X,Y\right)=\left(\kappa_{1}-\lambda-1\right)g\left(X,Y\right)-\left(\kappa_{1}+\mu-1\right)\eta\left(X\right)\eta\left(Y\right). \tag{31}$$

Thus, we can state the following theorem.

Theorem 3.1. *If (g,ξ,λ,μ) is an η-Ricci soliton on Kenmotsu manifold with respect to general connection M, then M is an η-Einstein manifold provided $\lambda \neq \kappa_1 - 1$ and $\mu \neq 1-\kappa_1$.*

Specifically, if $\lambda \neq \kappa_1 - 1$ and $\mu = 1-\kappa_1$, Kenmotsu manifold with respect to general connection M admitting η-Ricci soliton reduces to Einstein manifold.

If we choose $Y=\xi$ in (31) , we have

$$S^{G}\left(X,\xi\right)=-\left(\lambda+\mu\right)\eta\left(X\right). \tag{32}$$

Setting $X=Y=\xi$ in (32) , we have

$$\lambda+\mu=\left(n-1\right)\left(1-\kappa_{1}\right)^{2}. \tag{33}$$

Then, we can give the following results by using the equation (33).

Corollary 3.2. *If (g,ξ,λ,μ) is an η-Ricci soliton on Kenmotsu manifold admitting quarter symmetric metric connection, then the following classfications are true:*
i) If $\mu < n-1$, then M is expanding.
ii) If $\mu = n-1$, then M is steady.
iii) If $\mu > n-1$, then M is shrinking.

Corollary 3.3. *If (g,ξ,λ,μ) is an η-Ricci soliton on Kenmotsu manifold with respect to Schouten-van Kampen, Tanaka Webster or Zamkovoy connection M, respectively, then the following is provided:*
i) If $\mu < 0$, then M is expanding.
ii) If $\mu = 0$, then M is steady.
iii) If $\mu > 0$, then M is shrinking.

Definition 3.4. *Let M be an $n=(2m+1)-$dimensional Kenmotsu manifold with respect to general connection. If $R^G \cdot S^G$ and $Q^G\left(g,S^G\right)$ are linearly dependent, then the M is said to be **Ricci pseudosymmetric**.*

In this case, there exists a function $\mathcal{L}_R$ on M such that

$$R^{G}\cdot S^{G}=\mathcal{L}_{R}Q^{G}\left(g,S^{G}\right).$$

In particular, if $\mathcal{L}_R = 0$, the M is said to be **Ricci semisymmetric**.

Theorem 3.5. *Let M be an $n=(2m+1)-$dimensional Kenmotsu manifold with respect to general connection and (g,ξ,λ,μ) be an η-Ricci soliton on M. If M is a Ricci pseudosymmetric, then at least one of the following is provided:*
i) $\mathcal{L}_R = -\left(1-\kappa_1\right)^2$,
ii) $\lambda = \left(\kappa_1 - 1\right)+\left(n-1\right)\left(\kappa_1-1\right)^2$,
iii) $\kappa_1 = 1$,
iv) $\kappa_2 = 0$.

Proof. Let's assume that $n = (2m+1)$-dimensional Kenmotsu manifold with respect to general connection M be a Ricci pseudosymmetric and (g, ξ, λ, μ) be almost $\eta-$Ricci soliton on M. That's mean

$$\left(R^G(X,Y)\cdot S^G\right)(U,V) = \mathcal{L}_R Q^G\left(g, S^G\right)(U,V;X,Y),$$

for all $X, Y, U, V \in \Gamma(TM)$. From the last equation, we can easily write

$$\begin{aligned}&S^G\left(R^G(X,Y)U,V\right) + S^G\left(U, R^G(X,Y)V\right)\\&= \mathcal{L}_R\left\{S^G\left(\left(X\wedge_g Y\right)U,V\right) + S^G\left(U,\left(X\wedge_g Y\right)V\right)\right\}.\end{aligned} \tag{34}$$

If we choose $V = \xi$ in (34), we get

$$\begin{aligned}&S^G\left(R^G(X,Y)U,\xi\right) + S^G\left(U, R^G(X,Y)\xi\right)\\&= \mathcal{L}_R\left\{S^G\left(\left(X\wedge_g Y\right)U,\xi\right) + S^G\left(U,\left(X\wedge_g Y\right)\xi\right)\right\}.\end{aligned}$$

