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Generalized Lupaş-Durrmeyer type operators involving Pólya-Eggenberger distribution

Sahil Berwal[a], Khursheed J. Ansari[b,*], Arun Kajla[a]

[a]*Department of Mathematics, Central University of Haryana, Haryana-123029, India*
[b]*Department of Mathematics, College of Science, King Khalid University, Abha, 61413, Saudi Arabia*

Abstract. Approximation by positive linear operators is a mathematical concept that deals with approximating functions using a class of operators that are linear and preserve positivity. These operators are typically defined on function spaces and are commonly used in approximation theory and numerical analysis. Taking this concept further, in this article we introduce a modification to Lupaş type operators, referred to as Durrmeyer type operators, which are constructed based on the Pólya-Eggenberger distribution. In the second section, we establish essential auxiliary results pertinent to these newly devised operators. Our subsequent analysis is twofold: firstly, we investigate a Voronovskaja-type asymptotic formula, and secondly, we deduce estimates for the rate of approximation, incorporating both the modulus of smoothness and the Ditzian-Totik modulus of smoothness. Moreover, we determine the rate at which convergence occurs for differential functions characterized by derivatives of bounded variation. Finally, we employ Maple software to visually demonstrate the operators' convergence towards a specific function.

1. Introduction

The theory of approximation by positive linear operators provides a mathematical foundation for understanding the convergence properties and error analysis of approximation methods. It has applications in various fields, including numerical analysis, signal processing, image reconstruction, and probability theory. A positive linear operator is an operator that maps functions to functions while preserving positivity. That is, if T is a positive linear operator defined on a function space, then for any non-negative function f, the image $T(f)$ is also non-negative. The advantage of using positive linear operators for approximation is that they ensure the approximations remain non-negative, which is often desirable in applications where the functions being approximated have a physical or probabilistic interpretation. One common example of positive linear operators used for approximation is the family of Bernstein operators. The Bernstein

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* Corresponding author: Khursheed J. Ansari
Email addresses: sahil211939@cuh.ac.in (Sahil Berwal), ansari.jkhursheed@gmail.com (Khursheed J. Ansari), rachitkajla47@gmail.com (Arun Kajla)
ORCID iDs: https://orcid.org/0000-0003-0275-1505 (Sahil Berwal), https://orcid.org/0000-0003-4564-6211 (Khursheed J. Ansari), https://orcid.org/0000-0003-4273-4830 (Arun Kajla)

operators [10], which are extensively studied attached to $\mathcal{G} \in C[0,1]$ (the space of all continuous functions on $[0,1]$), are given by

$$B_n(\mathcal{G};\ell) = \sum_{\imath=0}^{n} b_{n,\imath}(\ell)\mathcal{G}\left(\frac{\imath}{n}\right) \quad (\ell \in [0,1], n \in \mathbb{N}), \tag{1}$$

where $b_{n,\imath}(\ell) = 0$ if $\imath < 0$ or $\imath > n$ and

$$b_{n,\imath}(\ell) = \left(\begin{array}{c} n \\ \imath \end{array}\right)\ell^{\imath}(1-\ell)^{n-\imath}.$$

Durrmeyer [15] derived the integral modification of the Bernstein operators (1) as follows:

$$M_n(\mathcal{G};\ell) = (n+1)\sum_{\imath=0}^{n} b_{n,\imath}(\ell)\int_0^1 b_{n,\imath}(t)\mathcal{G}(t)\mathrm{d}t.$$

Numerous recent developments involve broadening and adapting these types of operators [8, 9, 28, 29]. Pólya and Eggenberger [16] created the initial Pólya-Eggenberger urn model in 1923. The Pólya-Eggenberger urn model consists b black balls and w white balls. A ball is randomly selected, then it is replaced with other s balls of the same colour. This process is done m times while observing how the random variable Y, which represents how frequently a white ball is drawn, is distributed. The formula for Y distribution is

$$Pr(Y=\jmath) = \binom{m}{\jmath}\frac{b(b+s)\cdot\ldots\cdot(b+\overline{m-\jmath-1}s)w(w+s)\cdot\ldots\cdot(w+\overline{\jmath-1}s)}{(b+w)(b+w+s)\cdot\ldots\cdot(b+w+\overline{m-1}s)}, \tag{2}$$

for $\jmath = 0,1,2,\cdots,m-1,m$ and $\overline{\jmath-1}s = (\jmath-1)s$. The distribution (2) is referred to as a Pólya-Eggenberger distribution with parameters (s,b,w,m) and it includes binomial and hypergeometric distributions as specific cases.

