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Probabilistic bivariate Fubini polynomials and probabilistic degenerate bivariate Fubini polynomials

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Abstract. In this paper, we consider the probabilistic extensions of bivariate Fubini polynomials and degenerate bivariate Fubini polynomials and investigate their properties. From our investigation, we derive new identities of bivariate Fubini polynomials and degenerate bivariate Fubini polynomials in terms of probabilistic properties.

1. Introduction

In the study of special polynomials and number theory, Stirling numbers, Euler polynomials, and their generalizations play a fundamental role. Since the time of Jacob Bernoulli, classical summation formulas and combinatorial identities have been continuously extended, often through the introduction of new probabilistic or algebraic tools. In this direction, Komatsu and Ramírez [11] introduced the Fubini polynomials and the restricted Fubini numbers, established new explicit combinatorial formulas for these sequences as well as for related sequences like modified Bernoulli and Cauchy numbers, by means of determinant expressions and Trudi's formula. Kim and Kim [12, 13] further investigated degenerate Fubini polynomials through a variety of methods, including generating functions, combinatorial methods, umbral calculus, p-adic analysis, differential equations, probability, and special functions. Su and He [37] considered two-variable Fubini polynomials and obtained several interesting results for these polynomials. Kim et al. [10, 14, 15] introduced and analyzed the two-variable higher-order Fubini polynomials and their degenerate versions, establishing their generating functions, explicit formulas, and various mathematical properties. Kilar and Simsek [16, 17] constructed unified and modified families of Fubini numbers and polynomials, and further introduced a new family of these polynomials connected to Apostol-Bernoulli numbers and polynomials, deriving their generating functions, combinatorial properties, and various identities. Moreover, Simsek and Kilar [36] obtained new formulas connecting Eulerian and Fubini numbers through moment computations for the geometric distribution. Kim and Kim [18] also presented the

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two-variable higher-order central Fubini polynomials, together with their generating functions, explicit expressions and properties. In recent years, some mathematicians have explored probabilistic versions of many classical special polynomials and numbers, which include the probabilistic Stirling numbers of the second kind, the probabilistic Bernstein polynomials, the probabilistic bivariate Bell polynomials (see [1, 2, 19–21] and the references therein). Gomaa and Magar [9] investigated a new generating function of generalized Fubini-type polynomials, from which they derived various identities and relations together with several probabilistic properties and applications. Subsequently, Kim and Kim [22] studied the probabilistic extensions of Bernoulli and Euler polynomials associated with Y. And they gave the probabilistic versions of several special polynomials such as the probabilistic two variable Fubini polynomials, probabilistic r-Stirling numbers of the second kind associated with Y, etc. In [39], we considered a probabilistic version of the degenerate Fubini polynomials associated with Y, and discovered the probabilistic Stirling numbers of the second kind and the probabilistic Fubini polynomials associated with Y.

Within this framework, Fubini polynomials have attracted increasing attention. These polynomials arise naturally in combinatorics, number theory, and probability theory, while their bivariate and degenerate generalizations capture richer algebraic structures. On the other hand, combining probabilistic versions with degenerate versions in the study of bivariate Fubini polynomials provides not only a natural extension of existing theory but also a promising framework for discovering new combinatorial identities and applications.

Motivated by these developments, this paper is devoted to the study of probabilistic bivariate Fubini polynomials and probabilistic degenerate bivariate Fubini polynomials. We introduce their generating functions, and derive explicit formulas and combinatorial identities. Furthermore, we investigate their properties within a probabilistic framework, thereby enriching the theory of Fubini-type polynomials and laying the groundwork for future applications in probability theory, combinatorics, and related fields. Now, we present some definitions and foundational concepts and properties.

For $\lambda \in \mathbb{R}$, the degenerate exponential function has been given by

$$e_\lambda^x(t) = \sum_{k=0}^{\infty} (x)_{k,\lambda} \frac{t^k}{k!}, \quad e_\lambda(t) = e_\lambda^1(t), \quad \text{(see [3, 6, 12, 13, 15, 24, 26–31, 33, 38, 39])}, \tag{1}$$

with the degenerate falling factorial defined recursively as $(x)_{0,\lambda} = 1, \quad (x)_{n,\lambda} = x(x-\lambda)\dots(x-(n-1)\lambda)$. Note that $\lim_{\lambda\to 0} e_\lambda^x(t) = e^{xt}$.

