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Fuzzy stability of a generalized 4-dimensional AQCQ functional equation

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Abstract. Functional equations are widely used in various fields for solving practical examples, exploring theoretical ideas, and modeling complex relationships and the study of their stability is essential for understanding how small changes in the inputs or functional form affect the solutions. This has both theoretical significances and practical applications across mathematics, science, and engineering. For this purpose, in this paper, we explore the Ulam-Hyers stability of

$$\begin{aligned}
&\Lambda(z_1+z_2+\lambda(z_3+z_4))+\Lambda(z_1+z_2-\lambda(z_3+z_4))\\
&\quad=\lambda^2\Big\{\Lambda(z_1+z_2+z_3+z_4)+\Lambda(z_1+z_2-z_3-z_4)\Big\}+2\left(1-\lambda^2\right)\Lambda(z_1+z_2)\\
&\qquad+\frac{\left(\lambda^4-\lambda^2\right)}{12}\Big\{\Lambda(2(z_3+z_4))+\Lambda(-2(z_3+z_4))-4(\Lambda(z_3+z_4)+\Lambda(-z_3-z_4))\Big\},
\end{aligned}$$

a generalized 4-dimensional AQCQ functional equation in fuzzy normed spaces using two different methods.

1. Introduction

During the past eight decades, many researchers have extensively studied the solutions of functional equations and their stability results using several techniques, one can refer to [2, 19, 20, 46, 49, 50, 53]. The generalized terminology Ulam-Hyers stability comes from these backgrounds. These terminologies are also applied to the case of other functional equations.

C. Park *et al.* [36] proved the generalized Hyers-Ulam stability of the following additive-quadratic-cubic-quartic functional equation briefly, AQCQ-functional equation

$$\begin{aligned}
f(x+2y)+f(x-2y)&=4[f(x+y)+f(x-y)]-6f(x)\\
&\quad+f(2y)+f(-2y)-4f(y)-4f(-y) \qquad (1)
\end{aligned}$$

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in Banach Spaces. Later, Hyers-Ulam stability results have been established of (1) in various normed spaces [17, 37–41].

In [51], Ravi *et al.* introduced a general mixed-type AQCQ- functional equation

$$f(x+ay)+f(x-ay)=a^2\left[f(x+y)+f(x-y)\right]+2\left(1-a^2\right)f(x)+\frac{\left(a^4-a^2\right)}{12}\left[f(2y)+f(-2y)-4f(y)-4f(-y)\right], \tag{2}$$

which is a generalized form of the AQCQ-functional equation (1) and obtained its general solution and generalized Hyers-Ulam stability for a fixed integer a with $a \neq 0, \pm 1$ in Banach spaces.

Various other types of functional equations including AQCQ functional equations were introduced and investigated the stability problems in several spaces including fuzzy normed spaces have been investigated in [1, 3–9, 12–14, 16, 21, 23, 33, 34, 42–45, 47, 48, 52] and references mentioned there in.

1.1. Basics of Fuzzy Banach Space

A.K. Katsaras [24] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [18, 26, 54]. In particular, T. Bag and S.K. Samanta [10], following S.C. Cheng and J.N. Mordeson [15], gave an idea of the fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [25]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [11]. We use the definition of fuzzy normed spaces given in [29–32].

Definition 1.1. *Let X be a real linear space. A function $E : X \times \mathbb{R} \to [0,1]$ is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,*

($FBS1$) $E(x,c) = 0$ *for* $c \leq 0$;

($FBS2$) $x = 0$ *if and only if* $E(x,c) = 1$ *for all* $c > 0$;

($FBS3$) $E(cx,t) = E\left(x, \frac{t}{|c|}\right)$ *if* $c \neq 0$;

($FBS4$) $E(x+y, s+t) \geq \min\{E(x,s), E(y,t)\}$;

($FBS5$) $E(x,\cdot)$ *is a non-decreasing function on* $\mathbb{R}$ *and* $\lim_{t\to\infty} E(x,t) = 1$;

($FBS6$) *for* $x \neq 0$, $E(x,\cdot)$ *is (upper semi) continuous on* $\mathbb{R}$.

The pair (X,E) is called a fuzzy normed linear space.

Example 1.2. *Let $(X, \|\cdot\|)$ be a normed linear space. Then*

$$E\ (x,t) = \begin{cases} \dfrac{t}{t+\|x\|}, & t>0, \quad x\in X, \\ 0, & t\leq 0, \quad x \in X \end{cases}$$

is a fuzzy norm on X.

Definition 1.3. *Let (X,E) be a fuzzy normed linear space. Let x_n be a sequence in X. Then x_n is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} E(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence x_n and we denote it by $E\left(\lim_{n\to\infty} x_n - x, t\right) = 1$.*

Definition 1.4. *A sequence x_n in X is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $E(x_{n+p} - x_n, t) > 1 - \epsilon$.*

Definition 1.5. *Every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.*

Now, we will recall the fundamental results in fixed point theory.

Theorem 1.6. *[27](The alternative of fixed point) Suppose that for a complete generalized metric space (X, d) and a strictly contractive mapping $T : X \to X$ with Lipschitz constant L. Then, for each given element $x \in X$, either*

$(F1)$ $$d(T^n x, T^{n+1}x) = \infty, \quad \forall\ n \geq 0,$$

or

$(F2)$ *there exists a natural number n_0 such that:*

$(FP1)$ $d(T^n x, T^{n+1}x) < \infty$ *for all* $n \geq n_0$;

$(FP2)$*The sequence* $(T^n x)$ *is convergent to a fixed point* y^* *of* T;

$(FP3)$ y^* *is the unique fixed point of* T *in the set* $Y = \{y \in X : d(T^{n_0}x, y) < \infty\}$;

$(FP4)$ $d(y^*, y) \leq \dfrac{1}{1-L}\ d(y, Ty)$ *for all* $y \in Y$.

In this paper, we explore the Ulam-Hyers stability of a generalized 4-dimensional AQCQ functional equation

$$\begin{aligned}
&\Lambda\left(z_1 + z_2 + \lambda\left(z_3 + z_4\right)\right) + \Lambda\left(z_1 + z_2 - \lambda\left(z_3 + z_4\right)\right) \\
&\quad = \lambda^2\Big\{\Lambda\left(z_1 + z_2 + z_3 + z_4\right) + \Lambda\left(z_1 + z_2 - z_3 - z_4\right)\Big\} + 2\left(1 - \lambda^2\right)\Lambda\left(z_1 + z_2\right) \\
&\qquad + \frac{\left(\lambda^4 - \lambda^2\right)}{12}\Big\{\Lambda\left(2\left(z_3 + z_4\right)\right) + \Lambda\left(-2\left(z_3 + z_4\right)\right) - 4\left(\Lambda\left(z_3 + z_4\right) + \Lambda\left(-z_3 - z_4\right)\right)\Big\},
\end{aligned} \tag{3}$$

for all positive integers λ with $\lambda \geq 2$ in fuzzy Banach Space using two different methods. It is easy to verify that (2) and (3) are equivalent to each other.

To provide stability analysis, assume that $(\mathcal{S}_1, E), (\mathcal{S}_2, E')$ and $(\mathcal{S}_3, E')$ are linear space, fuzzy Banach space and fuzzy normed space, respectively. For, notation we may assume a mapping $\Lambda_{aqcq} : \mathcal{S}_1 \to \mathcal{S}_2$ by

$$\begin{aligned}
&\Lambda_{aqcq}\left(z_1, z_2, z_3, z_4\right) \\
&\quad = \Lambda\left(z_1 + z_2 + \lambda\left(z_3 + z_4\right)\right) + \Lambda\left(z_1 + z_2 - \lambda\left(z_3 + z_4\right)\right) \\
&\quad - \lambda^2\Big\{\Lambda\left(z_1 + z_2 + z_3 + z_4\right) + \Lambda\left(z_1 + z_2 - z_3 - z_4\right)\Big\} - 2\left(1 - \lambda^2\right)\Lambda\left(z_1\right) \\
&\quad - \frac{\left(\lambda^4 - \lambda^2\right)}{12}\Big\{\Lambda\left(2\left(z_3 + z_4\right)\right) + \Lambda\left(-2\left(z_3 + z_4\right)\right) - 4\left(\Lambda\left(z_3 + z_4\right) + \Lambda\left(-z_3 - z_4\right)\right)\Big\}.
\end{aligned}$$

Also, we take non zero substitutions for proving the results.

2. Fuzzy Stability Results: Λ is Even

The generalized Ulam-Hyers stability of 4-dimensional AQCQ functional equation (3) when Λ is even is analyzed in this section.

2.1. Quadratic Case Fuzzy Stability Result

Theorem 2.1. *An even function $\Lambda_{aqcq} : \mathcal{S}_1 \to \mathcal{S}_2$ satisfies the functional inequality*

$$E\left(\Lambda_{aqcq}\left(z_1, z_2, z_3, z_4\right), \Theta\right) \geq E'\left(\Phi\left(z_1, z_2, z_3, z_4\right), \Theta\right), \tag{4}$$

where $\Phi : \mathcal{S}_1^4 \to \mathcal{S}_3$ be a mapping satisfying

$$\begin{aligned}
&E'\left(\Lambda\left(2^{\pi\varpi}z_1, 2^{\pi\varpi}z_2, 2^{\pi\varpi}z_3, 2^{\pi\varpi}z_4\right), \Theta\right) \\
&\qquad \geq E'\left(\theta^{\pi\varpi}\Phi\left(z_1, z_2, z_3, z_4\right), \Theta\right); \qquad \pi \in \{-1, 1\}
\end{aligned} \tag{5}$$

where $\theta > 0$ with $0 < \left(\dfrac{\theta}{4}\right)^{\pi} < 1$ and the condition

$$\lim_{\varpi\to\infty} E'\left(\Phi\left(2^{\pi\varpi}z_1, 2^{\pi\varpi}z_2, 2^{\pi\varpi}z_3, 2^{\pi\varpi}z_4\right), 4^{\pi\varpi}\Theta\right) = 1, \tag{6}$$

for all $z_1, z_2, z_3, z_4 \in \mathcal{S}_1$ *and all* $\Theta > 0$ *. Then there exists a unique quadratic mapping* $\Omega_{Q2}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ *such that*

$$E\left(\Lambda(2z) - 16\Lambda(z) - \Omega_{Q2}(z), \Theta\right) \geq E'\left(\Phi_{EVEN}(z,z,z,z), \left(\frac{\lambda^4 - \lambda^2}{19 + 12\lambda^2}\right)|4 - \theta|\Theta\right), \tag{7}$$

where

$$E'\left(\Phi_{EVEN}(z,z,z,z),\Theta\right) \geq \min\left\{E'\left(\Phi\left(\frac{z}{2}, -\frac{z}{2}, \frac{z}{2}, \frac{z}{2}\right), \Theta\right), E'\left(\Phi(z,-z,z,z),\Theta\right), E'\left(\Phi\left(\frac{z}{2}, \frac{z}{2}, \frac{z}{2}, \frac{z}{2}\right), \Theta\right), E'\left(\Phi\left(\frac{\lambda z}{2}, \frac{\lambda z}{2}, \frac{z}{2}, \frac{z}{2}\right), \Theta\right)\right\}, \tag{8}$$

and the mapping

$$E\left(\lim_{\varpi\to\infty} \frac{1}{4^{\pi\varpi}}\left\{\Lambda\left(2^{(\varpi+1)\pi} z\right) - 16\Lambda\left(2^{\pi\varpi} z\right)\right\} - \Omega_{Q2}(z), \Theta\right) = 1, \tag{9}$$

for all $z \in \mathcal{S}_1$ *and all* $\Theta > 0$.

Proof. Setting (z_1, z_2, z_3, z_4) by $\left(\frac{z}{2}, \frac{-z}{2}, \frac{z}{2}, \frac{z}{2}\right)$ in (4) and using evenness of Λ, one can have

$$E\left(2\Lambda(\lambda z) - 2\lambda^2\Lambda(z) - \frac{\left(\lambda^4 - \lambda^2\right)}{12}\left(2\Lambda(2z) - 8\Lambda(z)\right), \Theta\right) \geq E'\left(\Phi\left(\frac{z}{2}, \frac{-z}{2}, \frac{z}{2}, \frac{z}{2}\right), \Theta\right), \tag{10}$$

and with the help of ($FBS3$), we have

$$E\left(24(1-\lambda^2)\Lambda(\lambda z) - 24(1-\lambda^2)\lambda^2\Lambda(z) - (1-\lambda^2)\left(\lambda^4 - \lambda^2\right)\left(2\Lambda(2z) - 8\Lambda(z)\right), 12(1-\lambda^2)\Theta\right) \geq E'\left(\Phi\left(\frac{z}{2}, \frac{-z}{2}, \frac{z}{2}, \frac{z}{2}\right), \Theta\right), \tag{11}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. Replacing z by $2z$ in (10), one can arrive

$$E\left(2\Lambda(2\lambda z) - 2\lambda^2\Lambda(2z) - \frac{\left(\lambda^4 - \lambda^2\right)}{12}\left(2\Lambda(4z) - 8\Lambda(2z)\right), \Theta\right) \geq E'\left(\Phi(z,-z,z,z), \Theta\right), \tag{12}$$

and with the help of ($FBS3$), we arrive

$$E\left(12\Lambda(2\lambda z) - 12\lambda^2\Lambda(2z) - \frac{\left(\lambda^4 - \lambda^2\right)}{2}\left(2\Lambda(4z) - 8\Lambda(2z)\right), 6\Theta\right) \geq E'\left(\Phi(z,-z,z,z), \Theta\right), \tag{13}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.

Again setting (z_1, z_2, z_3, z_4) by $\left(\frac{z}{2}, \frac{z}{2}, \frac{z}{2}, \frac{z}{2}\right)$ in (4) and using evenness of Λ, one can obtain

$$E\Big(\Lambda((\lambda+1)z) + \Lambda((\lambda-1)z) - \lambda^2\Lambda(2z) - 2\left(1-\lambda^2\right)\Lambda(z) - \frac{\left(\lambda^4 - \lambda^2\right)}{12}\left(2\Lambda(2z) - 8\Lambda(z)\right), \Theta\Big) \geq E'\left(\Phi\left(\frac{z}{2}, \frac{z}{2}, \frac{z}{2}, \frac{z}{2}\right), \Theta\right), \tag{14}$$

and with the help of ($FBS3$), we obtain

$$E\Big(12\lambda^2\Lambda\left((\lambda+1)z\right)+12\lambda^2\Lambda\left((\lambda-1)z\right)-12\lambda^4\Lambda\left(2z\right)-24\lambda^2\left(1-\lambda^2\right)\Lambda(z)$$
$$-\lambda^2\left(\lambda^4-\lambda^2\right)\left(2\Lambda\left(2z\right)-8\Lambda\left(z\right)\right),12\lambda^2\Theta\Big)\geq E'\left(\Phi\left(\frac{z}{2},\frac{z}{2},\frac{z}{2},\frac{z}{2}\right),\Theta\right), \tag{15}$$

for all $z\in\mathcal{S}_1$ and all $\Theta>0$.

