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On Euler-Genocchi numbers and polynomials

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Abstract. We study number theoretic and analytic properties of the recently defined Euler-Genocchi polynomials and numbers. In particular, we give several divisibility properties of the Euler-Genocchi numbers and integerness of the values of the Euler-Genocchi polynomials. We also consider zeta and p-adic zeta function representations for these polynomials and numbers.

1. Introduction

An Appell sequence $\{P_n(x)\}$ is defined formally by an exponential generating function of the form

$$G(x,t) = A(t)e^{xt} = \sum_{n=0}^{\infty} P_n(x)\frac{t^n}{n!},$$

where x and t are indeterminates, and $A(t)$ is a formal power series with $A(0) \neq 0$. Since any such generating function has polynomial coefficient satisfying

$$P'_n(x) = nP_{n-1}(x)$$

for $n \geq 1$, the members of an Appell sequence are called the Appell polynomials. The Appell polynomials have been well studied because of their remarkable applications in number theory and mathematical analysis. They include many types, the most famous of which are $B_n(x)$ Bernoulli, $E_n(x)$ Euler, and $G_n(x)$ Genocchi polynomials defined by

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!},\quad \frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!},\quad \frac{2te^{xt}}{e^t+1} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!},$$

respectively, where $|t| < 2\pi$ for the Bernoulli polynomials and $|t| < \pi$ for the Euler and Genocchi polynomials.

A new family of the Appell polynomials that generalizes both Euler and Genocchi polynomials were introduced and studied in [3] and [4]. This family is called the Euler-Genocchi polynomials and defined as

$$\frac{2t^r e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} A_n^{(r)}(x)\frac{t^n}{n!}, \tag{1.1}$$

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where $|t| < \pi$ and $A_n^{(r)}(x) = 0$ for $n < r$. It then follows that $E_n(x) = A_n^{(0)}(x)$ and $G_n(x) = A_n^{(1)}(x)$. The Euler-Genocchi numbers are defined to be $A_n^{(r)} = A_n^{(r)}(0)$.

Combinatorial identities involving nested sums expression, a determinantial approach, a multiplication formula, and recurrence and difference equations for the Euler-Genocchi polynomials, and alternating sums of powers formula in terms of the Euler-Genocchi polynomials have been discussed in [3] and [4]. Moreover, these polynomials and numbers emerge from some general families. These families, called unified generalizations of Bernoulli, Euler and Genocchi polynomials, are defined and studied in, for example, [2, 14–16]. In particular, generating functions of these families are defined in [15], their extensions and further generalizations are studied in [2] and [14], and p-adic distributions of them are presented in [16].

In this paper, we consider fundamental arithmetical properties of the Euler-Genocchi polynomials and numbers. We obtain numerous number theoretic results for the Euler-Genocchi numbers involving Staudt-Clausen-like theorem and Kummer-like congruences. We also present Almkvist-Meurman-type result for the Euler-Genocchi polynomials regarding integerness of them for special arguments as well. We study the relationships between Euler-Genocchi numbers and zeta, and p-adic zeta functions.

The content of this study is organized as follows. Section 2 is a preliminary section where we summarize the necessary contexts in the paper. In Section 3, we consider number theoretic properties of the Euler-Genocchi numbers. In Section 4, we study the values of the Euler-Genocchi polynomials for which they are integers. Section 5 is devoted to the investigation of the Euler-Genocchi numbers in the content of zeta and p-adic zeta functions.

2. Preliminaries

For a nonnegative integer n, the Bernoulli numbers B_n can be defined by the generating function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$$

have been extensively studied over the last two centuries. It is easy to find the values $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, and $B_n = 0$ for all odd $n \geq 3$. The Bernoulli numbers also occur as $B_n = B_n(0)$, where $B_n(x)$ denotes the nth Bernoulli polynomial defined by

$$\frac{te^x}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

One of the most important properties of Bernoulli numbers is the von Staudt-Clausen theorem, which explicitly determines the denominators of even indexed Bernoulli numbers. There are several versions, and we like to use the following one. If p is a prime number with $p \leq 2n + 1$, then

$$pB_{2n} \equiv \begin{cases} 0 \pmod{p}, & \text{if } (p-1) \nmid 2n, \\ -1 \pmod{p}, & \text{if } (p-1) \mid 2n. \end{cases}$$