Let $\log_\lambda t$ be the compositional inverse function of $e_\lambda(t)$ such that $\log_\lambda(e_\lambda(t)) = e_\lambda(\log_\lambda(t)) = t$. It is called the degenerate logarithm and given by

$$\log_\lambda(1+t) = \sum_{k=1}^{\infty} \frac{\lambda^{k-1}(1)_{k,\frac{1}{\lambda}}}{k!} t^k = \frac{1}{\lambda}\left((1+t)^\lambda - 1\right), \quad \text{(see [13, 24, 30])}. \tag{2}$$

Note that $\lim_{\lambda\to 0} \log_\lambda(1+t) = \log(1+t)$ and $\lim_{\lambda\to 0} e_\lambda(t) = e^t$.

The degenerate Euler polynomials are defined as

$$\frac{2}{e_\lambda(t)+1} e_\lambda^x(t) = \sum_{n=0}^{\infty} \varepsilon_{n,\lambda}(x) \frac{t^n}{n!}, \quad \text{(see [6, 30])}, \tag{3}$$

where

$$\varepsilon_{0,\lambda}(x) = 1,$$

$$\varepsilon_{1,\lambda}(x) = x - \frac{1}{2},$$

$$\varepsilon_{2,\lambda}(x) = x^2 - x(1+\lambda) + \frac{\lambda}{2}, \cdots .$$

When $x = 0$, $\varepsilon_{n,\lambda}$ are called as the degenerate Euler numbers.

We easily know that $\lim_{\lambda\to 0}\varepsilon_{n,\lambda}(x) = E_n(x)$, where $E_n(x)$ are ordinary Euler polynomials which are given by

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \quad \text{(see [6, 8, 30, 38])}. \tag{4}$$

where

$$E_0(x) = 1,$$
$$E_1(x) = x - \frac{1}{2},$$
$$E_2(x) = x^2 - x,$$
$$E_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{4}, \cdots.$$

The Stirling numbers of the first kind $S_1(n,k)$ are defined via the falling factorial expansion:

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k, \quad (n \geq 0), \quad (x)_0 = 1, \quad (x)_n = x(x-1)(x-2)\cdots(x-n+1), \quad \text{(see [5, 6, 8, 24])}, \tag{5}$$

where n is a positive integer.

Conversely, the inverse relation yields the Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$:

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (x)_k, \quad (n \geq 0). \tag{6}$$

The degenerate Stirling numbers of the second kind are introduced via the expansion:

$$(x)_{n,\lambda} = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_{\lambda} (x)_k, \quad (n \geq 0), \quad \text{(see [2, 24, 27])}. \tag{7}$$

Observe that

$$\lim_{\lambda\to 0} \left\{ {n \atop k} \right\}_{\lambda} = \left\{ {n \atop k} \right\}, \quad (n \geq k \geq 0).$$

It's widely known that the degenerate Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{k=0}^{n} S_{1,\lambda}(n,k)(x)_{k,\lambda}, \quad \text{(see [13, 24])}. \tag{8}$$

For any nonnegative integer r, the degenerate r-Stirling numbers of the second kind $\left\{ {n+r \atop k+r} \right\}_{r,\lambda}$:

$$(x+r)_{n,\lambda} = \sum_{k=0}^{n} \left\{ {n+r \atop k+r} \right\}_{r,\lambda} (x)_k, \quad (n \geq 0), \quad \text{(see [26, 27])}. \tag{9}$$

From (9), we note that

$$\frac{1}{k!}(e_\lambda(t)-1)^k e_\lambda^r(t) = \sum_{n=k}^{\infty} \left\{ {n+r \atop k+r} \right\}_{r,\lambda} \frac{t^n}{n!}, \tag{10}$$

where k is a nonnegative integer.