Further setting (z_1,z_2,z_3,z_4) by $\left(\frac{\lambda z}{2},\frac{\lambda z}{2},\frac{z}{2},\frac{z}{2}\right)$ in (4) and using evenness of Λ, one can get

$$E\Big(\Lambda\left(2\lambda z\right)-\lambda^2\left(\Lambda\left((\lambda+1)z\right)+\Lambda\left((\lambda-1)z\right)\right)-2\left(1-\lambda^2\right)\Lambda\left(\lambda z\right)$$
$$-\frac{\left(\lambda^4-\lambda^2\right)}{12}\left(2\Lambda\left(2z\right)-8(\Lambda z)\right),\Theta\Big)\geq E'\left(\Phi\left(\frac{\lambda z}{2},\frac{\lambda z}{2},\frac{z}{2},\frac{z}{2}\right),\Theta\right), \tag{16}$$

and with the help of ($FBS3$), we get

$$E\Big(12\Lambda\left(2\lambda z\right)-12\lambda^2\left(\Lambda\left((\lambda+1)z\right)+\Lambda\left((\lambda-1)z\right)\right)-24\left(1-\lambda^2\right)\Lambda\left(\lambda z\right)$$
$$-\left(\lambda^4-\lambda^2\right)\left(2\Lambda\left(2z\right)-8(\Lambda z)\right),12\Theta\Big)\geq E'\left(\Phi\left(\frac{\lambda z}{2},\frac{\lambda z}{2},\frac{z}{2},\frac{z}{2}\right),\Theta\right), \tag{17}$$

for all $z\in\mathcal{S}_1$ and all $\Theta>0$. With the help of ($FBS4$) and it follows from (11), (13), (15), (17), we achieve

$$E\left(\left\{\Lambda(4z)-20\Lambda(2z)+64\Lambda(z)\right\},\left(\frac{19+12\lambda^2}{\lambda^4-\lambda^2}\right)\Theta\right)$$
$$\geq\min\Big\{E\Big(24(1-\lambda^2)\,\Lambda(\lambda z)-24(1-\lambda^2)\lambda^2\Lambda(z)$$
$$-\left(1-\lambda^2\right)\left(\lambda^4-\lambda^2\right)\left\{2\Lambda(2z)-8\Lambda(z)\right\},12(1-\lambda^2)\,\Theta\Big),$$
$$E\left(12\Lambda(2\lambda z)-12\lambda^2\Lambda\left(2z\right)-\left(\lambda^4-\lambda^2\right)\left\{\Lambda(4z)-4\Lambda(2z)\right\},6\,\Theta\right),$$
$$E\Big(12\lambda^2\Lambda\left((\lambda+1)z\right)+12\lambda^2\Lambda\left((\lambda-1)z\right)-12\lambda^4\Lambda\left(2z\right)-24\lambda^2\left(1-\lambda^2\right)\Lambda(z)$$
$$-\lambda^2\left(\lambda^4-\lambda^2\right)\left\{2\Lambda\left(2z\right)-8\Lambda\left(z\right)\right\},12\lambda^2\,\Theta\Big),$$
$$E\Big(12\Lambda(2\lambda z)-12\lambda^2\left\{\Lambda\left((\lambda+1)z\right)+\Lambda\left((\lambda-1)z\right)\right\}-24\left(1-\lambda^2\right)\Lambda(\lambda z)$$
$$-\left(\lambda^4-\lambda^2\right)\left\{2\Lambda(2z)-8(\Lambda z)\right\},12\,\Theta\Big)\Big\}$$
$$\geq\min\left\{E'\left(\Phi\left(\frac{z}{2},-\frac{z}{2},\frac{z}{2},\frac{z}{2}\right),\Theta\right),E'\left(\Phi\left(z,-z,z,z\right),\Theta\right),E'\left(\Phi\left(\frac{z}{2},\frac{z}{2},\frac{z}{2},\frac{z}{2}\right),\Theta\right),\right.$$
$$\left.E'\left(\Phi\left(\frac{\lambda z}{2},\frac{\lambda z}{2},\frac{z}{2},\frac{z}{2}\right),\Theta\right)\right\}=E'\left(\Phi_{EVEN}\left(z,z,z,z\right),\Theta\right), \tag{18}$$

for all $z\in\mathcal{S}_1$ and all $\Theta>0$. Define a function $\Lambda_{Q2}:\mathcal{S}_1\to\mathcal{S}_2$ by

$$\Lambda_{Q2}\left(z\right)=\Lambda(2z)-16\Lambda(z), \tag{19}$$

for all $z\in\mathcal{S}_1$. It follows from (18) and (19), we reach

$$E\left(\Lambda_{Q2}(2z)-4\Lambda_{Q2}(z),\left(\frac{19+12\lambda^2}{\lambda^4-\lambda^2}\right)\Theta\right)$$
$$=E\left(\left\{\Lambda(4z)-16\Lambda(2z)\right\}-4\left\{\Lambda(2z)-16\Lambda(z)\right\},\left(\frac{19+12\lambda^2}{\lambda^4-\lambda^2}\right)\Theta\right)$$
$$\geq E'\left(\Phi_{EVEN}(z,z,z,z),\Theta\right), \tag{20}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. With the help of (*FBS*3) and it follows from (20) that

$$E\left(\frac{1}{4}\,\Lambda_{Q2}(2z) - \Lambda_{Q2}(z), \left(\frac{19+12\lambda^2}{\lambda^4-\lambda^2}\right)\cdot\frac{1}{4}\,\Theta\right) \geq E'\left(\Phi_{EVEN}(z,z,z,z),\Theta\right), \tag{21}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. With the help of (*FBS*3) and (*FBS*4), changing z by $2^{\varpi} z$ in (21) and again changing Θ by $\theta^{\varpi}\,\Theta$ in the resulting inequality, we land

$$\begin{aligned} E\left(\frac{1}{4^{\varpi+1}}\Lambda_{Q2}\left(2^{\varpi+1}z\right) - \frac{1}{4^{\varpi}}\Lambda_{Q2}\left(2^{\varpi}z\right),\ \frac{1}{4}\cdot\left(\frac{19+12\lambda^2}{\lambda^4-\lambda^2}\right)\cdot\left[\frac{\theta}{4}\right]^{\varpi}\Theta\right) \\ \geq E'\left(\Phi_{EVEN}(z,z,z,z),\Theta\right), \end{aligned} \tag{22}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. One can easy to verify from (22) that

$$\begin{aligned} & E\left(\Lambda_{Q2}(z) - \frac{1}{4^{\varpi}}\,\Lambda_{Q2}\left(2^{\varpi}z\right), \frac{1}{4}\cdot\left(\frac{19+12\lambda^2}{\lambda^4-\lambda^2}\right)\cdot\sum_{\rho=0}^{\varpi}\left[\frac{\theta}{4}\right]^{\rho}\Theta\right) \\ &= E\left(\sum_{\rho=0}^{\varpi}\left\{\frac{1}{4^{\rho+1}}\,\Lambda_{Q2}\left(2^{\rho+1}z\right) - \frac{1}{4^{\rho}}\,\Lambda_{Q2}\left(2^{\rho}z\right)\right\}, \frac{1}{4}\cdot\left(\frac{19+12\lambda^2}{\lambda^4-\lambda^2}\right)\cdot\sum_{\rho=0}^{\varpi}\left[\frac{\theta}{4}\right]^{\rho}\Theta\right) \\ &\geq \bigcup_{\rho=0}^{\varpi}\min\left\{E\left(\left\{\frac{1}{4^{\rho+1}}\,\Lambda_{Q2}\left(2^{\rho+1}z\right) - \frac{1}{4^{\rho}}\,\Lambda_{Q2}\left(2^{\rho}z\right)\right\}, \frac{1}{4}\cdot\left(\frac{19+12\lambda^2}{\lambda^4-\lambda^2}\right)\cdot\left[\frac{\theta}{4}\right]^{\rho}\Theta\right)\right\} \\ &\geq \bigcup_{\rho=0}^{\varpi}\min\left\{E'\left(\Phi_{EVEN}(z,z,z,z),\Theta\right)\right\} = E'\left(\Phi_{EVEN}(z,z,z,z),\Theta\right), \end{aligned} \tag{23}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. With the help of (*FBS*3), changing z by $2^{w} z$ in (23) and again changing Θ by $\theta^{w}\,\Theta$ in the resulting inequality, we obtain

$$\begin{aligned} E\left(\frac{1}{4^{w}}\Lambda_{Q2}\left(2^{w}z\right) - \frac{1}{4^{\varpi+w}}\,\Lambda_{Q2}\left(2^{\varpi}2^{w}z\right), \frac{1}{4}\cdot\left(\frac{19+12\lambda^2}{\lambda^4-\lambda^2}\right)\cdot\sum_{\rho=0}^{\varpi}\left[\frac{\theta}{4}\right]^{\rho+w}\Theta\right) \\ \geq E'\left(\Phi_{EVEN}(z,z,z,z),\Theta\right), \end{aligned} \tag{24}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. With the help of (*FBS*3), it follows from (24) that

$$\begin{aligned} & E\left(\frac{1}{4^{w}}\Lambda_{Q2}\left(2^{w}z\right) - \frac{1}{4^{\varpi+w}}\,\Lambda_{Q2}\left(2^{\varpi}2^{w}z\right),\ \Theta\right) \\ &\geq E'\left(\Phi_{EVEN}(z,z,z,z), \frac{\Theta}{\frac{1}{4}\cdot\left(\frac{19+12\lambda^2}{\lambda^4-\lambda^2}\right)\cdot\sum\limits_{\rho=0}^{\varpi}\left[\frac{\theta}{4}\right]^{\rho+w}}\right), \end{aligned} \tag{25}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. By data and the Cauchy criterion for convergence, (*FBS*5), implies that the sequence $\left\{\frac{1}{4^{\varpi}}\,\Lambda_{Q2}\left(2^{\varpi}z\right)\right\}$ is a Cauchy sequence in fuzzy Banach space $(\mathcal{S}_2, E')$ and this sequence converges to $\Omega_{Q2}(z)$. So, by notation, we write

$$\begin{aligned} & E\left(\Omega_{Q2}(z) - \lim_{\varpi\to\infty}\frac{1}{4^{\varpi}}\,\Lambda_{Q2}\left(2^{\varpi}z\right),\Theta\right) \\ &= E\left(\lim_{\varpi\to\infty}\Omega_{Q2}(z) - \frac{1}{4^{\varpi}}\left\{\Lambda\left(2^{\varpi+1}z\right)(2z) - 16\Lambda\left(2^{\varpi}z\right)\right\},\Theta\right) = 1, \end{aligned} \tag{26}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. Taking $w = 0$ and ϖ tends to infinity in (25), we arrive

$$E\left(\Lambda_{Q2}(z) - \Omega_{Q2}(z),\ \Theta\right) \geq E'\left(\Phi_{EVEN}\left(z, z, z, z\right), \frac{\left(\lambda^4 - \lambda^2\right)(4-\theta)\ \Theta}{(19 + 12\lambda^2)}\right), \tag{27}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. Now, we have to prove the existence of $\Omega_{Q2}(z)$ satisfies (3), substituting (z_1, z_2, z_3, z_4) by $(2^\varpi z_1, 2^\varpi z_2, 2^\varpi z_3, 2^\varpi z_4)$ in (4) and using $(FBS3)$, we get

$$E\ \left(\frac{1}{4^\varpi}\ \Lambda_{aqcq}\left(2^\varpi z_1, 2^\varpi z_2, 2^\varpi z_3, 2^\varpi z_4\right), \Theta\right) \geq E'\ \left(\Phi\left(2^\varpi z_1, 2^\varpi z_2, 2^\varpi z_3, 2^\varpi z_4\right), 4^\varpi\ \Theta\right), \tag{28}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. Now, from (6), (26), (27), (28) and taking ϖ tends to infinity, we arrive

$$\begin{aligned}
&E\Big(\Omega_{Q2}\left(z_1 + z_2 + \lambda\left(z_3 + z_4\right)\right) + \Omega_{Q2}\left(z_1 + z_2 - \lambda\left(z_3 + z_4\right)\right) \\
&\quad - \lambda^2\Big\{\Omega_{Q2}\left(z_1 + z_2 + z_3 + z_4\right) + \Omega_{Q2}\left(z_1 + z_2 - z_3 - z_4\right)\Big\} - 2\left(1 - \lambda^2\right)\Omega_{Q2}\left(z_1\right) \\
&\quad - \frac{\left(\lambda^4 - \lambda^2\right)}{12}\Big\{\Omega_{Q2}\left(2\left(z_3 + z_4\right)\right) + \Omega_{Q2}\left(-2\left(z_3 + z_4\right)\right) \\
&\qquad\qquad - 4\Big\{\Omega_{Q2}\left(z_3 + z_4\right) + \Omega_{Q2}\left(-z_3 - z_4\right)\Big\}\Big\}\Big) \\
&\geq \min\Big\{E\Big(\Omega_{Q2}\left(z_1 + z_2 + \lambda(z_3 + z_4)\right) - \frac{1}{4^\varpi}\Lambda\left(2^\varpi\left(z_1 + z_2 + \lambda(z_3 + z_4)\right)\right)\Big), \\
&\qquad E\Big(\Omega_{Q2}\left(z_1 + z_2 - \lambda(z_3 + z_4)\right) - \frac{1}{4^\varpi}\Lambda\left(2^\varpi\left(z_1 + z_2 - \lambda(z_3 + z_4)\right)\right)\Big), \\
&\qquad E\Big(- \lambda^2\Big\{\Omega_{Q2}\left(z_1 + z_2 + z_3 + z_4\right) + \Omega_{Q2}\left(z_1 + z_2 - z_3 - z_4\right)\Big\} \\
&\qquad\qquad + \frac{\lambda^2}{4^\varpi}\left\{\Lambda\left(2^\varpi\left(z_1 + z_2 + z_3 + z_4\right)\right) + \Lambda\left(2^\varpi\left(z_1 + z_2 - z_3 - z_4\right)\right)\right\}\Big), \\
&\qquad E\left(-2\left(1 - \lambda^2\right)\Omega_{Q2}\left(z_1\right) + \frac{2\left(1 - \lambda^2\right)}{4^\varpi}\Lambda\left(2^\varpi z_1\right)\right), \\
&\qquad E\Big(- \frac{\left(\lambda^4 - \lambda^2\right)}{12}\Big\{\Omega_{Q2}\left(2\left(z_3 + z_4\right)\right) + \Omega_{Q2}\left(-2\left(z_3 + z_4\right)\right) \\
&\qquad\qquad - 4\Big\{\Omega_{Q2}\left(z_3 + z_4\right) + \Omega_{Q2}\left(-z_3 - z_4\right)\Big\}\Big\} \\
&\qquad\quad + \frac{\left(\lambda^4 - \lambda^2\right)}{12\ 4^\varpi}\Big\{\Lambda\left(2\ 2^\varpi\ \left(z_3 + z_4\right)\right) + \Lambda\left(-2\ 2^\varpi\ \left(z_3 + z_4\right)\right) \\
&\qquad\qquad - 4\Big\{\Lambda\left(2^\varpi\ \left(z_3 + z_4\right)\right) + \Lambda\left(2^\varpi\ \left(-z_3 - z_4\right)\right)\Big\}\Big\}\Big), \\
&\qquad E\Big(\frac{1}{4^\varpi}\Lambda\left(2^\varpi\left(z_1 + z_2 + \lambda(z_3 + z_4)\right)\right) + \frac{1}{4^\varpi}\Lambda\left(2^\varpi\left(z_1 + z_2 - \lambda(z_3 + z_4)\right)\right) \\
&\qquad + \frac{\lambda^2}{4^\varpi}\left\{\Lambda\left(2^\varpi\left(z_1 + z_2 + z_3 + z_4\right)\right) + \Lambda\left(2^\varpi\left(z_1 + z_2 - z_3 - z_4\right)\right)\right\} + \frac{2\left(1 - \lambda^2\right)}{4^\varpi}\Lambda\left(2^\varpi z_1\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\left(\lambda^4-\lambda^2\right)}{12\,4^{\varpi}}\Big\{\Lambda\left(2\,2^{\varpi}\ (z_3+z_4)\right)+\Lambda\left(-2\,2^{\varpi}\ (z_3+z_4)\right)\\
&\qquad\qquad -4\Big\{\Lambda\left(2^{\varpi}\ (z_3+z_4)\right)+\Lambda\left(2^{\varpi}\ (-z_3-z_4)\right)\Big\}\Big\}\Big)\Big\}\\
&\geq \min\left\{1,1,1,1,1,E'\ \left(\Phi\left(2^{\varpi}z_1,2^{\varpi}z_2,2^{\varpi}z_3,2^{\varpi}z_4\right),4^{\varpi}\,\Theta\ \right)\right\}\\
&=\min\{1,1,1,1,1,1\}\\
&=1, \qquad (29)
\end{aligned}
$$

for all $z_1,z_2,z_3,z_4\in\mathcal{S}_1$ and all $\Theta>0$. Using ($FBS2$) in the above inequality, which gives the existence of $\Omega_{Q2}(z)$ satisfies (3). It is easy to see that the existence of $\Omega_{Q2}(z)$ is unique. Indeed, from ($FBS3$) and taking ϖ tends to infinity and using ($FBS5$), we obtain

