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On some Berezin number inequalities via the Moore-Penrose inverse

Yonghui Ren[a], Mohamed Amine Ighachane[b,*], Otmane Benchiheb[c]

[a]*School of Mathematics and Statistics, Zhoukou Normal University, Zhoukou 466001, China*
[b]*Sciences and Technologies Team (ESTE), Higher School of Education and Training of El Jadida, Chouaib Doukkali University, El Jadida, Morocco*
[c]*Department of Mathematics, Faculty of Science, Chouaib Doukkali University, El Jadida, Morocco*

Abstract. This work presents new insights into the behavior of Berezin numbers for bounded linear operators on reproducing kernel Hilbert spaces (RKHS). Focusing on operators that admit a Moore Penrose inverse, we derive a variety of refined inequalities that extend classical Berezin-type bounds. Our approach incorporates generalized mean functions, convexity techniques, and operator-theoretic tools to establish tighter upper estimates involving both the operator and its generalized inverse. The analysis further employs interpolational methods and positivity of block operator matrices to sharpen known results and produce novel estimates. Additionally, we utilize structural properties of doubly convex functions and variants of inner product inequalities, including those inspired by the Buzano and Schwarz inequalities. The results offer a unified framework for comparing Berezin numbers, numerical radii, and related quantities, with potential applications in operator theory and functional analysis.

1. Introduction and preliminaries

In the framework of operator theory on reproducing kernel Hilbert spaces (RKHS), the Berezin set and the Berezin number serve as refined tools to analyze the localized behavior of bounded linear operators. These concepts stem from Berezin's program on quantization and are fundamental in exploring the connections between operator theory and complex function spaces.

Let $\mathcal{H}$ be a reproducing kernel Hilbert space (RKHS) over a subset $\Omega \subset \mathbb{C}^n$, and let $\mathcal{T} \in \mathcal{B}(\mathcal{H})$ be a bounded linear operator. A Hilbert space of complex-valued functions on Ω is called a functional Hilbert space if, for every point $\lambda \in \Omega$, the evaluation mapping $f \mapsto f(\lambda)$ is continuous on $\mathcal{H}$. By the Riesz representation theorem, for each $\lambda \in \Omega$, there exists a unique vector $\xi_\lambda \in \mathcal{H}$ such that

$$f(\lambda) = \langle f, \xi_\lambda \rangle \quad \text{for all } f \in \mathcal{H}.$$

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* Corresponding author: Mohamed Amine Ighachane
Email addresses: yonghuiren1992@163.com (Yonghui Ren), mohamedamineighachane@gmail.com (Mohamed Amine Ighachane), otmane.benchiheb@gmail.com (Otmane Benchiheb)
ORCID iDs: https://orcid.org/0000-0003-3946-047X (Yonghui Ren), https://orcid.org/0000-0002-4089-5617 (Mohamed Amine Ighachane), https://orcid.org/0000-0002-6759-2368 (Otmane Benchiheb)

The function $\xi(z,\lambda) := \xi_\lambda(z)$ is known as the reproducing kernel of $\mathcal{H}$. It can be expanded as

$$\xi_\lambda(z) = \sum_{n=1}^{\infty} e_n(\lambda) e_n(z),$$

where $\{e_n\}_{n\geq 1}$ is any orthonormal basis of $\mathcal{H}$. For example, in the Hardy space $H^2(\mathbb{D})$ on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, the standard basis $\{z^n\}_{n\geq 0}$ yields the kernel

$$\xi_\lambda(z) = \frac{1}{1-\overline{\lambda}z}, \quad \lambda \in \mathbb{D}.$$

Let $\hat{\xi}_\lambda = \dfrac{\xi_\lambda}{\|\xi_\lambda\|}$ denote the normalized kernel at the point $\lambda \in \Omega$. The *Berezin symbol* of an operator $\mathcal{T}$ is the function

$$\lambda \mapsto \langle \mathcal{T}\hat{\xi}_\lambda, \hat{\xi}_\lambda \rangle,$$

defined for $\lambda \in \Omega$. Using this function, we define the *Berezin range* of $\mathcal{T}$ as

$$\mathrm{Ber}(\mathcal{T}) := \left\{ \langle \mathcal{T}\hat{\xi}_\lambda, \hat{\xi}_\lambda \rangle : \lambda \in \Omega \right\}.$$

The *Berezin number* is the maximal modulus of the Berezin symbol:

$$\mathrm{ber}(\mathcal{T}) := \sup_{\lambda\in\Omega} \left| \langle \mathcal{T}\hat{\xi}_\lambda, \hat{\xi}_\lambda \rangle \right|.$$

A further generalization is given by the *Berezin norm*, defined by

$$\|\mathcal{T}\|_{\mathrm{ber}} := \sup_{\lambda,\mu\in\Omega} \left| \langle \mathcal{T}\hat{\xi}_\lambda, \hat{\xi}_\mu \rangle \right|.$$

It is clear that

$$\mathrm{ber}(\mathcal{T}) \leq \|\mathcal{T}\|_{\mathrm{ber}}.$$

A significant property of the Berezin symbol is its injectivity: for $\mathcal{T}, \mathcal{S} \in \mathcal{B}(\mathcal{H})$, if

$$\langle \mathcal{T}\hat{\xi}_\lambda, \hat{\xi}_\lambda \rangle = \langle \mathcal{S}\hat{\xi}_\lambda, \hat{\xi}_\lambda \rangle \quad \text{for all } \lambda \in \Omega,$$

then it follows that $\mathcal{T} = \mathcal{S}$, at least in spaces of analytic functions (see Zhu [26]). Hence, the mapping $\mathcal{T} \mapsto \langle \mathcal{T}\hat{\xi}_\lambda, \hat{\xi}_\lambda \rangle$ is injective on a large class of operators. Further discussions and extensions of this property can be found in [3–6].

The Berezin number and norm enjoy several structural properties:

- $\mathrm{ber}(\alpha\mathcal{T}) = |\alpha|\,\mathrm{ber}(\mathcal{T})$ for all $\alpha \in \mathbb{C}$;
- $\mathrm{ber}(\mathcal{T}+\mathcal{S}) \leq \mathrm{ber}(\mathcal{T}) + \mathrm{ber}(\mathcal{S})$;
- $\|\alpha\mathcal{T}\|_{\mathrm{ber}} = |\alpha|\|\mathcal{T}\|_{\mathrm{ber}}$;
- $\|\mathcal{T}+\mathcal{S}\|_{\mathrm{ber}} \leq \|\mathcal{T}\|_{\mathrm{ber}} + \|\mathcal{S}\|_{\mathrm{ber}}$;
- $\mathrm{ber}(\mathcal{T}) = \mathrm{ber}(\mathcal{T}^*)$, and $\|\mathcal{T}\|_{\mathrm{ber}} = \|\mathcal{T}^*\|_{\mathrm{ber}}$.

Moreover, if $\mathcal{T}$ is a positive operator, then $\text{ber}(\mathcal{T}) = \|\mathcal{T}\|_{\text{ber}}$ [1].

Another closely related concept is the numerical range, given for any bounded operator $\mathcal{T}$ on $\mathcal{H}$ by

$$W(\mathcal{T}) := \{\langle \mathcal{T}x, x\rangle : x \in \mathcal{H}, \|x\| = 1\},$$

and the numerical radius,

$$w(\mathcal{T}) := \sup\{|\langle \mathcal{T}x, x\rangle| : \|x\| = 1\}.$$

While the numerical radius captures a global property of the operator, the Berezin number emphasizes its behavior on reproducing kernel vectors, which often provides a more nuanced understanding, especially in analytic function spaces.