In 1968 a new set of positive linear operators connected to a real-valued function $\mathcal{G}:[0,1]\to\mathbb{R}$ was presented by Stancu [36]. Using the Pólya-Eggenberger distribution (2) provided by

$$P_n^{[\alpha]}(\mathcal{G};\ell) = \sum_{\jmath=0}^{n} p_{n,\jmath}^{[\alpha]}(\ell)\mathcal{G}\left(\frac{\jmath}{n}\right) = \sum_{\jmath=0}^{n}\binom{n}{\jmath}\frac{\prod_{\nu=0}^{n-\jmath-1}(1-\ell+\nu\alpha)\prod_{\mu=0}^{\jmath-1}(\ell+\mu\alpha)}{(1+\alpha)(1+2\alpha)\cdots(1+(n-1)\alpha)}\mathcal{G}\left(\frac{\jmath}{n}\right). \tag{3}$$

The fundamental Stancu polynomials $p_{n,\jmath}^{[\alpha]}$ are present and $\alpha \geq 0$ is a parameter that may depend on the natural number n. For the case of $\alpha = 0$, operators (3) become the original Bernstein operators (1) [10] and for $\alpha = \frac{m}{n}, m > 0, n \in \mathbb{N}$, we have a particular case

$$P_n^{\left[\frac{m}{n}\right]}(\mathcal{G};\ell) = \sum_{\jmath=0}^{n} q_{n,\jmath}^{\left[\frac{m}{n}\right]}(\ell)\mathcal{G}\left(\frac{\jmath}{n}\right), \tag{4}$$

where $q_{n,\jmath}^{\left[\frac{m}{n}\right]}(\ell) = \frac{1}{(n)_{n,m}}\left(\begin{array}{c} n \\ \jmath \end{array}\right)(n\ell)_{\jmath,m}(n-n\ell)_{n-\jmath,m}$ and $(x)_{n,m} = x(x+m)(x+2m)...(x+(n-1)m)$ introduced by Yilmaz et al. [37].

The space of bounded Lebesgue integral functions on $[0,1]$ is denoted by $L_B[0,1]$, and the space of polynomials with degrees up to $n \in \mathbb{N}$ is denoted by Π_n. In 2007, the following class of operators $U_{n,\varrho}: L_B[0,1] \to \Pi_n$ was introduced by Păltănea [34] as:

$$\begin{aligned} U_{n,\varrho}(\mathcal{G};\ell) &= \sum_{\jmath=0}^{n} p_{n,\jmath}(\ell)F_{n,\jmath}^{\varrho}(\mathcal{G}) \\ &= (1-\ell)^n f(0) + \ell^n f(1) + \sum_{\jmath=1}^{n-1} p_{n,\jmath}(\ell)\left(\int_0^1 \frac{(1-\imath)^{(n-\jmath)\varrho-1}\imath^{\jmath\varrho-1}}{\beta(\jmath\varrho,(n-\jmath)\varrho)}\mathcal{G}(\imath)\mathrm{d}\imath\right), \end{aligned} \tag{5}$$

where $\varrho > 0$, $F_{n,\jmath}^{\varrho}(\mathcal{G}) = \int_0^1 \frac{(1-\imath)^{(n-\jmath)\varrho-1}\imath^{\jmath\varrho-1}}{\beta(\jmath\varrho,(n-\jmath)\varrho)}\mathcal{G}(\imath)\mathrm{d}\imath$, the Bernstein basis polynomial $p_{n,\jmath}(\ell) = \binom{n}{\jmath}\ell^{\jmath}(1-\ell)^{n-\jmath}$, and Euler's Beta function $\beta(\ell,x) = \int_0^1 \imath^{\ell-1}(1-\imath)^{x-1}d\imath$ for $\ell, x > 0$. Gonska and Păltănea [17] have conducted further research on the operators mentioned in (5). They have created a recursive formula for computing moments and estimating derivatives simultaneously. The author demonstrated that the operators $U_{n,\varrho}g$ can link the commonly used Bernstein operators with their real Bernstein-Durrmeyer counterparts.

Now, we are presenting here a new type sequence of operators called Lupaş-Durrmeyer operators $U_{n,\varrho}^{[\frac{m}{n}]} : L_B[0,1] \to \Pi_n$, which is defined as follows:

$$\begin{aligned} U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G};\ell) &= \sum_{\jmath=0}^{n} q_{n,\jmath}^{[\frac{m}{n}]}(\ell)\int_0^1 z_{n,\jmath}^{[\varrho]}\mathcal{G}(\imath)\mathrm{d}\imath \\ &= q_{n,0}^{[\frac{m}{n}]}(\ell)\mathcal{G}(0) + q_{n,n}^{[\frac{m}{n}]}(\ell)\mathcal{G}(1) + \sum_{\jmath=1}^{n-1} q_{n,\jmath}^{[\frac{m}{n}]}(\ell)\left(\int_0^1 z_{n,\jmath}^{[\varrho]}\mathcal{G}(\imath)\mathrm{d}\imath\right), \end{aligned} \tag{6}$$

where $z_{n,\jmath}^{[\varrho]}(\ell) = \frac{\imath^{\jmath\varrho-1}(1-\imath)^{(n-\jmath)\varrho-1}}{\beta(\jmath\varrho,(n-\jmath)\varrho)}$.