$$
\begin{aligned}
&E\left(\Omega_{Q2}(z)-\Omega^{Q}_{Q2}(z),2\,\Theta\right)\\
&=E\left(\frac{1}{4^{\varpi}}\left\{\Omega_{Q2}(2^{\varpi}z)-\Lambda(2^{\varpi}z)+\Lambda(2^{\varpi}z)-\Omega^{Q}_{Q2}(2^{\varpi}z)\right\},2\,\Theta\right)\\
&\geq\min\left\{E\left(\Omega_{Q2}(2^{\varpi}z)-\Lambda(2^{\varpi}z),4^{\varpi}\,\Theta\right),E\left(\Lambda(2^{\varpi}z)-\Omega^{Q}_{Q2}(2^{\varpi}z),4^{\varpi}\,\Theta\right)\right\}\\
&\geq\min\left\{E'\left(\Phi_{EVEN}(2^{\varpi}z,2^{\varpi}z,2^{\varpi}z,2^{\varpi}z),\frac{\left(\lambda^4-\lambda^2\right)\,(4-\theta)\,4^{\varpi}\,\Theta}{(19+12\lambda^2)}\right),\right.\\
&\qquad\qquad\left.E'\left(\Phi_{EVEN}(2^{\varpi}z,2^{\varpi}z,2^{\varpi}z,2^{\varpi}z),\frac{\left(\lambda^4-\lambda^2\right)\,(4-\theta)\,4^{\varpi}\,\Theta}{(19+12\lambda^2)}\right)\right\}\\
&=E'\left(\Phi_{EVEN}(2^{\varpi}z,2^{\varpi}z,2^{\varpi}z,2^{\varpi}z),\frac{\left(\lambda^4-\lambda^2\right)\,(4-\theta)\,4^{\varpi}\,\Theta}{(19+12\lambda^2)}\right)\\
&=E'\left(\Phi_{EVEN}(z,z,z,z),\frac{\left(\lambda^4-\lambda^2\right)\,(4-\theta)\,4^{\varpi}\,\Theta}{\theta^{\varpi}\,(19+12\lambda^2)}\right)\\
&=1, \qquad (30)
\end{aligned}
$$

for all $z\in\mathcal{S}_1$ and all $\Theta>0$. By ($FBS2$) and (30), we see $\Omega_{Q2}(z)$ is unique for all $z\in\mathcal{S}_1$ and all $\Theta>0$. Thus the theorem holds for $\pi=1$.

On the other hand, changing z by $\frac{z}{2}$ in (20) and using (5), ($FBS3$), we achieve

$$
\begin{aligned}
E\ \left(\Lambda_{Q2}(z)-4\Lambda_{Q2}\left(\frac{z}{2}\right),\left(\frac{19+12\lambda^2}{\lambda^4-\lambda^2}\right)\Theta\right)&\geq E'\ \left(\Phi_{EVEN}\left(\frac{z}{2},\frac{z}{2},\frac{z}{2},\frac{z}{2}\right),\Theta\ \right)\\
&=E'\ \left(\Phi_{EVEN}(z,z,z,z),\theta\,\Theta\ \right), \qquad (31)
\end{aligned}
$$

for all $z\in\mathcal{S}_1$ and all $\Theta>0$. With the help of ($FBS3$) and ($FBS4$), changing z by $\frac{z}{2^{\varpi}}$ in (31) and again changing Θ by $\frac{\Theta}{\theta^{\varpi}}$ in the resulting inequality, we land

$$
E\left(4^{\varpi}\Lambda_{Q2}\left(\frac{z}{2^{\varpi}}\right)-4^{\varpi+1}\Lambda_{Q2}\left(\frac{z}{2^{\varpi+1}}\right),\left(\frac{19+12\lambda^2}{\lambda^4-\lambda^2}\right)\left[\frac{4}{\theta}\right]^{\varpi}\Theta\right)\geq E'\left(\Phi_{EVEN}(z,z,z,z),\Theta\right), \qquad (32)
$$

for all $z\in\mathcal{S}_1$ and all $\Theta>0$. It follows from the above inequality that

$$
E\left(4^{\varpi}\Lambda_{Q2}\left(\frac{z}{2^{\varpi}}\right)-4^{\varpi+1}\Lambda_{Q2}\left(\frac{z}{2^{\varpi+1}}\right),\Theta\right)\geq E'\left(\Phi_{EVEN}(z,z,z,z),\frac{\Theta}{\left(\frac{19+12\lambda^2}{\lambda^4-\lambda^2}\right)\left[\frac{4}{\theta}\right]^{\varpi}}\right), \qquad (33)
$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. The rest of the proof is similar to that of the previous case. Hence the proof is complete. □

Corollary 2.2. *An even function $\Lambda_{aqcq} : \mathcal{S}_1 \to \mathcal{S}_2$ satisfies the functional inequality*

$$E\left(\Lambda_{aqcq}(z_1, z_2, z_3, z_4), \Theta\right) \geq \begin{cases} E'(\Delta, \Theta), \\ E'\left(\Delta \sum_{\tau=1}^{4} |z_\tau|^{\Psi}, \Theta\right), \\ E'\left(\Delta \prod_{\tau=1}^{4} |z_\tau|^{\Psi}, \Theta\right), \\ E'\left(\Delta \left\{\sum_{\tau=1}^{4} |z_\tau|^{4\Psi} + \prod_{\tau=1}^{4} |z_\tau|^{\Psi}\right\}, \Theta\right), \end{cases} \tag{34}$$

for all $z_1, z_2, z_3, z_4 \in \mathcal{S}_1$ and all $\Theta > 0$ with Δ a positive constant. Then there exists a unique quadratic mapping $\Omega_{Q2}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ such that

$$E\left(\Lambda(2z) - 16\Lambda(z) - \Omega_{Q2}(z), \Theta\right)$$
$$\geq \begin{cases} E'\left(\Delta, \left(\dfrac{\lambda^4 - \lambda^2}{19 + 12\lambda^2}\right)|3|\Theta\right); \\ E'\left(\left\{6\Delta + 8\,\Delta\, 2^{\Psi} + 2\,\Delta\,\lambda^{\Psi}\right\}\left|\dfrac{z}{2}\right|^{\Psi}, 4\left(\dfrac{\lambda^4 - \lambda^2}{19 + 12\lambda^2}\right)\left|4 - 2^{\Psi}\right|\Theta\right); \Psi \neq 2 \\ E'\left(\left\{\Delta + 2\,\Delta\, 2^{4\Psi} + \Delta\,\lambda^{4\Psi}\right\}\left|\dfrac{z}{2}\right|^{4\Psi}, 4\left(\dfrac{\lambda^4 - \lambda^2}{19 + 12\lambda^2}\right)\left|4 - 2^{4\Psi}\right|\Theta\right); \Psi \neq \frac{1}{2} \\ E'\left(\left\{7\Delta + 10\,\Delta\, 2^{4\Psi} + 3\,\Delta\,\lambda^{4\Psi}\right\}\left|\dfrac{z}{2}\right|^{4\Psi}, 4\left(\dfrac{\lambda^4 - \lambda^2}{19 + 12\lambda^2}\right)\left|4 - 2^{4\Psi}\right|\Theta\right); \Psi \neq \frac{1}{2} \end{cases} \tag{35}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.

2.2. Quartic Case Fuzzy Stability Result

Theorem 2.3. *An even function $\Lambda_{aqcq} : \mathcal{S}_1 \to \mathcal{S}_2$ satisfies the functional inequality (4) where $\Phi : \mathcal{S}_1^4 \to \mathcal{S}_3$ be a mapping satisfying (5) where $\theta > 0$ with $0 < \left(\frac{\theta}{16}\right)^{\pi} < 1$ and the condition*

$$\lim_{\varpi \to \infty} E'\left(\Phi\left(2^{\pi\varpi} z_1, 2^{\pi\varpi} z_2, 2^{\pi\varpi} z_3, 2^{\pi\varpi} z_4\right), 16^{\pi\varpi}\Theta\right) = 1, \tag{36}$$

for all $z_1, z_2, z_3, z_4 \in \mathcal{S}_1$ and all $\Theta > 0$. Then there exists a unique quartic mapping $\Omega_{Q4}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ such that

$$E\left(\Lambda(2z) - 4\Lambda(z) - \Omega_{Q4}(z), \Theta\right) \geq E'\left(\Phi_{EVEN}(z, z, z, z), \left(\frac{\lambda^4 - \lambda^2}{19 + 12\lambda^2}\right)|16 - \theta|\,\Theta\right), \tag{37}$$

where $E'(\Phi_{EVEN}(z, z, z, z), \Theta)$ is given in (8) and the mapping

$$E\left(\lim_{\varpi \to \infty} \frac{1}{16^{\pi\varpi}}\left\{\Lambda\left(2^{(\varpi+1)\pi} z\right) - 4\Lambda\left(2^{\pi\varpi} z\right)\right\} - \Omega_{Q4}(z), \Theta\right) = 1, \tag{38}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.

Proof. Now, Define a function $\Lambda_{Q4} : \mathcal{S}_1 \to \mathcal{S}_2$ by

$$\Lambda_{Q4}(z) = \Lambda(2z) - 4\Lambda(z), \tag{39}$$

for all $z \in \mathcal{S}_1$. It follows from (18) and (39), we reach

$$\begin{aligned} & E\left(\Lambda_{Q4}(2z) - 16\Lambda_{Q4}(z), \left(\frac{19+12\lambda^2}{\lambda^4-\lambda^2}\right)\Theta\right) \\ & \qquad = E\left(\left\{\Lambda(4z) - 4\Lambda(2z)\right\} - 16\left\{\Lambda(2z) - 4\Lambda(z)\right\}, \left(\frac{19+12\lambda^2}{\lambda^4-\lambda^2}\right)\Theta\right) \\ & \qquad \geq E'\left(\Phi_{EVEN}(z,z,z,z), \Theta\right), \end{aligned} \tag{40}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. With the help of ($FBS3$) and it follows from (40) that

$$E\left(\frac{1}{16}\Lambda_{Q4}\big(2z\big) - \Lambda_{Q4}\big(z\big), \left(\frac{19+12\lambda^2}{\lambda^4-\lambda^2}\right)\cdot\frac{1}{16}\Theta\right) \geq E'\left(\Phi_{EVEN}\big(z,z,z,z\big), \Theta\right), \tag{41}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. The rest of the proof is similar to that of Theorem 2.1. Hence the proof is complete. □

Corollary 2.4. *An even function $\Lambda_{aqcq} : \mathcal{S}_1 \to \mathcal{S}_2$ satisfies the functional inequality (34) for all $z_1, z_2, z_3, z_4 \in \mathcal{S}_1$ and all $\Theta > 0$ with Δ a positive constant. Then there exists a unique quartic mapping $\Omega_{Q4}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ such that*

$$E\left(\Lambda(2z) - 4\Lambda(z) - \Omega_{Q4}(z), \Theta\right)$$
$$\geq \begin{cases} E'\left(\Delta, \left(\frac{\lambda^4-\lambda^2}{19+12\lambda^2}\right)|15|\,\Theta\right); \\ E'\left(\left\{6\Delta + 8\,\Delta\,2^{\Psi} + 2\,\Delta\,\lambda^{\Psi}\right\}\left|\frac{z}{2}\right|^{\Psi}, 4\left(\frac{\lambda^4-\lambda^2}{19+12\lambda^2}\right)\left|16-2^{\Psi}\right|\Theta\right); \Psi \neq 4 \\ E'\left(\left\{\Delta + 2\,\Delta\,2^{4\Psi} + \Delta\,\lambda^{4\Psi}\right\}\left|\frac{z}{2}\right|^{4\Psi}, 4\left(\frac{\lambda^4-\lambda^2}{19+12\lambda^2}\right)\left|16-2^{4\Psi}\right|\Theta\right); \Psi \neq 1 \\ E'\left(\left\{7\Delta + 10\,\Delta\,2^{4\Psi} + 3\,\Delta\,\lambda^{4\Psi}\right\}\left|\frac{z}{2}\right|^{4\Psi}, 4\left(\frac{\lambda^4-\lambda^2}{19+12\lambda^2}\right)\left|16-2^{4\Psi}\right|\Theta\right); \Psi \neq 1 \end{cases} \tag{42}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.

2.3. *Quadratic-Quartic Case Fuzzy Stability Result*

Theorem 2.5. *An even function $\Lambda_{aqcq} : \mathcal{S}_1 \to \mathcal{S}_2$ satisfies the functional inequality (4) where $\Phi : \mathcal{S}_1^4 \to \mathcal{S}_3$ be a mapping satisfying (5) where $\theta > 0$ with $0 < \left(\frac{\theta}{4}\right)^{\pi} < 1$, $0 < \left(\frac{\theta}{16}\right)^{\pi} < 1$ and the conditions (6), (36) for all $z_1, z_2, z_3, z_4 \in \mathcal{S}_1$ and all $\Theta > 0$. Then there exists a unique quadratic mapping $\Omega_{Q2}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ and a unique quartic mapping $\Omega_{Q4}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ such that*

$$\begin{aligned} & E\left(\Lambda(z) - \Omega_{Q2}(z) - \Omega_{Q4}(z), \Theta\right) \\ & \qquad \geq E'\left(\Phi_{EVEN}(z,z,z,z), 3\left(\frac{\lambda^4-\lambda^2}{19+12\lambda^2}\right)\left\{|4-\theta| + |16-\theta|\right\}\Theta\right), \end{aligned} \tag{43}$$

where $E'\left(\Phi_{EVEN}(z,z,z,z), \Theta\right)$, $\Omega_{Q2}(z)$ and $\Omega_{Q4}(z)$ are given in (8), (9) and (38), respectively for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.

Proof. By Theorem 2.1, there exists a unique quadratic mapping $\Omega_{Q2}^2(z).: \mathcal{S}_1 \to \mathcal{S}_2$ such that

$$E\left(\Lambda(2z) - 16\Lambda(z) - \Omega_{Q2}^2(z), \Theta\right) \geq E'\left(\Phi_{EVEN}(z,z,z,z), \left(\frac{\lambda^4-\lambda^2}{19+12\lambda^2}\right)|4-\theta|\,\Theta\right), \tag{44}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.