If we make use of (2), (23), (24) in the last equality, we have

$$\begin{aligned}&(1-n)(1-\kappa_1)^4 g(\eta(Y)X - \eta(X)Y, U)\\&-\kappa_2(1-\kappa_1)^3(1-n)g\left(\eta(X)\phi Y - \eta(Y)\phi X, U\right)\\&+(1-\kappa_1)^2 S^G(\eta(X)Y - \eta(Y)X, U)\\&+\kappa_2(\kappa_1-1)S^G\left(\eta(Y)\phi X - \eta(X)\phi Y, U\right)\\&= \mathcal{L}_R\left\{(1-n)(1-\kappa_1)^2 g(\eta(X)Y - \eta(Y)X, U)\right.\\&\left.+\ S^G(\eta(Y)X - \eta(X)Y, U)\right\}.\end{aligned} \tag{35}$$

If we use (31) in (35) and make use of (4), we get

$$\begin{aligned}&\left[(1-\kappa_1)^2 + \mathcal{L}_R\right]\left[(n-1)(1-\kappa_1)^2 + (\kappa_1 - \lambda - 1)\right]U\\&+\kappa_2(1-\kappa_1)\left[(n-1)(\kappa_1-1)^2 + (\kappa_1-\lambda-1)\right]\phi U = 0.\end{aligned} \tag{36}$$

Since the vector fields U and ϕU are linearly independent, we can easily write (36) equation as

$$\left[(1-\kappa_1)^2 + \mathcal{L}_R\right]\left[(n-1)(1-\kappa_1)^2 + (\kappa_1-\lambda-1)\right], \tag{37}$$

and

$$\kappa_2(1-\kappa_1)\left[(n-1)(\kappa_1-1)^2 + (\kappa_1-\lambda-1)\right]. \tag{38}$$

It is clear from (37) and (38) that

$$\mathcal{L}_R = -(1-\kappa_1)^2,$$

$$\text{or } \lambda = (\kappa_1 - 1) + (n-1)(\kappa_1-1)^2,$$

$$\text{or } \kappa_1 - 1 = 0,$$

$$\text{or } \kappa_2 = 0.$$

This completes the proof. □

We can give some important consequences of this theorem as follows.

Corollary 3.6. *Let M be an $n = (2m+1)$ −dimensional Kenmotsu manifold with respect to general connection and (g, ξ, λ, μ) be an η-Ricci soliton on M. If M is a Ricci pseudosymmetric, then the following is provided:*
i) If $(n-1)(\kappa_1 - 1)^2 > 1 - \kappa_1$, then M is expanding.
ii) If $(n-1)(\kappa_1 - 1)^2 = 1 - \kappa_1$, then M is steady.
ii) If $(n-1)(\kappa_1 - 1)^2 < 1 - \kappa_1$, then M is shrinking.

Corollary 3.7. *Let M be an $n = (2m+1)$ −dimensional Kenmotsu manifold with respect to general connection and (g, ξ, λ, μ) be an η-Ricci soliton on M. If M is a Ricci semisymmetric, then the following is provided:*
i) $\kappa_1 = 1$,
ii) $\mu = 1 - \kappa_1$,
iii) $\lambda = (\kappa_1 - 1) + (n-1)(\kappa_1 - 1)^2$,
iv) M is an η-Einstein manifold,
v) If $(n-1)(\kappa_1 - 1)^2 > 1 - \kappa_1$, then M is expanding,
vi) If $(n-1)(\kappa_1 - 1)^2 = 1 - \kappa_1$, then M is steady,
vii) If $(n-1)(\kappa_1 - 1)^2 < 1 - \kappa_1$, then M is shrinking.

Corollary 3.8. *Let M be an $n = (2m+1)$ −dimensional Kenmotsu manifold with respect to quarter-symmetric metric connection and (g, ξ, λ, μ) be an η-Ricci soliton on M. If M is a Ricci pseudosymmetric, then the following is provided:*
i) $\mathcal{L}_R = -1$ or $\lambda = n - 2$.
ii) M is an η-Einstein manifold.
iii) M is always an expanding.

Corollary 3.9. *Let M be an $n = (2m+1)$ −dimensional Kenmotsu manifold with respect to Schouten-van Kampen, Tanaka Webster or Zamkovoy connection, respectively, and (g, ξ, λ, μ) be an η-Ricci soliton on M. If M is a Ricci pseudosymmetric, then M is either Ricci semisymmetric or always a steady.*

For an $n = (2m+1)$-dimensional semi-Riemann manifold M, the projective curvature tensor is defined as

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2m}\left[S(Y,Z)X - S(X,Z)Y\right].$$

Then, for an $n = (2m+1)$-dimensional Kenmotsu manifold with respect to general connection, the projective curvature tensor is defined as

$$P^G(X,Y)Z = R^G(X,Y)Z - \frac{1}{2m}\left[S^G(Y,Z)X - S^G(X,Z)Y\right]. \tag{39}$$