In this study, a new Durrmeyer-type modification of Lupaş type operators based on the Pólya-Eggenberger distribution is presented. For the case $m = 0$ and $\varrho \to \infty$ for each $\mathcal{G} \in C[0,1]$, the operator (6) converges uniformly to the Bernstein polynomials (1). Several essential auxiliary outcomes are established in the second part for these new operators. The asymptotic behavior and uniform convergence of these operators are the main topics of our current research. We will create some quantitative theorems in order to determine the degree of approximation.

Despite the fact that these operators (3) were introduced a while ago, there is still a lot of interest in studying them and they have been subject to generalizations up to the present day. Some examples of these kinds of generalizations and modifications in operators and their associated approximation properties can be seen in the papers of Agrawal et al. [5, 6] , Kajla et al. [21, 23], Gupta et al. [18, 19], Neer et al. [31] and Deo et al [12]. In the literature survey, the authors also recommend papers [7, 20, 22, 24–27, 30, 32, 33, 35].

2. Auxiliary Results

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{N}$ be the collection of positive numbers. The test functions, also known as the monomials $e_\jmath(\ell) = \ell^\jmath$ for $\jmath \in \mathbb{N}_0$, are important in linear positive operator uniform approximation. We offer a suitable form of these operators for determining the images of the monomials by the Lupaş-Durrmeyer type operators (6).

Lemma 2.1. *For the Lupaş-Durrmeyer type operators hold;*

$$U_{n,\varrho}^{[\frac{m}{n}]}(e_0;\ell) = 1; \qquad U_{n,\varrho}^{[\frac{m}{n}]}(e_1;\ell) = \ell; \qquad U_{n,\varrho}^{[\frac{m}{n}]}(e_2;\ell) = \frac{(-1+n)n\ell^2\varrho}{(m+n)(1+n\varrho)} + \frac{\ell(m+n+n\varrho+mn\varrho)}{(m+n)(1+n\varrho)};$$

$$\begin{aligned} U_{n,\varrho}^{[\frac{m}{n}]}(e_3;\ell) &= \frac{(-2+n)(-1+n)n^2\ell^3\varrho^2}{(m+n)(2m+n)(1+n\varrho)(2+n\varrho)} + \frac{\ell^2\big(3(-1+n)n^2\varrho+3(-1+n)n^2\varrho^2+3m(-1+n)n\varrho(2+n\varrho)\big)}{(m+n)(2m+n)(1+n\varrho)(2+n\varrho)} \\ &+ \frac{\ell\big(2n^2+3n^2\varrho+n^2\varrho^2+3mn(2+n\varrho)+3mn\varrho(2+n\varrho)+2m^2(1+n\varrho)(2+n\varrho)\big)}{(m+n)(2m+n)(1+n\varrho)(2+n\varrho)}; \end{aligned}$$

$$\begin{aligned} U_{n,\varrho}^{[\frac{m}{n}]}(e_4;\ell) &= \frac{(-3+n)(-2+n)(-1+n)n^3\ell^4\varrho^3}{(m+n)(2m+n)(3m+n)(1+n\varrho)(2+n\varrho)(3+n\varrho)} + \frac{\ell^3\big(6(-2+n)(-1+n)n^2(3m+n)\varrho^2+6(1+m)(-2+n)(-1+n)n^3\varrho^3\big)}{(m+n)(2m+n)(3m+n)(1+n\varrho)(2+n\varrho)(3+n\varrho)} \\ &+ \frac{\ell^2\big(11(-1+n)n(2m+n)(3m+n)\varrho+18(-1+n)n^2(3m+n)\varrho^2+18m(-1+n)n^2(3m+n)\varrho^2+(1+m)(-1+n)n^2(7n+m(-1+11n))\varrho^3\big)}{(m+n)(2m+n)(3m+n)(1+n\varrho)(2+n\varrho)(3+n\varrho)} \\ &+ \frac{\ell\big(11mn(2m+n)(3m+n)\varrho+6n^2(3m+n)\varrho^2+18mn^2(3m+n)\varrho^2+12m^2n^2(3m+n)\varrho^2+(1+m)n^2(n+m(-1+6(1+m)n))\varrho^3\big)}{(m+n)(2m+n)(3m+n)(1+n\varrho)(2+n\varrho)(3+n\varrho)} + \frac{\ell(6(m+n)(2m+n)(3m+n)+11n(2m+n)(3m+n)\varrho)}{(m+n)(2m+n)(3m+n)(1+n\varrho)(2+n\varrho)(3+n\varrho)}. \end{aligned}$$

For the purpose of conciseness, we will write $M_{n,\varrho,r}^{[\frac{m}{n}]}(\ell) := U_{n,\varrho}^{[\frac{m}{n}]}((e_1-\ell)^r;\ell)$, where $n \in \mathbb{N}$, $r \in \mathbb{N}_0$, in order to calculate the central moments of the operators (6).