Also, by Theorem 2.3, there exists a unique quartic mapping $\Omega_{Q4}^4(z) : \mathcal{S}_1 \to \mathcal{S}_2$ such that

$$E\left(\Lambda(2z) - 4\Lambda(z) - \Omega_{Q4}^4(z), \Theta\right) \geq E'\left(\Phi_{EVEN}(z,z,z,z), \left(\frac{\lambda^4 - \lambda^2}{19 + 12\lambda^2}\right)|16 - \theta|\Theta\right), \tag{45}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. Now,

$$\begin{aligned}
&E\left(12\Lambda(z) + \Omega_{Q2}^2(z) - \Omega_{Q4}^4(z), 2\Theta\right)\\
&= E\left(\Lambda(2z) - 4\Lambda(z) - \Omega_{Q4}^4(z) - \Lambda(2z) + 16\Lambda(z) + \Omega_{Q2}^2(z), 2\Theta\right)\\
&\geq \min\left\{E\left(\Lambda(2z) - 4\Lambda(z) - \Omega_{Q4}^4(z), \Theta\right), E\left(\Lambda(2z) - 16\Lambda(z) - \Omega_{Q2}^2(z), \Theta\right)\right\}\\
&= E'\left(2\,\Phi_{EVEN}(z,z,z,z), \left(\frac{\lambda^4 - \lambda^2}{19 + 12\lambda^2}\right)\left\{|4 - \theta| + |16 - \theta|\right\}\Theta\right),
\end{aligned} \tag{46}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. It follows from (46) that

$$\begin{aligned}
&E\left(\Lambda(z) + \frac{1}{12}\Omega_{Q2}^2(z) - \frac{1}{12}\Omega_{Q4}^4(z), \Theta\right)\\
&\quad\geq E'\left(2\Phi_{EVEN}(z,z,z,z), 6\left(\frac{\lambda^4 - \lambda^2}{19 + 12\lambda^2}\right)\left\{|4 - \theta| + |16 - \theta|\right\}\Theta\right),
\end{aligned} \tag{47}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. So, if we take

$$\Omega_{Q2}(z) = -\frac{1}{12}\Omega_{Q2}^2(z), \qquad \text{and} \qquad \Omega_{Q4}(z) = \frac{1}{12}\Omega_{Q4}^4(z),$$

we arrived at our desired result. Hence the proof is complete. □

Corollary 2.6. *An even function $\Lambda_{aqcq} : \mathcal{S}_1 \to \mathcal{S}_2$ satisfies the functional inequality (34) for all $z_1, z_2, z_3, z_4 \in \mathcal{S}_1$ and all $\Theta > 0$ with Δ a positive constant. Then there exists a unique quadratic mapping $\Omega_{Q2}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ and a unique quartic mapping $\Omega_{Q4}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ such that*

$$\begin{aligned}
&E\left(\Lambda(z) - \Omega_{Q2}(z) - \Omega_{Q4}(z), \Theta\right)\\
&\geq \begin{cases}
E'\left(\Delta, 3\left(\dfrac{\lambda^4 - \lambda^2}{19 + 12\lambda^2}\right)\{|3| + |15|\}\Theta\right);\\
E'\left(\left\{6\Delta + 8\,\Delta\,2^{\Psi} + 2\,\Delta\,\lambda^{\Psi}\right\}\left|\dfrac{z}{2}\right|^{\Psi},\right.\\
\qquad\qquad \left. 3\left(\dfrac{\lambda^4 - \lambda^2}{19 + 12\lambda^2}\right)\left\{\left|4 - 2^{\Psi}\right| + \left|16 - 2^{\Psi}\right|\right\}\Theta\right); \Psi \neq 2, 4\\
E'\left(\left\{\Delta + 2\,\Delta\,2^{4\Psi} + \Delta\,\lambda^{4\Psi}\right\}\left|\dfrac{z}{2}\right|^{4\Psi},\right.\\
\qquad\qquad \left. 3\left(\dfrac{\lambda^4 - \lambda^2}{19 + 12\lambda^2}\right)\left\{\left|4 - 2^{4\Psi}\right| + \left|16 - 2^{4\Psi}\right|\right\}\Theta\right); \Psi \neq \frac{1}{2}, 1\\
E'\left(\left\{7\Delta + 10\,\Delta\,2^{4\Psi} + 3\,\Delta\,\lambda^{4\Psi}\right\}\left|\dfrac{z}{2}\right|^{4\Psi},\right.\\
\qquad\qquad \left. 3\left(\dfrac{\lambda^4 - \lambda^2}{19 + 12\lambda^2}\right)\left\{\left|4 - 2^{4\Psi}\right| + \left|16 - 2^{4\Psi}\right|\right\}\Theta\right); \Psi \neq \frac{1}{2}, 1
\end{cases}
\end{aligned} \tag{48}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.

3. Fuzzy Stability Results: Λ is Odd

The generalized Ulam-Hyers stability of 4 D AQCQ functional equation (3) when Λ is odd is analyzed in this section.

3.1. Additive Case Fuzzy Stability Result

Theorem 3.1. *An odd function $\Lambda_{aqcq} : \mathcal{S}_1 \to \mathcal{S}_2$ satisfies the functional inequality*

$$E\left(\Lambda_{aqcq}(z_1, z_2, z_3, z_4), \Theta\right) \geq E'\left(\Phi(z_1, z_2, z_3, z_4), \Theta\right) \tag{49}$$

where $\Phi : \mathcal{S}_1^4 \to \mathcal{S}_3$ be a mapping satisfying

$$E'\left(\Lambda\left(2^{\pi\varpi} z_1, 2^{\pi\varpi} z_2, 2^{\pi\varpi} z_3, 2^{\pi\varpi} z_4\right), \Theta\right) \geq E'\left(\theta^{\pi\varpi}\Phi(z_1, z_2, z_3, z_4), \Theta\right); \pi \in \{-1, 1\} \tag{50}$$

where $\theta > 0$ with $0 < \left(\frac{\theta}{2}\right)^{\pi} < 1$ and the condition

$$\lim_{\varpi\to\infty} E'\left(\Phi\left(2^{\pi\varpi} z_1, 2^{\pi\varpi} z_2, 2^{\pi\varpi} z_3, 2^{\pi\varpi} z_4\right), 2^{\pi\varpi}\Theta\right) = 1, \tag{51}$$

for all $z_1, z_2, z_3, z_4 \in \mathcal{S}_1$ and all $\Theta > 0$. Then there exists a unique additive mapping $\Omega_{A1}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ such that

$$E\left(\Lambda(2z) - 8\Lambda(z) - \Omega_{A1}(z), \Theta\right) \geq E'\left(\Phi_{ODD}(z, z, z, z), \left(\frac{\lambda^4 - \lambda^2}{16 - 3\lambda^2}\right) |2 - \theta|\ \Theta\right), \tag{52}$$

where

$$\begin{aligned}
&E'\left(\Phi_{ODD}(z, z, z, z), \Theta\right)\\
&\geq \min\left\{E'\left(\Phi\left(\frac{z}{2}, \frac{z}{2}, \frac{z}{2}, \frac{z}{2}\right), \Theta\right), E'\left(\Phi\left(\frac{z}{2}, \frac{z}{2}, z, z\right), \Theta\right), E'\left(\Phi\left(z, z, \frac{z}{2}, \frac{z}{2}\right), \Theta\right),\right.\\
&\qquad E'\left(\Phi\left(\frac{(1+\lambda)z}{2}, \frac{(1+\lambda)z}{2}, \frac{z}{2}, \frac{z}{2}\right), \Theta\right), E'\left(\Phi\left(\frac{(1-\lambda)z}{2}, \frac{(1-\lambda)z}{2}, \frac{z}{2}, \frac{z}{2}\right), \Theta\right)\\
&\qquad E'\left(\Phi\left(\frac{z}{2}, \frac{z}{2}, \frac{z}{2}, \frac{z}{2}\right), \Theta\right), E'\left(\Phi(z, z, z, z), \Theta\right),\\
&\qquad E'\left(\Phi\left(\frac{z}{2}, \frac{z}{2}, z, z\right), \Theta\right), E'\left(\Phi\left(\frac{z}{2}, \frac{z}{2}, \frac{3z}{2}, \frac{3z}{2}\right), \Theta\right),\\
&\qquad \left. E'\left(\Phi\left(\frac{(1+2\lambda)z}{2}, \frac{(1+2\lambda)z}{2}, \frac{z}{2}, \frac{z}{2}\right), \Theta\right), E'\left(\Phi\left(\frac{(1-2\lambda)z}{2}, \frac{(1-2\lambda)z}{2}, \frac{z}{2}, \frac{z}{2}\right), \Theta\right)\right\},
\end{aligned} \tag{53}$$

and the mapping

$$E\left(\lim_{\varpi\to\infty} \frac{1}{2^{\pi\varpi}}\left\{\Lambda\left(2^{(\varpi+1)\pi} z\right) - 8\Lambda\left(2^{\pi\varpi} z\right)\right\} - \Omega_{A1}(z), \Theta\right) = 1, \tag{54}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.

Proof. Setting (z_1, z_2, z_3, z_4) by $\left(\frac{z}{2}, \frac{z}{2}, \frac{z}{2}, \frac{z}{2}\right)$ in (49) and using oddness of Λ, one can have

$$\begin{aligned}
&E\left(\Lambda((1+\lambda)z) + \Lambda((1-\lambda)z) - \lambda^2\Lambda(2z) - 2\left(1-\lambda^2\right)\Lambda(z), \Theta\right)\\
&\qquad\qquad \geq E'\left(\Phi\left(\frac{z}{2}, \frac{z}{2}, \frac{z}{2}, \frac{z}{2}\right), \Theta\right),
\end{aligned} \tag{55}$$

and with the help of $(FBS3)$, we have

$$\begin{aligned}
&E\Big(2(1-\lambda^2)\Lambda((1+\lambda)z) + 2(1-\lambda^2)\Lambda((1-\lambda)z) - 2\lambda^2(1-\lambda^2)\Lambda(2z)\\
&\qquad - 4\left(1-\lambda^2\right)^2\Lambda(z), 2(1-\lambda^2)\Theta\Big) \geq E'\left(\Phi\left(\frac{z}{2}, \frac{z}{2}, \frac{z}{2}, \frac{z}{2}\right), \Theta\right),
\end{aligned} \tag{56}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.

Again setting (z_1, z_2, z_3, z_4) by $\left(z, z, \frac{z}{2}, \frac{z}{2}\right)$ in (49) and using oddness of Λ, one can obtain

$$E\Big(\Lambda\left((2+\lambda)z\right) + \Lambda\left((2-\lambda)z\right) - \lambda^2\Lambda(3z) - \lambda^2\Lambda(z) - 2\left(1-\lambda^2\right)\Lambda(2z), \Theta\Big) \geq E'\left(\Phi\left(z, z, \frac{z}{2}, \frac{z}{2}\right), \Theta\right), \tag{57}$$

and with the help of ($FBS3$), we obtain

$$E\Big(\lambda^2\Lambda\left((2+\lambda)z\right) + \lambda^2\Lambda\left((2-\lambda)z\right) - \lambda^4\Lambda(3z) - \lambda^4\Lambda(z) - 2\lambda^2\left(1-\lambda^2\right)\Lambda(2z), \lambda^2\,\Theta\Big) \geq E'\left(\Phi\left(z, z, \frac{z}{2}, \frac{z}{2}\right), \Theta\right), \tag{58}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.

Further setting (z_1, z_2, z_3, z_4) by $\left(\frac{z}{2}, \frac{z}{2}, z, z\right)$ in (49) and using oddness of Λ, one can get

$$E\Big(\Lambda\left((1+2\lambda)z\right) + \Lambda\left((1-2\lambda)\right) - \lambda^2\Lambda(3z) + \lambda^2\Lambda(z) - 2\left(1-\lambda^2\right)\Lambda(z), \Theta\Big) \geq E'\left(\Phi\left(\frac{z}{2}, \frac{z}{2}, z, z\right), \Theta\right), \tag{59}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.

Once again setting (z_1, z_2, z_3, z_4) by $\left(\frac{(1+\lambda)z}{2}, \frac{(1+\lambda)z}{2}, \frac{z}{2}, \frac{z}{2}\right)$ in (49) and using oddness of Λ, one can arrive

$$E\Big(\Lambda\left((1+2\lambda)z\right) + \Lambda(z) - \lambda^2\Lambda\left((2+\lambda)z\right) - \lambda^2\Lambda(\lambda z) - 2\left(1-\lambda^2\right)\Lambda\left((1+\lambda)z\right), \Theta\Big) \geq E'\left(\Phi\left(\frac{(1+\lambda)z}{2}, \frac{(1+\lambda)z}{2}, \frac{z}{2}, \frac{z}{2}\right), \Theta\right), \tag{60}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.

Finally setting (z_1, z_2, z_3, z_4) by $\left(\frac{(1-\lambda)z}{2}, \frac{(1-\lambda)z}{2}, \frac{z}{2}, \frac{z}{2}\right)$ in (49) and using oddness of Λ, one can reach

$$E\Big(\Lambda\left((1-2\lambda)z\right) + \Lambda(z) - \lambda^2\Lambda\left((2-\lambda)z\right) + \lambda^2\Lambda(\lambda z) - 2\left(1-\lambda^2\right)\Lambda\left((1-\lambda)z\right), \Theta\Big) \geq E'\left(\Phi\left(\frac{(1-\lambda)z}{2}, \frac{(1-\lambda)z}{2}, \frac{z}{2}, \frac{z}{2}\right), \Theta\right), \tag{61}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.

With the help of ($FBS4$) and it follows from (56), (58), (59), (60), (61), we achieve

$$\begin{aligned} &E\left(\left(\lambda^4 - \lambda^2\right)\left\{\Lambda(3z) - 4\,\Lambda(2z) + 5\Lambda(z)\right\}, \left(5-\lambda^2\right)\Theta\right) \\ &\geq \min\Big\{E'\left(\Phi\left(\frac{z}{2}, \frac{z}{2}, \frac{z}{2}, \frac{z}{2}\right), \Theta\right), E'\left(\Phi\left(\frac{z}{2}, \frac{z}{2}, z, z\right), \Theta\right), E'\left(\Phi\left(z, z, \frac{z}{2}, \frac{z}{2}\right), \Theta\right), \\ &\quad E'\left(\Phi\left(\frac{(1+\lambda)z}{2}, \frac{(1+\lambda)z}{2}, \frac{z}{2}, \frac{z}{2}\right), \Theta\right), E'\left(\Phi\left(\frac{(1-\lambda)z}{2}, \frac{(1-\lambda)z}{2}, \frac{z}{2}, \frac{z}{2}\right), \Theta\right)\Big\}, \end{aligned} \tag{62}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. Put z by $2z$ in (55) , we have

$$E\left(\Lambda\left(2\left(1+\lambda\right)z\right) + \Lambda\left(2\left(1-\lambda\right)z\right) - \lambda^2\,\Lambda(4z) - 2\left(1-\lambda^2\right)\Lambda(2z), \Theta\right) \geq E'\left(\Phi(z, z, z, z), \Theta\right) \tag{63}$$

and with the help of ($FBS3$), we have

$$E\left(\lambda^2\Lambda\left(2\left(1+\lambda\right)z\right)+\lambda^2\,\Lambda\left(2\left(1-\lambda\right)z\right)-\lambda^4\,\Lambda\left(4z\right)-2\lambda^2\,\left(1-\lambda^2\right)\Lambda\left(2z\right),\lambda^2\,\Theta\,\right)\\ \geq E'\left(\Phi(z,z,z,z),\Theta\right),\tag{64}$$

for all $z\in\mathcal{S}_1$ and all $\Theta>0$. From (59) with the help of ($FBS3$), we obtain

$$E\Big(2\left(1-\lambda^2\right)\Lambda\left(\left(1+2\lambda\right)z\right)+2\left(1-\lambda^2\right)+\Lambda\left(\left(1-2\lambda\right)\right)-2\lambda^2\left(1-\lambda^2\right)\Lambda\left(3z\right)\\ +2\lambda^2\left(1-\lambda^2\right)\Lambda\left(z\right)-4\left(1-\lambda^2\right)^2\Lambda\left(z\right),2\left(1-\lambda^2\right)\,\Theta\,\Big)\geq E'\left(\Phi\left(\frac{z}{2},\frac{z}{2},z,z\right),\Theta\right),\tag{65}$$

for all $z\in\mathcal{S}_1$ and all $\Theta>0$.

Again putting (z_1,z_2,z_3,z_4) by $\left(\frac{z}{2},\frac{z}{2},\frac{3z}{2},\frac{3z}{2}\right)$ in (49) and using oddness of Λ, one can arrive

$$E\left(\Lambda\left(\left(1+3\lambda\right)z\right)+\Lambda\left(\left(1-3\lambda\right)z\right)-\lambda^2\Lambda\left(4z\right)+\lambda^2\Lambda\left(2z\right)-2\left(1-\lambda^2\right)\Lambda\left(z\right),\,\Theta\right)\\ \geq E'\left(\Phi\left(\frac{z}{2},\frac{z}{2},\frac{3z}{2},\frac{3z}{2}\right),\Theta\right),\tag{66}$$

for all $z\in\mathcal{S}_1$ and all $\Theta>0$.

Once again putting (z_1,z_2,z_3,z_4) by $\left(\frac{(1+2\lambda)z}{2},\frac{(1+2\lambda)z}{2},\frac{z}{2},\frac{z}{2}\right)$ in (49) and using oddness of Λ, one can arrive

$$E\left(\Lambda\left((1+3\lambda)z\right)+\Lambda\left((1+\lambda)z\right)-\lambda^2\Lambda\left(2(1+\lambda)z\right)-\lambda^2\Lambda\left(2\lambda z\right)-2\left(1-\lambda^2\right)\Lambda\left((1+2\lambda)z\right),\,\Theta\right)\\ \geq E'\left(\Phi\left(\frac{(1+2\lambda)z}{2},\frac{(1+2\lambda)z}{2},\frac{z}{2},\frac{z}{2}\right),\Theta\right),\tag{67}$$

for all $z\in\mathcal{S}_1$ and all $\Theta>0$.