If we choose $Z = \xi$ in (39), we can write

$$P^G(X,Y)\xi = \kappa_2(\kappa_1 - 1)\left[\eta(Y)\phi X - \eta(X)\phi Y\right], \tag{40}$$

and similarly if we take the inner product of both sides of (39) by ξ, we get

$$\eta\left(P^G(X,Y)Z\right) = \kappa_2(\kappa_1 - 1)\, g\left(\eta(X)\phi Y - \eta(Y)\phi X, Z\right). \tag{41}$$

Definition 3.10. *Let M be an $n = (2m+1)$-dimensional Kenmotsu manifold with respect to general connection. If $P^G \cdot S^G$ and $Q^G\left(g, S^G\right)$ are linearly dependent, then M is said to be* ***projective Ricci pseudosymmetric.***

In this case, there exists a function $\mathcal{L}_P$ on M such that

$$P^G \cdot S^G = \mathcal{L}_P Q^G\left(g, S^G\right).$$

In particular, if $\mathcal{L}_P = 0$, the M is said to be **projective Ricci semisymmetric.**

Theorem 3.11. *Let M be an $n=(2m+1)-$dimensional Kenmotsu manifold with respect to general connection and (g,ξ,λ,μ) be an η-Ricci soliton on M. If M is a projectively Ricci pseudosymmetric, then at least one of the following is provided:*
i) M is a projectively Ricci semisymmetric manifold,
ii) M is an η-Einstein manifold,
iii) $\kappa_1=1$,
iv) $\kappa_2=0$,
v) $\lambda=(1-n)(1-\kappa_1)^2-(1-\kappa_1)$.

Proof. Let's assume that $n=(2m+1)$-dimensional Kenmotsu manifold with respect to general connection M be a projectively Ricci pseudosymmetric and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M. That's mean

$$\left(P^G(X,Y)\cdot S^G\right)(U,V)=\mathcal{L}_P Q^G\left(g,S^G\right)(U,V;X,Y),$$

for all $X,Y,U,V\in\Gamma(TM)$. From the last equation, we can easily write

$$\begin{aligned}&S^G\left(P^G(X,Y)U,V\right)+S^G\left(U,P^G(X,Y)V\right)\\&=\mathcal{L}_P\left\{S^G\left(\left(X\wedge_g Y\right)U,V\right)+S^G\left(U,\left(X\wedge_g Y\right)V\right)\right\}.\end{aligned}\tag{42}$$

If we choose $V=\xi$ in (42), we get

$$\begin{aligned}&S^G\left(P^G(X,Y)U,\xi\right)+S^G\left(U,P^G(X,Y)\xi\right)\\&=\mathcal{L}_P\left\{S^G\left(\left(X\wedge_g Y\right)U,\xi\right)+S^G\left(U,\left(X\wedge_g Y\right)\xi\right)\right\}.\end{aligned}$$

If we make use of (23) and (40) in the last equality, we have

$$\begin{aligned}&(1-n)(1-\kappa_1)^2\eta\left(P^G(X,Y)U\right)\\&+\kappa_2(\kappa_1-1)S^G\left(\eta(Y)\phi X-\eta(X)\phi Y,U\right)\\&=\mathcal{L}_P\left\{(1-n)(1-\kappa_1)^2g(\eta(X)Y-\eta(Y)X,U)\right.\\&\left.+S^G(\eta(Y)X-\eta(X)Y,U)\right\}.\end{aligned}\tag{43}$$

If we use (41) in (43), we get

$$\begin{aligned}&\kappa_2(n-1)(1-\kappa_1)^3g\left(\eta(X)\phi Y-\eta(Y)\phi X,U\right)\\&+\kappa_2(\kappa_1-1)S^G\left(\eta(Y)\phi X-\eta(X)\phi Y,U\right)\\&=\mathcal{L}_P\left\{(1-n)(1-\kappa_1)^2g(\eta(X)Y-\eta(Y)X,U)\right.\\&\left.+S^G(\eta(Y)X-\eta(X)Y,U)\right\}.\end{aligned}\tag{44}$$

If we use (31) in (44), we have

$$\begin{aligned}&\kappa_2(\kappa_1-1)\left[(n-1)(1-\kappa_1)^2+(\kappa_1-\lambda-1)\right]g\left(\eta(X)Y-\eta(Y)X,\phi U\right)\\&+\mathcal{L}_P\left[(n-1)(1-\kappa_1)^2+(\kappa_1-\lambda-1)\right]g(\eta(X)Y-\eta(Y)X,U)=0,\end{aligned}$$

and so

$$\begin{aligned}&\kappa_2(\kappa_1-1)\left[(n-1)(1-\kappa_1)^2+(\kappa_1-\lambda-1)\right]\phi U\\&+\mathcal{L}_P\left[(n-1)(1-\kappa_1)^2+(\kappa_1-\lambda-1)\right]U=0.\end{aligned} \tag{45}$$