The Berezin number is a significant numerical invariant associated with bounded linear operators on Hilbert spaces, defined via the Berezin transform. It plays an important role in operator theory due to its close relation to the numerical radius and norm of operators. Berezin number inequalities provide refined bounds that improve classical inequalities, offering deeper insights into operator behavior. These inequalities are valuable tools for analyzing positivity, spectral properties, and functional calculus of operators. Recent literature has focused on establishing sharper Berezin number bounds and extending their applications to block operators and operator matrices. The study of these inequalities also intersects with matrix analysis, quantum information, and numerical range theory. Such developments enhance both theoretical understanding and computational methods in operator theory. Consequently, Berezin number inequalities continue to attract interest for their rich mathematical structure and practical relevance.

For some recent inequalities on the Berezin number, the reader is referred to [13–15, 17, 18, 20, 24, 25].

A fundamental concept in operator theory is the Moore-Penrose inverse, which generalizes the notion of operator inverses to more general settings.

For an operator $\mathcal{T} \in \mathcal{B}(\mathcal{H})$, the Moore-Penrose inverse of $\mathcal{T}$, denoted by $\mathcal{T}^\dagger$, is the unique operator satisfying the following four Penrose equations [19],

$$\mathcal{T}\mathcal{T}^\dagger\mathcal{T} = \mathcal{T}, \quad \mathcal{T}^\dagger\mathcal{T}\mathcal{T}^\dagger = \mathcal{T}^\dagger, \quad (\mathcal{T}\mathcal{T}^\dagger)^* = \mathcal{T}\mathcal{T}^\dagger, \quad (\mathcal{T}^\dagger\mathcal{T})^* = \mathcal{T}^\dagger\mathcal{T}. \tag{1}$$

These conditions ensure the existence and uniqueness of $\mathcal{T}^\dagger$ whenever $\mathcal{T}$ has a closed range.

We define the set of all operators in $\mathcal{B}(\mathcal{H})$ that admit a Moore-Penrose inverse as follows:

$$\mathbb{CR}(\mathcal{H}) = \{\mathcal{T} \in \mathcal{B}(\mathcal{H}) \mid \text{range}(\mathcal{T}) \text{ is closed}\}.$$

For any $\mathcal{T} \in \mathbb{CR}(\mathcal{H})$, its Moore-Penrose inverse $\mathcal{T}^\dagger$ is also a bounded operator in $\mathcal{B}(\mathcal{H})$.

For an operator $\mathcal{T} \in \mathbb{CR}(\mathcal{H})$, the Moore-Penrose inverse satisfies the following important properties:

- **Self-adjoint property:** If $\mathcal{T}$ is self-adjoint, i.e., $\mathcal{T} = \mathcal{T}^*$, then $\mathcal{T}^\dagger$ is also self-adjoint:

$$(\mathcal{T}^\dagger)^* = \mathcal{T}^\dagger.$$

- **Involutivity:** The Moore-Penrose inverse satisfies

$$(\mathcal{T}^\dagger)^\dagger = \mathcal{T}.$$

- **Order preservation:** If $\mathcal{T}_1 \le \mathcal{T}_2$ in the operator order, then

$$\mathcal{T}_2^\dagger \le \mathcal{T}_1^\dagger.$$

- **Product formula:** If $\mathcal{T}_1, \mathcal{T}_2 \in \mathbb{CR}(\mathcal{H})$ satisfy $\text{range}(\mathcal{T}_1) \subseteq \text{range}(\mathcal{T}_2)$, then

$$(\mathcal{T}_2\mathcal{T}_1)^\dagger = \mathcal{T}_1^\dagger\mathcal{T}_2^\dagger.$$

We refer the interested readers to [10, 12] for additional insights into the Moore-Penrose inverses of operators in $\mathcal{B}(\mathcal{H})$.

The Schwarz inequality applied to positive operators states that if $\mathcal{T}$ is a positive operator in $\mathcal{B}(\mathcal{H})$, then for any $x, y \in \mathcal{H}$,

$$|\langle \mathcal{T} x, y\rangle|^2 \leq \langle \mathcal{T} x, x\rangle\langle \mathcal{T} y, y\rangle. \tag{2}$$

Very recently, Sababheh et al. [23] established an interesting version of the Cauchy-Schwarz inequality involving the Moore-Penrose inverse, stated as follows: For $\mathcal{T} \in \mathbb{CR}(\mathcal{H})$, and for any $x, y \in \mathcal{H}$,

$$\left|\langle \mathcal{T} x, y\rangle\right| \leq \sqrt{\langle |\mathcal{T}|^2 x, x\rangle\langle \mathcal{T}\mathcal{T}^{\dagger} y, y\rangle}. \tag{3}$$

As a direct consequence of the previous inequality, we obtain the following berezin bound involving the Moore-Penrose inverse, which can be found in [16]. Specifically, for $\mathcal{T} \in \mathbb{CR}(\mathcal{H})$ and $r \geq 1$, we have:

$$\mathrm{ber}^r(\mathcal{T}) \leq \frac{1}{2}\,\mathrm{ber}\left(|\mathcal{T}|^{2r} + \mathcal{T}\mathcal{T}^{\dagger}\right). \tag{4}$$

Another norm inequality established in the same paper involves the Berezin norm and is stated as follows.

Theorem 1.1 ([16]). *Let $\mathcal{T}, \mathcal{S} \in \mathbb{CR}(\mathcal{H})$. Then*

$$\mathrm{ber}(\mathcal{T} + \mathcal{S}) \leq \mathrm{ber}^{\frac{1}{2}}\left(|\mathcal{T}|^2 + \mathcal{S}^{\dagger}\mathcal{S}\right) \cdot \mathrm{ber}^{\frac{1}{2}}\left(|\mathcal{S}^*|^2 + \mathcal{T}\mathcal{T}^{\dagger}\right). \tag{5}$$

Further developments of inequalities involving the numerical radius and the Berezin number through the use of the Moore-Penrose inverse are discussed in the following references [9, 11, 16, 22, 23].

A binary function $\sigma : [0, \infty) \times [0, \infty) \to [0, \infty)$ is called a mean if it satisfies the following properties for all $u, v, w, z \geq 0$ and all $t > 0$:

1. If $u \leq v$, then $u \leq u\,\sigma\,v \leq v$;
2. If $u \leq w$ and $v \leq z$, then $u\,\sigma\,v \leq w\,\sigma\,z$;
3. σ is continuous in both variables;
4. $t(u\,\sigma\,v) = (tu)\,\sigma\,(tv)$.

It is immediate from these conditions that $u\,\sigma\,u = u$. Several important classes of means arise in this framework, particularly the weighted means, which depend on a parameter $\mu \in (0, 1)$. The most common examples include:

- The **weighted arithmetic mean**:

  $$u\nabla_{\mu}v = (1-\mu)u + \mu v;$$

- The **weighted geometric mean**:

  $$u\sharp_{\mu}v = u^{1-\mu}v^{\mu};$$

- The **weighted harmonic mean**:

  $$u!_{\mu}v = \left((1-\mu)u^{-1} + \mu v^{-1}\right)^{-1}.$$

These means satisfy the inequality:

$$u!_{\mu}v \leq u\sharp_{\mu}v \leq u\nabla_{\mu}v \quad \text{for all } u, v > 0.$$

A mean σ is said to be symmetric if $u\,\sigma\,v = v\,\sigma\,u$ for all $u, v > 0$. Furthermore, an interpolational path $\{\sigma_t\}_{t\in[0,1]}$ for a symmetric mean σ is a family of means satisfying:

1. $u\,\sigma_0\,v = u$, $u\,\sigma_{1/2}\,v = u\,\sigma\,v$, and $u\,\sigma_1\,v = v$;
2. $(u\,\sigma_p\,v)\,\sigma_r\,(u\,\sigma_q\,v) = u\,\sigma_{rp+(1-r)q}\,v$;
3. The map $t \mapsto u\,\sigma_t\,v$ is continuous;
4. σ_t is increasing in each variable.