The degenerate Fubini polynomials are given by

$$\frac{1}{1-y(e_\lambda(t)-1)}=\sum_{n=0}^{\infty}F_{n,\lambda}(y)\frac{t^n}{n!},\quad \text{(see [12, 36])}. \tag{11}$$

Let $\lambda\to 0$, the Fubini polynomials are defined by

$$\frac{1}{1-y(e^t-1)}=\sum_{n=0}^{\infty}F_n(y)\frac{t^n}{n!},\quad \text{(see [13, 16, 17, 25, 32, 36])}. \tag{12}$$

From (11), we have

$$F_{n,\lambda}(y)=\sum_{k=0}^{n}y^k\left\{\begin{matrix}n\\k\end{matrix}\right\}_\lambda k!,\quad (n\geq 0). \tag{13}$$

The classical bivariate Fubini polynomials $F_n(x,y)$ are given by

$$\frac{1}{1-y(e^t-1)}e^{xt}=\sum_{n=0}^{\infty}F_n(x,y)\frac{t^n}{n!},\quad \text{(see [10, 14–17, 23])}. \tag{14}$$

The degenerate bivariate Fubini polynomials are defined by

$$\frac{1}{1-y(e_\lambda(t)-1)}e_\lambda^x(t)=\sum_{n=0}^{\infty}F_{n,\lambda}(x,y)\frac{t^n}{n!},\quad \text{(see [10, 14, 15, 23])}. \tag{15}$$

Note that $F_{n,\lambda}(0,y)=F_{n,\lambda}(y),\quad (n\geq 0)$.

For any positive integer r, the degenerate Fubini polynomials of order r are given by

$$\left(\frac{1}{1-y(e_\lambda(t)-1)}\right)^r=\sum_{n=0}^{\infty}F_{n,\lambda}^{(r)}(y)\frac{t^n}{n!},\quad \text{(see [12, 16, 17, 39])}. \tag{16}$$

Note that $\lim_{\lambda\to 0}F_{n,\lambda}^{(r)}(y)=F_n^{(r)}(y)$ are called the Fubini polynomials of order r.

So, by (16), we have

$$F_{n,\lambda}^{(r)}(y)=\sum_{k=0}^{n}\binom{k+r-1}{k}y^k\left\{\begin{matrix}n\\k\end{matrix}\right\}_\lambda k!. \tag{17}$$

Consider the random variable Y whose moment generating function

$$E[e^{tY}]=\sum_{n=0}^{\infty}E[Y^n]\frac{t^n}{n!},\quad (|t|<r),\quad \text{(see [1, 2, 19, 20, 30, 39])} \tag{18}$$

exists for some $r>0$, where E denotes mathematical expectation.

Let $(Y_k)_{k\geq 1}$ be a sequence of independent identically distributed copies of Y, and define the sums $S_k=\sum_{i=1}^{k}Y_i$ for $k\geq 1$ with $S_0=0$.

The probabilistic degenerate Stirling numbers of the first kind are defined by

$$\frac{1}{k!}(log_{-\lambda}(E[e_\lambda^Y(t)]))^k=\sum_{n=k}^{\infty}(-1)^{n-k}S_{1,\lambda}^Y(n,k)\frac{t^n}{n!},\quad \text{(see [2, 30])}. \tag{19}$$

The probabilistic degenerate Stirling numbers of the second kind are given by

$$\frac{1}{k!}(E[e_\lambda^Y(t)]-1)^k=\sum_{n=k}^{\infty}{n \brace k}_{Y,\lambda}\frac{t^n}{n!},\quad \text{(see [2, 30, 39]).} \tag{20}$$

Observe that the probabilistic degenerate Stirling numbers satisfy ${n \brace k}_{Y,\lambda}={n \brace k}_{\lambda}$ when $Y=1$, with the limit property $\lim_{\lambda\to 0}{n \brace k}_{Y,\lambda}={n \brace k}_{Y}$ for $n\geq k\geq 0$. Here ${n \brace k}_{Y}$ denotes the probabilistic Stirling numbers of the second kind, defined by

$$\frac{1}{k!}(E[e^{Yt}]-1)^k=\sum_{n=k}^{\infty}{n \brace k}_{Y}\frac{t^n}{n!},\quad \text{(see [1, 2, 19, 20, 22]).} \tag{21}$$