Let k and n be nonnegative integers with $n > 0$, and consider the sums of powers of first n natural numbers

$$S_{k,n} = 1^k + 2^k + \cdots + n^k.$$

Since

$$\sum_{k=0}^{\infty} S_{k,n} \frac{t^k}{k!} = \frac{e^{(n+1)t} - e^t}{e^t - 1},$$

$S_{k,n}$ are related to Bernoulli numbers in that

$$S_{k,n} = \frac{1}{k+1}\sum_{m=0}^{k}\binom{k+1}{m}B_m n^{k+1-m}.$$

In [8], Howard studied the alternating sums of powers

$$T_{k,n} = 1^k - 2^k + 3^k - \cdots + (-1)^{n-1} n^k.$$

Analyzing the identity

$$\sum_{k=0}^{\infty} T_{k,n}\frac{t^k}{k!} = \frac{e^t + (-1)^{n-1} e^{(n+1)t}}{e^t + 1},$$

he deduced that these sums are related to Genocchi numbers $G_n(0) = G_n$ in that

$$T_{k,n} = \frac{(-1)^{n-1}}{2}n^k + \frac{(-1)^n}{2}\sum_{m=0}^{k-1}\binom{k}{m}\frac{G_{k-m+1}}{k-m+1}n^m - \frac{G_{k+1}}{2(k+1)},$$

where the Genocchi polynomials $G_n(x)$ are defined respectively by

$$\frac{2te^{xt}}{e^t+1} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}.$$

Note that the Genocchi polynomials are closely related to the Euler polynomials $E_n(x)$ defined by

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}.$$

There are numerous number sequences related to the Bernoulli numbers. One of which is the sequence known as the Stirling numbers of the second kind, denoted by $S(n,m)$. They can be defined by means of the generating function

$$\frac{(e^t-1)^m}{m!} = \sum_{n=m}^{\infty} S(n,m)\frac{t^n}{n!}, \tag{2.1}$$

and count the ways to divide a set of n elements into m nonempty sets. In particular, we have $S(n,0) = 0$ for $n \neq 0$, $S(n,1) = S(n,n) = 1$, and $S(p,m) \equiv 0 \pmod{p}$, where p is an odd prime and $m = 2, 3, \ldots, p-1$ (c.f. [9])

The Stirling numbers of the second kind are particular examples of Hurwitz series. A Hurwitz series is a power series

$$\sum_{n=0}^{\infty} a_n \frac{t^n}{n!},$$

for which each coefficient a_n is an integer. It has been defined by Hurwitz [10] in connection with the coefficients of the lemniscate function (see also [5]). The set of all Hurwitz series is closed under coefficient-wise addition and multiplication, and the reciprocal of a Hurwitz series with constant term 1 is also Hurwitz series.

3. Arithmetic properties of the Euler-Genocchi numbers

In this section, we present some congruences for the Euler-Genocchi numbers modulo a prime p extended to the ring of rational numbers with denominators not divisible by p. For such fractions,

$$\frac{a}{b} \equiv \frac{c}{d} \pmod p \Leftrightarrow ad \equiv bc \pmod p,$$

and the residue class of a/b is the residue class of ab', where b' is the inverse of b modulo p.

We start with a relationship between the Euler-Genocchi and Bernoulli numbers.

Lemma 3.1. *We have*

$$A_n^{(r)} = 2(n)_{r-1}(1 - 2^{n-r+1})B_{n-r+1},$$

where B_n is the nth Bernoulli number and

$$(n)_r = \begin{cases} n(n-1)\cdots(n-r+1), & \text{if } n \geq r > 0, \\ 1, & \text{if } r = 0, \\ 0, & \text{if } r > n, \end{cases}$$

is the falling factorial.

Proof. By (1.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} A_n^{(r)} \frac{t^n}{n!} &= \frac{2t^r}{e^t + 1} = \frac{2t^r(e^t - 1)}{e^{2t} - 1} = \frac{2t^r(e^t + 1 - 2)}{e^{2t} - 1} \\ &= \frac{2t^r}{e^t - 1} - \frac{4t^r}{e^{2t} - 1} = 2t^{r-1} \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} - 2t^{r-1} \sum_{n=0}^{\infty} 2^n B_n \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} 2(1 - 2^n) B_n \frac{t^{n+r-1}}{n!} = \sum_{n=r-1}^{\infty} 2(n)_{r-1}(1 - 2^{n-r+1}) B_{n-r+1} \frac{t^n}{n!} \\ &= \sum_{n=r}^{\infty} 2(n)_{r-1}(1 - 2^{n-r+1}) B_{n-r+1} \frac{t^n}{n!}.\end{aligned}$$

Since $A_n^{(r)} = 0$ for $n < r$, the result follows by equating the coefficients of t^n on the both sides. □

Lemma 3.1 particularly implies that if $(n - r)$ is a non-zero even integer, then $A_n^{(r)} = 0$. Lemma 3.1 also provides a von Staudt-Clausen-type theorem for the Euler-Genocchi numbers.