Finally putting (z_1,z_2,z_3,z_4) by $\left(\frac{(1-2\lambda)z}{2},\frac{(1-2\lambda)z}{2},\frac{z}{2},\frac{z}{2}\right)$ in (49) and using oddness of Λ, one can arrive

$$E\left(\Lambda\left((1-3\lambda)z\right)+\Lambda\left((1-\lambda)z\right)-\lambda^2\Lambda\left(2(1-\lambda)z\right)+\lambda^2\Lambda\left(2\lambda z\right)-2\left(1-\lambda^2\right)\Lambda\left((1-2\lambda)z\right),\,\Theta\,\right)\\ \geq E'\left(\Phi\left(\frac{(1-2\lambda)z}{2},\frac{(1-2\lambda)z}{2},\frac{z}{2},\frac{z}{2}\right),\Theta\right),\tag{68}$$

for all $z\in\mathcal{S}_1$ and all $\Theta>0$.

With the help of ($FBS4$) and it follows from (55), (63), (65), (66), (67), (68), we achieve

$$E\left(\left(\lambda^4-\lambda^2\right)\{\Lambda\left(4z\right)-2\Lambda\left(3z\right)-2\Lambda\left(2z\right)+6\Lambda\left(z\right)\},\left(6-\lambda^2\right)\Theta\right)\\ \geq\min\left\{E'\left(\Phi\left(\frac{z}{2},\frac{z}{2},\frac{z}{2},\frac{z}{2}\right),\Theta\right),E'\left(\Phi\left(z,z,z,z\right),\Theta\right),E'\left(\Phi\left(\frac{z}{2},\frac{z}{2},z,z\right),\Theta\right),\right.\\ E'\left(\Phi\left(\frac{z}{2},\frac{z}{2},\frac{3z}{2},\frac{3z}{2}\right),\Theta\right),E'\left(\Phi\left(\frac{(1+2\lambda)z}{2},\frac{(1+2\lambda)z}{2},\frac{z}{2},\frac{z}{2}\right),\Theta\right),\\ \left.E'\left(\Phi\left(\frac{(1-2\lambda)z}{2},\frac{(1-2\lambda)z}{2},\frac{z}{2},\frac{z}{2}\right),\Theta\right)\right\},\tag{69}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. With the help of (*FBS*3) (*FBS*4) and it follows from (62), (69) , we reach

$$\begin{aligned}
&E\left(\left(\lambda^4-\lambda^2\right)\{\Lambda(4z)-10\Lambda(2z)+16\Lambda(z)\},\left(16-3\lambda^2\right)\Theta\right)\\
&\geq \min\left\{E'\left(\Phi\left(\frac{z}{2},\frac{z}{2},\frac{z}{2},\frac{z}{2}\right),\Theta\right),E'\left(\Phi\left(\frac{z}{2},\frac{z}{2},z,z\right),\Theta\right),E'\left(\Phi\left(z,z,\frac{z}{2},\frac{z}{2}\right),\Theta\right),\right.\\
&\qquad E'\left(\Phi\left(\frac{(1+\lambda)z}{2},\frac{(1+\lambda)z}{2},\frac{z}{2},\frac{z}{2}\right),\Theta\right),E'\left(\Phi\left(\frac{(1-\lambda)z}{2},\frac{(1-\lambda)z}{2},\frac{z}{2},\frac{z}{2}\right),\Theta\right)\\
&\qquad E'\left(\Phi\left(\frac{z}{2},\frac{z}{2},\frac{z}{2},\frac{z}{2}\right),\Theta\right),E'\left(\Phi(z,z,z,z),\Theta\right),\\
&\qquad E'\left(\Phi\left(\frac{z}{2},\frac{z}{2},z,z\right),\Theta\right),E'\left(\Phi\left(\frac{z}{2},\frac{z}{2},\frac{3z}{2},\frac{3z}{2}\right),\Theta\right),\\
&\qquad \left.E'\left(\Phi\left(\frac{(1+2\lambda)z}{2},\frac{(1+2\lambda)z}{2},\frac{z}{2},\frac{z}{2}\right),\Theta\right),E'\left(\Phi\left(\frac{(1-2\lambda)z}{2},\frac{(1-2\lambda)z}{2},\frac{z}{2},\frac{z}{2}\right),\Theta\right)\right\}\\
&= E'\left(\Phi_{ODD}(z,z,z,z),\Theta\right),
\end{aligned}\tag{70}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. With the help of (*FBS*3) it follows from (70), that

$$E\left(\Lambda(4z)-10\Lambda(2z)+16\Lambda(z),\left(\frac{16-3\lambda^2}{\lambda^4-\lambda^2}\right)\Theta\right)\geq E'\left(\Phi_{ODD}(z,z,z,z),\Theta\right),\tag{71}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. Define a function $\Lambda_{A1} : \mathcal{S}_1 \to \mathcal{S}_2$ by

$$\Lambda_{A1}(z)=\Lambda(2z)-8\Lambda(z),\tag{72}$$

for all $z \in \mathcal{S}_1$. It follows from (71) and (72), we reach

$$\begin{aligned}
&E\left(\Lambda_{A1}(2z)-2\Lambda_{A1}(z),\left(\frac{16-3\lambda^2}{\lambda^4-\lambda^2}\right)\Theta\right)\\
&\qquad = E\left(\left\{\Lambda(4z)-8\Lambda(2z)\right\}-2\left\{\Lambda(2z)-8\Lambda(z)\right\},\left(\frac{16-3\lambda^2}{\lambda^4-\lambda^2}\right)\Theta\right)\\
&\qquad \geq E'\left(\Phi_{ODD}(z,z,z,z),\Theta\right),
\end{aligned}\tag{73}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. With the help of (*FBS*3) and it follows from (63) that

$$E\left(\frac{1}{2}\Lambda_{A1}(2z)-2\Lambda_{A1}(z),\left(\frac{16-3\lambda^2}{\lambda^4-\lambda^2}\right)\cdot\frac{1}{2}\Theta\right)\geq E'\left(\Phi_{ODD}(z,z,z,z),\Theta\right)\tag{74}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. The rest of the proof is similar to that of Theorem 2.1. Hence the proof is complete. □

Corollary 3.2. *An odd function $\Lambda_{aqcq} : \mathcal{S}_1 \to \mathcal{S}_2$ satisfies the functional inequality*

$$E\left(\Lambda_{aqcq}(z_1,z_2,z_3,z_4),\Theta\right)\geq\begin{cases}E'(\Delta,\Theta),\\ E'\left(\Delta\sum_{\tau=1}^{4}|z_\tau|^{\Psi},\Theta\right),\\ E'\left(\Delta\prod_{\tau=1}^{4}|z_\tau|^{\Psi},\Theta\right),\\ E'\left(\Delta\left\{\sum_{\tau=1}^{4}|z_\tau|^{4\Psi}+\prod_{\tau=1}^{4}|z_\tau|^{\Psi}\right\},\Theta\right),\end{cases}\tag{75}$$

for all $z_1, z_2, z_3, z_4 \in \mathcal{S}_1$ and all $\Theta > 0$ with Δ a positive constant. Then there exists a unique additive mapping $\Omega_{A1}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ such that

$$E\left(\Lambda(2z) - 8\Lambda(z) - \Omega_{A1}(z), \Theta\right) \geq \begin{cases} E'\left(\Delta, \left(\dfrac{\lambda^4 - \lambda^2}{16 - 3\lambda^2}\right)|1|\Theta\right); \\ E'\Big(\Delta\Big\{24 + 10 \cdot 2^{\Psi} + 2 \cdot 3^{\Psi} + 2 \cdot (1+\lambda)^{\Psi} + 2 \cdot (1-\lambda)^{\Psi} \\ \qquad +2 \cdot (1+2\lambda)^{\Psi} + 2 \cdot (1-2\lambda)^{\Psi}\Big\} \left|\frac{z}{2}\right|^{\Psi}, 11\left(\dfrac{\lambda^4 - \lambda^2}{16 - 3\lambda^2}\right)\left|2 - 2^{\Psi}\right|\Theta\Big); \Psi \neq 1 \\ E'\Big(\Delta\Big\{2 + 3 \cdot 2^{2\Psi} + 2^{4\Psi} + 3^{2\Psi} + (1+\lambda)^{2\Psi} + (1-\lambda)^{2\Psi} \\ \qquad +(1+2\lambda)^{2\Psi} + (1-2\lambda)^{2\Psi}\Big\} \left|\frac{z}{2}\right|^{4\Psi}, 11\left(\dfrac{\lambda^4 - \lambda^2}{16 - 3\lambda^2}\right)\left|2 - 2^{4\Psi}\right|\Theta\Big); \Psi \neq \dfrac{1}{4} \\ E'\Big(\Delta\Big\{26 + 11 \cdot 2^{4\Psi} + 2 \cdot 3^{4\Psi} + 2 \cdot (1+\lambda)^{4\Psi} + 2 \cdot (1-\lambda)^{4\Psi} \\ \qquad +2 \cdot (1+2\lambda)^{4\Psi} + 2 \cdot (1-2\lambda)^{4\Psi} + 3 \cdot 2^{2\Psi} + 3^{2\Psi} + (1+\lambda)^{2\Psi} + (1-\lambda)^{2\Psi} \\ \qquad +(1+2\lambda)^{2\Psi} + (1-2\lambda)^{2\Psi}\Big\}, 11\left(\dfrac{\lambda^4 - \lambda^2}{16 - 3\lambda^2}\right)\left|2 - 2^{4\Psi}\right|\Theta\Big); \Psi \neq \frac{1}{4} \end{cases} \tag{76}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.

3.2. Cubic Case Fuzzy Stability Result

Theorem 3.3. *An odd function $\Lambda_{aqcq} : \mathcal{S}_1 \to \mathcal{S}_2$ satisfies the functional inequality (49) where $\Phi : \mathcal{S}_1^4 \to \mathcal{S}_3$ be a mapping satisfying (50) where $\theta > 0$ with $0 < \left(\frac{\theta}{8}\right)^{\pi} < 1$ and the condition*

$$\lim_{\varpi \to \infty} E'\left(\Phi\left(2^{\pi\varpi}z_1, 2^{\pi\varpi}z_2, 2^{\pi\varpi}z_3, 2^{\pi\varpi}z_4\right), 8^{\pi\varpi}\Theta\right) = 1, \tag{77}$$

for all $z_1, z_2, z_3, z_4 \in \mathcal{S}_1$ and all $\Theta > 0$. Then there exists a unique cubic mapping $\Omega_{C3}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ such that

$$E\left(\Lambda(2z) - 2\Lambda(z) - \Omega_{C3}(z), \Theta\right) \geq E'\left(\Phi_{ODD}(z,z,z,z), \left(\frac{\lambda^4 - \lambda^2}{16 - 3\lambda^2}\right)|8 - \theta|\Theta\right), \tag{78}$$

where $E'(\Phi_{ODD}(z,z,z,z), \Theta)$ is given in (53) and the mapping

$$E\left(\lim_{\varpi \to \infty} \frac{1}{8^{\pi\varpi}}\left\{\Lambda\left(2^{(\varpi+1)\pi}z\right) - 2\Lambda\left(2^{\pi\varpi}z\right)\right\} - \Omega_{C3}(z), \Theta\right) = 1, \tag{79}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.

Proof. Define a function $\Lambda_{C3} : \mathcal{S}_1 \to \mathcal{S}_2$ by

$$\Lambda_{C3}(z) = \Lambda(2z) - 2\Lambda(z), \tag{80}$$

for all $z \in \mathcal{S}_1$. It follows from (71) and (80), we reach

$$\begin{aligned} E&\left(\Lambda_{C3}(2z) - 8\Lambda_{C3}(z), \left(\frac{16 - 3\lambda^2}{\lambda^4 - \lambda^2}\right)\Theta\right) \\ &= E\left(\Big\{\Lambda(4z) - 2\Lambda(2z)\Big\} - 8\Big\{\Lambda(2z) - 8\Lambda(z)\Big\}, \left(\frac{16 - 3\lambda^2}{\lambda^4 - \lambda^2}\right)\Theta\right) \\ &\geq E'\left(\Phi_{ODD}(z,z,z,z), \Theta\right), \end{aligned} \tag{81}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. With the help of ($FBS3$) and it follows from (81) that

$$E\left(\frac{1}{8}\Lambda_{C3}(2z) - \Lambda_{C3}(z), \left(\frac{16-3\lambda^2}{\lambda^4-\lambda^2}\right)\cdot\frac{1}{8}\Theta\right) \geq E'\left(\Phi_{ODD}(z,z,z,z),\Theta\right), \tag{82}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. The rest of the proof is similar to that of Theorem 2.1. Hence the proof is complete. □

Corollary 3.4. *An odd function $\Lambda_{aqcq} : \mathcal{S}_1 \to \mathcal{S}_2$ satisfies the functional inequality (75) for all $z_1, z_2, z_3, z_4 \in \mathcal{S}_1$ and all $\Theta > 0$ with Δ a positive constant. Then there exists a unique cubic mapping $\Omega_{C3}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ such that*

$$\begin{aligned}
&E\left(\Lambda(2z) - 2\Lambda(z) - \Omega_{C3}(z), \Theta\right) \\
&\geq \begin{cases}
E'\left(\Delta, \left(\dfrac{\lambda^4-\lambda^2}{16-3\lambda^2}\right)|7|\Theta\right); \\
E'\Big(\Delta\Big\{24 + 10\cdot 2^{\Psi} + 2\cdot 3^{\Psi} + 2\cdot(1+\lambda)^{\Psi} + 2\cdot(1-\lambda)^{\Psi} \\
\qquad +2\cdot(1+2\lambda)^{\Psi} + 2\cdot(1-2\lambda)^{\Psi}\Big\}\left|\frac{z}{2}\right|^{\Psi}, 11\left(\dfrac{\lambda^4-\lambda^2}{16-3\lambda^2}\right)\left|8-2^{\Psi}\right|\Theta\Big); \Psi \neq 3 \\
E'\Big(\Delta\Big\{2 + 3\cdot 2^{2\Psi} + 2^{4\Psi} + 3^{2\Psi} + (1+\lambda)^{2\Psi} + (1-\lambda)^{2\Psi} \\
\qquad +(1+2\lambda)^{2\Psi} + (1-2\lambda)^{2\Psi}\Big\}\left|\frac{z}{2}\right|^{4\Psi}, 11\left(\dfrac{\lambda^4-\lambda^2}{16-3\lambda^2}\right)\left|8-2^{4\Psi}\right|\Theta\Big); \Psi \neq \dfrac{3}{4} \\
E'\Big(\Delta\Big\{26 + 11\cdot 2^{4\Psi} + 2\cdot 3^{4\Psi} + 2\cdot(1+\lambda)^{4\Psi} + 2\cdot(1-\lambda)^{4\Psi} \\
\qquad +2\cdot(1+2\lambda)^{4\Psi} + 2\cdot(1-2\lambda)^{4\Psi} + 3\cdot 2^{2\Psi} + 3^{2\Psi} + (1+\lambda)^{2\Psi} + (1-\lambda)^{2\Psi} \\
\qquad +(1+2\lambda)^{2\Psi} + (1-2\lambda)^{2\Psi}\Big\}, 11\left(\dfrac{\lambda^4-\lambda^2}{16-3\lambda^2}\right)\left|8-2^{4\Psi}\right|\Theta\Big); \Psi \neq \frac{3}{4}
\end{cases}
\end{aligned} \tag{83}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.