A prototypical example of an interpolational path is the family of power means defined by

$$u\,m_{r,t}\,v = ((1-t)u^r + tv^r)^{1/r}, \quad r \in [-1,1],\ t \in [0,1],$$

with the limits

$$\begin{aligned} u\,m_{1,t}\,v &= u\nabla_t v, \\ \lim_{r\to 0} u\,m_{r,t}\,v &= u\sharp_t v, \\ u\,m_{-1,t}\,v &= u!_t v. \end{aligned}$$

Very recently, M. Bakherad et al. [7] established a refinement of the classical Cauchy–Schwarz inequality by employing the concept of means. This approach provides a more precise inequality and highlights the role of various interpolating means in strengthening classical results. The refined inequality they obtained is given as follows.

Theorem 1.2 ([7]). *Let $x, y \in \mathcal{H}$, $r \geq 0$ and let σ, τ, ρ be three arbitrary means on $[0,\infty)$. Then we have the following inequality:*

$$|\langle x, y\rangle|^r \leq (|\langle x, y\rangle|^r\,\sigma\,\|x\|^r\|y\|^r)\ \rho\ (|\langle x, y\rangle|^r\,\tau\,\|x\|^r\|y\|^r) \leq \|x\|^r\|y\|^r. \tag{6}$$

Another interesting refinement of the Cauchy–Schwarz inequality was established by the same authors, providing a sharper result using the framework of means. This contribution offers an alternative perspective on improving classical inequalities. The result is stated as follows.

Theorem 1.3 ([7]). *Let $x, y \in \mathcal{H}$ and let σ, τ, ρ be three arbitrary means on $[0,\infty)$. Then the following inequalities holds:*

$$|\langle x, y\rangle| \leq \sqrt{(|\langle x, y\rangle|^2\,\sigma\,|\langle x, y\rangle|\|x\|\|y\|)\ \rho\ (|\langle x, y\rangle|\|x\|\|y\|\,\tau\,\|x\|^2\|y\|^2)} \leq \|x\|\|y\|, \tag{7}$$

and

$$|\langle x, y\rangle| \leq \sqrt{(|\langle x, y\rangle|^2\,\sigma\,|\langle x, y\rangle|\|x\|\|y\|)}\,\rho\,\sqrt{(|\langle x, y\rangle|\|x\|\|y\|\,\tau\,\|x\|^2\|y\|^2)} \leq \|x\|\|y\|. \tag{8}$$

In this work, we establish several refined Berezin number inequalities for bounded linear operators on reproducing kernel Hilbert spaces. By employing generalized inner product inequalities and exploring the positivity of certain block operator matrices involving the Moore–Penrose inverse, we derive new upper bounds for Berezin-type quantities. These results provide sharpened forms of known inequalities and contribute to the structural understanding of operators through the lens of reproducing kernels. In particular, our refinements extend classical Berezin number and triangle inequalities using tools such as convexity, operator means, and generalized inverses.

This paper is organized as follows. In Section 1, we provide the necessary background on Berezin symbols, Berezin numbers, and the Moore-Penrose inverse, along with essential preliminary results. Section 2 is devoted to establishing new Berezin number inequalities through the use of scalar means and interpolational techniques. These results include refined operator bounds involving the Moore-Penrose inverse. In Section 3, we employ generalized inner product inequalities, particularly those of Buzano type, to derive sharper Berezin-type estimates. Finally, concluding remarks are presented, highlighting the main contributions and potential directions for future work.

2. Inequalities involving the Berezin number derived via the Moore-Penrose inverse.

In this section, we present new inequalities related to the Berezin number. By interpolating various arbitrary means, we extend and unify several recent results. Our approach offers a broader perspective on existing inequalities. These findings contribute to the ongoing development in this area.

To establish our main results, we employ the following variant of McCarthy's inequality, which applies to operators.

Lemma 2.1 ([8]). *Let $\mathcal{T} \in \mathcal{B}(\mathcal{H})$ be self-adjoint with spectrum in the interval J. If $f : J \to \mathbb{R}$ is convex, then*

$$f(\langle \mathcal{T}x, x\rangle) \le \langle f(\mathcal{T})x, x\rangle \tag{9}$$

for all $x \in \mathcal{H}$, with $\|x\| = 1$. Furthermor, when f is concave, then the inequality is reversed. In particular, for $r \ge 1$, we have

$$\langle \mathcal{T}x, x\rangle^r \le \langle \mathcal{T}^r x, x\rangle, \tag{10}$$

where $\mathcal{T}$ is a positive operator and x is a unit vector.

Our first main theorem presents a significant refinement of inequality (4) by incorporating various means into the framework. This result not only strengthens the existing bound but also highlights the role of mean-based techniques in Berezin-type inequalities.

Theorem 2.2. *Let $\mathcal{T} \in \mathbb{CR}(\mathcal{H})$ and let σ, τ, ρ be three arbitrary means on $[0, \infty)$. Then, for all $r \ge 1$, the following inequality holds:*

$$\begin{aligned} \mathrm{ber}^r(\mathcal{T}) &\le \left(\mathrm{ber}^r(\mathcal{T})\, \sigma\, \mathrm{ber}\left(\frac{|\mathcal{T}|^{2r} + \mathcal{T}\mathcal{T}^{\dagger}}{2}\right)\right) \rho \left(\mathrm{ber}^r(\mathcal{T})\, \tau\, \mathrm{ber}\left(\frac{|\mathcal{T}|^{2r} + \mathcal{T}\mathcal{T}^{\dagger}}{2}\right)\right) \\ &\le \mathrm{ber}\left(\frac{|\mathcal{T}|^{2r} + \mathcal{T}\mathcal{T}^{\dagger}}{2}\right). \end{aligned}$$

Proof. Assume $\hat{\xi}_\lambda \in \mathcal{H}$ and $r \ge 1$. By substituting $x = \mathcal{T}\hat{\xi}_\lambda$ and $y = \mathcal{T}\mathcal{T}^{\dagger}\hat{\xi}_\lambda$ into the first inequality of (6), we have

$$\begin{aligned} &|\langle \mathcal{T}\hat{\xi}_\lambda, \hat{\xi}_\lambda\rangle|^r = |\langle \mathcal{T}\mathcal{T}^{\dagger}\mathcal{T}\hat{\xi}_\lambda, \hat{\xi}_\lambda\rangle|^r = |\langle \mathcal{T}\hat{\xi}_\lambda, \mathcal{T}\mathcal{T}^{\dagger}\hat{\xi}_\lambda\rangle|^r \\ &\le \left(|\langle \mathcal{T}\hat{\xi}_\lambda, \hat{\xi}_\lambda\rangle|^r\, \sigma\, \|\mathcal{T}\hat{\xi}_\lambda\|^r \|\mathcal{T}\mathcal{T}^{\dagger}\hat{\xi}_\lambda\|^r\right) \rho \left(|\langle \mathcal{T}\hat{\xi}_\lambda, \hat{\xi}_\lambda\rangle|^r\, \tau\, \|\mathcal{T}\hat{\xi}_\lambda\|^r \|\mathcal{T}\mathcal{T}^{\dagger}\hat{\xi}_\lambda\|^r\right). \end{aligned}$$

Also, by the arithmetic-geometric mean inequality,

$$\begin{aligned} \|\mathcal{T}\hat{\xi}_\lambda\|^r\, \|\mathcal{T}\mathcal{T}^{\dagger}\hat{\xi}_\lambda\|^r &= \left(\langle |\mathcal{T}|^2\hat{\xi}_\lambda, \hat{\xi}_\lambda\rangle\right)^{\frac{r}{2}} \left(\langle \mathcal{T}\mathcal{T}^{\dagger}\hat{\xi}_\lambda, \hat{\xi}_\lambda\rangle\right)^{\frac{r}{2}} \\ &\le \left(\langle |\mathcal{T}|^{2r}\hat{\xi}_\lambda, \hat{\xi}_\lambda\rangle\right)^{\frac{1}{2}} \left(\langle \mathcal{T}\mathcal{T}^{\dagger}\hat{\xi}_\lambda, \hat{\xi}_\lambda\rangle\right)^{\frac{1}{2}} \\ &\le \frac{1}{2}\langle \left(|\mathcal{T}|^{2r} + \mathcal{T}\mathcal{T}^{\dagger}\right)\hat{\xi}_\lambda, \hat{\xi}_\lambda\rangle. \end{aligned}$$