Theorem 3.2. *If p is an odd prime such that $p \leq n - r + 2$, then $pA_n^{(r)} \equiv 0 \pmod p$.*

Proof. By Lemma 3.1, we write

$$pA_n^{(r)} = 2(n)_{r-1}(1 - 2^{n-r+1})pB_{n-r+1}.$$

Since p is odd, $2^{n-r+1} \equiv 1 \pmod p$ by Fermat's theorem when $(p - 1) \mid (n - r + 1)$, so $pA_n^{(r)} \equiv 0 \pmod p$. If $(p - 1) \nmid (n - r + 1)$, then by the von Staudt-Clausen theorem $pB_{n-r+1} \equiv 0 \pmod p$, which again implies that $pA_n^{(r)} \equiv 0 \pmod p$. □

When $p = 2$, Lemma 3.1 and the von Staudt-Clausen theorem for the Bernoulli numbers yield to the congruences

$$A_{2n+1}^{(2)} \equiv A_{2n}^{(1)} \equiv -1 \pmod 2.$$

One of the versions for the von Staudt-Clausen theorem gives divisibility property for the sums of powers of integers. A similar result for the alternating sums of powers of integers is related to the Euler-Genocchi numbers.

Lemma 3.3. *Let p be an odd prime. For integers r and n with $0 \leq r \leq n$, we have*

$$A_n^{(r)} \equiv (n)_r \sum_{m=0}^{p-1} (-1)^m m^{n-r} (\operatorname{mod} p).$$

Proof. It is not difficult to see that

$$A_n^{(r)}(x+1) + A_n^{(r)}(x) = 2(n)_r x^{n-r}.$$

Since

$$A_n^{(r)}(x) = \sum_{s=0}^{n} \binom{n}{s} A_s^{(r)} x^{n-s} = A_n^{(r)} + \sum_{s=0}^{n-1} \binom{n}{s} A_s^{(r)} x^{n-s},$$

we have

$$A_n^{(r)}(k) \equiv A_n^{(r)} (\operatorname{mod} k), \tag{3.1}$$

where k is a natural number. Now,

$$(-1)^{k+1} A_n^{(r)}(k) + A_n^{(r)} = \sum_{m=0}^{k-1} \left((-1)^m A_n^{(r)}(m) - (-1)^{m+1} A_n^{(r)}(m+1) \right)$$

gives for an odd positive integer k that

$$A_n^{(r)}(k) + A_n^{(r)} = \sum_{m=0}^{k-1} (-1)^m \left(A_n^{(r)}(m+1) + A_n^{(r)}(m) \right) = \sum_{m=0}^{k-1} (-1)^m 2(n)_r m^{n-r}.$$

We then have

$$A_n^{(r)} \equiv (n)_r \sum_{m=0}^{k-1} (-1)^m m^{n-r} (\operatorname{mod} k).$$

Choosing $k = p$ results in the desired identity. □

Corollary 3.4. *Let p be an odd prime.*
a) If n is a multiple of p, then $A_n^{(r)} \equiv 0 (\operatorname{mod} p)$.
b) If $p > n - r$, then $A_n^{(r)} \equiv 0 (\operatorname{mod} p)$.
c) If $(p-1) | (n-r)$, then $A_n^{(r)} \equiv (n)_r (\operatorname{mod} p)$. In particular, we have $A_{p-1}^{(p-1)} \equiv -1 (\operatorname{mod} p)$.

Proof. Part (a) and part (b) immediately follow from Lemma 3.3 because of the presence of the factor $(n)_r$ on the right. For part (c), we note that

$$\sum_{m=0}^{p-1} (-1)^m m^{n-r} \equiv \sum_{m=0}^{p-1} (-1)^m = 1 (\operatorname{mod} p)$$

by Fermat's theorem and since p is odd. □

Next result is an analogue of Kummer's congruences for the Bernoulli numbers.

