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Existence of solutions for ϕ-Caputo fractional hybrid boundary value problem on time scales

Najat Chefnaj[a,*], Abdellah Taqbibt[a], Khalid Hilal[a], M'hamed Elomari[a]

[a] *Applied Mathematics and Scientific Computing Laboratory & Sultan Moulay Slimane University Slimane, BP 523, 23000 Beni Mellal, Morocco*

Abstract. This study explores the existence of solutions for a class of fractional hybrid differential equations incorporating maxima and ϕ-Caputo derivatives on time scales. The analysis employs Dhage's fixed point theorem alongside key methods from ϕ-fractional calculus on time scales. To highlight the practical relevance and effectiveness of the findings, an illustrative example is provided. The proofs are derived using Dhage's fixed point theorem and essential techniques from ϕ-fractional calculus on time scales. To demonstrate the applicability and effectiveness of the results, an illustrative example is presented.

1. Introduction

In recent years, the theory of time-scale differential problems has undergone intense development, as can be seen from references [4, 11]. At the same time, recent years have seen a growing interest in fractional-order differential equations, due to the variety of their applications. These equations play a significant role in fields such as physiology, rheology, control, viscoelasticity, electrochemistry, electromagnetism, and many others. Further details are available in references [6–8, 10, 12–14]. A number of researchers have studied fractional differential equations including maxima, as can be seen in[1, 5, 17].

In [16], Otrocol particularly explored the following problem:

$$\begin{cases} v'(\lambda) = \mathcal{F}(\lambda, v(\lambda)) + \mathcal{G}\left(\lambda, \max\limits_{\ell\in[0,\lambda]} v(\ell)\right), \\ v(0) = \varphi, \end{cases}$$

where $\lambda \in [0,d]$, $d \in \mathbb{R}$, $\varphi \in \mathbb{R}^p$, and $\mathcal{F}, \mathcal{G} \in [0,d] \times \mathbb{R}^p \longrightarrow \mathbb{R}^p$.

In [15], the authors investigated the existence and uniqueness of solutions to initial value problems for Caputo Δ-fractional differential equations with maxima on the time scales $\mathbb{T}_S$ of the form:

$$\begin{cases} {}^{C}\Delta_c^{\gamma} v(\lambda) = \xi\left(\lambda, v(\lambda), V(\lambda)\right), \quad \lambda \in \Sigma = [c,d]_{\mathbb{T}_S} := [c,d] \cap \mathbb{T}_S, \quad 0 < \gamma < 1, \\ v(c) = \varphi, \end{cases}$$

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* Corresponding author: Najat Chefnaj
Email addresses: najatchefnajj@gmail.com (Najat Chefnaj), abdellah.taqbibt@usms.ma (Abdellah Taqbibt), hilalkhalid2005@yahoo.fr (Khalid Hilal), e.elomari@usms.ma (M'hamed Elomari)
ORCID iDs: https://orcid.org/0000-0003-1153-4932 (Najat Chefnaj), https://orcid.org/0000-0001-5650-1988 (Abdellah Taqbibt), https://orcid.org/0000-0002-0806-2623 (Khalid Hilal), https://orcid.org/0000-0002-4256-1434 (M'hamed Elomari)

where $v : [c,d]_{\mathbb{T}_S} \longrightarrow \mathbb{R}$, $V(\lambda) = \max_{\ell\in[c,\lambda]} v(\ell)$, $d > c$, ${}^C\Delta_c^{\gamma}$ is the Caputo Δ-fractional derivative operator of order γ, $\xi : [c,d]_{\mathbb{T}_S} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a function, and φ represents a real number.

In this article, we investigate the existence of a solution for ϕ-Caputo's Δ-fractional nonlinear hybrid differential equations with maxima on the time scales:

$$\begin{cases} {}^C\Delta_c^{\phi,\gamma}\left(\frac{v(\lambda)}{\mathcal{F}(\lambda,v(\lambda))}\right) = \xi\left(\lambda, v(\lambda), V(\lambda)\right), \quad \lambda \in \Sigma = [c,d]_{\mathbb{T}_S} := \mathbb{T}_S \cap [c,d], \quad 0 < \gamma < 1, \\ \frac{v(c)}{\mathcal{F}(c,v(c))} = \varphi, \end{cases} \tag{1}$$

where ${}^C\Delta_c^{\phi,\gamma}$ is ϕ-Caputo Δ-fractional derivative operator of order γ, $c < d$, $\varphi \in \mathbb{R}$, $v : [c,d]_{\mathbb{T}_S} \longrightarrow \mathbb{R}$, $V(\lambda) = \max_{\ell\in[c,\lambda]} v(\ell)$, $\xi : [c,d]_{\mathbb{T}_S} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a function.
The structure of this paper unfolds as: in the section 2, we present some preliminary concepts related to fractional calculus. In section 3, we establish criteria on the existence of solutions to the problem above. In section 4, provides an example to illustrate the practical applications of of these results.