Lemma 2.2. *For the Lupaş-Durrmeyer type operators hold;*

$M^{[\frac{m}{n}]}_{n,\varrho,1}(\ell) = 0; \qquad M^{[\frac{m}{n}]}_{n,\varrho,2}(\ell) = \frac{\ell(1-\ell)(m+n+(1+m)n\varrho)}{(m+n)(1+n\varrho)};$

$$M^{[\frac{m}{n}]}_{n,\varrho,4}(\ell) = \frac{n^3\ell^4\left(-\frac{18m^3(1+n\varrho)(2+n\varrho)(3+n\varrho)}{n^3}-18(1+\varrho)+3(-6+n)\varrho(1+\varrho)+3(-2+n)\varrho^2(1+\varrho)+\frac{3m^2(2+n\varrho)(3+n\varrho)(-11+(-12+n)\varrho)}{n^2}+\frac{6m(3+n\varrho)(1+\varrho)(-6+(-4+n)\varrho)}{n}\right)}{(m+n)(2m+n)(3m+n)(1+n\varrho)(2+n\varrho)(3+n\varrho)}$$
$$+\frac{n^3\ell^3(18+18\varrho-3(-6+n)\varrho-3(-6+n)\varrho^2-3(-2+n)\varrho^2-3(-2+n)\varrho^3}{((m+n)(2m+n)(3m+n)(1+n\varrho)(2+n\varrho)(3+n\varrho))}$$
$$+\frac{n^3\ell^3(18(1+\varrho)-3(-6+n)\varrho(1+\varrho)-3(-2+n)\varrho^2(1+\varrho)-\frac{12m(1+\varrho)(-6+(-4+n)\varrho)(3+n\varrho)}{n}-\frac{6m^2(-11+(-12+n)\varrho)(2+n\varrho)(3+n\varrho)}{n^2}+\frac{36m^3(1+n\varrho)(2+n\varrho)(3+n\varrho)}{n^3})}{((m+n)(2m+n)(3m+n)(1+n\varrho)(2+n\varrho)(3+n\varrho))}$$
$$+\frac{n^3\ell^2(-18-18\varrho+3(-6+n)\varrho+3(-6+n)\varrho^2+3(-2+n)\varrho^2+3(-2+n)\varrho^3-6(1+\varrho)-5\varrho(1+\varrho)-\varrho^2(1+\varrho))}{((m+n)(2m+n)(3m+n)(1+n\varrho)(2+n\varrho)(3+n\varrho))}$$
$$+\frac{n^3\ell^2\left(-\frac{24m^3(1+n\varrho)(2+n\varrho)(3+n\varrho)}{n^3}+\frac{m(1+\varrho)(-144+\varrho(-91+\varrho+n(-29-31\varrho+6n\varrho)))}{n}+\frac{m^2(-264+\varrho(-282+n(-202+\varrho(-234+\varrho+3n(-10+(-16+n)\varrho)))))}{n^2}\right)}{((m+n)(2m+n)(3m+n)(1+n\varrho)(2+n\varrho)(3+n\varrho))}$$
$$+\frac{n^3\ell\left(6+11\varrho+6\varrho^2+\varrho^3+\frac{36m(1+\varrho)}{n}+\frac{6m^3(1+n\varrho)(2+n\varrho)(3+n\varrho)}{n^3}+\frac{m\varrho(1+\varrho)(19+11n-\varrho+7n\varrho)}{n}+\frac{m^2(1+\varrho)(66+n\varrho(55-\varrho+12n\varrho))}{n^2}\right)}{(m+n)(2m+n)(3m+n)(1+n\varrho)(2+n\varrho)(3+n\varrho)}.$$

Lemma 2.3. *For $\varrho > 0$ and $\ell \in (0,1)$, we get*

$$U^{[\frac{m}{n}]}_{n,\varrho}(\mathcal{G};\ell) = \frac{1}{\beta\left(\frac{\ell n}{m}, \frac{(1-\ell)n}{m}\right)}\int_0^1 \imath^{\frac{\ell n}{m}-1}(1-\imath)^{\frac{(1-\ell)n}{m}-1}U_{n,\varrho}(\mathcal{G};\imath)\mathrm{d}\imath,$$

where $U_{n,\varrho}(\mathcal{G},\imath)$ is defined in equation (5).