Theorem 3.5. *Let p be an odd prime. For nonnegative integers n and r with $p \leq n - r$ and $\gcd(p, n) = 1$, we have*

$$\frac{A_{n+p-1}^{(r)}}{n-r} \equiv \frac{A_n^{(r)}}{n} (\operatorname{mod} p).$$

Proof. In Lemma 3.3, we write $n+p-1$ instead of n to obtain

$$A_{n+p-1}^{(r)} \equiv (n+p-1)_r \sum_{m=0}^{p-1} (-1)^m m^{n+p-1-r} (\operatorname{mod} p).$$

Since $(n+p-1)_r \equiv (n-1)_r (\operatorname{mod} p)$, we have by Fermat's theorem that

$$A_{n+p-1}^{(r)} \equiv \sum_{m=0}^{p-1} (-1)^m m^{n-r} (\operatorname{mod} p).$$

Since $n(n-r)_r = (n-r)(n)_r$ and $\gcd(n,p) = 1$, the results is obtained. □

A prime shift in the index yields to the following congruence.

Theorem 3.6. *For an odd prime p and integer r with $0 \leq r < p$, we have*

$$A_{p+r}^{(r)} \equiv -\frac{r!}{2} (\operatorname{mod} p).$$

Proof. By (1.1) and (2.1), we deduce for $|t| < \pi$ that

$$\begin{aligned}
\sum_{n=0}^{\infty} A_n^{(r)} \frac{t^n}{n!} &= \frac{2t^r}{e^t+1} = \frac{t^r}{\frac{e^t+1}{2}} = \frac{t^r}{1-\frac{1-e^t}{2}} = t^r \sum_{m=0}^{\infty} \left(\frac{1-e^t}{2}\right)^m \\
&= t^r \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m} \left(e^t - 1\right)^m = t^r \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m} m! \sum_{n=m}^{\infty} S(n,m) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \frac{(-1)^m}{2^m} m! S(n,m) \right) \frac{t^{n+r}}{n!} = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n-r} \frac{(-1)^m}{2^m} m! S(n-r,m) \right) (n)_r \frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of t^n gives

$$A_n^{(r)} = (n)_r \sum_{m=0}^{n-r} \frac{(-1)^m}{2^m} m! S(n-r,m). \tag{3.2}$$

Let $n-r=p$ be an odd prime in (3.2). Then, by the properties of the Stirling numbers of the second kind, we find that

$$A_{p+r}^{(r)} = (p+r)_r \sum_{m=0}^{p} \frac{(-1)^m}{2^m} m! S(p,m) = (p+r)_r \left(\frac{-1}{2} - \frac{p!}{2^p} + \sum_{m=2}^{p-1} \frac{(-1)^m}{2^m} m! S(p,m) \right) \equiv -\frac{r!}{2} (\operatorname{mod} p)$$

which is the desired result. □

As a final result in this section, we consider the value of the Euler-Genocchi polynomial at 1/2.

Theorem 3.7. *Let p be an odd prime and n and r be integers with $0 \leq r \leq n$. If $(p-1)|(n-r)$, then*

$$A_n^{(r)}\left(\frac{1}{2}\right) \equiv (-4|p)\,(n)_r (\operatorname{mod} p),$$

where $(\cdot|p)$ is the Legendre symbol.

Proof. The exponential generating function of the alternating sum

$$\sum_{m=0}^{k-1}(-1)^m(2m+1)^n$$

is

$$\frac{(-1)^{k-1}e^{(2k+1)t}+e^t}{e^{2t}+1}.$$

Since

$$\begin{aligned}\frac{(-1)^{k-1}e^{(2k+1)t}+e^t}{e^{2t}+1} &= \frac{(-1)^{k-1}}{2(2t)^r}\frac{2(2t)^r e^{(k+\frac{1}{2})2t}}{e^{2t}+1}+\frac{1}{2(2t)^r}\frac{2(2t)^r e^{(\frac{1}{2})2t}}{e^{2t}+1}\\ &= \frac{(-1)^{k-1}}{2^{r+1}t^r}\sum_{n=0}^{\infty}A_n^{(r)}\left(k+\frac{1}{2}\right)2^n\frac{t^n}{n!}+\frac{1}{2^{r+1}t^r}\sum_{n=0}^{\infty}A_n^{(r)}\left(\frac{1}{2}\right)2^n\frac{t^n}{n!}\\ &= \frac{(-1)^{k-1}}{2^{r+1}}\sum_{n=0}^{\infty}\frac{2^{n+r}}{(n+r)_r}A_{n+r}^{(r)}\left(k+\frac{1}{2}\right)\frac{t^n}{n!}+\frac{1}{2^{r+1}}\frac{2^r}{(n+r)_r}\sum_{n=0}^{\infty}2^nA_{n+r}^{(r)}\left(\frac{1}{2}\right)\frac{t^n}{n!},\end{aligned}$$