2. Preliminaries

In this section, we present definitions, notations and results that will be used consistently throughout this paper.

Definition 2.1. ***(ϕ-Caputo Δ-fractional integral operator on the time scales)*** *Given that $\mathbb{T}_S$ denotes a time scale such that $[c,d]$ is an interval of $\mathbb{T}_S$ and v is an integrable function on the interval $[c,d]$. Let $\phi \in C^n([c,d],\mathbb{R})$ with $\phi'(t) > 0$ for each t in $[c,d]$. Consider $\gamma > 0$. The Δ-fractional integral at order γ in the sense of ϕ-Caputo for the function v is given by the following expression*

$${}^{\mathbb{T}}J_c^{\gamma,\phi}v(\lambda) := \frac{1}{\Gamma(\gamma)}\int_c^{\lambda}\phi'(\ell)(\phi(\lambda)-\phi(\ell))^{\gamma-1}v(\ell)\Delta\ell.$$

where $\Gamma(\gamma)$ is the gamma function, defined as

$$\Gamma(\gamma) = \int_0^{\infty}\ell^{\gamma-1}e^{-\ell}\,d\ell \quad \textit{pour } Re(\gamma) > 0.$$

Theorem 2.2. *(semigroup property) Suppose that the function ξ is integrable on the interval $[c,d]$, and that β and α are two positive real constants. Then,*

$${}^{\mathbb{T}_S}J_c^{\alpha,\phi}\,{}^{\mathbb{T}_S}J_c^{\beta,\phi}\xi(\lambda) = {}^{\mathbb{T}_S}J_c^{\alpha+\beta,\phi}\xi(\lambda).$$

Definition 2.3. ***(ϕ-Caputo Δ-fractional derivative operator on the time scales)*** *Given that $\mathbb{T}_S$ denotes a time scale such that $[c,d]$ is an interval of $\mathbb{T}_S$ and $v : \mathbb{T}_S \longrightarrow \mathbb{R}$ is a continuous function. Let $\phi \in C^n([c,d],\mathbb{R})$ with $\phi'(t) > 0$ for every t in $[c,d]$. The Δ-fractional derivative of order γ in the sens of ϕ-Caputo for the function v is defined by de following expression*

$${}^C\Delta_{c^+}^{\gamma,\phi}v(\lambda) = \frac{1}{\Gamma(n-\gamma)}\int_c^{\lambda}\phi'(\ell)(\phi(\lambda)-\phi(\ell))^{n-\gamma-1}v_{\phi}^{\Delta^{[n]}}(\ell)\Delta\ell, \tag{2}$$

where

$$v_{\phi}^{\Delta^{[n]}}(\ell) = \left(\frac{1}{\phi'(\ell)}\frac{d}{d\ell}\right)^{\Delta^n} v(\ell) \quad \textit{and} \quad n = [\gamma] + 1.$$

The symbol $[\cdot]$ denotes the integer part.

A time scale, represented by $\mathbb{T}_\mathbf{S}$, is a nonempty and closed subset of $\mathbb{R}$ (see[2, 3]).

Example 2.4. *Let*

1. $\mathbb{T}_{\mathbf{S}2} = \{\sqrt{r} : r \in \mathbb{N}_0\}$,
2. $\mathbb{T}_{\mathbf{S}1} = \{2^r : r \in \mathbb{Z}\} \cup \{0\}$,

$\mathbb{T}_{\mathbf{S}1}$ *and* $\mathbb{T}_{\mathbf{S}2}$ *are both time scales.*

Definition 2.5. *Let* $\mathbb{T}_\mathbf{S}$ *be a time scale. For each* $\varsigma \in \mathbb{T}_\mathbf{S}$, *we define two operators* $\alpha : \mathbb{T}_\mathbf{S} \to \mathbb{T}_\mathbf{S}$ *and* $\theta : \mathbb{T}_\mathbf{S} \to \mathbb{T}_\mathbf{S}$, *by the following formulas:*

$$\theta(\lambda) = \sup\{\varsigma \in \mathbb{T}_\mathbf{S} : \varsigma < \lambda\},$$

and

$$\alpha(\lambda) = \inf\{\varsigma \in \mathbb{T}_\mathbf{S} : \varsigma > \lambda\}.$$

The operators θ *and* α, *are called backward jump and forward jump, respectively.*

In the previous definition, we specify that

1. $\sup \emptyset = \inf \mathbb{T}_\mathbf{S}$ (which means $\theta(\lambda) = \lambda$ if the set $\mathbb{T}_\mathbf{S}$ contains a minimum element λ),
2. $\inf \emptyset = \sup \mathbb{T}_\mathbf{S}$ (which means $\alpha(\lambda) = \lambda$ if the set $\mathbb{T}_\mathbf{S}$ contains a maximum element λ),

where $\emptyset$ represents the empty set.