Since the vector fields U and ϕU are linearly independent, we can easily write (45) equation as

$$\kappa_2(\kappa_1-1)\left[(n-1)(1-\kappa_1)^2+(\kappa_1-\lambda-1)\right]=0, \tag{46}$$

and

$$\mathcal{L}_P\left[(n-1)(1-\kappa_1)^2+(\kappa_1-\lambda-1)\right]=0. \tag{47}$$

It is clear from (46) and (47) that

$\mathcal{L}_P=0$,
or $\kappa_1=1$,
or $\kappa_2=0$,
or $\lambda=(n-1)(1-\kappa_1)^2+(\kappa_1-1)$.

This completes the proof. □

We can give some important results of this theorem as follows.

Corollary 3.12. *Let M be an $n=(2m+1)-$dimensional Kenmotsu manifold with respect to general connection and (g,ξ,λ,μ) be an η-Ricci soliton on M. If M is a projectively Ricci pseudosymmetric, then the following is provided:*
i) If $(n-1)(1-\kappa_1)^2>1-\kappa_1$, then M is expanding.
ii) If $(n-1)(1-\kappa_1)^2=1-\kappa_1$, then M is steady.
ii) If $(n-1)(1-\kappa_1)^2<1-\kappa_1$, then M is shrinking.

Corollary 3.13. *Let M be an $n=(2m+1)-$dimensional Kenmotsu manifold with respect to quarter-symmetric metric connection and (g,ξ,λ,μ) be an η-Ricci soliton on M. If M is a projectively Ricci pseudosymmetric, then M is either projectively Ricci semisymmetric or always an expanding or $\kappa_2=0$.*

Corollary 3.14. *Let M be an $n=(2m+1)-$dimensional Kenmotsu manifold with respect to Schouten-van Kampen, Tanaka Webster or Zamkovoy connection, respectively, and (g,ξ,λ,μ) be an η-Ricci soliton on M. If M is a projectively Ricci pseudosymmetric, then M is either projectively Ricci semisymmetric or always a shrinking.*

For an $n=(2m+1)$-dimensional semi-Riemann manifold M, the concircular curvature tensor is defined as

$$C(X,Y)Z=R(X,Y)Z-\frac{r}{n(n-1)}\left[g(Y,Z)X-g(X,Z)Y\right].$$

For an $n-$dimensional Kenmotsu manifold with respect to general connection M, the concircular curvature tensor is defined as

$$C^G(X,Y)Z=R^G(X,Y)Z-\frac{r}{n(n-1)}\left[g(Y,Z)X-g(X,Z)Y\right]. \tag{48}$$

If we choose $Z=\xi$ in (48), we can write

$$\begin{aligned}C^G(X,Y)\xi &=\left[(1-\kappa_1)^2-\tfrac{r}{n(n-1)}\right]\left[\eta(X)Y-\eta(Y)X\right]\\&\quad+\kappa_2(\kappa_1-1)\left[\eta(Y)\phi X-\eta(X)\phi Y\right],\end{aligned} \tag{49}$$

and similarly if we take the inner product of both sides of (48) by ξ, we get

$$\eta\left(C^G(X,Y)Z\right) = \left[(1-\kappa_1)^2 - \tfrac{r}{n(n-1)}\right] g\left(\eta(Y)X - \eta(X)Y, Z\right) + \kappa_2(\kappa_1 - 1)\, g\left(\eta(X)\phi Y - \eta(Y)\phi X, Z\right). \tag{50}$$

Definition 3.15. *Let M be an $n = (2m+1)$-dimensional Kenmotsu manifold with respect to general connection. If $C^G \cdot S^G$ and $Q^G\left(g, S^G\right)$ are linearly dependent, then M is said to be* ***concircular Ricci pseudosymmetric.***

In this case, there exists a function $\mathcal{L}_C$ on M such that

$$C^G \cdot S^G = \mathcal{L}_C Q^G\left(g, S^G\right).$$

In particular, if $\mathcal{L}_C = 0$, the M is said to be **concircular Ricci semisymmetric.**

Theorem 3.16. *Let M be an $n = (2m+1)$ −dimensional Kenmotsu manifold with respect to general connection and (g, ξ, λ, μ) be an η-Ricci soliton on M. If M is a concircular Ricci pseudosymmetric, then the following is provided:*
i) M is an η-Einstein manifold.
ii) $\mathcal{L}_C = \frac{r}{n(n-1)} - (1-\kappa_1)^2$.
iii) $\lambda = (\kappa_1 - 1) - (1-n)(1-\kappa_1)^2$.
iv) $\kappa_1 = 1$.
v) $\mu = 1 - \kappa_1$.