we have

$$\frac{2^{n-1}}{(n+r)_r}\left((-1)^{k-1}A_{n+r}^{(r)}\left(k+\frac{1}{2}\right)+A_{n+r}^{(r)}\left(\frac{1}{2}\right)\right)=\sum_{m=0}^{k-1}(-1)^m(2m+1)^n$$

or equivalently

$$2^{n-r}\left((-1)^{k-1}A_n^{(r)}\left(k+\frac{1}{2}\right)+A_n^{(r)}\left(\frac{1}{2}\right)\right)=2\,(n)_r\sum_{m=0}^{k-1}(-1)^m\,(2m+1)^{n-r}.$$

Let $k=p$ be an odd prime. Then, by (3.1), we obtain that

$$2^{n-r}A_n^{(r)}\left(\frac{1}{2}\right)\equiv(n)_r\sum_{m=0}^{p-1}(-1)^m\,(2m+1)^{n-r}\,(\operatorname{mod}p). \tag{3.3}$$

Now, suppose that $(p-1)|(n-r)$. Since

$$\sum_{m=0}^{p-1}(-1)^m\,(2m+1)^{n-r}\equiv\begin{cases}0\,(\operatorname{mod}p), & \text{if } p\equiv 1\,(\operatorname{mod}4),\\ 2\,(\operatorname{mod}p) & \text{if } p\equiv 3\,(\operatorname{mod}4),\end{cases}$$

or equivalently

$$\sum_{m=0}^{p-1}(-1)^m\,(2m+1)^{n-r}\equiv(-4|p)\,(\operatorname{mod}p),$$

we reach at the desired result by Fermat's theorem. □

4. The Almkvist-Meurman theorem for the Euler-Genocchi polynomials

In 1991, Almkvist and Meurman [1] showed that if h and k are integers with $k\neq 0$

$$k^n\left(B_n\left(\frac{h}{k}\right)-B_n\right)$$

is also integer. In 1999 Fox [7] showed that

$$k^n\left(E_n\left(\frac{h}{k}\right)-(-1)^{hk}E_n(0)\right)$$

is an integer whenever $h,k\in\mathbb{Z}$, $k\neq 0$, and in 2001 Sury [19] showed that for an arbitrary integer h, $k^nE_n\left(\frac{h}{k}\right)$ is an integer if k is even, and

$$k^n\left(E_n\left(\frac{h}{k}\right)+(-1)^{h-1}E_n(0)\right)$$

is an integer if k is odd. In this section we obtain similar results for the Euler-Genocchi polynomials.

Theorem 4.1. *If $k\neq 0$ is an even integer, then $k^{n-r}A_n^{(r)}\left(\frac{h}{k}\right)$ is an integer for all integers h and natural numbers r.*

Proof. By (1.1), we have

$$\sum_{n=0}^{\infty}k^{n-r}A_n^{(r)}\left(\frac{h}{k}\right)\frac{t^n}{n!}=\frac{2t^r}{e^{kt}+1}e^{ht}.$$

Since k is even,

$$\frac{1}{2}\left(e^{kt}+1\right)=1+\sum_{n=1}^{\infty}\frac{1}{2}k^n\frac{t^n}{n!}$$

has integer coefficients and constant term 1. Thus, its reciprocal $\frac{2}{e^{kt}+1}$ is a Hurwitz series. On the other hand, we have

$$t^re^{ht}=\sum_{n=0}^{\infty}(n)_r\,h^{n-r}\frac{t^n}{n!}.$$

So, t^re^{ht} is a Hurwitz series as well, and the result follows. □

Theorem 4.2. *If k is odd, then*

$$\frac{1}{2}k^{n-r}\left(A_n^{(r)}\left(\frac{h}{k}\right)-(-1)^hA_n^{(r)}\right)$$

is an integer for any integers h and r with $n\geq r\geq 0$.