Example 2.6. *Let's briefly examine some examples:* $\mathbb{T}_\mathbf{S} = \mathbb{R}$, *and* $\mathbb{T}_\mathbf{S} = \mathbb{Z}$.

(1) *If* $\mathbb{T}_\mathbf{S} = \mathbb{Z}$, *for any number* ς *in the set* $\mathbb{Z}$, *we have*

$$\begin{cases} \alpha(\lambda) = \inf\{\varsigma \in \mathbb{Z} : \varsigma > \lambda\} = inf\{\lambda+1, \lambda+2, \lambda+3, \lambda+4, \lambda+5, \cdots\} = \lambda+1, \\ \theta(\lambda) = \lambda - 1. \end{cases}$$

(2) *If* $\mathbb{T}_\mathbf{S} = \mathbb{R}$, *then for any number* ς *in the set* $\mathbb{R}$, *we have*

$$\begin{cases} \alpha(\lambda) = \inf\{\varsigma \in \mathbb{R} : \varsigma > \lambda\} = inf(\lambda, \infty) = \lambda, \\ \theta(\lambda) = \lambda. \end{cases}$$

Definition 2.7. *Here are some other definitions that we need in this paper*

1. ***Left-Scattered:*** *A point* ς *is said to be left-scattered if* $\theta(\varsigma) < \varsigma$.
2. ***Right-Scattered:*** *A point* ς *is said to be right-scattered if* $\alpha(\varsigma) > \varsigma$.
3. ***Isolated:*** *A point* ς *is called isolated if it is both left-scattered and right-scattered simultaneously.*
4. ***Left-Dense:*** *A point* ς *is said to be left-dense if* $\theta(\varsigma) = \varsigma$ *and* $\varsigma > \inf \mathbb{T}_\mathbf{S}$.
5. ***Right-Dense:*** *A point* ς *is said to be right-dense if* $\alpha(\varsigma) = \varsigma$ *and* $\varsigma < \sup \mathbb{T}_\mathbf{S}$.
6. ***Dense:*** *A point* ς *is called dense if it is both left-dense and right-dense simultaneously.*

Definition 2.8. *Let* $g : \mathbb{T}_\mathbf{S} \longrightarrow \mathbb{R}$ *be a function. The function* g *is termed rd-continuous if it satisfies two conditions:*

1. *It is continuous at all dense points in* $\mathbb{T}_\mathbf{S}$ *when approaching from the right.*
2. *It has finite left-side limits at all left-dense points in* $\mathbb{T}_\mathbf{S}$.

Let C_{rd} denote the set of all functions $g : \mathbb{T}_\mathbf{S} \longrightarrow \mathbb{R}$ that are *rd*-continuous.

Definition 2.9. *[5](**Delta Derivative**) Let $\Psi : \mathbb{T}_{\mathbf{S}} \longrightarrow \mathbb{R}$ be a function and λ be an element of $\mathbb{T}_{\mathbf{S}}$. The Δ-derivative of the function Ψ at the point λ, denoted $\Psi^{\Delta}(\lambda)$ (if it exists), is given such that for all $\kappa > 0$, there exists a neighborhood Υ of $\lambda \in \mathbb{T}_{\mathbf{S}}$ satisfying:*

$$\left|\Psi(\eta(\lambda)) - \Psi(\ell) - \Psi^{\Delta}(\lambda)[\eta(\lambda) - \ell]\right| \leq \kappa|\eta(\lambda) - \ell|, \quad \text{for each } \ell \in \Upsilon.$$

Definition 2.10. *[5] Given a function Ψ defined from $\mathcal{I}$ to $\mathbb{R}$, where $\mathcal{I}$ is a bounded closed interval of $\mathbb{T}_{\mathbf{S}}$. We say that Ψ is a Δ-antiderivative of the function $\psi : [d, r) \longrightarrow \mathbb{R}$ if the following conditions are satisfied:*