Thus, by applying the monotonicity property of the means, we obtain

$$\begin{aligned} &|\langle \mathcal{T}\hat{\xi}_\lambda, \hat{\xi}_\lambda\rangle|^r \le \left(|\langle \mathcal{T}\hat{\xi}_\lambda, \hat{\xi}_\lambda\rangle|^r\, \sigma\, \|\mathcal{T}\hat{\xi}_\lambda\|^r \|\mathcal{T}\mathcal{T}^{\dagger}\hat{\xi}_\lambda\|^r\right) \rho \left(|\langle \mathcal{T}\hat{\xi}_\lambda, \hat{\xi}_\lambda\rangle|^r\, \tau\, \|\mathcal{T}\hat{\xi}_\lambda\|^r \|\mathcal{T}\mathcal{T}^{\dagger}\hat{\xi}_\lambda\|^r\right) \\ &\le \left(|\langle \mathcal{T}\hat{\xi}_\lambda, \hat{\xi}_\lambda\rangle|^r\, \sigma\, \frac{1}{2}\langle \left(|\mathcal{T}|^{2r} + \mathcal{T}\mathcal{T}^{\dagger}\right)\hat{\xi}_\lambda, \hat{\xi}_\lambda\rangle\right) \rho \left(|\langle \mathcal{T}\hat{\xi}_\lambda, \hat{\xi}_\lambda\rangle|^r\, \tau\, \frac{1}{2}\langle \left(|\mathcal{T}|^{2r} + \mathcal{T}\mathcal{T}^{\dagger}\right)\hat{\xi}_\lambda, \hat{\xi}_\lambda\rangle\right), \end{aligned}$$

by taking supremum over all $\hat{\xi}_\lambda \in \mathcal{H}$ gives the first part of the inequality.

For the second part, applying similar arguments and properties of means gives

$$
\begin{aligned}
\operatorname{ber}^r(\mathcal{T}) &\leq \left(\operatorname{ber}^r(\mathcal{T})\,\sigma\,\operatorname{ber}\left(\frac{|\mathcal{T}|^{2r}+\mathcal{T}\mathcal{T}^{\dagger}}{2}\right)\right)\rho\left(\operatorname{ber}^r(\mathcal{T})\,\tau\,\operatorname{ber}\left(\frac{|\mathcal{T}|^{2r}+\mathcal{T}\mathcal{T}^{\dagger}}{2}\right)\right)\\
&\leq \left(\operatorname{ber}\left(\frac{|\mathcal{T}|^{2r}+\mathcal{T}\mathcal{T}^{\dagger}}{2}\right)\sigma\,\operatorname{ber}\left(\frac{|\mathcal{T}|^{2r}+\mathcal{T}\mathcal{T}^{\dagger}}{2}\right)\right)\rho\left(\operatorname{ber}\left(\frac{|\mathcal{T}|^{2r}+\mathcal{T}\mathcal{T}^{\dagger}}{2}\right)\tau\,\operatorname{ber}\left(\frac{|\mathcal{T}|^{2r}+\mathcal{T}\mathcal{T}^{\dagger}}{2}\right)\right)\\
&= \operatorname{ber}\left(\frac{|\mathcal{T}|^{2r}+\mathcal{T}\mathcal{T}^{\dagger}}{2}\right),
\end{aligned}
$$

completing the proof.

□

By choosing $\sigma = \sharp_{\mu}$, $\rho = \nabla_{\frac{1}{q}}$ (the mean with parameter $\frac{1}{q}$), and $\tau = \sharp_{\nu}$ in Theorem 2.2, we obtain the following special case as an immediate consequence.

Corollary 2.3. *Let $\mathcal{T} \in \mathbb{CR}(\mathcal{H})$. For any $r \geq 1$, $\mu, \nu \in [0,1]$, and positive p, q with $\frac{1}{p} + \frac{1}{q} = 1$, the following refinement holds:*

$$
\begin{aligned}
\operatorname{ber}^r(\mathcal{T}) &\leq \frac{1}{2^{r\mu}p}\operatorname{ber}^{r(1-\mu)}(\mathcal{T})\operatorname{ber}^{r\mu}\left((\mathcal{T}\mathcal{T}^{\dagger}) + |\mathcal{T}|^2\right)\\
&\quad + \frac{1}{2^{r(1-\nu)}q}\operatorname{ber}_{\nu}^{r\nu}(\mathcal{T})\operatorname{ber}^{2r(1-\nu)}\left(|\mathcal{T}|^2 + (\mathcal{T}\mathcal{T}^{\dagger})\right)\\
&\leq \operatorname{ber}\left(\frac{|\mathcal{T}|^{2r}+\mathcal{T}\mathcal{T}^{\dagger}}{2}\right). \qquad (11)
\end{aligned}
$$

An additional enhancement of inequality (4) can be established through the next theorem. This result offers a sharper bound and extends the existing framework. The following statement precisely captures this refinement.

Theorem 2.4. *Let $\mathcal{T} \in \mathbb{CR}(\mathcal{H})$ and let σ, τ, ρ be three arbitrary means on $[0, \infty)$. Then the following inequalities holds:*

$$
\begin{aligned}
&\operatorname{ber}(\mathcal{T})\\
&\leq \left(\operatorname{ber}^2(\mathcal{T})\,\sigma\,\operatorname{ber}(\mathcal{T})\operatorname{ber}\left(\frac{|\mathcal{T}|^{2}+\mathcal{T}\mathcal{T}^{\dagger}}{2}\right)\right)\rho\left(\operatorname{ber}(\mathcal{T})\operatorname{ber}\left(\frac{|\mathcal{T}|^{2}+\mathcal{T}\mathcal{T}^{\dagger}}{2}\right)\tau\,\operatorname{ber}\left(\frac{|\mathcal{T}|^{4}+\mathcal{T}\mathcal{T}^{\dagger}}{2}\right)\right)\\
&\leq \operatorname{ber}\left(\frac{|\mathcal{T}|^{2r}+\mathcal{T}\mathcal{T}^{\dagger}}{2}\right).
\end{aligned}
$$

Proof. Assume $\hat{\xi}_{\lambda} \in \mathcal{H}$ and $r \geq 1$. By substituting $x = \mathcal{T}\hat{\xi}_{\lambda}$ and $y = \mathcal{T}\mathcal{T}^{\dagger}\hat{\xi}_{\lambda}$ into the inequality of (7), we have

$$
\begin{aligned}
&|\langle \mathcal{T}\hat{\xi}_{\lambda}, \hat{\xi}_{\lambda}\rangle| = |\langle \mathcal{T}\mathcal{T}^{\dagger}\mathcal{T}\hat{\xi}_{\lambda}, \hat{\xi}_{\lambda}\rangle| = |\langle \mathcal{T}\hat{\xi}_{\lambda}, \mathcal{T}\mathcal{T}^{\dagger}\hat{\xi}_{\lambda}\rangle|\\
&\leq \sqrt{\left(|\langle \mathcal{T}\hat{\xi}_{\lambda}, \hat{\xi}_{\lambda}\rangle|^2\,\sigma\,|\langle \mathcal{T}\hat{\xi}_{\lambda}, \hat{\xi}_{\lambda}\rangle|\|\mathcal{T}\hat{\xi}_{\lambda}\|\|\mathcal{T}\mathcal{T}^{\dagger}\hat{\xi}_{\lambda}\|\right)\rho\left(|\langle \mathcal{T}\hat{\xi}_{\lambda}, \hat{\xi}_{\lambda}\rangle|\|\mathcal{T}\hat{\xi}_{\lambda}\|\|\mathcal{T}\mathcal{T}^{\dagger}\hat{\xi}_{\lambda}\|\,\tau\,\|\mathcal{T}\hat{\xi}_{\lambda}\|^2\|\mathcal{T}\mathcal{T}^{\dagger}\hat{\xi}_{\lambda}\|^2\right)}.
\end{aligned}
$$