Proof. Letting

$$a_n^{(r)}=\frac{1}{2}k^{n-r}\left(A_n^{(r)}\left(\frac{h}{k}\right)-(-1)^hA_n^{(r)}\right),$$

we may write

$$\sum_{n=0}^{\infty}a_n^{(r)}\frac{t^n}{n!}=\frac{t^r}{e^{kt}+1}\left(e^{ht}-(-1)^h\right),$$

from which we have

$$\frac{e^{kt}+1}{e^t+1}\sum_{n=0}^{\infty}a_n^{(r)}\frac{t^n}{n!}=t^r\frac{\left(e^{ht}-(-1)^h\right)}{e^t+1},$$

or equivalently have

$$\sum_{m=0}^{k-1}(-1)^m e^{mt}\sum_{n=0}^{\infty}a_n^{(r)}\frac{t^n}{n!}=t^r(-1)^{h-1}\sum_{m=0}^{h-1}(-1)^m e^{mt}.$$

Now, the left hand side can be written as

$$\begin{aligned}\sum_{m=0}^{k-1}(-1)^m\left(\sum_{n=0}^{\infty}m^n\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}a_n^{(r)}\frac{t^n}{n!}\right)&=\sum_{m=0}^{k-1}(-1)^m\sum_{n=0}^{\infty}\left(\sum_{s=0}^{n}\binom{n}{s}m^{n-s}a_s^{(r)}\right)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{s=0}^{n}\binom{n}{s}a_s^{(r)}\sum_{m=0}^{k-1}(-1)^m m^{n-s}\right)\frac{t^n}{n!}\\&=-\sum_{n=0}^{\infty}\left(\sum_{s=0}^{n}\binom{n}{s}a_s^{(r)}T_{n-s,k-1}\right)\frac{t^n}{n!},\end{aligned}$$

while we have

$$\begin{aligned}t^r(-1)^{h-1}\sum_{m=0}^{h-1}(-1)^m e^{mt}&=t^r(-1)^{h-1}\sum_{m=0}^{h-1}(-1)^m\sum_{n=0}^{\infty}m^n\frac{t^n}{n!}\\&=(-1)^{h-1}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{h-1}(-1)^m m^{n-r}\right)(n)_r\frac{t^n}{n!}\\&=-(-1)^{h-1}\sum_{n=0}^{\infty}(n)_r T_{n-r,h-1}\frac{t^n}{n!}.\end{aligned}$$

Thus, we find that

$$\sum_{s=0}^{n}\binom{n}{s}a_s^{(r)}T_{n-s,k-1}=(-1)^{h-1}(n)_r T_{n-r,h-1}.$$

Since k is odd, we have $T_{0,k-1}=1$, so

$$a_n^{(r)}=(-1)^{h-1}(n)_r T_{n-r,h-1}-\sum_{s=0}^{n-1}\binom{n}{s}a_s^{(r)}T_{n-s,k-1}.$$

$a_r^{(r)}=0$ or $r!$ according as h is even or odd, so by induction $a_r^{(r)}\in\mathbb{Z}$ for all $n>r$. This completes the proof. □

5. Zeta functions and Euler-Genocchi numbers and polynomials

In 2017, Young [22] defined the general zeta function

$$Z_{m,k}(s,a)=\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}\mathrm{Li}_k\left(\frac{1-e^{-t}}{m}\right)\frac{me^{-at}}{1-e^{-t}}\mathrm{d}t,$$

for $\mathrm{Re}(s)>0$, $\mathrm{Re}(a)>0$, and $|r|\geq 1$, where $\Gamma(s)$ is the gamma function and

$$\mathrm{Li}_k(z)=\sum_{r=1}^{\infty}\frac{z^r}{r^k}$$

is the polylogarithm function of order k. Important special cases of $Z_{m,k}(s,a)$ include

$$Z_{1,1}(s,a) = s\zeta(s+1,a),$$

where

$$\zeta(s,a) = \sum_{k=0}^{\infty} \frac{1}{(k+a)^s}$$

is the Hurwitz zeta function,

$$Z_{2,0}\left(s,\frac{1}{2}\right) = 2^{s+1}\beta(s),$$

where

$$\beta(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^s}$$

is the Dirichlet beta function,

$$Z_{2,0}(s,1) = 2\eta(s),$$

where

$$\eta(s) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^s}$$

is the alternating zeta function, and

$$Z_{1,1}\left(s,\frac{1}{2}\right) = s\left(2^{s+1}-1\right)\zeta(s+1),$$

where

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

is the Riemann zeta function.

The Riemann zeta function and Bernoulli numbers share a particular relationship in that

$$\zeta(1-n) = -\frac{B_n}{n} + \delta_{n,1},$$

where n is a positive integer and $\delta_{n,1}$ is the Kronecker symbol defined as

$$\delta_{n,1} = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases}$$

Moreover, we have

$$\zeta(1-n,a) = -\frac{B_n(a)}{n},$$

and

$$\zeta_E(-n,a) = E_n(a),$$

where

$$\zeta_E(s,a) = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+a)^s}$$

is the Hurwitz-type Euler zeta function defined in [11].