1. *Ψ is continuous on $[d, r]$,*
2. *$\Psi^{\Delta}(\lambda) = g(\lambda)$ for all $\lambda \in [d, r)$,*
3. *Ψ is delta differentiable on $[d, r)$.*

The Δ-integral of ψ from d to r is defined as follows

$$\int_d^r \psi(\lambda)\Delta\lambda = \Psi(r) - \Psi(d).$$

Lemma 2.11. *[5] Given that $\mathbb{T}_{\mathbf{S}}$ denotes a time scale and ψ is an increasing and continuous function on $[d, r]$ within this time scale. Define ϖ as the extension of the function ψ to the real interval $[d, r]$ using the following expression*

$$\varpi(\lambda) = \begin{cases} \psi(\lambda) & \text{if } \lambda \in \mathbb{T}_{\mathbf{S}}, \\ \psi(\ell) & \text{if } \lambda \in (\ell, \eta(\ell)) \notin \mathbb{T}_{\mathbf{S}}. \end{cases}$$

Then,

$$\int_d^r \psi(\lambda)\Delta\lambda \leq \int_d^r \varpi(\lambda)d\lambda.$$

Theorem 2.12. *(Dhage theorem)[9] we consider S as a bounded, closed, convex and non-empty subset of Ξ, the Banach algebra. Let $\mathcal{G}_1 : \Xi \longrightarrow \Xi$ and $\mathcal{G}_2 : S \longrightarrow \Xi$ be two operators which satisfy the following properties:*

(a) $\mathcal{G}_2$ is completely continuous,

(b) $\mathcal{G}_1$ is Lipschitzian with a Lipschitz constant κ,

(c) $u = \mathcal{G}_1(u)\mathcal{G}_2(v)$ implies that u in S for every v in S, and

(d) $\eta\kappa < 1$, where $\eta = \|\mathcal{G}_2(S)\| = \sup\{\|\mathcal{G}_2(u)\| : u \in S\}$.

Then, the operator $\Psi u = \mathcal{G}_1(u)\mathcal{G}_2(u)$ has a fixed point.

3. Results

In this section, in order to demonstrate the existence of solutions for the Δ-fractional hybrid problem (1), the following assumptions are required:

(H1) The function $\mathcal{F} \in C(\Sigma \times \mathbb{R}, \mathbb{R}\backslash\{0\})$ where it satisfies the conditions that:

- **(i)** $|\mathcal{F}(\lambda, v) - \mathcal{F}(\lambda, u)| \leq L|v - u|, \quad L > 0$ for all $u,\ v \in C_{rd}(\Sigma \times \mathbb{R})$.
- **(ii)** The mapping $v \longrightarrow \frac{v}{\mathcal{F}(\lambda, v)}$ is increasing in $\mathbb{R}$ a.e., for $\lambda \in \Sigma$.

(H2) $\xi \in C(\Sigma \times \mathbb{R}^2, \mathbb{R})$ is a function such that $|\xi(\lambda, v(\lambda), u(\lambda)| \leq h(\lambda)$ a.e., $\lambda \in \Sigma$, $h \in C(\Sigma, \mathbb{R}^+)$.

Definition 3.1. *A function $v \in C^1_{rd}(\Sigma, \mathbb{R})$ is a solution to the Δ-fractional hybrid problem (1), if it fulfills the initial condition $v(c) = \varphi$ and satisfies the fractional equations ${}^{C}\Delta_c^{\phi,\gamma} v(\lambda) = \xi\left(\lambda, v(\lambda), V(\lambda)\right)$ on Σ.*

Lemma 3.2. *Let $\xi : \Sigma \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a rd-continuous and $0 < \gamma < 1$. Then, the function $v \in C^1_{rd}(\Sigma, \mathbb{R})$ serves as a solution to the Δ-fractional hybrid problem* (1) *if and only if it verifies the following integral equation:*

$$v(\lambda) = \mathcal{F}(\lambda, v(\lambda))\left(\varphi + \frac{1}{\Gamma(\gamma)}\int_c^\lambda \phi'(\ell)(\phi(\lambda) - \phi(\ell))^{\gamma-1}\xi(\ell, v(\ell), V(\ell))\,\Delta\ell\right), \tag{3}$$

where $V(\ell) = \max_{\ell \in [c,\lambda]} v(\ell)$.