By (20), we derive the combinatorial expression:

$${n \brace k}_{Y,\lambda}=\frac{1}{k!}\sum_{j=0}^{k}\binom{n}{j}(-1)^{k-j}E[(S_j)_{n,\lambda}]. \tag{22}$$

Applying binomial inversion, this is equivalent to

$$E[(S_k)_{n,\lambda}]=\sum_{j=0}^{k}\binom{k}{j}j!{n \brace j}_{Y,\lambda}. \tag{23}$$

Kim-Kim defined the probabilistic bivariate Fubini polynomials associated with Y

$$\frac{1}{1-y(E[e^{Yt}]-1)}(E[e^{Yt}])^x=\sum_{n=0}^{\infty}F_n^Y(x,y)\frac{t^n}{n!},\quad \text{(see [19, 22]),} \tag{24}$$

Note that $F_n^Y(x,y)$ are the classical bivariate Fubini polynomials $F_n(x,y)$ when $Y=1$.

In this work, we develop a probabilistic extension for bivariate Fubini polynomials and their degenerate versions, constructed via moment generating functions. We further establish explicit expressions and combinatorial identities for these probabilistic bivariate and probabilistic degenerate bivariate Fubini polynomials associated with random variables.

2. Probabilistic bivariate Fubini polynomials

From (12), (21) and (24), we note that

$$\begin{aligned}\frac{(E[e^{Yt}]+1-1)^x}{1-y(E[e^{Yt}]-1)} &= \frac{1}{1-y(E[e^{Yt}]-1)}\sum_{m=0}^{\infty}\frac{(x)_m}{m!}(E[e^{Yt}]-1)^m \\ &= \sum_{l=0}^{\infty}F_l^Y(y)\frac{t^l}{l!}\sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}{m \brace k}_{Y}(x)_k\right)\frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}F_{n-m}^Y(y)\sum_{k=0}^{m}{m \brace k}_{Y}(x)_k\right)\frac{t^n}{n!}.\end{aligned} \tag{25}$$

Consequently, combining identity (24) with (25) yields:

Theorem 2.1. *For $n\geq 0$,*

$$F_n^Y(x,y)=\sum_{m=0}^{n}\binom{n}{m}F_{n-m}^Y(y)\sum_{k=0}^{m}{m \brace k}_{Y}(x)_k.$$

Motivated by the framework in [28], we formally define the probabilistic Stirling polynomials of the second kind associated with Y as

$$\frac{1}{k!}(E[e^{Yt}]-1)^k(E[e^{Yt}])^x=\sum_{n=k}^{\infty}S_{2,Y}(n,k|x)\frac{t^n}{n!}. \tag{26}$$

where k is nonnegative integer.

From (26), we note that

$$\begin{aligned}\frac{1}{k!}(E[e^{Yt}]-1)^k(E[e^{Yt}])^x &= \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(E[e^{Yt}])^j(E[e^{Yt}])^x \\ &= \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(E[e^{Yt}])^{j+x} \\ &= \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}E[e^{S_{j+x}t}] \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}E[S_{j+x}^n]\right)\frac{t^n}{n!}.\end{aligned} \tag{27}$$

Thus, we get

$$S_{2,Y}(n,k|x)=\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}E[S_{j+x}^n]. \tag{28}$$

From (24) and (26), we get

$$\begin{aligned}\sum_{n=0}^{\infty}F_n^Y(x,y)\frac{t^n}{n!} &= \frac{1}{1-y(E[e^{Yt}]-1)}(E[e^{Yt}])^x \\ &= \sum_{k=0}^{\infty}y^k k!\frac{1}{k!}\left(E[e^{Yt}]-1\right)^k(E[e^{Yt}])^x \\ &= \sum_{k=0}^{\infty}y^k k!\sum_{n=k}^{\infty}S_{2,Y}(n,k|x)\frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}y^k k! S_{2,Y}(n,k|x)\right)\frac{t^n}{n!}.\end{aligned} \tag{29}$$

By comparing the coefficients on both sides of (29), we derive Theorem 2.2.