These relationships motivate a possible expression of the Euler-Genocchi numbers in terms of zeta functions. For such a relation we note that

$$\mathrm{Li}_0\left(\frac{1-e^{-t}}{2}\right)\frac{2e^{-at}}{1-e^{-t}} = \frac{2e^{-at}}{1+e^{-t}} = \frac{2e^{(1-a)t}}{1+e^t}.$$

So

$$Z_{2,0}(s,a) = \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}\frac{2e^{(1-a)t}}{1+e^t}\mathrm{d}t,$$

and hence we find that

$$Z_{2,0}(s+r,1) = \frac{1}{\Gamma(s+r)}\int_0^{\infty} t^{s-1}\frac{2t^r}{1+e^t}\mathrm{d}t = \frac{1}{\Gamma(s+r)}\int_0^{\infty} t^{s-1}\sum_{k=0}^{\infty} A_k^{(r)}\frac{t^k}{k!}\mathrm{d}t.$$

Since $Z_{2,0}(s,1) = 2\eta(s)$, we conclude that

$$2\eta(s+r) = \frac{1}{\Gamma(s+r)}\int_0^{\infty} t^{s-1}\sum_{k=0}^{\infty} A_k^{(r)}\frac{t^k}{k!}\mathrm{d}t.$$

By the residue theorem, we find that

$$2\eta(s+r) = \frac{1}{\Gamma(s+r)}\int_0^{\infty} t^{-n-1}\sum_{k=0}^{\infty} A_k^{(r)}\frac{t^k}{k!}\mathrm{d}t = \frac{(n-r)!}{(-1)^{n-r}}\frac{A_n^{(r)}}{n!},$$

or equivalently,

$$\eta(-n+r) = \frac{(-1)^{n-r}}{2}\frac{A_n^{(r)}}{(n)_r}.$$

Let p be an odd prime and $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic numbers, and the completion of an algebraic closure of $\mathbb{Q}_p$, respectively. We say that $f : \mathbb{Z}_p \to \mathbb{C}_p$ is uniformly differentiable function at a point $a \in \mathbb{Z}_p$ if $\frac{f(x)-f(y)}{x-y}$ have limit as $(x,y) \to (a,a)$, provided that $x \neq y$. The p-adic functions with nice properties are powerful tools for studying many results of classical number theory in a straightforward manner. They strength almost all the arithmetic results on the Bernoulli and Euler numbers (see, for example, [6, Chapter 11], [17]). In particular, we have

$$B_n = \int_{\mathbb{Z}_p} a^n da,$$

where

$$\int_{\mathbb{Z}_p} f(x)\,dx = \lim_{N\to\infty}\frac{1}{p^N}\sum_{x=0}^{p^N-1} f(x)$$

is the Volkenborn integral ([17, p.264]).

There are different ways to describe Euler numbers in the p-adic context (e.g. [11], [12], [13], [18], and [22], and references therein). In particular, Osipov [13] modified the Volkenborn integral as

$$\int_{\mathbb{Z}_p} f(a)d\mu_{\varepsilon}(a) = \lim_{N\to\infty} \sum_{a=0}^{p^{mN}-1} f(a)\varepsilon^a,$$

where $f : \mathbb{Z}_p \to \mathbb{C}_p$ is uniformly differentiable function, $\varepsilon^k = 1$, $\varepsilon \neq 1$, $\gcd(k, p) = 1$ and $k|(p^m - 1)$. If $k = 2$ and $m = 1$, we have

$$\int_{\mathbb{Z}_p} f(a)d\mu_{-1}(a) = \lim_{N\to\infty} \sum_{a=0}^{p^N-1} (-1)^a f(a). \tag{5.1}$$

We note that equation (5.1) is defined and studied in detail by Kim [12] within the concept of symmetric p-adic invariant integrals on $\mathbb{Z}_p$. Since these integrals are related to the Euler polynomials, we prefer them here to investigate the Euler-Genocchi polynomials over the set of p-adic numbers.