3.3. Additive - Cubic Case Fuzzy Stability Result

Theorem 3.5. *An odd function $\Lambda_{aqcq} : \mathcal{S}_1 \to \mathcal{S}_2$ satisfies the functional inequality (49) where $\Phi : \mathcal{S}_1^4 \to \mathcal{S}_3$ be a mapping satisfying (50) where $\theta > 0$ with $0 < \left(\frac{\theta}{2}\right)^{\pi} < 1$, $0 < \left(\frac{\theta}{8}\right)^{\pi} < 1$ and the conditions (51), (77) for all $z_1, z_2, z_3, z_4 \in \mathcal{S}_1$ and all $\Theta > 0$. Then there exists a unique additive mapping $\Omega_{A1}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ and a unique cubic mapping $\Omega_{C3}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ such that*

$$\begin{aligned}
&E\left(\Lambda(z) - \Omega_{A1}(z) - \Omega_{C3}(z), \Theta\right) \\
&\qquad \geq E'\left(\Phi_{ODD}(z,z,z,z), 3\left(\frac{\lambda^4-\lambda^2}{16-3\lambda^2}\right)\Big(|2-\theta| + |8-\theta|\Big)\Theta\right)
\end{aligned} \tag{84}$$

where $E'(\Phi_{ODD}(z,z,z,z),\Theta)$, $\Omega_{A1}(z)$ and $\Omega_{C3}(z)$ are given in (53), (54) and (79), respectively for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.

Proof. By Theorem 3.1, there exists a unique additive mapping $\Omega^1_{A1}(z)\,. : \mathcal{S}_1 \to \mathcal{S}_2$ such that

$$E\left(\Lambda(2z) - 8\Lambda(z) - \Omega^1_{A1}(z), \Theta\right) \geq E'\left(\Phi_{ODD}(z,z,z,z), \left(\frac{\lambda^4-\lambda^2}{16-3\lambda^2}\right)|2-\theta|\Theta\right) \tag{85}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.

Also, by Theorem 3.3, there exists a unique cubic mapping $\Omega^3_{C3}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ such that

$$E\left(\Lambda(2z) - 2\Lambda(z) - \Omega^3_{C3}(z), \Theta\right) \geq E'\left(\Phi_{ODD}(z,z,z,z), \left(\frac{\lambda^4-\lambda^2}{16-3\lambda^2}\right)|8-\theta|\Theta\right), \tag{86}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. Now,

$$\begin{aligned}
& E\left(6\Lambda(z) + \Omega^1_{A1}(z) - \Omega^3_{C3}(z), 2\Theta\right) \\
&= E\left(\Lambda(2z) - 2\Lambda(z) - \Omega^3_{C3}(z) - \Lambda(2z) + 8\Lambda(z) + \Omega^1_{A1}(z), 2\Theta\right) \\
&\geq \min\left\{E\left(\Lambda(2z) - 2\Lambda(z) - \Omega^3_{C3}(z), \Theta\right), E\left(\Lambda(2z) - 8\Lambda(z) - \Omega^1_{A1}(z), \Theta\right)\right\} \\
&\geq \min\left\{E'\left(\Phi_{ODD}(z,z,z,z), \left(\frac{\lambda^4-\lambda^2}{16-3\lambda^2}\right)|2-\theta|\,\Theta\right),\right. \\
&\qquad\qquad \left. E'\left(\Phi_{ODD}(z,z,z,z), \left(\frac{\lambda^4-\lambda^2}{16-3\lambda^2}\right)|8-\theta|\,\Theta\right)\right\} \\
&= E'\left(2\Phi_{ODD}(z,z,z,z), \left(\frac{\lambda^4-\lambda^2}{16-3\lambda^2}\right)(|2-\theta| + |8-\theta|)\,\Theta\right),
\end{aligned} \tag{87}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. It follows from (87) that

$$\begin{aligned}
& E\left(\Lambda(z) + \frac{1}{6}\Omega^1_{A1}(z) - \frac{1}{6}\Omega^3_{C3}(z), \Theta\right) \\
&\qquad \geq E'\left(2\Phi_{ODD}(z,z,z,z), 6\left(\frac{\lambda^4-\lambda^2}{16-3\lambda^2}\right)(|2-\theta| + |8-\theta|)\,\Theta\right),
\end{aligned} \tag{88}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. So, if we take

$$\Omega_{A1}(z) = -\frac{1}{6}\Omega^1_{A1}(z), \qquad \text{and} \qquad \Omega_{C3}(z) = \frac{1}{6}\Omega^3_{C3}(z),$$

we arrive our desired result. Hence the proof is complete. □

Corollary 3.6. *An odd function $\Lambda_{aqcq} : \mathcal{S}_1 \to \mathcal{S}_2$ satisfies the functional inequality (75) for all $z_1, z_2, z_3, z_4 \in \mathcal{S}_1$ and all $\Theta > 0$ with Δ a positive constant. Then there exists a unique additive mapping $\Omega_{A1}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ and a unique cubic mapping $\Omega_{C3}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ such that*

$$E\left(\Lambda(z) - \Omega_{A1}(z) - \Omega_{C3}(z), \Theta\right)$$
$$\geq \begin{cases}
E'\left(\Delta, 3\left(\frac{\lambda^4-\lambda^2}{16-3\lambda^2}\right)(|1| + |7|)\,\Theta\right); \\
E'\Big(\Delta\Big\{24 + 10\cdot 2^\Psi + 2\cdot 3^\Psi + 2\cdot(1+\lambda)^\Psi + 2\cdot(1-\lambda)^\Psi \\
\qquad\qquad +2\cdot(1+2\lambda)^\Psi + 2\cdot(1-2\lambda)^\Psi\Big\}\left|\frac{z}{2}\right|^\Psi, \\
\qquad 3\left(\frac{\lambda^4-\lambda^2}{16-3\lambda^2}\right)\left\{\left|2-2^\Psi\right| + \left|8-2^\Psi\right|\right\}\Theta\Big); \quad \Psi \neq 1, 3 \\
E'\Big(\Delta\Big\{2 + 3\cdot 2^{2\Psi} + 2^{4\Psi} + 3^{2\Psi} + (1+\lambda)^{2\Psi} + (1-\lambda)^{2\Psi} \\
\qquad\qquad +(1+2\lambda)^{2\Psi} + (1-2\lambda)^{2\Psi}\Big\}\left|\frac{z}{2}\right|^{4\Psi}, \\
\qquad 3\left(\frac{\lambda^4-\lambda^2}{16-3\lambda^2}\right)\left\{\left|2-2^{4\Psi}\right| + \left|8-2^{4\Psi}\right|\right\}\Theta\Big); \quad \Psi \neq \frac{1}{4}, \frac{3}{4} \\
E'\Big(\Delta\Big\{26 + 11\cdot 2^{4\Psi} + 2\cdot 3^{4\Psi} + 2\cdot(1+\lambda)^{4\Psi} + 2\cdot(1-\lambda)^{4\Psi} \\
\qquad +2\cdot(1+2\lambda)^{4\Psi} + 2\cdot(1-2\lambda)^{4\Psi} + 3\cdot 2^{2\Psi} + 3^{2\Psi} \\
\qquad +(1+\lambda)^{2\Psi} + (1-\lambda)^{2\Psi} + (1+2\lambda)^{2\Psi} + (1-2\lambda)^{2\Psi}\Big\}, \\
\qquad 3\left(\frac{\lambda^4-\lambda^2}{16-3\lambda^2}\right)\left\{\left|2-2^{4\Psi}\right| + \left|8-2^{4\Psi}\right|\right\}\Theta\Big); \quad \Psi \neq \frac{1}{4}, \frac{3}{4}
\end{cases} \tag{89}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.

4. Fuzzy Stability Results: Λ is Odd and Even

The generalized Ulam-Hyers stability of 4-D AQCQ functional equation (3) when Λ is odd and Even is analyzed in this section.

Theorem 4.1. *A function $\Lambda_{aqcq} : \mathcal{S}_1 \to \mathcal{S}_2$ satisfies the functional inequality*

$$E\left(\Lambda_{aqcq}(z_1, z_2, z_3, z_4), \Theta\right) \geq E'\left(\Phi(z_1, z_2, z_3, z_4), \Theta\right) \tag{90}$$

where $\Phi : \mathcal{S}_1^4 \to \mathcal{S}_3$ be a mapping satisfying (5) and (50) where $\theta > 0$ with $0 < \left(\frac{\theta}{2}\right)^{\pi} < 1$, $0 < \left(\frac{\theta}{4}\right)^{\pi} < 1$, $0 < \left(\frac{\theta}{8}\right)^{\pi} < 1$, $0 < \left(\frac{\theta}{16}\right)^{\pi} < 1$ and the conditions (6), (36), (51), (77) for all $z_1, z_2, z_3, z_4 \in \mathcal{S}_1$ and all $\Theta > 0$. Then there exists a unique additive mapping $\Omega_{A1}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ a unique quadratic mapping $\Omega_{Q2}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ a unique cubic mapping $\Omega_{C3}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ a unique quartic mapping $\Omega_{Q4}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ such that

$$\begin{aligned}
&E\left(\Lambda(z) - \Omega_{A1}(z) - \Omega_{Q2}(z) - \Omega_{C3}(z) - \Omega_{Q4}(z), 4\Theta\right) \\
&\geq E'\Bigg(\Phi_{EVEN}(z,z,z,z) + \Phi_{EVEN}(-z,-z,-z,-z) + \Phi_{ODD}(z,z,z,z) + \Phi_{ODD}(-z,-z,-z,-z), \\
&\qquad 6\Theta\left\{\left(\frac{\lambda^4 - \lambda^2}{19 + 12\lambda^2}\right)(|4-\theta| + |16-\theta|) + \left(\frac{\lambda^4 - \lambda^2}{16 - 3\lambda^2}\right)(|2-\theta| + |8-\theta|)\right\}\Bigg),
\end{aligned} \tag{91}$$

where $E'(\Phi_{EVEN}(z,z,z,z),\Theta)$, $\Omega_{Q2}(z)$, $\Omega_{Q4}(z)$, $E'(\Phi_{ODD}(z,z,z,z),\Theta)$, $\Omega_{A1}(z)$, $\Omega_{C3}(z)$ are given in (8), (9), (38), (53), (54) and (79), respectively for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.

Proof. Assume a function $\Lambda_{EVEN}(z)$ by

$$\Lambda_{EVEN}(z) = \frac{1}{2}\left(\Lambda(z) + \Lambda(-z)\right). \tag{92}$$

From this we see

$$\Lambda_{EVEN}(0) = 0, \qquad \Lambda_{EVEN}(-z) = \Lambda_{EVEN}(z), \tag{93}$$

for all $z \in \mathcal{S}_1$. Now,

$$\begin{aligned}
&E\left(\Lambda_{EVENaqcq}(z_1, z_2, z_3, z_4), 2\Theta\right) \\
&\qquad \geq \min\left\{E\left(\Lambda_{aqcq}(z_1, z_2, z_3, z_4), \Theta\right), E\left(\Lambda_{aqcq}(-z_1, -z_2, -z_3, -z_4), \Theta\right)\right\} \\
&\qquad \geq \min\left\{E'\left(\Phi(z_1, z_2, z_3, z_4), \Theta\right), E'\left(\Phi(-z_1, -z_2, -z_3, -z_4), \Theta\right)\right\},
\end{aligned} \tag{94}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. From (94) and by Theorem 2.5, there exists a unique quadratic mapping $\Omega_{Q2}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ and a unique quartic mapping $\Omega_{Q4}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ such that

$$\begin{aligned}
&E\left(\Lambda_{EVEN}(z) - \Omega_{Q2}(z) - \Omega_{Q4}(z), 2\Theta\right) \\
&\geq \min\Bigg\{E'\left(\Phi_{EVEN}(z,z,z,z), 3\left(\frac{\lambda^4 - \lambda^2}{19 + 12\lambda^2}\right)(|4-\theta| + |16-\theta|)\Theta\right) \\
&\qquad E'\left(\Phi_{EVEN}(-z,-z,-z,-z), 3\left(\frac{\lambda^4 - \lambda^2}{19 + 12\lambda^2}\right)(|4-\theta| + |16-\theta|)\Theta\right)\Bigg\},
\end{aligned} \tag{95}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.

Assume a function $\Lambda_{ODD}(z)$ by

$$\Lambda_{ODD}(z) = \frac{1}{2}\left(\Lambda(z) - \Lambda(-z)\right). \tag{96}$$

From this we see

$$\Lambda_{ODD}(0) = 0 \qquad \Lambda_{ODD}(-z) = -\Lambda_{ODD}(z), \tag{97}$$

for all $z \in \mathcal{S}_1$. Now,

$$\begin{aligned} &E\left(\Lambda_{ODDaqcq}(z_1, z_2, z_3, z_4), 2\Theta\right) \\ &\qquad \geq \min\left\{E\left(\Lambda_{aqcq}(z_1, z_2, z_3, z_4), \Theta\right), E\left(\Lambda_{aqcq}(-z_1, -z_2, -z_3, -z_4), \Theta\right)\right\} \\ &\qquad \geq \min\left\{E'\left(\Phi(z_1, z_2, z_3, z_4), \Theta\right), E'\left(\Phi(-z_1, -z_2, -z_3, -z_4), \Theta\right)\right\}, \end{aligned} \tag{98}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. From (98) and by Theorem 3.5, there exists a unique additive mapping $\Omega_{A1}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ and a unique cubic mapping $\Omega_{C3}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ such that

$$\begin{aligned} &E\left(\Lambda_{ODD}(z) - \Omega_{A1}(z) - \Omega_{C3}(z), 2\Theta\right) \\ &\geq \min\left\{E'\left(\Phi_{ODD}(z,z,z,z), 3\left(\frac{\lambda^4 - \lambda^2}{16 - 3\lambda^2}\right)\left(|2-\theta| + |8-\theta|\right)\Theta\right)\right. \\ &\qquad \left. E'\left(\Phi_{ODD}(-z,-z,-z,-z), 3\left(\frac{\lambda^4 - \lambda^2}{16 - 3\lambda^2}\right)\left(|2-\theta| + |8-\theta|\right)\Theta\right)\right\}, \end{aligned} \tag{99}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. Define a mapping $\Lambda(z)$ by

$$\Lambda(z) = \Lambda_{EVEN}(z) + \Lambda_{ODD}(z), \tag{100}$$

for all $z \in \mathcal{S}_1$. Using (95), (99) in (100) it follows that

$$\begin{aligned} &E\left(\Lambda(z) - \Omega_{A1}(z) - \Omega_{Q2}(z) - \Omega_{C3}(z) - \Omega_{Q4}(z), 4\Theta\right) \\ &= E\left(\Lambda_{EVEN}(z) + \Lambda_{ODD}(z) - \Omega_{A1}(z) - \Omega_{Q2}(z) - \Omega_{C3}(z) - \Omega_{Q4}(z), 4\Theta\right) \\ &\geq \min\left\{E\left(\Lambda_{EVEN}(z) - \Omega_{Q2}(z) - \Omega_{Q4}(z), 2\Theta\right), E\left(\Lambda_{ODD}(z) - \Omega_{A1}(z) - \Omega_{C3}(z), 2\Theta\right)\right\} \\ &\geq \min\left\{E'\left(\Phi_{EVEN}(z,z,z,z), 3\left(\frac{\lambda^4 - \lambda^2}{19 + 12\lambda^2}\right)\left(|4-\theta| + |16-\theta|\right)\Theta\right)\right. \\ &\qquad E'\left(\Phi_{EVEN}(-z,-z,-z,-z), 3\left(\frac{\lambda^4 - \lambda^2}{19 + 12\lambda^2}\right)\left(|4-\theta| + |16-\theta|\right)\Theta\right) \\ &\qquad E'\left(\Phi_{ODD}(z,z,z,z), 3\left(\frac{\lambda^4 - \lambda^2}{16 - 3\lambda^2}\right)\left(|2-\theta| + |8-\theta|\right)\Theta\right) \\ &\qquad \left. E'\left(\Phi_{ODD}(-z,-z,-z,-z), 3\left(\frac{\lambda^4 - \lambda^2}{16 - 3\lambda^2}\right)\left(|2-\theta| + |8-\theta|\right)\Theta\right)\right\} \\ &\geq E'\Bigg(\Phi_{EVEN}(z,z,z,z) + \Phi_{EVEN}(-z,-z,-z,-z) \\ &\qquad\qquad + \Phi_{ODD}(z,z,z,z) + \Phi_{ODD}(-z,-z,-z,-z), \\ &\qquad 6\Theta\left\{\left(\frac{\lambda^4 - \lambda^2}{19 + 12\lambda^2}\right)\left(|4-\theta| + |16-\theta|\right) + \left(\frac{\lambda^4 - \lambda^2}{16 - 3\lambda^2}\right)\left(|2-\theta| + |8-\theta|\right)\right\}\Bigg). \end{aligned} \tag{101}$$