Theorem 2.2. *For $n \geq 0$,*

$$F_n^Y(x,y)=\sum_{k=0}^{n}y^k k! S_{2,Y}(n,k|x).$$

From (21) and (24), we note that

$$\begin{aligned}\sum_{n=0}^{\infty} F_n^Y(x,y)\frac{t^n}{n!} &= \frac{1}{1-y(E[e^{Yt}]-1)}(E[e^{Yt}])^x \\ &= \frac{1}{1+y}\frac{1}{1-\frac{y}{1+y}E[e^{Yt}]}\sum_{m=0}^{\infty}\binom{x}{m}(E[e^{Yt}]-1)^m \\ &= \frac{1}{1+y}\sum_{k=0}^{\infty}\Big(\frac{y}{1+y}\Big)^k(E[e^{Yt}])^k\sum_{m=0}^{\infty}\Big(\sum_{l=0}^{m}\left\{ {m \atop l} \right\}_Y(x)_l\Big)\frac{t^m}{m!} \\ &= \frac{1}{1+y}\sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\Big(\frac{y}{1+y}\Big)^k E[(Y_1+\cdots+Y_k)_s]\frac{t^s}{s!}\sum_{m=0}^{\infty}\Big(\sum_{l=0}^{m}\left\{ {m \atop l} \right\}_Y(x)_l\Big)\frac{t^m}{m!} \\ &= \frac{1}{1+y}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}\Big(\frac{y}{1+y}\Big)^k E[(S_k)_{n-m}]\left\{ {m \atop l} \right\}_Y(x)_l\frac{t^n}{n!}.\end{aligned} \tag{30}$$

Accordingly, equating coefficients in (30) reveals Theorem 2.3.

Theorem 2.3. *For $n \geq 0$,*

$$F_n^Y(x,y) = \sum_{k=0}^{\infty}\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}\left\{ {m \atop l} \right\}_Y\frac{1}{(1+y)^{k+1}}y^k E[(S_k)_n - m](x)_l.$$

We know the probabilistic Euler polynomials which are defined by

$$\frac{2}{E[e^{Yt}]+1}(E[e^{Yt}])^x = \sum_{n=0}^{\infty}\mathcal{E}_n^Y(x)\frac{t^n}{n!}, \quad \text{(see [22])}. \tag{31}$$

Taking $y = -\frac{1}{2}$ in (24), we observe that

$$\sum_{n=0}^{\infty}F_n^Y(x,-\frac{1}{2})\frac{t^n}{n!} = \sum_{n=0}^{\infty}\mathcal{E}_n^Y(x)\frac{t^n}{n!}. \tag{32}$$

We have

$$F_n^Y(x,-\frac{1}{2}) = \mathcal{E}_n^Y(x), \tag{33}$$

when $Y = 1$, it is easy to find that $F_n(x,-\frac{1}{2}) = \mathcal{E}_n(x)$.

From (31), we observe

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{E}_n^Y(x)\frac{t^n}{n!} &= \frac{2}{E[e^{Yt}]+1}(E[e^{Yt}]-1+1)^x \\ &= \frac{2}{E[e^{Yt}]+1}\sum_{k=0}^{\infty}\binom{x}{k}(E[e^{Yt}]-1)^k \\ &= \sum_{l=0}^{\infty}\mathcal{E}_l^Y\frac{t^l}{l!}\sum_{m=0}^{\infty}\sum_{k=0}^{m}(x)_k\left\{ {m \atop k} \right\}_Y\frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\mathcal{E}_{n-m}^Y\sum_{k=0}^{m}(x)_k\left\{ {m \atop k} \right\}_Y\frac{t^n}{n!}.\end{aligned} \tag{34}$$

Combining identities (33) and (34), we establish Theorem 2.4.