Let X be an arbitrary subset of $\mathbb{C}_p$ closed under the shift $x \to x + a$ for $a \in \mathbb{Z}_p$ and $x \in X$. Suppose $f : X \to \mathbb{C}_p$ is uniformly differentiable on X, so that for fixed $x \in X$, the function $a \to f(x + a)$ is uniformly differentiable on $\mathbb{Z}_p$. Using (5.1), we obtain that

$$\begin{aligned}\int_{\mathbb{Z}_p} e^{at}d\mu_{-1}(a) + \int_{\mathbb{Z}_p} e^{(a-1)t}d\mu_{-1}(a) &= (1 + e^{-t}) \lim_{N\to\infty} \sum_{a=0}^{p^N-1} (-1)^a e^{at}\\ &= (1 + e^{-t}) \lim_{N\to\infty} \frac{1 + (-1)^{p^N-1}e^{p^N t}}{1 + e^t} = e^{-t} \lim_{N\to\infty} (1 + e^{p^N t}) = 2e^{-t}.\end{aligned}$$

Thus,

$$(1 + e^{-t}) \int_{\mathbb{Z}_p} e^{at}d\mu_{-1}(a) = 2e^{-t}$$

yields to

$$\int_{\mathbb{Z}_p} e^{at}d\mu_{-1}(a) = \frac{2}{e^t + 1}.$$

By (1.1), we immediately deduce that

$$t^r e^{xt} \int_{\mathbb{Z}_p} e^{at}d\mu_{-1}(a) = \sum_{n=0}^{\infty} A_n^{(r)} \frac{t^n}{n!}, \tag{5.2}$$

where $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$. Therefore, by (5.2), we have

$$\int_{\mathbb{Z}_p} (a + x)^{n-r} d\mu_{-1}(a) = \frac{A_n^{(r)}(x)}{(n)_r},$$

and particularly

$$\int_{\mathbb{Z}_p} a^{n-r}d\mu_{-1}(a) = \frac{A_n^{(r)}}{(n)_r}.$$

Given $a \in \mathbb{Z}_p$, $p \nmid a$ and $p > 2$, there is a $(p-1)$th root of unity $\omega(a) \in \mathbb{Z}_p$ such that

$$a \equiv \omega(a) (\operatorname{mod} \mathbb{Z}_p).$$

Let $\langle a \rangle = \omega^{-1}(a)a$, so $\langle a \rangle \equiv 1 (\operatorname{mod} p)$. The projection $\langle a \rangle$ can be extended to $a \in \mathbb{C}_p^{\times}$ ([20]), where $\mathbb{C}_p^{\times}$ is the group of units in $\mathbb{C}_p$, so that for $a \in \mathbb{C}_p^{\times}$ and $s \in \mathbb{C}_p$, we have

$$\langle a \rangle^s = \sum_{n=0}^{\infty} \binom{s}{n} (\langle a \rangle - 1)^n,$$

provided that the series converges.

For $x \in \mathbb{C}_p - \mathbb{Z}_p$, we define the p-adic Hurwitz type zeta function $\eta_p(s, x)$ by

$$\eta_p(s, x) = \int_{\mathbb{Z}_p} \langle a + x \rangle^{1-s} d\mu_{-1}(a).$$

Since $|x|_p > 1$ and $|a|_p \leq 1$, we have $\omega(a + x) = \omega(x)$, and hence for $s = 1 - n + r$, we obtain that

$$\eta_p(1 - n + r, x) = \int_{\mathbb{Z}_p} \langle a + x \rangle^{n-r} d\mu_{-1}(a) = \omega^{-n+r}(x) \frac{A_n^{(r)}(x)}{(n)_r}.$$

Therefore, the function $\eta_p(s, x)$ can be regarded as the p-adic interpolating function for the Euler-Genocchi polynomial $A_n^{(r)}(x)$.

6. Conclusion

In this study, some properties provided by the Eulerâ€“Genocchi numbers and polynomials have been addressed. Specifically, the arithmetic properties of Eulerâ€“Genocchi numbers (divisibility properties and analogues of the von Staudtâ€“Clausen theorem provided by Bernoulli numbers, as well as Kummer's congruences) have been obtained. Integer values assumed by the Euler-Genocchi polynomials by means of the classical Almkvist-Meurman theorem are examined as well. The relationships between zeta functions and p-adic integral representations of these polynomials and numbers have also been studied.

Altogether, these results not only extend known properties of the classical Euler and Genocchi polynomials and numbers but also pave the way for further exploration in combinatorics, special functions, and arithmetic analysis within the framework of the Appell polynomials.

Future research may focus on the unified extensions of these numbers and polynomials [2, 14–16]. In particular, it would be interesting to examine the analogues of the von Staudt-Clausen theorem and Kummer-type congruences for the unified extensions.

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