Proof. Let's assume that $n = (2m+1)$-dimensional Kenmotsu manifold with respect to general connection M be a concircular Ricci pseudosymmetric and (g, ξ, λ, μ) be almost η−Ricci soliton on M. That's mean

$$\left(C^G(X,Y) \cdot S^G\right)(U,V) = \mathcal{L}_C Q^G\left(g, S^G\right)(U,V;X,Y),$$

for all $X, Y, U, V \in \Gamma(TM)$. From the last equation, we can easily write

$$S^G\left(C^G(X,Y)U, V\right) + S^G\left(U, C^G(X,Y)V\right) = \mathcal{L}_C\left\{S^G\left(\left(X \wedge_g Y\right)U, V\right) + S^G\left(U, \left(X \wedge_g Y\right)V\right)\right\}. \tag{51}$$

If we choose $V = \xi$ in (51), we get

$$S^G\left(C^G(X,Y)U, \xi\right) + S^G\left(U, C^G(X,Y)\xi\right) = \mathcal{L}_C\left\{S^G\left(\left(X \wedge_g Y\right)U, \xi\right) + S^G\left(U, \left(X \wedge_g Y\right)\xi\right)\right\}.$$

If we make use of (23) and (49) in the last equality, we have

$$\begin{aligned} &(1-n)(1-\kappa_1)^2\,\eta\left(C^G(X,Y)U\right) \\ &+\left[(\kappa_1-1)^2 - \tfrac{r}{n(n-1)}\right] S^G\left(\eta(X)Y - \eta(Y)X, U\right) \\ &+\kappa_2(\kappa_1-1)\, S^G\left(\eta(Y)\phi X - \eta(X)\phi Y, U\right) \\ &= \mathcal{L}_C\left\{(1-n)(1-\kappa_1)^2\, g\left(\eta(X)Y - \eta(Y)X, U\right)\right. \\ &\left.+\, S^G\left(\eta(Y)X - \eta(X)Y, U\right)\right\}. \end{aligned} \tag{52}$$

If we use (50) in (52), we get

$$\begin{aligned}
&(1-n)(1-\kappa_1)^2\left[(1-\kappa_1)^2-\tfrac{r}{n(n-1)}\right]g(\eta(Y)X-\eta(X)Y,U)\\
&-(1-n)(1-\kappa_1)^3\kappa_2 g\left(\eta(X)\phi Y-\eta(Y)\phi X,U\right)\\
&+\left[(1-\kappa_1)^2-\tfrac{r}{n(n-1)}\right]S^G(\eta(X)Y-\eta(Y)X,U)\\
&+\kappa_2(\kappa_1-1)S^G\left(\eta(Y)\phi X-\eta(X)\phi Y,U\right)\\
&=\mathcal{L}_C\left\{(1-n)(1-\kappa_1)^2 g(\eta(X)Y-\eta(Y)X,U)\right.\\
&\left.+\ S^G(\eta(Y)X-\eta(X)Y,U)\right\}.
\end{aligned} \tag{53}$$

If we use (31) in (53), we have

$$\begin{aligned}
&\left\{\mathcal{L}_C+\left[(1-\kappa_1)^2-\tfrac{r}{n(n-1)}\right]\right\}\left[(1-n)(1-\kappa_1)^2-(\kappa_1-\lambda-1)\right]U\\
&-\kappa_2(1-\kappa_1)\left[(1-n)(1-\kappa_1)^2-(\kappa_1-\lambda-1)\right]\phi U=0.
\end{aligned} \tag{54}$$

Since the vector fields U and ϕU are linearly independent, we can easily write (54) equation as

$$\left\{\mathcal{L}_C+\left[(1-\kappa_1)^2-\frac{r}{n(n-1)}\right]\right\}\left[(1-n)(1-\kappa_1)^2-(\kappa_1-\lambda-1)\right]=0, \tag{55}$$

and

$$-\kappa_2(1-\kappa_1)\left[(1-n)(1-\kappa_1)^2-(\kappa_1-\lambda-1)\right]=0. \tag{56}$$

It is clear from (55) and (56) that

$$\begin{aligned}
\mathcal{L}_C&=\tfrac{r}{n(n-1)}-(1-\kappa_1)^2,\\
\text{or } \lambda&=(1-n)(1-\kappa_1)^2-(\kappa_1-1),\\
&\text{or } \kappa_1=1,\\
&\text{or } \kappa_2=0.
\end{aligned}$$

This completes the proof. □

We can give some important consequences of this theorem as follows.