Theorem 2.4. *For $n \geq 0, \quad y = -\frac{1}{2}$, we have*

$$F_n^Y(x, -\frac{1}{2}) = \sum_{m=0}^{n} \sum_{k=0}^{m} \binom{n}{m} \varepsilon_{n-m}^Y(x)_k \left\{ \begin{matrix} m \\ k \end{matrix} \right\}_Y$$

Let Y be the Poisson random variable with parameter $\alpha \geq 0$, we have

$$E[e^{Yt}] = e^{\alpha(e^t-1)}. \tag{35}$$

From (31) and (35), we see

$$\begin{aligned} \sum_{n=0}^{\infty} F_n^Y(x, -\frac{1}{2}) \frac{t^n}{n!} &= \frac{2}{E[e^{Yt}] + 1} (E[e^{Yt}])^x \\ &= \frac{2}{e^{\alpha(e^t-1)} + 1} (e^{\alpha z(e^t-1)}) \\ &= \sum_{k=0}^{\infty} \varepsilon_k(x) \frac{\alpha^k}{k!} (e^t - 1)^k \\ &= \sum_{k=0}^{\infty} \varepsilon_k(x) \alpha^k \sum_{n=k}^{\infty} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \alpha^k \varepsilon_k(x) \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \frac{t^n}{n!}. \end{aligned} \tag{36}$$

Consequently, equating coefficients on both sides of (36) yields Theorem 2.5.

Theorem 2.5. *Let Y be the Poisson random variable with parameter $\alpha \geq 0$. For $n \geq 0$, we have*

$$F_n^Y(x, -\frac{1}{2}) = \sum_{k=0}^{n} \alpha^k F_k(x, -\frac{1}{2}) \left\{ \begin{matrix} n \\ k \end{matrix} \right\}.$$

For any positive integer r, Kim et al. have defined the probabilistic bivariate Fubini polynomials of order r

$$\frac{1}{(1 - y(E[e^{Yt}] - 1))^r} (E[e^{Yt}])^x = \sum_{n=0}^{\infty} F_n^{(r,Y)}(x, y) \frac{t^n}{n!}, \quad \text{(see [22])}. \tag{37}$$

From (37), we get

$$\begin{aligned} \frac{(E[e^{Yt}])^x}{(1 - y(E[e^{Yt}] - 1))^r} &= (E[e^{Yt}])^x (1 - y(E[e^{Yt}] - 1))^{-r} \\ &= \sum_{k=0}^{\infty} (r + k - 1)_k y^k \frac{1}{k!} (E[e^{Yt}] - 1)^k \sum_{l=0}^{\infty} \binom{x}{l} (E[e^{Yt}] - 1)^l \\ &= \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} (r + k - 1)_k y^k \frac{1}{k!} \binom{x}{m-k} \right) (E[e^{Yt}] - 1)^m \\ &= \sum_{m=0}^{\infty} \sum_{k=0}^{m} (r + k - 1)_k y^k \frac{m!}{k!} \binom{x}{m-k} \sum_{n=0}^{\infty} \left\{ \begin{matrix} n \\ m \end{matrix} \right\}_Y \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \sum_{k=0}^{m} \binom{r + k - 1}{k} y^k m! \binom{x}{m-k} \left\{ \begin{matrix} n \\ m \end{matrix} \right\}_Y \right) \frac{t^n}{n!}. \end{aligned} \tag{38}$$

By comparing the coefficients on both sides of (38), we derive Theorem 2.6.

Theorem 2.6. *For $n \geq 0$, we have*

$$F_n^{(r,Y)}(x,y) = \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{r+k-1}{k}\binom{x}{m-k}y^k m!\left\{ {n \atop m} \right\}_Y.$$

We define the probabilistic Frobenius-Euler polynomials of order r associated with random variable Y

$$\left(\frac{1-u}{E[e^{Yt}]-u}\right)^r (E[e^{Yt}])^x = \sum_{n=0}^{\infty} H_n^{(r,Y)}(u;x)\frac{t^n}{n!}. \tag{39}$$

For $y \neq 0$, we get

$$\begin{aligned}\sum_{n=0}^{\infty} F_n^{(r,Y)}(x,y)\frac{t^n}{n!} &= \frac{1}{(1-y(E[e^{Yt}]-1))^r}(E[e^{Yt}])^x \\ &= \left(\frac{1-\frac{y+1}{y}}{E[e^{Yt}]-\frac{y+1}{y}}\right)^r (E[e^{Yt}])^x \\ &= \sum_{n=0}^{\infty} H_n^{(r,Y)}\left(\frac{y+1}{y};x\right)\frac{t^n}{n!}.\end{aligned} \tag{40}$$

So, we obtain the relationship between the probabilistic bivariate Fubini polynomials of order r and the probabilistic Frobenius-Euler polynomials of order r.