Proof. From (2), we have

$$\begin{aligned}{}^{C}\Delta^{\gamma,\phi}_{c^+}\Big(\frac{v(\lambda)}{\mathcal{F}(\lambda, v(\lambda))}\Big) &= \frac{1}{\Gamma(1-\gamma)}\int_c^\lambda \phi'(\ell)(\phi(\lambda)-\phi(\ell))^{-\gamma}\Big(\frac{v(\lambda)}{\mathcal{F}(\lambda, v(\lambda))}\Big)^{\Delta}_{\phi}(\ell)\Delta\ell \\ &= {}^{\mathbb{T}}_{\mathbf{S}}J^{1-\gamma,\phi}_{c}\Big(\frac{v(\lambda)}{\mathcal{F}(\lambda, v(\lambda))}\Big)^{\Delta}_{\phi}.\end{aligned}$$

Since $\gamma \in (0,1)$. Then, the proof can be concluded from the relations

$$\begin{aligned}{}^{\mathbb{T}_{\mathbf{S}}}J^{\gamma,\phi}_{c}\,{}^{C}\Delta^{\phi,\gamma}_{c}\left(\frac{v(\lambda)}{\mathcal{F}(\lambda, v(\lambda))}\right) &= {}^{\mathbb{T}_{\mathbf{S}}}J^{\gamma,\phi}_{c}\,{}^{\mathbb{T}_{\mathbf{S}}}J^{1-\gamma,\phi}_{c}\Big(\frac{v(\lambda)}{\mathcal{F}(\lambda, v(\lambda))}\Big)^{\Delta}_{\phi} \\ &= \frac{v(\lambda)}{\mathcal{F}(\lambda, v(\lambda))} - \frac{v(c)}{\mathcal{F}(c, v(c))} \\ &= \frac{v(\lambda)}{\mathcal{F}(\lambda, v(\lambda))} - \varphi,\end{aligned}$$

it follows that

$$v(\lambda) = \mathcal{F}(\lambda, v(\lambda))\left(\varphi + \frac{1}{\Gamma(\gamma)}\int_c^\lambda \phi'(\ell)(\phi(\lambda) - \phi(\ell))^{\gamma-1}\xi(\ell, v(\ell), V(\ell))\,\Delta\ell\right),$$

where $V(\ell) = \max_{\ell \in [c,\lambda]} v(\ell)$. □

Theorem 3.3. *Assuming that the conditions* ***(H1)*** *and* ***(H2)*** *are met. Then, under the following condition*

$$L\left(M_\varphi + \frac{\|h\|_\infty}{\Gamma(1+\gamma)}(\phi(d) - \phi(c))^\gamma\right) < 1, \tag{4}$$

the Δ-fractional hybrid problem (1) *has a solution.*

Proof. Let $M_\varphi = |\varphi|$ and $\Xi = (C(\Sigma, \mathbb{R}), \|.\|)$, where $\|v\| = \sup_{\lambda\in\Sigma}|v(\lambda)|$. It is evident that Ξ forms a Banach algebra, where multiplication is defined as follows

$$(uv)(\lambda) = u(\lambda)v(\lambda), \quad \lambda \in \Sigma, \quad u, v \in \Xi.$$

we define a subset S of Ξ as

$$S = \{v \in \Xi : \|v\| \le r\},$$

where

$$r = \frac{M_{\mathcal{F}}\left(M_\varphi + \frac{\|h\|_\infty}{\Gamma(1+\gamma)}(\phi(d) - \phi(c))^\gamma\right)}{1 - L\left(M_\varphi + \frac{\|h\|_\infty}{\Gamma(\gamma+1)}(\phi(d) - \phi(c))^\gamma\right)}.$$

1. S is Bounded

By definition, S consists of all functions $v \in \Xi$ such that $\|v\| \le r$. This means that for every $v \in S$, the supremum of $|v(\lambda)|$ over $\lambda \in \Sigma$ is bounded by r. Thus, S is contained in the closed ball of radius r centered

at the zero function in Ξ. This directly implies that S is **bounded**.
2. S is Closed
To show that S is closed, we need to prove that the limit of any convergent sequence in S also lies in S.
Let $\{v_n\}$ be a sequence in S such that $v_n \to v$ in Ξ. This means $\|v_n - v\| \to 0$ as $n \to \infty$.
Since $v_n \in S$, we have $\|v_n\| \leq r$ for all n.
By the continuity of the norm, we know that $\|v\| \leq \|v_n - v\| + \|v_n\|$.
Taking the limit as $n \to \infty$, we get:

$$\|v\| \leq \lim_{n\to\infty} \|v_n - v\| + \lim_{n\to\infty} \|v_n\| \leq 0 + r = r.$$