Corollary 3.17. *Let M be an $n=(2m+1)$ −dimensional Kenmotsu manifold with respect to general connection and (g,ξ,λ,μ) be an η-Ricci soliton on M. If M is a concircular Ricci semisymmetric, then the following is provided:*
i) $\kappa_1=1$.
ii) $\lambda=(\kappa_1-1)-(1-n)(1-\kappa_1)^2-1$.
iii) M is a scalar curvature manifold with $r=n(n-1)(1-\kappa_1)^2$.
iv) If $(1-n)(1-\kappa_1)^2>(\kappa_1-1)$, *then M is expanding.*
v) If $(1-n)(1-\kappa_1)^2=(\kappa_1-1)$, *then M is steady.*
vi) If $(1-n)(1-\kappa_1)^2<(\kappa_1-1)$, *then M is shrinking.*
vii) M is an η-Einstein manifold.

Corollary 3.18. *Let M be an $n=(2m+1)$ −dimensional Kenmotsu manifold with respect to quarter-symmetric metric connection and (g,ξ,λ,μ) be an η-Ricci soliton on M. If M is a concircular Ricci pseudosymmetric, then M is either always an expanding or* $\mathcal{L}_C=\dfrac{r-n(n-1)}{n(n-1)}$.

Corollary 3.19. *Let M be an $n = (2m+1)$ −dimensional Kenmotsu manifold with respect to quarter-symmetric metric connection and (g, ξ, λ, μ) be an η-Ricci soliton on M. If M is a concircular Ricci semisymmetric, then M is either always an expanding or a scalar curvature manifold with $r = n(n-1)$.*

Corollary 3.20. *Let M be an $n = (2m+1)$ −dimensional Kenmotsu manifold with respect to Schouten-van Kampen, Tanaka Webster or Zamkovoy connection, respectively, and (g, ξ, λ, μ) be an η-Ricci soliton on M. If M is a concircular Ricci pseudosymmetric, then M is either always a steady or $\mathcal{L}_C = \frac{r}{n(n-1)}$.*

Corollary 3.21. *Let M be an $n = (2m+1)$ −dimensional Kenmotsu manifold with respect to Schouten-van Kampen, Tanaka Webster or Zamkovoy connection, respectively, and (g, ξ, λ, μ) be an η-Ricci soliton on M. If M is a concircular Ricci semisymmetric, then M is either always a steady or a scalar curvature manifold with $r = 0$.*

Now let's give an important example given earlier by A. Biswas and K. K. Baishya which supports the above theorems and corollaries [1].

Example 3.22. *By the help of [1] we introduce an example of 3-dimensional Kenmotsu manifold with respect to Generalised Tanaka-Webster connection. Choosing the linearly independent vector field as*

$$e_1 = e^{-z}\frac{\partial}{\partial x}, e_2 = e^{-z}\frac{\partial}{\partial y}, e_3 = \frac{\partial}{\partial z},$$

at each point of 3-dimensional manifold M, where

$$M = \left\{(x, y, z) \in \mathbb{R}^3 \middle| x \neq 0\right\}.$$

Let g be the Riemannian metric. The 1-form η is defined by

$$g(Y, e_3) = \eta(Y),$$

and the (1, 1)-type tensor field φ is defined by

$$\phi e_1 = -e_2, \phi(e_2) = e_1 \text{ and } \phi(e_3) = 0.$$

Let D be the Levi-Civita connection with respect to the Riemannian metric g. Then we have