Theorem 2.7. *For $y \neq 0$,*

$$F_n^{(r,Y)}(x,y) = H_n^{(r,Y)}\left(\frac{y+1}{y};x\right).$$

3. Probabilistic degenerate bivariate Fubini polynomials

We know the Probabilistic degenerate bivariate Fubini polynomials are defined by

$$\frac{1}{1-y(E[e_\lambda^Y(t)]-1)}(E[e_\lambda^Y(t)])^x = \sum_{n=0}^{\infty} F_{n,\lambda}^Y(x,y)\frac{t^n}{n!}, \quad \text{(see [13, 31])}. \tag{41}$$

We define the probabilistic degenerate Stirling polynomials of the second kind $S_{2,\lambda}^Y(n,k|x)$

$$\frac{1}{k!}(E[e_\lambda^Y(t)]-1)^k (E[e_\lambda^Y(t)])^x = \sum_{n=k}^{\infty} S_{2,\lambda}^Y(n,k|x)\frac{t^n}{n!}. \tag{42}$$

By the definition (42), we derive that

$$\begin{aligned}\frac{1}{k!}(E[e_\lambda^Y(t)]-1)^k (E[e_\lambda^Y(t)])^x &= \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(E[e_\lambda^Y(t)])^j (E[e_\lambda^Y(t)])^x \\ &= \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(E[e_\lambda^Y(t)])^{j+x} \\ &= \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}E[e_\lambda^{S_{j+x}}(t)] \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}E[(S_{j+x})_{n,\lambda}]\right)\frac{t^n}{n!}.\end{aligned} \tag{43}$$

Thus, we get

$$S^{Y}_{2,\lambda}(n,k|x) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}E[(S_{j+x})_{n,\lambda}]. \tag{44}$$

From (41) and (42), we get

$$\begin{aligned}\sum_{n=0}^{\infty}F^{Y}_{n,\lambda}(x,y)\frac{t^n}{n!} &= \frac{(E[e^{Y}_{\lambda}(t)])^x}{1-y(E[e^{Y}_{\lambda}(t)]-1)} \\ &= \sum_{k=0}^{\infty}y^k k!\frac{1}{k!}(E[e^{Y}_{\lambda}(t)]-1)^k(E[e^{Y}_{\lambda}(t)])^x \\ &= \sum_{k=0}^{\infty}y^k k!\sum_{n=k}^{\infty}S^{Y}_{2,\lambda}(n,k|x)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}y^k k! S^{Y}_{2,\lambda}(n,k|x)\Big)\frac{t^n}{n!}.\end{aligned} \tag{45}$$

By comparing the coefficients on both sides of (45), we derive Theorem 3.1.