Thus, $\|v\| \leq r$, which means $v \in S$. Since the limit of any convergent sequence in S also lies in S, S is **closed**.
3. S is Convex
To show that S is convex, we need to prove that for any $v_1, v_2 \in S$ and any $t \in [0,1]$, the function $tv_1 + (1-t)v_2$ also lies in S.
Let $v_1, v_2 \in S$, so $\|v_1\| \leq r$ and $\|v_2\| \leq r$.
For any $t \in [0,1]$, consider the function $tv_1 + (1-t)v_2$.
Using the triangle inequality and the fact that the norm is homogeneous, we have:

$$\|tv_1 + (1-t)v_2\| \leq t\,\|v_1\| + (1-t)\,\|v_2\| \leq tr + (1-t)r = r.$$

Thus, $\|tv_1 + (1-t)v_2\| \leq r$, which means $tv_1 + (1-t)v_2 \in S$.
Since S is closed under convex combinations, it is **convex**.
Thus, S is a **bounded**, **closed**, and **convex** subset of the Banach algebra Ξ.
Let's consider the operators $\mathcal{G}_1 : \Xi \longrightarrow \Xi$ and $\mathcal{G}_2 : S \longrightarrow \Xi$, which are defined as follows

$$\mathcal{G}_1 v(\lambda) = \mathcal{F}\left(\lambda, v(\lambda)\right), \tag{5}$$

$$\mathcal{G}_2 v(\lambda) = \varphi + \frac{1}{\Gamma(\gamma)} \int_c^\lambda \phi'(\ell)(\phi(\lambda) - \phi(\ell))^{\gamma-1} \xi\left(\ell, v(\ell), V(\ell)\right) \Delta\ell. \tag{6}$$

The operator equation form representing the equivalent integral equation Eq. (3) corresponding to the fractional hybrid problem (1) is expressed as

$$v = \mathcal{G}_1 v\, \mathcal{G}_2 v, \quad v \in \Xi.$$

We establish that the operators $\mathcal{G}1$ and $\mathcal{G}2$ verify the conditions stated in Theorem 2.12.The demonstration for this has been provided in the subsequent steps.

Step 1: We prove that $\mathcal{G}_1$ is Lipschitz.
Applying the Lipschitz condition to $\mathcal{F}$, with $v, u \in \Xi$ and $\lambda \in \Sigma$, we obtain

$$\begin{aligned} |\mathcal{G}_1 v(\lambda) - \mathcal{G}_1 u(\lambda)| &= |\mathcal{F}(\lambda, v(\lambda)) - \mathcal{F}(\lambda, u(\lambda))| \\ &\leq L|v(\lambda) - u(\lambda)|. \end{aligned}$$

This gives,

$$\|\mathcal{G}_1 v - \mathcal{G}_1 u\| \leq L\|v - u\|, \quad u, v \in \Xi.$$

Step 2: In this step, we will show the complete continuity of the operator $\mathcal{G}_2$. We demonstrate that $\mathcal{G}_2 : S \longrightarrow \Xi$ is a continuous and compact operator on S into Ξ.
To begin, we establish the continuity of $\mathcal{G}_2$ on S. Let $v_n \in S$ converging to $v \in S$. For each $\lambda \in \Sigma$, we have

$$\begin{aligned} &|\mathcal{G}_2 v_n(\lambda) - \mathcal{G}_2 v(\lambda)| \\ &\qquad \leq \frac{1}{\Gamma(\gamma)} \int_c^\lambda \phi'(\ell)(\phi(\lambda) - \phi(\ell))^{\gamma-1} \left|\xi\left(\ell, v_n(\ell), V_n(\ell)\right) - \xi\left(\ell, v(\ell), V(\ell)\right)\right| \Delta\ell, \end{aligned}$$

using Lemma 2.11, we get

$$\begin{aligned}&|\mathcal{G}_2 v_n(\lambda) - \mathcal{G}_2 v(\lambda)| \\ &\qquad \leq \frac{1}{\Gamma(\gamma)} \int_c^\lambda \phi'(\ell)(\phi(\lambda) - \phi(\ell))^{\gamma-1} \left|\xi\left(\ell, v_n(\ell), V_n(\ell)\right) - \xi\left(\ell, v(\ell), V(\ell)\right)\right| d\ell.\end{aligned}$$

On the other hand, from **(H2)**, we deduce that

$$\phi'(\ell)(\phi(\lambda) - \phi(\ell))^{\gamma-1} \left|\xi\left(\ell, v_n(\ell), V_n(\ell)\right) - \xi\left(\ell, v(\ell), V(\ell)\right)\right| \leq 2\phi'(\ell)(\phi(\lambda) - \phi(\ell))^{\gamma-1} \|h\|_\infty,$$

and

$$\lim_{n\to\infty} \xi\left(\ell, v_n(\ell), V_n(\ell)\right) = \xi\left(\ell, v(\ell), V(\ell)\right).$$