Also, by the arithmetic-geometric mean inequality,

$$
\begin{aligned}
\|\mathcal{T}\hat{\xi}_{\lambda}\|^2\,\|\mathcal{T}\mathcal{T}^{\dagger}\hat{\xi}_{\lambda}\|^2 &= \left(\langle |\mathcal{T}|^2\hat{\xi}_{\lambda}, \hat{\xi}_{\lambda}\rangle\right)\left(\langle \mathcal{T}\mathcal{T}^{\dagger}\hat{\xi}_{\lambda}, \hat{\xi}_{\lambda}\rangle\right)\\
&\leq \frac{1}{2}\left(\langle |\mathcal{T}|^2\hat{\xi}_{\lambda}, \hat{\xi}_{\lambda}\rangle^2 + \langle \mathcal{T}\mathcal{T}^{\dagger}\hat{\xi}_{\lambda}, \hat{\xi}_{\lambda}\rangle^2\right)\\
&\leq \frac{1}{2}\langle\left(|\mathcal{T}|^4 + \mathcal{T}\mathcal{T}^{\dagger}\right)\hat{\xi}_{\lambda}, \hat{\xi}_{\lambda}\rangle.
\end{aligned}
$$

Thus, by utilizing the monotonicity property of operator means and taking the supremum over all vectors $\hat{\xi}_\lambda \in \mathcal{H}$, we obtain the first part of the inequality.

For the second part, similar reasoning based on the same monotonicity and properties of operator means as used in the proof of the preceding theorem leads to the desired conclusion, thereby completing the proof. □

To establish further results, we incorporate the concept of doubly convex functions. Recall that a function $f : [0, \infty) \to [0, \infty)$ is said to be doubly convex if it is convex in the usual sense and satisfies the inequality

$$f\left(a^{1-\alpha} b^{\alpha}\right) \leq f^{1-\alpha}(a) f^{\alpha}(b), \quad \text{for all } a, b \geq 0 \text{ and } 0 \leq \alpha \leq 1. \tag{12}$$

Examples of such functions on $[0, \infty)$ include $f(t) = \sinh t$ and $f(t) = \cosh t$.

We also require the following lemma, which is a direct consequence of the well-known Young's inequality.

Lemma 2.5. *Let $u, v \in \mathbb{R}$ be positive, let J be a set such that $(0, 1) \subset J \subset \mathbb{R}$, let ζ be a mapping $\zeta : J \to [0, 1]$ be such that $\zeta(\alpha) + \zeta(1 - \alpha) = 1$. Then, we have*

$$uv \leq (\zeta(\alpha)u + \zeta(1 - \alpha)v)(\zeta(1 - \alpha)u + \zeta(\alpha)v).$$

The final lemma is stated as follows.

Lemma 2.6 ([2]). *Let $f : [0, \infty) \to [0, \infty)$ be a convex function, and let $\mathcal{T}, \mathcal{S} \in \mathcal{B}(\mathcal{H})$ be positive operators. If $0 \leq \alpha \leq 1$, then*

$$\| f\left((1 - \alpha)\mathcal{T} + \alpha\mathcal{S}\right) \| \leq \|(1 - \alpha) f(\mathcal{T}) + \alpha f(\mathcal{S})\|.$$

In the subsequent theorems, we derive new Berezin number inequalities that provide refined upper bounds for the Berezin number of operators on Hilbert spaces, which generalize the inequality (4). These bounds are obtained through the use of the Moore-Penrose inverse and extend existing results by introducing a more precise framework for estimating operator behavior.

Theorem 2.7. *Let $\mathcal{T} \in \mathbb{CR}(\mathcal{H})$, let J be a set such that $(0, 1) \subset J \subset \mathbb{R}$, and let $\zeta : J \to [0, 1]$ be a mapping such that $\zeta(\alpha) + \zeta(1 - \alpha) = 1$ for $0 \leq \alpha \leq 1$. If $f : [0, \infty) \to [0, \infty)$ is an increasing doubly convex function, then*

$$f\Big(ber(\mathcal{T})\Big) \leq \sqrt{\left\|\zeta(\alpha) f(\mathcal{T}\mathcal{T}^{\dagger}) + \zeta(1 - \alpha) f(|\mathcal{T}|^2)\right\|_{\text{ber}} \cdot \left\|\zeta(1 - \alpha) f(\mathcal{T}\mathcal{T}^{\dagger}) + \zeta(\alpha) f(|\mathcal{T}|^2)\right\|_{\text{ber}}}.$$

In particular, for any $r \geq 1$,

$$ber^r(\mathcal{T}) \leq \sqrt{\left\|\zeta(\alpha)\mathcal{T}\mathcal{T}^{\dagger} + \zeta(1 - \alpha)|\mathcal{T}|^{2r}\right\|_{\text{ber}} \cdot \left\|\zeta(1 - \alpha)\mathcal{T}\mathcal{T}^{\dagger} + \zeta(\alpha)|\mathcal{T}|^{2r}\right\|_{\text{ber}}}. \tag{13}$$

Proof. Let $\hat{\xi}_\lambda \in \mathcal{H}$ be a normalized reproducing kernel (unit) vector. Applying the well-known Cauchy-Schwarz inequality together with Lemma 2.5, we obtain

$$\begin{aligned}
&|\langle \mathcal{T}\hat{\xi}_\lambda, \hat{\xi}_\lambda \rangle| = |\langle \mathcal{T}\mathcal{T}^{\dagger}\mathcal{T}\hat{\xi}_\lambda, \hat{\xi}_\lambda \rangle| = |\langle \mathcal{T}\hat{\xi}_\lambda, \mathcal{T}\mathcal{T}^{\dagger}\hat{\xi}_\lambda \rangle| \\
&\leq \sqrt{\|\mathcal{T}\mathcal{T}^{\dagger}\hat{\xi}_\lambda\|^2 \cdot \|\mathcal{T}\hat{\xi}_\lambda\|^2} \\
&= \sqrt{\langle \mathcal{T}\mathcal{T}^{\dagger}\hat{\xi}_\lambda, \hat{\xi}_\lambda \rangle \cdot \langle |\mathcal{T}|^2\hat{\xi}_\lambda, \hat{\xi}_\lambda \rangle} \\
&\leq \sqrt{\Big(\zeta(\alpha)\langle \mathcal{T}\mathcal{T}^{\dagger}\hat{\xi}_\lambda, \hat{\xi}_\lambda \rangle + \zeta(1 - \alpha)\langle |\mathcal{T}|^2\hat{\xi}_\lambda, \hat{\xi}_\lambda \rangle\Big) \cdot \Big(\zeta(1 - \alpha)\langle \mathcal{T}\mathcal{T}^{\dagger}\hat{\xi}_\lambda, \hat{\xi}_\lambda \rangle + \zeta(\alpha)\langle |\mathcal{T}|^2\hat{\xi}_\lambda, \hat{\xi}_\lambda \rangle\Big)} \\
&= \sqrt{\Big\langle \left(\zeta(\alpha)\mathcal{T}\mathcal{T}^{\dagger} + \zeta(1 - \alpha)|\mathcal{T}|^2\right)\hat{\xi}_\lambda, \hat{\xi}_\lambda \Big\rangle \cdot \Big\langle \left(\zeta(1 - \alpha)\mathcal{T}\mathcal{T}^{\dagger} + \zeta(\alpha)|\mathcal{T}|^2\right)\hat{\xi}_\lambda, \hat{\xi}_\lambda \Big\rangle}.
\end{aligned}$$