Proof. Applying the relationship between gamma and beta functions

$$\beta(\ell, x) = \frac{\Gamma(\ell)\Gamma(x)}{\Gamma(\ell+x)},$$

where $\Gamma(s)$ is Gamma function defined by

$$\Gamma(s) = \int_0^\infty u^{s-1}e^{-u}\mathrm{d}u, \quad s > 0,$$

with $\Gamma(s+n) = s(s+1)\cdot\ldots\cdot(s+n-1)\Gamma(s)$, for $n \in \mathbb{N}$, then we get

$$\beta\left(\frac{\ell n}{m}+\jmath, \frac{(1-\ell)n}{m}+n-\jmath\right) = \frac{\Gamma\left(\frac{\ell n}{m}+\jmath\right)\Gamma\left(\frac{(1-\ell)n}{m}+n-\jmath\right)}{\Gamma\left(\frac{n}{m}+n\right)} = q^{[\frac{m}{n}]}_{n,\jmath}(\ell)\binom{n}{\jmath}^{-1}\beta\left(\frac{\ell n}{m}, \frac{(1-\ell)n}{m}\right).$$

Hence

$$q^{[\frac{m}{n}]}_{n,\jmath}(\ell) = \binom{n}{\jmath}\left(\beta\left(\frac{\ell n}{m}, \frac{(1-\ell)n}{m}\right)\right)^{-1}\beta\left(\frac{\ell n}{m}+\jmath, \frac{(1-\ell)n}{m}+n-\jmath\right),$$

and it follows

$$
\begin{aligned}
U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G};\ell) &= \sum_{\jmath=1}^{n-1}\binom{n}{\jmath}\frac{\beta\left(\frac{\ell n}{m}+\jmath,\frac{(1-\ell)n}{m}+n-\jmath\right)}{\beta\left(\frac{\ell n}{m},\frac{(1-\ell)n}{m}\right)}\frac{1}{\beta(\jmath\varrho,(n-\jmath)\varrho)}\int_0^1 s^{\jmath\varrho-1}(1-s)^{(n-\jmath)\varrho-1}\mathcal{G}(s)\mathrm{d}s \\
&\quad+\frac{\beta\left(\frac{\ell n}{m},\frac{(1-\ell)n}{m}+n\right)}{\beta\left(\frac{\ell n}{m},\frac{(1-\ell)n}{m}\right)}\mathcal{G}(0)+\frac{\beta\left(\frac{\ell n}{m}+n,\frac{(1-\ell)n}{m}\right)}{\beta\left(\frac{\ell n}{m},\frac{(1-\ell)n}{m}\right)}\mathcal{G}(1) \\
&= \frac{1}{\beta\left(\frac{\ell n}{m},\frac{(1-\ell)n}{m}\right)}\left(\sum_{\jmath=1}^{n-1}\binom{n}{\jmath}\int_0^1 \imath^{\frac{\ell n}{m}+\jmath-1}(1-\imath)^{\frac{(1-\ell)n}{m}+n-\jmath-1}\mathrm{d}\imath\right. \\
&\quad\times\frac{1}{\beta(\jmath\varrho,(n-\jmath)\varrho)}\int_0^1 s^{\jmath\varrho-1}(1-s)^{(n-\jmath)\varrho-1}\mathcal{G}(s)\mathrm{d}s+\mathcal{G}(0)\int_0^1 \imath^{\frac{\ell n}{m}-1}(1-\imath)^{\frac{(1-\ell)n}{m}+n-1}\mathrm{d}\imath \\
&\quad\left.+\mathcal{G}(1)\int_0^1 \imath^{\frac{\ell n}{m}+n-1}(1-\imath)^{\frac{(1-\ell)n}{m}-1}\mathrm{d}\imath\right) \\
&= \frac{1}{\beta\left(\frac{\ell n}{m},\frac{(1-\ell)n}{m}\right)}\int_0^1 \imath^{\frac{\ell n}{m}-1}(1-\imath)^{\frac{(1-\ell)n}{m}-1}U_{n,\varrho}(\mathcal{G};\imath)\mathrm{d}\imath.
\end{aligned}
$$

□

We provide four results involving Lupaş-Durrmeyer type operators (6) below without providing any justification because all that is required to obtain them is mechanical work. The following provides the images of the test functions created by operators (6) for $\jmath \in \mathbb{N}_0$ and $e_\jmath(\ell) = \ell^\jmath$.