Applying the Lebesgue's dominated convergence theorem leads us to following limit

$$\lim_{n\to\infty} \int_c^\lambda \phi'(\ell)(\phi(\lambda) - \phi(\ell))^{\gamma-1} \left|\xi\left(\ell, v_n(\ell), V_n(\ell)\right) - \xi\left(\ell, v(\ell), V(\ell)\right)\right| d\ell = 0, \quad \text{for all} \quad \lambda \in \Sigma.$$

This prove that $\mathcal{G}_2$ is a continuous on S.
Based on assumption **(H2)**, for every $v \in S$ and $\lambda \in \Sigma$, we obtain

$$\begin{aligned}|\mathcal{G}_2 v(\lambda)| &\leq |\varphi| + \frac{1}{\Gamma(\gamma)} \int_c^\lambda \phi'(\ell)(\phi(\lambda) - \phi(\ell))^{\gamma-1} |\xi\left(\ell, v(\ell), V(\ell)\right)| \Delta\ell \\ &\leq M_\varphi + \frac{1}{\Gamma(\gamma)} \int_c^\lambda \phi'(\ell)(\phi(\lambda) - \phi(\ell))^{\gamma-1} |h(\ell)| \Delta\ell \\ &\leq M_\varphi + \frac{\|h\|_\infty}{\Gamma(\gamma)} \int_c^\lambda \phi'(\ell)(\phi(\lambda) - \phi(\ell))^{\gamma-1} \Delta\ell\end{aligned}$$

From Lemma 2.11, we derive that

$$\begin{aligned}|\mathcal{G}_2 v(\lambda)| &\leq M_\varphi + \frac{\|h\|_\infty}{\Gamma(\gamma)} \int_c^b \phi'(\ell)(\phi(\lambda) - \phi(\ell))^{\gamma-1} d\ell \\ &\leq M_\varphi + \frac{\|h\|_\infty}{\Gamma(\gamma+1)} (\phi(d) - \phi(c))^\gamma.\end{aligned}$$

This gives,

$$\|\mathcal{G}_2 v\| \leq M_\varphi + \frac{\|h\|_\infty}{\Gamma(\gamma+1)} (\phi(d) - \phi(c))^\gamma, \tag{7}$$

This demonstrates that the operator $\mathcal{G}_2$ is uniformly bounded on Σ.

Furthermore, we establish that the operator $\mathcal{G}_2(S)$ is an equicontinuous set in Ξ. Let $v \in S$ and λ_1, $\lambda_2 \in \Sigma$ such that $\lambda_1 < \lambda_2$. Then, we have

$$\begin{aligned}&|\mathcal{G}_2 v(\lambda_2) - \mathcal{G}_2 v(\lambda_1)| \\ &\quad \leq \frac{1}{\Gamma(\gamma)} \int_c^{\lambda_1} \phi'(\ell) \left((\phi(\lambda_2) - \phi(\ell))^{\gamma-1} - (\phi(\lambda_1) - \phi(\ell))^{\gamma-1}\right) |\xi\left(\ell, v(\ell), V(\ell)\right)| \Delta\ell \\ &\quad + \frac{1}{\Gamma(\gamma)} \int_{\lambda_1}^{\lambda_2} \phi'(\ell)(\phi(\lambda_2) - \phi(\ell))^{\gamma-1} |\xi\left(\ell, v(\ell), V(\ell)\right)| \Delta\ell.\end{aligned}$$

According to Lemma 2.11, we can obtain the following

$$\begin{aligned}
&|\mathcal{G}_2 v(\lambda_2) - \mathcal{G}_2 v(\lambda_1)| \\
&\quad\le \frac{1}{\Gamma(\gamma)} \int_c^{\lambda_1} \phi'(\ell)\left((\phi(\lambda_2) - \phi(\ell))^{\gamma-1} - (\phi(\lambda_1) - \phi(\ell))^{\gamma-1}\right)|h(\ell)|d\ell \\
&\quad+ \frac{1}{\Gamma(\gamma)} \int_{\lambda_1}^{\lambda_2} \phi'(\ell)(\phi(\lambda_2) - \phi(\ell))^{\gamma-1}|h(\ell)|d\ell \\
&\quad\le \frac{\|h\|_\infty}{\Gamma(\gamma+1)}\left((\phi(\lambda_2) - \phi(c))^\gamma - (\phi(\lambda_1) - \phi(c))^\gamma\right).
\end{aligned}$$

This demonstrates that $\mathcal{G}_2(S)$ is equicontinious set in Ξ. Since $\mathcal{G}_2(S)$ is equicontinious and uniformly bounded set in Ξ. Then, by apply Ascoli-Arzela theorem, we get $\mathcal{G}_2$ is completely continuous.