Theorem 3.1. *For $n \geq 0$,*

$$F^{Y}_{n,\lambda}(x,y) = \sum_{k=0}^{n}y^k k! S^{Y}_{2,\lambda}(n,k|x).$$

From (19) and (41), we have

$$\begin{aligned}\sum_{n=0}^{\infty}F^{Y}_{n,\lambda}(x,y)\frac{t^n}{n!} &= \frac{1}{1-y(E[e^{Y}_{\lambda}(t)-1])}(E[e^{Y}_{\lambda}(t)])^x \\ &= \sum_{j=0}^{\infty}F^{Y}_{j,\lambda}(y)\frac{t^j}{j!}e^{x}_{-\lambda}(log_{-\lambda}(E[e^{Y}_{\lambda}(t)])) \\ &= \sum_{j=0}^{\infty}F^{Y}_{j,\lambda}(y)\frac{t^j}{j!}\sum_{k=0}^{\infty}(x)_{k,-\lambda}\frac{1}{k!}(log_{-\lambda}(E[e^{Y}_{\lambda}(t)]))^k \\ &= \sum_{j=0}^{\infty}F^{Y}_{j,\lambda}(y)\frac{t^j}{j!}\sum_{k=0}^{\infty}(x)_{k,-\lambda}\sum_{m=k}^{\infty}(-1)^{m-k}S^{Y}_{1,\lambda}(m,k)\frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{m}(-1)^{m-k}\binom{n}{m}(x)_{k,-\lambda}S^{Y}_{1,\lambda}(m,k)F^{Y}_{n-m,\lambda}(y)\right)\frac{t^n}{n!}.\end{aligned} \tag{46}$$

By (46), we obtain the relationship between the probabilistic degenerate bivariate Fubini polynomials and $S^{Y}_{1,\lambda}(n,k)$.

Theorem 3.2. *For $y \geq 0$,*

$$F^{Y}_{n,\lambda}(x,y) = \sum_{m=0}^{n}\sum_{k=0}^{m}(-1)^{m-k}\binom{n}{m}(x)_{k,-\lambda}S^{Y}_{1,\lambda}(m,k)F^{Y}_{n-m,\lambda}(y).$$

For any positive integer r, the probabilistic degenerate bivariate Fubini polynomials of order r are given by

$$\frac{1}{(1-y(E[e_\lambda^Y(t)]-1))^r}(E[e_\lambda^Y(t)])^x=\sum_{n=0}^\infty F_{n,\lambda}^{(r,Y)}(x,y)\frac{t^n}{n!},\quad \text{(see [30])}. \tag{47}$$

From (47), we observe that

$$\begin{aligned}\frac{(E[e_\lambda^Y(t)])^x}{(1-y(E[e_\lambda^Y(t)]-1))^r} &= (E[e_\lambda^Y(t)])^x(1-y(E[e_\lambda^Y(t)]-1))^{-r} \\ &= \sum_{l=0}^\infty\binom{r+l-1}{l}y^l(E[e_\lambda^Y(t)]-1)^l\sum_{m=0}^\infty\binom{x}{m}(E[e_\lambda^Y(t)]-1)^m \\ &= \sum_{k=0}^\infty\sum_{l=0}^k\binom{r+l-1}{l}y^l\binom{x}{k-l}(E[e_\lambda^Y(t)]-1)^k \\ &= \sum_{k=0}^\infty\sum_{l=0}^k\binom{r+l-1}{l}y^lk!\binom{x}{k-l}\sum_{n=0}^\infty\left\{\begin{matrix}n\\k\end{matrix}\right\}_{Y,\lambda}\frac{t^n}{n!} \\ &= \sum_{n=0}^\infty\left(\sum_{k=0}^n\sum_{l=0}^k\binom{r+l-1}{l}y^lk!\binom{x}{k-l}\left\{\begin{matrix}n\\k\end{matrix}\right\}_{Y,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{48}$$

Accordingly, by comparing the coefficients on both sides of (48) reveals Theorem 3.3.

Theorem 3.3. *For $n\geq 0$, we have*

$$F_{n,\lambda}^{(r,Y)}(x,y)=\sum_{k=0}^n\sum_{l=0}^k\binom{r+l-1}{l}y^lk!\binom{x}{k-l}\left\{\begin{matrix}n\\k\end{matrix}\right\}_{Y,\lambda}.$$

4. Conclusion

In this paper, we introduced the probabilistic bivariate Fubini polynomials and their degenerate versions, which are the probabilistic extensions of bivariate Fubini and degenerate bivariate Fubini polynomials. We investigated several identities, properties and relationships associated with these polynomials. We derived new extended identities of degenerate bivariate Fubini polynomials, degenerate Euler polynomials, degenerate Stirling numbers of the first and second kind in the view of probabilistic properties. Future work will focus on developing probabilistic extensions of other special polynomials and numbers, exploring their mathematical implications and potential physical applications.

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