Using the fact that f is an increasing doubly convex function, we obtain

$$\begin{aligned} f\left(|\langle \mathcal{T}\hat{\xi}_\lambda, \hat{\xi}_\lambda\rangle|\right) &\leq f\left(\sqrt{\left\langle \left(\zeta(\alpha)\mathcal{T}\mathcal{T}^{\dagger} + \zeta(1-\alpha)|\mathcal{T}|^2\right)\hat{\xi}_\lambda, \hat{\xi}_\lambda\right\rangle \cdot \left\langle \left(\zeta(1-\alpha)\mathcal{T}\mathcal{T}^{\dagger} + \zeta(\alpha)|\mathcal{T}|^2\right)\hat{\xi}_\lambda, \hat{\xi}_\lambda\right\rangle}\right) \\ &\leq \sqrt{f\left(\left\langle \left(\zeta(\alpha)\mathcal{T}\mathcal{T}^{\dagger} + \zeta(1-\alpha)|\mathcal{T}|^2\right)\hat{\xi}_\lambda, \hat{\xi}_\lambda\right\rangle\right) \cdot f\left(\left\langle \left(\zeta(1-\alpha)\mathcal{T}\mathcal{T}^{\dagger} + \zeta(\alpha)|\mathcal{T}|^2\right)\hat{\xi}_\lambda, \hat{\xi}_\lambda\right\rangle\right)} \\ &\leq \sqrt{\left\langle f\left(\zeta(\alpha)\mathcal{T}\mathcal{T}^{\dagger} + \zeta(1-\alpha)|\mathcal{T}|^2\right)\hat{\xi}_\lambda, \hat{\xi}_\lambda\right\rangle \cdot \left\langle f\left(\zeta(1-\alpha)\mathcal{T}\mathcal{T}^{\dagger} + \zeta(\alpha)|\mathcal{T}|^2\right)\hat{\xi}_\lambda, \hat{\xi}_\lambda\right\rangle} \\ &\leq \sqrt{\left\|f(\zeta(\alpha)\mathcal{T}\mathcal{T}^{\dagger} + \zeta(1-\alpha)|\mathcal{T}|^2)\right\|_{\text{ber}} \cdot \left\|f(\zeta(1-\alpha)\mathcal{T}\mathcal{T}^{\dagger} + \zeta(\alpha)|\mathcal{T}|^2)\right\|_{\text{ber}}} \\ &= \sqrt{\left\|\zeta(\alpha)f(\mathcal{T}\mathcal{T}^{\dagger}) + \zeta(1-\alpha)f(|\mathcal{T}|^2)\right\|_{\text{ber}} \cdot \left\|\zeta(1-\alpha)f(\mathcal{T}\mathcal{T}^{\dagger}) + \zeta(\alpha)f(|\mathcal{T}|^2)\right\|_{\text{ber}}}. \end{aligned}$$

Taking the supremum over all $\lambda \in \Omega$, we obtain the desired inequality involving $\text{ber}(\mathcal{T})$. □

In the next theorem, we present a valid alternative bound that complements the previously established estimates.

Theorem 2.8. *Let $\mathcal{T} \in \mathbb{CR}(\mathcal{H})$, let J be a set such that $(0,1) \subset J \subset \mathbb{R}$, and let $\zeta : J \to [0,1]$ be a mapping such that $\zeta(\alpha) + \zeta(1-\alpha) = 1$ with $0 \leq \alpha \leq 1$. If $f : [0,\infty) \to [0,\infty)$ is an increasing doubly convex function, then*

$$f(\text{ber}(\mathcal{T})) \leq \sqrt{\|\zeta(\alpha)f(\mathcal{T}^{\dagger}\mathcal{T}) + \zeta(1-\alpha)f(|\mathcal{T}^*|^2)\|_{\text{ber}} \cdot \|\zeta(1-\alpha)f(\mathcal{T}^{\dagger}\mathcal{T}) + \zeta(\alpha)f(|\mathcal{T}^*|^2)\|_{\text{ber}}}. \tag{14}$$

In particular, for any $r \geq 1$,

$$\text{ber}^r(\mathcal{T}) \leq \sqrt{\|\zeta(\alpha)\mathcal{T}^{\dagger}\mathcal{T} + \zeta(1-\alpha)|\mathcal{T}^*|^{2r}\|_{\text{ber}} \cdot \|\zeta(1-\alpha)\mathcal{T}^{\dagger}\mathcal{T} + \zeta(\alpha)|\mathcal{T}^*|^{2r}\|_{\text{ber}}}. \tag{15}$$

Proof. Let $\hat{\xi}_\lambda \in \mathcal{H}$ be a unit vector. We compute:

$$\begin{aligned} |\langle \mathcal{T}\hat{\xi}_\lambda, \hat{\xi}_\lambda\rangle| = |\langle \mathcal{T}^{\dagger}\mathcal{T}\hat{\xi}_\lambda, \mathcal{T}^*\hat{\xi}_\lambda\rangle| &\leq \sqrt{\|\mathcal{T}^{\dagger}\mathcal{T}\hat{\xi}_\lambda\|^2 \cdot \|\mathcal{T}^*\hat{\xi}_\lambda\|^2} \\ &= \sqrt{\langle \mathcal{T}^{\dagger}\mathcal{T}\hat{\xi}_\lambda, \hat{\xi}_\lambda\rangle \cdot \langle |\mathcal{T}^*|^2\hat{\xi}_\lambda, \hat{\xi}_\lambda\rangle} \\ &\leq \sqrt{\left(\zeta(\alpha)\langle \mathcal{T}^{\dagger}\mathcal{T}\hat{\xi}_\lambda, \hat{\xi}_\lambda\rangle + \zeta(1-\alpha)\langle |\mathcal{T}^*|^2\hat{\xi}_\lambda, \hat{\xi}_\lambda\rangle\right)} \\ &\quad \times \sqrt{\left(\zeta(1-\alpha)\langle \mathcal{T}^{\dagger}\mathcal{T}\hat{\xi}_\lambda, \hat{\xi}_\lambda\rangle + \zeta(\alpha)\langle |\mathcal{T}^*|^2\hat{\xi}_\lambda, \hat{\xi}_\lambda\rangle\right)} \\ &= \sqrt{\left\langle (\zeta(\alpha)\mathcal{T}^{\dagger}\mathcal{T} + \zeta(1-\alpha)|\mathcal{T}^*|^2)\hat{\xi}_\lambda, \hat{\xi}_\lambda\right\rangle \cdot \left\langle (\zeta(1-\alpha)\mathcal{T}^{\dagger}\mathcal{T} + \zeta(\alpha)|\mathcal{T}^*|^2)\hat{\xi}_\lambda, \hat{\xi}_\lambda\right\rangle}. \end{aligned}$$

Now applying the monotonicity and convexity of f, we get

$$\begin{aligned} f\left(|\langle \mathcal{T}\hat{\xi}_\lambda, \hat{\xi}_\lambda\rangle|\right) &\leq \sqrt{\left\langle f\left(\zeta(\alpha)\mathcal{T}^{\dagger}\mathcal{T} + \zeta(1-\alpha)|\mathcal{T}^*|^2\right)\hat{\xi}_\lambda, \hat{\xi}_\lambda\right\rangle} \\ &\quad \cdot \sqrt{\left\langle f\left(\zeta(1-\alpha)\mathcal{T}^{\dagger}\mathcal{T} + \zeta(\alpha)|\mathcal{T}^*|^2\right)\hat{\xi}_\lambda, \hat{\xi}_\lambda\right\rangle} \\ &\leq \sqrt{\|\zeta(\alpha)f(\mathcal{T}^{\dagger}\mathcal{T}) + \zeta(1-\alpha)f(|\mathcal{T}^*|^2)\|_{\text{ber}} \cdot \|\zeta(1-\alpha)f(\mathcal{T}^{\dagger}\mathcal{T}) + \zeta(\alpha)f(|\mathcal{T}^*|^2)\|_{\text{ber}}}. \end{aligned}$$

Taking the supremum over $\lambda \in \Omega$ gives the desired Berezin number inequality. □

3. Berezin Number Bounds via Positivity of Matrix Operators

In this section, we employed characterizations of matrix positivity to derive new inner product inequalities involving the Moore-Penrose inverse. These results enabled us to establish refined inequalities for the Berezin number. By leveraging operator-theoretic techniques and partial isometries, we extended known estimates in a sharper form. The approach blends structural insights from generalized inverses with functional inequalities. This framework offers new tools for analyzing operator behavior in reproducing kernel Hilbert spaces. Our findings contribute to the refinement of Berezin-type bounds and deepen the understanding of operator inequalities within this context.