$$[e_1, e_2] = 0, [e_1, e_3] = e_1, [e_2, e_3] = e_2.$$

Considering $e_3 = \xi$ and Koszul's formula we get

$$D_{e_1}e_3 = e_1, \quad D_{e_1}e_2 = 0, \quad D_{e_1}e_3 = -e_3,$$

$$D_{e_2}e_1 = 0, \quad D_{e_2}e_2 = -e_3, \quad D_{e_2}e_3 = e_2,$$

$$D_{e_3}e_1 = 0, \quad D_{e_3}e_2 = 0, \quad D_{e_3}e_3 = 0.$$

By the help of the last two equations above, we obtain the following

$$R(e_1, e_2)e_3 = 0, \quad R(e_1, e_2)e_2 = -e_1,$$

$$R(e_1, e_3)e_3 = -e_1, \quad R(e_2, e_2)e_3 = 0,$$

$$R(e_2, e_3)e_3 = -e_2, \quad R(e_2, e_1)e_1 = -e_2$$

$$R(e_3, e_2)e_2 = -e_3, \quad R(e_3, e_1)e_2 = 0,$$

$$R(e_3, e_1)e_1 = -e_3, \quad R(e_3, e_2)e_1 = 0,$$

$$R(e_2, e_1)e_3 = 0, \quad R(e_1, e_3)e_2 = 0.$$

Using (15) *and* (19) *, we can easily calculate the following*

$$D^G_{e_1}e_1 = (\kappa_1 - 1)\, e_3, \quad D^G_{e_1}e_2 = 0, \quad D^G_{e_1}e_3 = (1 - \kappa_1)\, e_1,$$

$$D^G_{e_2}e_1 = 0, \quad D^G_{e_2}e_2 = (1 - \kappa_1)\, e_3, \quad D^G_{e_2}e_3 = (1 - \kappa_1)\, e_2,$$

$$D^G_{e_3}e_1 = -\kappa_2 e_2, \quad D^G_{e_3}e_2 = \kappa_2 e_1, \quad D^G_{e_3}e_3 = 0,$$

$$R^G(e_1, e_2)\, e_2 = \left(\kappa_1^2 + 2\kappa_1 - 1\right) e_1,$$

$$R^G(e_1, e_2)\, e_3 = 0,$$

$$R^G(e_2, e_3)\, e_3 = -\kappa_2 (1 - \kappa_1)\, e_1 - (1 - \kappa_1)\, e_2,$$

$$R^G(e_3, e_1)\, e_1 = (\kappa_1 - 1)\, e_3,$$

$$R^G(e_3, e_2)\, e_2 = (\kappa_1 - 1)\, e_3,$$

$$R^G(e_2, e_1)\, e_1 = -\left(\kappa_1^2 - 2\kappa_1 + 1\right) e_2,$$

$$R^G(e_1, e_3)\, e_3 = \kappa_2 (1 - \kappa_1)\, e_2 - (1 - \kappa_1)\, e_1,$$

$$R^G(e_1, e_3)\, e_2 = \kappa_2 (\kappa_1 - 1)\, e_3,$$

$$R^G(e_2, e_1)\, e_3 = 0,$$

$$R^G(e_1, e_3)\, e_3 = \kappa_2 (1 - \kappa_1)\, e_2 - (1 - \kappa_1)\, e_1,$$

$$S^G(e_1, e_1) = \kappa_1^2 + 3\kappa_1 - 2,$$

$$S^G(e_2, e_2) = \kappa_1^2 + 3\kappa_1 - 2,$$

$$S^G(e_3, e_3) = 2(\kappa_1 - 1),$$

and

$$r^G = 2\kappa_1^2 + 8\kappa_1 - 6.$$

Thus it can be seen that equation (24) *is satisfied. Now from* 31, *we get*

$$\lambda + \mu = 2(1 - \kappa_1).$$

Hence the manifold under consideration satisfies above theorems and corollaries.

References

[1] A. Biswas, K.K. Baishya, *Study on generalized pseudo (Ricci) symmetric Sasakian manifold admitting general connection*, Bulletin of the Transilvania University of Brasov. **12(61)** (2) (2019), 233-246.

[2] A. Biswas, K.K. Baishya, *A general connection on Sasakian manifolds andthe case of almost pseudo symmetric Sasakian manifolds*, Scientific Studies and Research Series Mathematics and Informatics. **29(1)** (2019), 59-72.

[3] S. Golab, *On semi-symmetric and quarter-symmetric linear connections*, Tensor(N.S.), **29** (1975), 249–254.

[4] A. Biswas, S. Das, K.K. Baishya, *On Sasakian manifolds satisfying curvature restrictions with respect to quarter symmetric metric connection*, ScientificStudies and Research Series Mathematics and Informatics. **28(1)** (2018), 29–40.

[5] J.A. Schouten, E.R. Van Kampen, *Zur Einbettungs-und Krummungsthe-orie nichtholonomer*, Gebilde Math. Ann. **103** (1930), 752–783.

[6] S. Tanno, *The automorphism groups of almost contact Riemannian manifold*,Tohoku Math. J. **21** (1969), 21–38.