Step 3: Let $v \in \Xi$ and $u \in S$ such that $v = \mathcal{G}_1 v\, \mathcal{G}_2 u$. Our objective is to demonstrate that $v \in S$. Utilizing hypothesis $(H1)$ and condition (7), we obtain

$$\begin{aligned}
|v(\lambda)| &= |\mathcal{G}_1 v(\lambda)\mathcal{G}_2 u(\lambda)| \\
&\le \left\{|\mathcal{F}(\lambda, v(\lambda)) - \mathcal{F}(\lambda, 0)| + |\mathcal{F}(\lambda, 0)|\right\}\left(M_\varphi + \frac{\|h\|_\infty}{\Gamma(\gamma+1)}(\phi(d) - \phi(c))^\gamma\right) \\
&\le \{L|v(\lambda)| + M_{\mathcal{F}}\}\left(M_\varphi + \frac{\|h\|_\infty}{\Gamma(\gamma+1)}(\phi(d) - \phi(c))^\gamma\right).
\end{aligned}$$

This gives,

$$\begin{aligned}
|v(\lambda)| &\le \frac{M_{\mathcal{F}}\left(M_\varphi + \frac{\|h\|_\infty}{\Gamma(\gamma+1)}(\phi(d) - \phi(c))^\gamma\right)}{1 - L\left(M_\varphi + \frac{\|h\|_\infty}{\Gamma(\gamma+1)}(\phi(d) - \phi(c))^\gamma\right)} \\
&= r, \quad \text{for all } \lambda \in \Sigma.
\end{aligned}$$

Therefore,

$$\|v\| \le r.$$

This establishes that $v \in S$.

Step 4: Now, we show that $\kappa\eta < 1$.
We have

$$\kappa = L \quad \text{and} \quad \eta = M_\varphi + \frac{\|h\|_\infty}{\Gamma(\gamma+1)}(\phi(d) - \phi(c))^\gamma.$$

By condition (4), we can get the following

$$\eta\kappa = L\left(M_\varphi + \frac{\|h\|_\infty}{\Gamma(\gamma+1)}(\phi(d) - \phi(c))^\gamma\right) < 1.$$

By proceeding through steps **1** through **4**, we can conclude that all the conditions outlined in Theorem 2.12 are met. Thus, the following equation

$$v = \mathcal{G}_1 v \mathcal{G}_2 v$$

possesses a solution in S, which is a solution to the Δ-fractional hybrid problem (1). This concludes the proof of the theorem.

□

4. Example

let us consider the following hybrid problem

$$\begin{cases} {}^{C}\Delta_{c}^{\phi,\gamma}\left(\frac{2|v(t)|+2|v(t)|^2}{e^{t-1}}\right) = \dfrac{\exp\{-\max_{\ell\in[0,t]}|v(\ell)|\}}{|v(t)|+1+t^2}, \quad t \in \Sigma = [0,1]_{\mathbb{T}_S}, \\ v(0) = 0, \end{cases} \tag{8}$$

where $\mathbb{T}_S$ represent any time scale that includes both 0 and 1. Here $c = 0$, $d = 1$, $\phi = t$, $\gamma = \frac{1}{2}$, $M_\varphi = 0$ and

$\mathcal{F}(t, v) = \frac{e^{t-1}}{2(1+v)}$,

$\xi(t, v, u) = \frac{e^{-u}}{|v|+t^2+1}$,

Let $t \in [0, 1]$ and u, v in $\mathbb{R}$, we have

$$\begin{aligned} |\mathcal{F}(t,v) - \mathcal{F}(t,u)| &\leq \frac{1}{2}\left|\frac{|u|-|v|}{(|v|+1)(1+|u|)}\right| \\ &\leq \frac{1}{2}|v-u|. \end{aligned}$$

Hence, the condition **(H1)** holds, with $L = \frac{1}{2}$. We also have the following inequality

$$|\xi(t, v, u)| \leq h(t),$$

where $h(t) = \frac{1}{t^2+1}$ and thus

$$\int_0^1 h(t)dt = \int_0^1 \frac{1}{t^2+1}dt = \frac{\pi}{4}.$$

Then, condition **(H2)** holds.
Now, we can show that

$$L\left(M_\varphi + \frac{\|h\|_\infty}{\Gamma(\gamma+1)}(\phi(d) - \phi(c))^\gamma\right) = 0.4431134627 < 1.$$

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