Lemma 2.4. *For any natural number n, we can express*

$$M_{n,\varrho,2}^{[\frac{m}{n}]}(\ell) = U_{n,\varrho}^{[\frac{m}{n}]}((e_1-\ell)^2;\ell) \le \frac{\mathcal{D}_\varrho^{[\frac{m}{n}]}\ \ell(1-\ell)}{1+n\varrho},$$

the equation involves a positive constant $\mathcal{D}_\varrho^{[\frac{m}{n}]}$*, which depends on both* ϱ *and m and can be taken as* $\mathcal{D}_\varrho^{[\frac{m}{n}]} = 1+\varrho+m\varrho$.

Lemma 2.5. *If* $\frac{m}{n} \to 0$ *as* $n \to \infty$*, and* m, ϱ *being constant, then*

$$\lim_{n\to\infty} nM_{n,\varrho,1}^{[\frac{m}{n}]}(\ell) = 0,$$

$$\lim_{n\to\infty} nM_{n,\varrho,2}^{[\frac{m}{n}]}(\ell) = \frac{(1+\varrho+m\varrho)\ell(1-\ell)}{\varrho},$$

$$\lim_{n\to\infty} n^2M_{n,\varrho,4}^{[\frac{m}{n}]}(\ell) = \frac{3\ell^2(1+\varrho)(1+\varrho+2m\varrho)}{\varrho^2} - \frac{6\ell^3(1+\varrho+m\varrho)^2}{\varrho^2} + \frac{\ell^4(3+3\varrho(2+\varrho+2m(1+\varrho+m\varrho)))}{\varrho^2}.$$

3. Theorems and local approximation

Our ongoing research focuses on the qualitative aspects of operators of the Lupaş-Durrmeyer type, including uniform convergence and asymptotic behavior.

Theorem 3.1. *Suppose* $\mathcal{G} \in C[0,1]$ *and* $\frac{m}{n} \ge 0$ *is a parameter that depends on* $n \in \mathbb{N}$*. If* $\frac{m}{n} \to 0$ *as n approaches infinity, and* m, ϱ *being constant, then* $\lim_{n\to\infty} U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G};\ell)$ *converges uniformly to* $\mathcal{G}(\ell)$ *over the interval* $[0,1]$.

Proof. Since $U_{n,\varrho}^{[\frac{m}{n}]}(1;\ell) = 1$, $U_{n,\varrho}^{[\frac{m}{n}]}(e_1;\ell) = \ell$ and $U_{n,\varrho}^{[\frac{m}{n}]}(e_2;\ell) = \frac{(-1+n)n\ell^2\varrho}{(m+n)(1+n\varrho)} + \frac{\ell(m+n+n\varrho+mn\varrho)}{(m+n)(1+n\varrho)}$, it follows

$$\lim_{n\to\infty} U_{n,\varrho}^{[\frac{m}{n}]}(e_\imath;\ell) = e_\imath(\ell), \text{ for } \imath = 0,1,2.$$

By utilizing the established Korovkin's theorem, we can derive the following

$$\lim_{n\to\infty} U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G};\ell) = \mathcal{G}(\ell) \text{ uniformly on } [0,1].$$

□

The following result provides a Voronovskaja-type result for the operators of the Lupaş-Durrmeyer type.

Theorem 3.2. *Let $\mathcal{G} : [0,1] \to \mathbb{R}$, $\frac{m}{n} \to 0$ as $n \to \infty$, and m, ϱ being constant. If $\mathcal{G} \in C^2[0,1]$, then*

$$\lim_{n\to\infty} n\left(U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G};\ell) - \mathcal{G}(\ell)\right) = \frac{(1+\varrho+m\varrho)\ell(1-\ell)}{2\varrho}\mathcal{G}''(\ell).$$

Proof. It is as follows using Taylor's expansion formula for the function $\mathcal{G}$

$$\mathcal{G}(\imath) = \mathcal{G}(\ell) + \mathcal{G}'(\ell)(\imath-\ell) + \frac{1}{2}\mathcal{G}''(\ell)(\imath-\ell)^2 + \varkappa(\imath,\ell)(\imath-\ell)^2, \tag{7}$$

where $\lim_{\imath\to\ell}\varkappa(\imath,\ell) = 0$ and the function $\varkappa(\imath,\ell) := \varkappa(\imath-\ell)$ is bounded. Given that Lupaş-Durrmeyer type operators are linear, and after applying the operators $U_{n,\varrho}^{[\frac{m}{n}]}$ to both sides of the previous equation (7), we obtain

$$U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G};\ell) - \mathcal{G}(\ell) = U_{n,\varrho}^{[\frac{m}{n}]}((e_1-\ell);\ell)\mathcal{G}'(\ell) + \frac{1}{2}U_{n,\varrho}^{[\frac{m}{n}]}\left((e_1-\ell)^2;\ell\right)\mathcal{G}''(\ell) + U_{n,\varrho}^{[\frac{m}{n}]}\left(\varkappa(\imath,\ell)\cdot(e_1-\ell)^2;\ell\right).$$