The following theorem was recently established by Kittaneh et al. [21].

Theorem 3.1. *Let $\mathcal{T}, \mathcal{A}, \mathcal{B} \in \mathcal{B}(\mathcal{H})$ with $\mathcal{A}, \mathcal{B} \geq O$, Let $\mathcal{T} \in \mathcal{B}(\mathcal{H})$ be represented by the polar decomposition $\mathcal{T} = V|\mathcal{T}|$. Then $\begin{bmatrix} \mathcal{A} & \mathcal{T}^* \\ \mathcal{T} & \mathcal{B} \end{bmatrix} \geq O$ if and only if for all $x, y \in \mathcal{H}$ and for a certain partial isometry $V \in \mathcal{B}(\mathcal{H})$, we have*

$$|\langle \mathcal{T}x, y\rangle| \leqslant \frac{1}{2}\left[\left\langle \mathcal{B}^{\frac{1}{2}} V \mathcal{A}^{\frac{1}{2}} x, y\right\rangle + \sqrt{\langle \mathcal{A}x, x\rangle\langle \mathcal{B}y.y\rangle}\right]. \tag{16}$$

By applying Theorem 3.1 in conjunction with the positivity of the following block operator matrix [23].

$$\begin{bmatrix} |\mathcal{T}|^2 & \mathcal{T}^* \\ \mathcal{T} & \mathcal{T}^{\dagger}\mathcal{T} \end{bmatrix} \geq 0,$$

we derive a refined version of inequality (3), stated below.

Theorem 3.2. *Let $\mathcal{T} \in \mathbb{CR}(\mathcal{H})$. Then, for all vectors $x, y \in \mathcal{H}$, there exists a partial isometry $V \in \mathcal{B}(\mathcal{H})$ such that*

$$|\langle \mathcal{T}x, y\rangle| \leq \frac{1}{2}\left[\left|\left\langle (\mathcal{T}\mathcal{T}^{\dagger})V|\mathcal{T}|x, y\right\rangle\right| + \sqrt{\langle |\mathcal{T}|^2 x, x\rangle \langle \mathcal{T}\mathcal{T}^{\dagger} y, y\rangle}\right]. \tag{17}$$

Remark 3.3. *Theorem 3.2 offers a refined enhancement of inequality* (3) *by exploiting the identity*

$$\left|\left\langle (\mathcal{T}\mathcal{T}^{\dagger})V|\mathcal{T}|x, y\right\rangle\right| = \left|\left\langle V|\mathcal{T}|x, (\mathcal{T}\mathcal{T}^{\dagger})y\right\rangle\right|.$$

This quantity admits the estimate

$$\begin{aligned}\left|\left\langle V|\mathcal{T}|x, (\mathcal{T}\mathcal{T}^{\dagger})y\right\rangle\right| &\leq \|V\| \cdot \||\mathcal{T}|x\| \cdot \|(\mathcal{T}\mathcal{T}^{\dagger})y\| \\ &= \sqrt{\langle |\mathcal{T}|^2 x, x\rangle \langle \mathcal{T}\mathcal{T}^{\dagger} y, y\rangle},\end{aligned}$$

since V is a partial isometry with norm at most 1. This result not only improves the previous inequality but also sharpens the interplay between the norm structure of the operator and its Moore–Penrose inverse. Such improvements are particularly valuable in the study of numerical radius bounds and other operator-theoretic inequalities.

Theorem 3.4. *Let $\mathcal{T} \in \mathbb{CR}(\mathcal{H})$, and let $f : [0, \infty) \to [0, \infty)$ be a doubly convex function. Then a partial isometry $V \in \mathcal{B}(\mathcal{H})$ exists such that for all normalized reproducing kernels $\hat{\xi}_\lambda, \hat{\xi}_\mu \in \mathcal{H}$,*

$$f(|\langle \mathcal{T}\hat{\xi}_\lambda, \hat{\xi}_\mu\rangle|) \leq \frac{1}{2}\left[f\left(\left|\left\langle (\mathcal{T}\mathcal{T}^{\dagger})V|\mathcal{T}|\hat{\xi}_\lambda, \hat{\xi}_\mu\right\rangle\right|\right) + \sqrt{\left\langle f\left(|\mathcal{T}|^2\right)\hat{\xi}_\lambda, \hat{\xi}_\lambda\right\rangle \left\langle f\left(\mathcal{T}\mathcal{T}^{\dagger}\right)\hat{\xi}_\mu, \hat{\xi}_\mu\right\rangle}\right]. \tag{18}$$

In particular, for $r \geq 1$, we have

$$|\langle \mathcal{T}\hat{\xi}_\lambda, \hat{\xi}_\mu\rangle|^r \leq \frac{1}{2}\left[\left|\left\langle (\mathcal{T}\mathcal{T}^{\dagger})V|\mathcal{T}|\hat{\xi}_\lambda, \hat{\xi}_\mu\right\rangle\right|^r + \sqrt{\left\langle |\mathcal{T}|^{2r}\hat{\xi}_\lambda, \hat{\xi}_\lambda\right\rangle \left\langle (\mathcal{T}\mathcal{T}^{\dagger})\hat{\xi}_\mu, \hat{\xi}_\mu\right\rangle}\right]. \tag{19}$$

Proof. Let $\hat{\xi}_\lambda, \hat{\xi}_\mu \in \mathcal{H}$ be normalized reproducing kernels. Starting from the first inequality in Theorem 3.2 and using the fact that f is an increasing doubly convex function, we obtain

$$\begin{aligned} f(|\langle \mathcal{T}\hat{\xi}_\lambda, \hat{\xi}_\mu\rangle|) &\leq f\left(\frac{1}{2}\left|\left\langle (\mathcal{T}\mathcal{T}^\dagger)V|\mathcal{T}|\hat{\xi}_\lambda, \hat{\xi}_\mu\right\rangle\right| + \frac{1}{2}\sqrt{\left\langle |\mathcal{T}|^2\hat{\xi}_\lambda, \hat{\xi}_\lambda\right\rangle \left\langle (\mathcal{T}\mathcal{T}^\dagger)\hat{\xi}_\mu, \hat{\xi}_\mu\right\rangle}\right) \\ &\leq \frac{1}{2}f\left(\left|\left\langle (\mathcal{T}\mathcal{T}^\dagger)V|\mathcal{T}|\hat{\xi}_\lambda, \hat{\xi}_\mu\right\rangle\right|\right) + \frac{1}{2}f\left(\sqrt{\left\langle |\mathcal{T}|^2\hat{\xi}_\lambda, \hat{\xi}_\lambda\right\rangle \left\langle (\mathcal{T}\mathcal{T}^\dagger)\hat{\xi}_\mu, \hat{\xi}_\mu\right\rangle}\right) \\ &\leq \frac{1}{2}f\left(\left|\left\langle (\mathcal{T}\mathcal{T}^\dagger)V|\mathcal{T}|\hat{\xi}_\lambda, \hat{\xi}_\mu\right\rangle\right|\right) + \frac{1}{2}\sqrt{\left\langle f(|\mathcal{T}|^2)\hat{\xi}_\lambda, \hat{\xi}_\lambda\right\rangle \left\langle f(\mathcal{T}\mathcal{T}^\dagger)\hat{\xi}_\mu, \hat{\xi}_\mu\right\rangle}, \end{aligned}$$

where the second inequality follows from convexity of f, the third from inequality (12), and the final step uses Lemma 9. □

In the following theorem, we present a further refinement of inequality (4) using the Berezin number.