Hence the proof is complete. □

Corollary 4.2. *A function $\Lambda_{aqcq} : \mathcal{S}_1 \to \mathcal{S}_2$ satisfies the functional inequality*

$$E\left(\Lambda_{aqcq}(z_1, z_2, z_3, z_4), \Theta\right) \geq \begin{cases} E'(\Delta, \Theta), \\ E'\left(\Delta \sum_{\tau=1}^{4} |z_\tau|^{\Psi}, \Theta\right), \\ E'\left(\Delta \prod_{\tau=1}^{4} |z_\tau|^{\Psi}, \Theta\right), \\ E'\left(\Delta \left\{\sum_{\tau=1}^{4} |z_\tau|^{4\Psi} + \prod_{\tau=1}^{4} |z_\tau|^{\Psi}\right\}, \Theta\right), \end{cases} \tag{102}$$

for all $z_1, z_2, z_3, z_4 \in \mathcal{S}_1$ and all $\Theta > 0$ with Δ a positive constant. Then there exists a unique additive mapping $\Omega_{A1}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ a unique quadratic mapping $\Omega_{Q2}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ a unique cubic mapping $\Omega_{C3}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ a unique quartic mapping $\Omega_{Q4}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ such that

$$\begin{aligned} & E\left(\Lambda(z) - \Omega_{A1}(z) - \Omega_{Q2}(z) - \Omega_{C3}(z) - \Omega_{Q4}(z), 4\Theta\right) \\ & \geq \begin{cases} E'\left(\Delta, 3\Theta\left\{\left(\dfrac{\lambda^4 - \lambda^2}{19 + 12\lambda^2}\right)(|3| + |15|) + \left(\dfrac{\lambda^4 - \lambda^2}{16 - 3\lambda^2}\right)(|1| + |7|)\right\}\right); \\ E'\Big(\Delta\Big\{30 + 18\, 2^{\Psi} + 2\, \lambda^{\Psi} + 2 \cdot 3^{\Psi} + 2 \cdot (1 + \lambda)^{\Psi} + 2 \cdot (1 - \lambda)^{\Psi} \\ \qquad + 2 \cdot (1 + 2\lambda)^{\Psi} + 2 \cdot (1 - 2\lambda)^{\Psi}\Big\} \left|\frac{z}{2}\right|^{\Psi}, \\ \quad 3\Theta\Big\{\left(\dfrac{\lambda^4 - \lambda^2}{19 + 12\lambda^2}\right)\left\{\left|4 - 2^{\Psi}\right| + \left|16 - 2^{\Psi}\right|\right\} \\ \qquad + \left(\dfrac{\lambda^4 - \lambda^2}{16 - 3\lambda^2}\right)\left\{\left|2 - 2^{\Psi}\right| + \left|8 - 2^{\Psi}\right|\right\}\Big\}\Big); \Psi \neq 1, 2, 3, 4 \\ E'\Big(\Delta\Big\{3 + 3\, 2^{4\Psi} + \lambda^{4\Psi} + 3\, 2^{2\Psi} + 3^{2\Psi} + (1 + \lambda)^{2\Psi} + (1 - \lambda)^{2\Psi} \\ \qquad + (1 + 2\lambda)^{2\Psi} + (1 - 2\lambda)^{2\Psi}\Big\} \left|\dfrac{z}{2}\right|^{4\Psi}, \\ \quad 3\Theta\Big\{\left(\dfrac{\lambda^4 - \lambda^2}{19 + 12\lambda^2}\right)\left\{\left|4 - 2^{\Psi}\right| + \left|16 - 2^{\Psi}\right|\right\} \\ \qquad + \left(\dfrac{\lambda^4 - \lambda^2}{16 - 3\lambda^2}\right)\left\{\left|2 - 2^{\Psi}\right| + \left|8 - 2^{\Psi}\right|\right\}\Big\}\Big); \Psi \neq \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, 1 \\ E'\Big(\Delta\Big\{33 + 21\, 2^{4\Psi} + 3\lambda^{4\Psi} + 2 \cdot 3^{4\Psi} + 2 \cdot (1 + \lambda)^{4\Psi} + 2 \cdot (1 - \lambda)^{4\Psi} \\ \quad + 2(1 + 2\lambda)^{4\Psi} + 2(1 - 2\lambda)^{4\Psi} + 3\, 2^{2\Psi} + 3^{2\Psi} \\ \qquad + (1 + \lambda)^{2\Psi} + (1 - \lambda)^{2\Psi} + (1 + 2\lambda)^{2\Psi} + (1 - 2\lambda)^{2\Psi}\Big\} \left|\dfrac{z}{2}\right|^{4\Psi}, \\ \quad 3\Theta\Big\{\left(\dfrac{\lambda^4 - \lambda^2}{19 + 12\lambda^2}\right)\left\{\left|4 - 2^{\Psi}\right| + \left|16 - 2^{\Psi}\right|\right\} \\ \qquad + \left(\dfrac{\lambda^4 - \lambda^2}{16 - 3\lambda^2}\right)\left\{\left|2 - 2^{\Psi}\right| + \left|8 - 2^{\Psi}\right|\right\}\Big\}\Big); \Psi \neq \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, 1 \end{cases} \end{aligned} \tag{103}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.

5. Fuzzy Stability Fixed Point Results: Λ is Odd

The generalized Ulam-Hyers stability of 4 D AQCQ functional equation (3) when Λ is odd is given in this section.

5.1. Additive Case Fuzzy Stability Result

Theorem 5.1. *An odd function $\Lambda_{aqcq} : \mathcal{S}_1 \to \mathcal{S}_2$ satisfies the functional inequality (49) where $\Phi : \mathcal{S}_1^4 \to \mathcal{S}_3$ be a mapping satisfying*

$$\lim_{\varpi \to \infty} E'\left(\Phi\left(\rho_\delta^{\varpi} z_1, \rho_\delta^{\varpi} z_2, \rho_\delta^{\varpi} z_3, \rho_\delta^{\varpi} z_4\right), \rho_\delta^{\varpi}\Theta\right) = 1; \qquad \rho_\delta = \begin{cases} 2 & \text{if } \delta = 0; \\ \frac{1}{2} & \text{if } \delta = 1; \end{cases} \tag{104}$$

for all $z_1, z_2, z_3, z_4 \in \mathcal{S}_1$ and all $\Theta > 0$. If there exists $L = L(\delta)$, such that the function has the property

$$E'\left(\Phi_{ODD}(z,z,z,z),\Theta\right) = E'\left(\Phi_{ODD}\left(\frac{z}{2},\frac{z}{2},\frac{z}{2},\frac{z}{2}\right),\Theta\right) \tag{105}$$

$$E'\left(L\,\frac{1}{\Delta_\delta}\Phi_{ODD}\left(\rho_\delta z,\rho_\delta z,\rho_\delta z,\rho_\delta z\right),\Theta\right) = E'\left(\Phi_{ODD}(z,z,z,z),\Theta\right), \tag{106}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. Then there exists a unique additive mapping $\Omega_{A1}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ such that

$$E\left(\Lambda(2z) - 8\Lambda(z) - \Omega_{A1}(z),\Theta\right) \geq E'\left(\left(\frac{L^{1-\delta}}{1-L}\right)\Phi_{ODD}(z,z,z,z),\left(\frac{\lambda^4-\lambda^2}{16-3\lambda^2}\right)\Theta\right), \tag{107}$$

where $E'\left(\Phi_{ODD}(z,z,z,z),\Theta\right)$ is given in (53) for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.

Proof. Consider a set

$$\Xi = \left\{\Lambda_1 \middle| \Lambda_1 : \mathcal{S}_1 \to \mathcal{S}_2, \Lambda_1(0) = 0\right\}. \tag{108}$$

Define a general metric D on Ξ by

$$D(\Lambda_1,\Lambda_2) = \inf\left\{K \in (0,\infty) \middle| E\left(\Lambda_1(z) - \Lambda_2(z),\Theta\right) \geq E'\left(\Phi_{ODD}(z,z,z,z),K\Theta\right)\right\}, \tag{109}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. It is easy to see that (Ξ, D) is complete.
Define $\Gamma : \Xi \to \Xi$ by

$$\Gamma\Lambda_1(z) = \frac{1}{\rho_\delta}\Lambda_1(\rho_\delta z), \tag{110}$$

for all $z \in \mathcal{S}_1$. For $\Lambda_1, \Lambda_2 \in \Xi$, we see

$$\begin{aligned}
&D(\Lambda_1,\Lambda_2) \leq K \\
&\Rightarrow E\left(\Lambda_1(z) - \Lambda_2(z),\Theta\right) \geq E'\left(\Phi_{ODD}(z,z,z,z),K\Theta\right) \\
&\Rightarrow E\left(\frac{1}{\rho_\delta}\Lambda_1(\rho_\delta z) - \frac{1}{\rho_\delta}\Lambda_2(\rho_\delta z),\Theta\right) \geq E'\left(\Phi_{ODD}\left(\rho_\delta z,\rho_\delta z,\rho_\delta z,\rho_\delta z\right),K\rho_\delta\Theta\right) \\
&\Rightarrow E\left(\Gamma\Lambda_1(z) - \Gamma\Lambda_1(z),\Theta\right) \geq E'\left(\Phi_{ODD}(z,z,z,z),KL\Theta\right) \\
&\Rightarrow D(\Gamma\Lambda_1,\Gamma\Lambda_2) \leq KL \\
&\Rightarrow D(\Gamma\Lambda_1,\Gamma\Lambda_2) \leq LD(\Lambda_1,\Lambda_2),
\end{aligned} \tag{111}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. Thus, Γ is strictly contractive mapping on Ξ with Lipschitz constant L. With the help of (106) when $\delta = 0$, it follows from (74), we reach

$$\begin{aligned}
&E\left(\frac{1}{2}\Lambda_{A1}(2z) - \Lambda_{A1}(z),\left(\frac{16-3\lambda^2}{\lambda^4-\lambda^2}\right)L\Theta\right) \geq E'\left(\Phi_{ODD}(z,z,z,z),\Theta\right) \\
&\qquad\qquad \Rightarrow D(\Gamma\Lambda_{A1},\Lambda_{A1}) \leq L = L^{1-\delta},
\end{aligned} \tag{112}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. Changing z by $\frac{z}{2}$ in (73) and with the help of (105), (106) when $\delta = 1$, it follows from , we land

$$\begin{aligned}
&E\left(\Lambda_{A1}(z) - 2\Lambda_{A1}\left(\frac{z}{2}\right),\left(\frac{16-3\lambda^2}{\lambda^4-\lambda^2}\right)\Theta\right) \geq E'\left(\Phi_{ODD}\left(\frac{z}{2},\frac{z}{2},\frac{z}{2},\frac{z}{2}\right),\Theta\right) \\
&\qquad\qquad \Rightarrow D(\Lambda_{A1},\Gamma\Lambda_{A1}) \leq 1 = L^{1-\delta},
\end{aligned} \tag{113}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. We conclude from (112) and (113)

$$D\Big(\Lambda_{A1}, \Gamma\Lambda_{A1}\Big) \leq L^{1-\delta}. \tag{114}$$

So, (FP1) of Theorem 1.6 holds. So by Theorem 1.6, we arrive at our desired result. Hence the proof is complete. □

Corollary 5.2. *An odd function $\Lambda_{aqcq} : \mathcal{S}_1 \to \mathcal{S}_2$ satisfies the functional inequality (75) for all $z_1, z_2, z_3, z_4 \in \mathcal{S}_1$ and all $\Theta > 0$ with Δ a positive constant. Then there exists a unique additive mapping $\Omega_{A1}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ such that (76) for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.*

Proof. Taking

$$E'\left(\Phi\left(z_1, z_2, z_3, z_4\right), \Theta\right) = \begin{cases} E'\left(\Delta, \Theta\right), \\ E'\left(\Delta \sum\limits_{\tau=1}^{4} |z_\tau|^{\Psi}, \Theta\right), \\ E'\left(\Delta \prod\limits_{\tau=1}^{4} |z_\tau|^{\Psi}, \Theta\right), \\ E'\left(\Delta \left\{\sum\limits_{\tau=1}^{4} |z_\tau|^{4\Psi} + \prod\limits_{\tau=1}^{4} |z_\tau|^{\Psi}\right\}, \Theta\right), \end{cases}$$

for all $z_1, z_2, z_3, z_4 \in \mathcal{S}_1$ and all $\Theta > 0$. If we change

$$(z_1, z_2, z_3, z_4) = \left(\Delta_\delta^{\varpi} z_1, \Delta_\delta^{\varpi} z_2, \Delta_\delta^{\varpi} z_3, \Delta_\delta^{\varpi} z_4\right),$$

in the above equation, using $(FBS3)$ and letting ϖ tends to infinity, we see (104) holds. By (105) and (53), we have

$$\begin{aligned} &E'\left(\Phi_{ODD}(z, z, z, z), \Theta\right) \\ &= E'\left(\Phi_{ODD}\left(\frac{z}{2}, \frac{z}{2}, \frac{z}{2}, \frac{z}{2}\right), \Theta\right) \\ &= \begin{cases} E'\left(\Delta, \Theta\right) \\ E'\Big(\dfrac{\Delta}{2^{\Psi}}\Big\{12 + 5\, 2^{\Psi} + 3^{\Psi} + (1+\lambda)^{\Psi} + (1-\lambda)^{\Psi} \\ \qquad\qquad + (1+2\lambda)^{\Psi} + (1-2\lambda)^{\Psi}\Big\} \left|\frac{z}{2}\right|^{\Psi}, 11\Theta\Big), \\ E'\Big(\dfrac{\Delta}{2^{4\Psi}}\Big\{2 + 3\, 2^{2\Psi} + 2^{4\Psi} + 3^{2\Psi} + (1+\lambda)^{2\Psi} + (1-\lambda)^{2\Psi} \\ \qquad\qquad + (1+2\lambda)^{2\Psi} + (1-2\lambda)^{2\Psi}\Big\} \left|\frac{z}{2}\right|^{4\Psi}, 11\Theta\Big), \\ E'\Big(\dfrac{\Delta}{2^{4\Psi}}\Big\{26 + 11\, 2^{4\Psi} + 2\, 3^{4\Psi} + 2(1+\lambda)^{4\Psi} + 2(1-\lambda)^{4\Psi} + 2\cdot(1+2\lambda)^{4\Psi} \\ \qquad + 2\cdot(1-2\lambda)^{4\Psi} + 3\cdot 2^{2\Psi} + 3^{2\Psi} + (1+\lambda)^{2\Psi} + (1-\lambda)^{2\Psi} \\ \qquad\qquad + (1+2\lambda)^{2\Psi} + (1-2\lambda)^{2\Psi}\Big\}, 11\Theta\Big), \end{cases} \end{aligned} \tag{115}$$

and

$$E'\left(\frac{1}{\rho_\delta}\Phi_{ODD}\left(\rho_\delta z, \rho_\delta z, \rho_\delta z, \rho_\delta z\right), \Theta\right) = \begin{cases} E'\left(\rho_\delta^{-1}\Phi_{ODD}(z, z, z, z), \Theta\right), \\ E'\left(\rho_\delta^{\Psi-1}\Phi_{ODD}(z, z, z, z), \Theta\right), \\ E'\left(\rho_\delta^{4\Psi-1}\Phi_{ODD}(z, z, z, z), \Theta\right), \\ E'\left(\rho_\delta^{4\Psi-1}\Phi_{ODD}(z, z, z, z), \Theta\right), \end{cases} \tag{116}$$

for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. Now from (107), we have the following cases.