By using Lemma 2.2, the result as follows

$$\lim_{n\to\infty} n\left(U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G};\ell) - \mathcal{G}(\ell)\right) = \frac{\ell(1-\ell)(1+\varrho+m\varrho)}{2\varrho}\mathcal{G}''(\ell) + \lim_{n\to\infty} n\left(U_{n,\varrho}^{[\frac{m}{n}]}\left(\varkappa(\imath,\ell)\cdot(e_1-\ell)^2;\ell\right)\right). \tag{8}$$

Using the Cauchy-Schwarz's inequality, we estimate the final component on the right-hand side of the previous equality, resulting in

$$nU_{n,\varrho}^{[\frac{m}{n}]}\left(\varkappa(\imath,\ell)\cdot(e_1-\ell)^2;\ell\right) \le \sqrt{U_{n,\varrho}^{[\frac{m}{n}]}(\varkappa^2(\imath,\ell);\ell)}\sqrt{n^2U_{n,\varrho}^{[\frac{m}{n}]}((e_1-\ell)^4;\ell)}. \tag{9}$$

Because $\varkappa^2(\ell,\ell) = 0$ and $\varkappa^2(\cdot,\ell) \in C[0,1]$, utilizing the convergence established in Theorem 3.1, the result is as follows

$$\lim_{n\to\infty} U_{n,\varrho}^{[\frac{m}{n}]}\left(\varkappa^2(\imath,\ell);\ell\right) = \varkappa^2(\ell,\ell) = 0. \tag{10}$$

Conclusion reached by using Lemma 2.5 in combination with equations (9) and (10)

$$\lim_{n\to\infty} n\left(U_{n,\varrho}^{[\frac{m}{n}]}\left(\varkappa(\imath,\ell)\cdot(e_1-\ell)^2;\ell\right)\right) = 0,$$

and by applying (8), we discover how the Lupaş-Durrmeyer type operators (6) behave asymptotically. □

Moduli of smoothness [14] are the basic parameters used to assess the degree of linear positive operators approximation to the identity operator. The first order and second order smoothness moduli of $\mathcal{G} \in C[0,1]$ and $\lambda \ge 0$ are defined as follows:

$$\omega_1(\mathcal{G},\lambda) := \sup\{|\mathcal{G}(\ell+h) - \mathcal{G}(\ell)| : \ell, \ell+h \in [0,1],\ 0 \le h \le \lambda\},$$

respectively,

$$\omega_2(\mathcal{G},\lambda) := \sup\{|\mathcal{G}(\ell+h) - 2\mathcal{G}(\ell) + \mathcal{G}(\ell-h)| \,:\, \ell, \ell \pm h \in [0,1],\ 0 \le h \le \lambda\}.$$

Also, let us define Peetre's K-functional [13]

$$K_2(\mathcal{G},\lambda) = \inf\{\|\mathcal{G} - g\| + \lambda\|g''\| \,:\, g \in C^2[0,1]\},\ \text{for } \lambda > 0. \tag{11}$$

There exists a constant $M > 0$, such that

$$K_2(\mathcal{G},\lambda) \le M\omega_2\left(\mathcal{G}, \sqrt{\lambda}\right). \tag{12}$$

Proposition 3.3. *If $\mathcal{G}$ is a continuous real-valued function that is bounded to the domain $[0,1]$, with $\|\mathcal{G}\| = \max_{\ell\in[0,1]} |\mathcal{G}(\ell)|$, then*

$$\left|U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G};\ell)\right| \le \|\mathcal{G}\|.$$

Proof. According to Lemma 2.1 and the definition of Lupaş-Durrmeyer type operators, it follows

$$\left|U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G};\ell)\right| = \left|\sum_{\jmath=0}^{n} p_{n,\jmath}^{[\frac{m}{n}]}(\ell)F_{n,\jmath}^{\varrho}(\mathcal{G})\right| \le \sum_{\jmath=0}^{n} p_{n,\jmath}^{[\frac{m}{n}]}(\ell)F_{n,\jmath}^{\varrho}(|\mathcal{G}|) \le \|\mathcal{G}\| U_{n,\varrho}^{[\frac{m}{n}]}(e_0;\ell) = \|\mathcal{G}\|.$$

□

The following discussion provides direct calculations utilizing Peetre's K-functional and moduli of smoothness.