Theorem 3.5. *Let $\mathcal{T} \in \mathbb{CR}(\mathcal{H})$, and let $f : [0, \infty) \to [0, \infty)$ be a doubly convex function. Then, there exists a partial isometry V such that*

$$f(\mathrm{ber}(\mathcal{T})) \leq \frac{1}{2} f\left(\mathrm{ber}\left((\mathcal{T}\mathcal{T}^\dagger)V|\mathcal{T}|\right)\right) + \frac{1}{2}\mathrm{ber}\left(f(|\mathcal{T}|^2) + f(\mathcal{T}\mathcal{T}^\dagger)\right). \tag{20}$$

In particular, for $r \geq 1$, we have

$$\mathrm{ber}^r(\mathcal{T}) \leq \frac{1}{2}\,\mathrm{ber}^r\left((\mathcal{T}\mathcal{T}^\dagger)V|\mathcal{T}|\right) + \frac{1}{2}\mathrm{ber}\left(|\mathcal{T}|^{2r} + \mathcal{T}\mathcal{T}^\dagger\right). \tag{21}$$

Proof. By Theorem 3.4, for any unit reproducing kernel $\hat{\xi}_\lambda \in \mathcal{H}$ and a suitable partial isometry V, we have

$$\begin{aligned} f\left(|\langle \mathcal{T}\hat{\xi}_\lambda, \hat{\xi}_\lambda\rangle|\right) &\leq \frac{1}{2}\left[f\left(\left|\left\langle (\mathcal{T}\mathcal{T}^\dagger)V|\mathcal{T}|\hat{\xi}_\lambda, \hat{\xi}_\lambda\right\rangle\right|\right) + \sqrt{\left\langle f\left(|\mathcal{T}|^2\right)\hat{\xi}_\lambda, \hat{\xi}_\lambda\right\rangle \left\langle f\left(\mathcal{T}\mathcal{T}^\dagger\right)\hat{\xi}_\lambda, \hat{\xi}_\lambda\right\rangle}\right] \\ &\leq \frac{1}{2}f\left(\left|\left\langle (\mathcal{T}\mathcal{T}^\dagger)V|\mathcal{T}|\hat{\xi}_\lambda, \hat{\xi}_\lambda\right\rangle\right|\right) + \frac{1}{2}\left\langle \left(f(|\mathcal{T}|^2) + f(\mathcal{T}\mathcal{T}^\dagger)\right)\hat{\xi}_\lambda, \hat{\xi}_\lambda\right\rangle. \end{aligned}$$

Taking the supremum over all $\lambda \in \Omega$ and noting that f is increasing yields the desired Berezin-type inequality. □

In the following theorem, we propose an additional refinement of inequality (4) in terms of the Berezin number.

Theorem 3.6. *Let $\mathcal{T} \in \mathbb{CR}(\mathcal{H})$, and let $p > q \geq 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, there exists a partial isometry V such that*

$$\mathrm{ber}^{2s}(\mathcal{T}) \leq \frac{1}{2}\,\mathrm{ber}^{2s}\left((\mathcal{T}\mathcal{T}^\dagger)V|\mathcal{T}|\right) + \frac{1}{2}\mathrm{ber}\left(\frac{1}{q}|\mathcal{T}|^{2qs} + \frac{1}{p}(\mathcal{T}\mathcal{T}^\dagger)\right). \tag{22}$$

Proof. Let $\hat{\xi}_\lambda$ be a unit reproducing kernel vector in $\mathcal{H}$. Then,

$$\begin{aligned} |\langle \mathcal{T}\hat{\xi}_\lambda, \hat{\xi}_\lambda\rangle|^s &\leq \frac{1}{2}\left|\left\langle (\mathcal{T}\mathcal{T}^\dagger)V|\mathcal{T}|\hat{\xi}_\lambda, \hat{\xi}_\lambda\right\rangle\right|^s \\ &\quad + \frac{1}{2}\sqrt{\left\langle |\mathcal{T}|^{2sq}\hat{\xi}_\lambda, \hat{\xi}_\lambda\right\rangle^{\frac{1}{q}} \left\langle (\mathcal{T}\mathcal{T}^\dagger)\hat{\xi}_\lambda, \hat{\xi}_\lambda\right\rangle^{\frac{1}{p}}} \quad \text{(by Lemma 9)} \\ &\leq \frac{1}{2}\left|\left\langle (\mathcal{T}\mathcal{T}^\dagger)V|\mathcal{T}|\hat{\xi}_\lambda, \hat{\xi}_\lambda\right\rangle\right|^s + \frac{1}{2}\left\langle\left(\frac{1}{q}|\mathcal{T}|^{2qs} + \frac{1}{p}(\mathcal{T}\mathcal{T}^\dagger)\right)\hat{\xi}_\lambda, \hat{\xi}_\lambda\right\rangle^{1/2}, \end{aligned}$$

where the last inequality follows from Young's inequality. Taking the supremum over all $\lambda \in \Omega$, we obtain

$$\mathrm{ber}^s(\mathcal{T}) \leq \frac{1}{2}\,\mathrm{ber}^s\left((\mathcal{T}\mathcal{T}^{+})V|\mathcal{T}|\right) + \frac{1}{2}\,\mathrm{ber}^{1/2}\left(\frac{1}{q}|\mathcal{T}|^{2qs} + \frac{1}{p}(\mathcal{T}\mathcal{T}^{+})\right).$$

Now, squaring both sides and using the convexity of the function $\xi(\lambda) = \lambda^2$, we get

$$\begin{aligned}\mathrm{ber}^{2s}(\mathcal{T}) &\leq \left(\frac{1}{2}\,\mathrm{ber}^s\left((\mathcal{T}\mathcal{T}^{+})V|\mathcal{T}|\right) + \frac{1}{2}\,\mathrm{ber}^{1/2}\left(\frac{1}{q}|\mathcal{T}|^{2qs} + \frac{1}{p}(\mathcal{T}\mathcal{T}^{+})\right)\right)^2 \\ &\leq \frac{1}{2}\,\mathrm{ber}^{2s}\left((\mathcal{T}\mathcal{T}^{+})V|\mathcal{T}|\right) + \frac{1}{2}\,\mathrm{ber}\left(\frac{1}{q}|\mathcal{T}|^{2qs} + \frac{1}{p}(\mathcal{T}\mathcal{T}^{+})\right).\end{aligned}$$

This completes the proof. □

Conclusions

In this paper, we have established several new Berezin number inequalities for bounded linear operators acting on reproducing kernel Hilbert spaces, with particular emphasis on operators possessing a Moore-Penrose inverse. By employing a combination of operator-theoretic tools, convexity arguments, and functional inequalities involving scalar means, we derived refined upper bounds that significantly enhance classical results. The introduction of doubly convex functions and interpolational frameworks enabled us to uncover deeper structural relationships between operator norms, numerical radii, and Berezin-type quantities.

Furthermore, by incorporating positivity criteria for block operator matrices and generalized Buzano-type inequalities, we presented sharper estimates that reflect the localized behavior of operators. These developments not only enrich the theoretical landscape of operator inequalities but also pave the way for further investigations into related quantities in analytic function spaces and other functional settings.

Future work may explore extensions of these inequalities to unbounded operators, non-Hilbertian settings, or connections with matrix inequalities and quantum information theory. The framework developed here provides a versatile and unified foundation for such explorations.

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