$$L = 2^{-1} \quad \text{if} \quad \delta = 0,$$

$$\begin{aligned}
E\left(\Lambda(2z) - 8\Lambda(z) - \Omega_{A1}(z), \Theta\right) &\geq E'\left(\left(\frac{(2^{-1})^{1-0}}{1-2^{-1}}\right)\Phi_{ODD}(z,z,z,z), \left(\frac{\lambda^4-\lambda^2}{16-3\lambda^2}\right)\Theta\right)\\
&= E'\left(\Delta, \left(\frac{\lambda^4-\lambda^2}{16-3\lambda^2}\right)(1)\Theta\right).
\end{aligned}$$

$$L = \frac{1}{2^{-1}} = 2 \quad \text{if} \quad \delta = 1,$$

$$\begin{aligned}
E\left(\Lambda(2z) - 8\Lambda(z) - \Omega_{A1}(z), \Theta\right) &\geq E'\left(\left(\frac{(2)^{1-1}}{1-2}\right)\Phi_{ODD}(z,z,z,z), \left(\frac{\lambda^4-\lambda^2}{16-3\lambda^2}\right)\Theta\right)\\
&= E'\left(\Delta, \left(\frac{\lambda^4-\lambda^2}{16-3\lambda^2}\right)(-1)\Theta\right).
\end{aligned}$$

$$L = 2^{\Psi-1} \quad \text{if} \quad \delta = 0,$$

$$\begin{aligned}
&E\left(\Lambda(2z) - 8\Lambda(z) - \Omega_{A1}(z), \Theta\right)\\
&\geq E'\left(\left(\frac{(2^{\Psi-1})^{1-0}}{1-2^{\Psi-1}}\right)\Phi_{ODD}(z,z,z,z), \left(\frac{\lambda^4-\lambda^2}{16-3\lambda^2}\right)\Theta\right)\\
&= E'\bigg(\frac{\Delta}{2^\Psi}\left(\frac{(2^\Psi)}{2-2^\Psi}\right)\Big\{12 + 5\,2^\Psi + 3^\Psi + (1+\lambda)^\Psi + (1-\lambda)^\Psi\\
&\qquad + (1+2\lambda)^\Psi + (1-2\lambda)^\Psi\Big\}\left|\frac{z}{2}\right|^\Psi, \left(\frac{\lambda^4-\lambda^2}{16-3\lambda^2}\right)11\Theta\bigg)\\
&= E'\bigg(\Delta\Big\{12 + 5\cdot 2^\Psi + 3^\Psi + (1+\lambda)^\Psi + (1-\lambda)^\Psi\\
&\qquad + (1+2\lambda)^\Psi + (1-2\lambda)^\Psi\Big\}\left|\frac{z}{2}\right|^\Psi, 11\left(\frac{\lambda^4-\lambda^2}{16-3\lambda^2}\right)\Theta\left(2-2^\Psi\right)\bigg).
\end{aligned}$$

$$L = \frac{1}{2^{\Psi-1}} = 2^{1-\Psi} \quad \text{if} \quad \delta = 1,$$

$$\begin{aligned}
&E\left(\Lambda(2z) - 8\Lambda(z) - \Omega_{A1}(z), \Theta\right)\\
&\geq E'\left(\left(\frac{(2^{1-\Psi})^{1-1}}{1-2^{1-\Psi}}\right)\Phi_{ODD}(z,z,z,z), \left(\frac{\lambda^4-\lambda^2}{16-3\lambda^2}\right)\Theta\right)\\
&= E'\bigg(\frac{\Delta}{2^\Psi}\left(\frac{2^\Psi}{2^\Psi-2}\right)\Big\{12 + 5\,2^\Psi + 3^\Psi + (1+\lambda)^\Psi + (1-\lambda)^\Psi\\
&\qquad + (1+2\lambda)^\Psi + (1-2\lambda)^\Psi\Big\}\left|\frac{z}{2}\right|^\Psi, \left(\frac{\lambda^4-\lambda^2}{16-3\lambda^2}\right)11\Theta\bigg)\\
&= E'\bigg(\Delta\Big\{12 + 5\,2^\Psi + 3^\Psi + (1+\lambda)^\Psi + (1-\lambda)^\Psi\\
&\qquad + (1+2\lambda)^\Psi + (1-2\lambda)^\Psi\Big\}\left|\frac{z}{2}\right|^\Psi, 11\left(\frac{\lambda^4-\lambda^2}{16-3\lambda^2}\right)\Theta\left(2^\Psi-2\right)\bigg).
\end{aligned}$$

Similarly one can prove the other cases. Hence the proof is completed. □

5.2. Cubic Case Fuzzy Stability Result

Theorem 5.3. *An odd function $\Lambda_{aqcq} : \mathcal{S}_1 \to \mathcal{S}_2$ satisfies the functional inequality (49) where $\Phi : \mathcal{S}_1^4 \to \mathcal{S}_3$ be a mapping satisfying (104) for all $z_1, z_2, z_3, z_4 \in \mathcal{S}_1$ and all $\Theta > 0$. If there exists $L = L(\delta)$, such that the function has the property (105), (106) for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. Then there exists a unique cubic mapping $\Omega_{C3}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ such that*

$$E\left(\Lambda(2z) - 2\Lambda(z) - \Omega_{C3}(z), \Theta\right) \geq E'\left(\left(\frac{L^{1-\delta}}{1-L}\right)\Phi_{ODD}(z,z,z,z), \left(\frac{\lambda^4 - \lambda^2}{16 - 3\lambda^2}\right)|8-\theta|\Theta\right), \tag{117}$$

where $E'(\Phi_{ODD}(z,z,z,z), \Theta)$ is given in (53) for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.

Proof. The proof follows by similar lines of Theorem 5.1, by define $\Gamma : \Xi \to \Xi$ by

$$\Gamma\Lambda_1(z) = \frac{1}{\rho_\delta^3}\Lambda_1(\rho_\delta z), \tag{118}$$

for all $z \in \mathcal{S}_1$. □

Corollary 5.4. *An odd function $\Lambda_{aqcq} : \mathcal{S}_1 \to \mathcal{S}_2$ satisfies the functional inequality (75) for all $z_1, z_2, z_3, z_4 \in \mathcal{S}_1$ and all $\Theta > 0$ with Δ a positive constant. Then there exists a unique cubic mapping $\Omega_{C3}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ such that (83) for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.*

5.3. Additive - Cubic Case Fuzzy Stability Result

Theorem 5.5. *An odd function $\Lambda_{aqcq} : \mathcal{S}_1 \to \mathcal{S}_2$ satisfies the functional inequality (49) where $\Phi : \mathcal{S}_1^4 \to \mathcal{S}_3$ be a mapping satisfying (104) for all $z_1, z_2, z_3, z_4 \in \mathcal{S}_1$ and all $\Theta > 0$. If there exists $L = L(\delta)$, such that the function has the property (105), (106) for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. Then there exists a unique additive mapping $\Omega_{A1}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ and a unique cubic mapping $\Omega_{C3}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ such that*

$$\begin{aligned} &E\left(\Lambda(z) - \Omega_{A1}(z) - \Omega_{C3}(z), \Theta\right) \\ &\qquad \geq E'\left(\left(\frac{L^{1-\delta}}{1-L}\right)\Phi_{ODD}(z,z,z,z), 3\left(\frac{\lambda^4 - \lambda^2}{16 - 3\lambda^2}\right)\Theta\right), \end{aligned} \tag{119}$$

where $E'(\Phi_{ODD}(z,z,z,z), \Theta)$ is given in (53) for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.

Proof. The proof follows by Theorem 3.5. □

Corollary 5.6. *An odd function $\Lambda_{aqcq} : \mathcal{S}_1 \to \mathcal{S}_2$ satisfies the functional inequality (75) for all $z_1, z_2, z_3, z_4 \in \mathcal{S}_1$ and all $\Theta > 0$ with Δ a positive constant. Then there exists a unique additive mapping $\Omega_{A1}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ and a unique cubic mapping $\Omega_{C3}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ such that (89) for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.*

6. Fuzzy Stability Fixed Point Results: Λ is Even

The generalized Ulam-Hyers stability of 4 D AQCQ functional equation (3) when Λ is even is given in this section.

6.1. Quadratic Case Fuzzy Stability Result

Theorem 6.1. *An even function $\Lambda_{aqcq} : \mathcal{S}_1 \to \mathcal{S}_2$ satisfies the functional inequality (4) where $\Phi : \mathcal{S}_1^4 \to \mathcal{S}_3$ be a mapping satisfying (104) for all $z_1, z_2, z_3, z_4 \in \mathcal{S}_1$ and all $\Theta > 0$. If there exists $L = L(\delta)$, such that the function has the property (105), (106) for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. Then there exists a unique quadratic mapping $\Omega_{Q2}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ such that*

$$E\left(\Lambda(2z) - 16\Lambda(z) - \Omega_{Q2}(z), \Theta\right) \geq E'\left(\left(\frac{L^{1-\delta}}{1-L}\right)\Phi_{EVEN}\Big(z,z,z,z\Big), \left(\frac{\lambda^4 - \lambda^2}{19 + 12\lambda^2}\right)\Theta\right) \tag{120}$$

. where $E'\left(\Phi_{EVEN}\Big(z,z,z,z\Big), \Theta\right)$ is given in (8) for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.

Proof. The proof follows by similar lines of Theorem 5.1, by define $\Gamma : \Xi \to \Xi$ by

$$\Gamma\Lambda_1(z) = \frac{1}{\rho_\delta^2}\Lambda_1(\rho_\delta z), \tag{121}$$

for all $z \in \mathcal{S}_1$. □

Corollary 6.2. *An even function $\Lambda_{aqcq} : \mathcal{S}_1 \to \mathcal{S}_2$ satisfies the functional inequality (34) for all $z_1, z_2, z_3, z_4 \in \mathcal{S}_1$ and all $\Theta > 0$ with Δ a positive constant. Then there exists a unique quadratic mapping $\Omega_{Q2}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ such that (35) for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.*

6.2. Quartic Case Fuzzy Stability Result

Theorem 6.3. *An even function $\Lambda_{aqcq} : \mathcal{S}_1 \to \mathcal{S}_2$ satisfies the functional inequality (4) where $\Phi : \mathcal{S}_1^4 \to \mathcal{S}_3$ be a mapping satisfying (104) for all $z_1, z_2, z_3, z_4 \in \mathcal{S}_1$ and all $\Theta > 0$. If there exists $L = L(\delta)$, such that the function has the property (105), (106) for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. Then there exists a unique quartic mapping $\Omega_{Q4}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ such that*

$$\begin{aligned} &E\left(\Lambda(2z) - 4\Lambda(z) - \Omega_{Q4}(z), \Theta\right) \\ &\qquad \geq E'\left(\left(\frac{L^{1-\delta}}{1-L}\right)\Phi_{EVEN}(z,z,z,z), \left(\frac{\lambda^4 - \lambda^2}{19 + 12\lambda^2}\right)|16 - \theta|\Theta\right), \end{aligned} \tag{122}$$

where $E'(\Phi_{EVEN}(z,z,z,z), \Theta)$ is given in (8) for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.

Proof. The proof follows by similar lines of Theorem 5.1, by define $\Gamma : \Xi \to \Xi$ by

$$\Gamma\Lambda_1(z) = \frac{1}{\rho_\delta^4}\Lambda_1(\rho_\delta z), \tag{123}$$

for all $z \in \mathcal{S}_1$. □

Corollary 6.4. *An even function $\Lambda_{aqcq} : \mathcal{S}_1 \to \mathcal{S}_2$ satisfies the functional inequality (34) for all $z_1, z_2, z_3, z_4 \in \mathcal{S}_1$ and all $\Theta > 0$ with Δ a positive constant. Then there exists a unique quartic mapping $\Omega_{Q4}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ such that (42) for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.*

6.3. Quadratic - Quartic Case Fuzzy Stability Result

Theorem 6.5. *An even function $\Lambda_{aqcq} : \mathcal{S}_1 \to \mathcal{S}_2$ satisfies the functional inequality (4) where $\Phi : \mathcal{S}_1^4 \to \mathcal{S}_3$ be a mapping satisfying (104) for all $z_1, z_2, z_3, z_4 \in \mathcal{S}_1$ and all $\Theta > 0$. If there exists $L = L(\delta)$, such that the function has the property (105), (106) for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. Then there exists a unique quadratic mapping $\Omega_{Q2}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ and a unique quartic mapping $\Omega_{Q4}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ such that*

$$\begin{aligned} &E\left(\Lambda(z) - \Omega_{Q2}(z) - \Omega_{Q4}(z), \Theta\right) \\ &\qquad\qquad \geq E'\left(\left(\frac{L^{1-\delta}}{1-L}\right)\Phi_{EVEN}(z,z,z,z), 3\left(\frac{\lambda^4 - \lambda^2}{19 + 12\lambda^2}\right)\Theta\right), \end{aligned} \tag{124}$$

where $E'(\Phi_{EVEN}(z,z,z,z), \Theta)$ is given in (8) for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.

Proof. The proof follows by Theorem 2.5. □

Corollary 6.6. *An even function $\Lambda_{aqcq} : \mathcal{S}_1 \to \mathcal{S}_2$ satisfies the functional inequality (34) for all $z_1, z_2, z_3, z_4 \in \mathcal{S}_1$ and all $\Theta > 0$ with Δ a positive constant. Then there exists a unique quadratic mapping $\Omega_{Q2}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ and a unique quartic mapping $\Omega_{Q4}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ such that (48) for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.*

7. Fuzzy Stability Fixed Point Results: Λ is Odd and Even

The generalized Ulam-Hyers stability of 4 D AQCQ functional equation (3) when Λ is odd and Even is given in this section.

Theorem 7.1. *A function $\Lambda_{aqcq} : \mathcal{S}_1 \to \mathcal{S}_2$ satisfies the functional inequality*

$$E\left(\Lambda_{aqcq}(z_1, z_2, z_3, z_4), \Theta\right) \geq E'\left(\Phi(z_1, z_2, z_3, z_4), \Theta\right), \tag{125}$$

where $\Phi : \mathcal{S}_1^4 \to \mathcal{S}_3$ be a mapping satisfying (104) for all $z_1, z_2, z_3, z_4 \in \mathcal{S}_1$ and all $\Theta > 0$. If there exists $L = L(\delta)$, such that the function has the property (105), (106) for all $z \in \mathcal{S}_1$ and all $\Theta > 0$. Then there exists a unique additive mapping $\Omega_{A1}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ a unique quadratic mapping $\Omega_{Q2}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ a unique cubic mapping $\Omega_{C3}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ a unique quartic mapping $\Omega_{Q4}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ such that

$$\begin{aligned} &E\left(\Lambda(z) - \Omega_{A1}(z) - \Omega_{Q2}(z) - \Omega_{C3}(z) - \Omega_{Q4}(z), 4\Theta\right) \\ &\geq E'\left(\left(\frac{L^{1-\delta}}{1-L}\right)\Big\{\Phi_{EVEN}(z,z,z,z) + \Phi_{EVEN}(-z,-z,-z,-z) \right. \\ &\qquad\qquad\qquad + \Phi_{ODD}(z,z,z,z) + \Phi_{ODD}(-z,-z,-z,-z)\Big\}, \\ &\qquad\qquad\qquad \left. 6\Theta\left\{\left(\frac{\lambda^4-\lambda^2}{19+12\lambda^2}\right) + \left(\frac{\lambda^4-\lambda^2}{16-3\lambda^2}\right)\right\}\right), \end{aligned} \tag{126}$$

where $E'(\Phi_{EVEN}(z,z,z,z), \Theta)$, $E'(\Phi_{ODD}(z,z,z,z), \Theta)$ are given in (8), (53), respectively for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.

Proof. The proof is similar lines to that of Theorem 4.1. □

Corollary 7.2. *A function $\Lambda_{aqcq} : \mathcal{S}_1 \to \mathcal{S}_2$ satisfies the functional inequality (102) for all $z_1, z_2, z_3, z_4 \in \mathcal{S}_1$ and all $\Theta > 0$ with Δ a positive constant. Then there exists a unique additive mapping $\Omega_{A1}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ a unique quadratic mapping $\Omega_{Q2}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ a unique cubic mapping $\Omega_{C3}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ a unique quartic mapping $\Omega_{Q4}(z) : \mathcal{S}_1 \to \mathcal{S}_2$ such that (103) for all $z \in \mathcal{S}_1$ and all $\Theta > 0$.*

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On behalf of all authors, the corresponding author states that there is no conflict of interest.

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Not applicable, as the study did not involve human or animal subjects and no sensitive data were used.

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My manuscript has no associated data.

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