[7] S. Zamkovoy, *Canonical connections on paracontact manifolds,* Ann. GlobalAnal. Geom. **36(1)** (2008), 37–60.
[8] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, http://arXiv.org/abs/math/0211159, (2002), 1–39.
[9] G. Perelman, *Ricci flow with surgery on three manifolds*, http://arXiv.org/abs/math/0303109, (2003), 1–22.
[10] R. Sharma, *Certain results on k-contact and* (k, μ)*-contact manifolds*, J. Geom. **89** (2008),138–147.
[11] S.R. Ashoka, C.S. Bagewadi, G. Ingalahalli, *Certain results on Ricci Solitons in* $\alpha-$*Sasakian manifolds*, Hindawi Publ. Corporation, Geometry, Vol.(2013), Article ID 573925,4 Pages.
[12] S.R. Ashoka, C.S. Bagewadi, G. Ingalahalli, *A geometry on Ricci solitons in* $(LCS)_n$ *manifolds*, Diff. Geom. Dynamical Systems. **16** (2014), 50–62.
[13] C.S. Bagewadi, G. Ingalahalli, *Ricci solitons in Lorentzian-Sasakian manifolds*, Acta Math. Acad. Paeda. Nyire. **28** (2012), 59-68.
[14] G. Ingalahalli, C.S. Bagewadi, *Ricci solitons in* $\alpha-$*Sasakian manifolds*, ISRN Geometry. Vol.(2012), Article ID 421384, 13 Pages.
[15] C.L. Bejan, M. Crasmareanu, *Ricci Solitons in manifolds with quasi-contact curvature,* Publ. Math. Debrecen. **78** (2011), 235-243.
[16] A.M. Blaga, $\eta-$*Ricci solitons on para-Kenmotsu manifolds*, Balkan J. Geom. Appl. **20** (2015), 1–13.
[17] S. Chandra, S.K. Hui, A.A. Shaikh, *Second order parallel tensors and Ricci solitons on* $(LCS)_n$*-manifolds*, Commun. Korean Math. Soc. **30** (2015), 123–130.
[18] B.Y. Chen, S. Deshmukh, *Geometry of compact shrinking Ricci solitons*, Balkan J. Geom. Appl. **19** (2014), 13–21.
[19] S. Deshmukh, H. Al-Sodais, H. Alodan, *A note on Ricci solitons*, Balkan J. Geom. Appl. **16** (2011), 48–55.
[20] C. He, M. Zhu, *Ricci solitons on Sasakian manifolds*, arxiv:1109.4407V2, [Math DG], (2011).
[21] M. Atçeken, T. Mert, P. Uygun, *Ricci-Pseudosymmetric* $(LCS)_n$*-manifolds admitting almost* η*-Ricci solitons*, Asian Journal of Math. and Computer Research. **29(2)** (2022), 23-32.
[22] H. Nagaraja, C.R. Premalatta, *Ricci solitons in Kenmotsu manifolds*, J. Math. Analysis, **3(2)** (2012), 18–24.
[23] M.M. Tripathi, *Ricci solitons in contact metric manifolds*, arxiv:0801,4221 V1, [Math DG], (2008).
[24] K.K. Baishya, P.R. Chowdhury, M. Josef, P. Peska, *On almost generalized weakly symmetric Kenmotsu manifolds*, Acta Univ. Palacki. Olomuc. Fac. rer.nat. Mathematica. **55(2)** (2016), 5–15.
[25] S. Eyasmin, P. Roy Chowdhury, K.K. Baishya, η*-Ricci solitons in Kenmotsu manifolds*, Honam Mathematical J. **40(2)** (2018), 383–392.
[26] K. Kenmotsu, *A class of almost contact Riemannian manifolds*, Tohoku Math.J. **24** (1972), 93–103.
[27] V. F. Kirichenko, *On the geometry of Kenmotsu manifolds*, Dokl. Akad. Nauk. Ross. Akad. Nauk. **380** (2001), 585–587.
[28] A.N. Siddiqui, M.D. Siddiqi, V. Vandana, *Ricci solitons on* α*-Sasakian manifolds with quarter symmetric metric connection*, Bulletin of theTransilvania University of Braşov Series III: Mathematics and Computer Science. **4(66)(1)** (2024), 175-190.
[29] A.H. Hakami, M.D. Siddiqi, A.N. Siddiqui, K. Ahmad, *Solitons Equipped with a Semi-Symmetric Metric Connection with Some Applications on Number Theory*, Mathematics. **11(21)** (2023), 44-52.
[30] M.D. Siddiqi, η*-Einstein solitons in an* (ε)*-Kenmotsu manifolds with a semi-symmetric metric connection*, Annales, Univ. Sci. Budapest. **62(LXII)** (2019), 5-25.
[31] M.D. Siddiqi, η*-Ricci solitons in* δ*-Lorentzian trans Sasakian manifolds with a semi-symmetric metric connection*, Kyungpook Mathematical Journal. **59(3)** (2019), 537-562.