Theorem 3.4. *Let $\mathcal{G}$ be a differentiable function on the interval $[0,1]$, and its derivative $\mathcal{G}' \in C_B[0,1]$. Then, for any value of $\ell \in [0,1]$, the following statement holds*

$$\left|U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G};\ell) - \mathcal{G}(\ell)\right| \le \frac{3\lambda}{4}\omega_1(\mathcal{G}',\lambda),\ \ \textit{with}\ \ \lambda = \sqrt{\frac{(1-\ell)\ell(m+n+(1+m)n\varrho)}{(m+n)(1+n\varrho)}}.$$

Proof. Using with the identity

$$\mathcal{G}(\imath) - \mathcal{G}(\ell) = \mathcal{G}'(\ell)(\imath-\ell) + \mathcal{G}(\imath) - \mathcal{G}(\ell) - \mathcal{G}'(\ell)(\imath-\ell),$$

we get for c between $\imath$ and ℓ

$$\mathcal{G}(\imath) - \mathcal{G}(\ell) - \mathcal{G}'(\ell)(\imath-\ell) = \mathcal{G}'(c) - \mathcal{G}'(\ell)(\imath-\ell),$$

using the Lagrange mean value theorem (there exists a c between $\imath$ and ℓ, such that $\mathcal{G}(\imath) - \mathcal{G}(\ell) = \mathcal{G}'(c)(\imath-\ell)$. Because $|c-\ell| \le |\imath-\ell|$, it follows

$$\mathcal{G}'(c) - \mathcal{G}'(\ell) \le \omega_1(\mathcal{G}',(\imath-\ell)) \le \left(1+\lambda^{-1}(\imath-\ell)^2\right)\omega_1(\mathcal{G}',\lambda)$$

and

$$\mathcal{G}(\imath) - \mathcal{G}(\ell) - \mathcal{G}'(\ell)(\imath-\ell) \le \left(\imath-\ell+\lambda^{-1}(\imath-\ell)^2\right)\omega_1(\mathcal{G}',\lambda).$$

Applying the linear positive Lupaş-Durrmeyer type operators to the inequality

$$\mathcal{G}(\imath) - \mathcal{G}(\ell) \le \mathcal{G}'(\ell)(\imath-\ell) + \left((\imath-\ell)+\lambda^{-1}(\imath-\ell)^2\right)\omega_1(\mathcal{G}',\lambda),$$

following from the above relationship, we obtain

$$\left|U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G};\ell) - \mathcal{G}(\ell)\right| \le |\mathcal{G}'(\ell)|\, U_{n,\varrho}^{[\frac{m}{n}]}(|e_1-\ell|;\ell) + \left(U_{n,\varrho}^{[\frac{m}{n}]}(|e_1-\ell|;\ell) + \frac{1}{\lambda}U_{n,\varrho}^{[\frac{m}{n}]}\left((e_1-\ell)^2;\ell\right)\right)\omega_1(\mathcal{G}',\lambda).$$

Using Cauchy-Schwarz for linear positive operators, we have

$$U_{n,\varrho}^{[\frac{m}{n}]}(|e_1-\ell|;\ell) \leq \sqrt{\left(U_{n,\varrho}^{[\frac{m}{n}]}(e_0;\ell)\right)}\sqrt{\left(U_{n,\varrho}^{[\frac{m}{n}]}((e_1-\ell)^2;\ell)\right)},$$

and the outcomes attained in Lemma 2.1 and Lemma 2.2 lead to

$$\begin{aligned}\left|U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G};\ell)-\mathcal{G}(\ell)\right| &\leq \left(\left(U_{n,\varrho}^{[\frac{m}{n}]}(e_0;\ell)\right)^{\frac{1}{2}}\left(U_{n,\varrho}^{[\frac{m}{n}]}\left((e_1-\ell)^2;\ell\right)\right)^{\frac{1}{2}}+\frac{1}{\lambda}U_{n,\varrho}^{[\frac{m}{n}]}\left((e_1-\ell)^2;\ell\right)\right)\omega_1(\mathcal{G}',\lambda)\\ &\leq \left(U_{n,\varrho}^{[\frac{m}{n}]}\left((e_1-\ell)^2;\ell\right)\right)^{\frac{1}{2}}\left(1+\frac{1}{\lambda}\left(U_{n,\varrho}^{[\frac{m}{n}]}\left((e_1-\ell)^2;\ell\right)\right)^{\frac{1}{2}}\right)\omega_1(\mathcal{G}',\lambda).\end{aligned}$$

Because

$$\left(M_{n,\varrho,2}^{[\frac{m}{n}]}(\ell)\right)^{\frac{1}{2}}=\left(U_{n,\varrho}^{[\frac{m}{n}]}\left((e_1-\ell)^2;\ell\right)\right)^{\frac{1}{2}}=\sqrt{\frac{(1-\ell)\ell(m+n+(1+m)n\varrho)}{(m+n)(1+\mathrm{n}\varrho)}}\leq\frac{1}{2}\sqrt{\frac{(m+n+(1+m)n\varrho)}{(m+n)(1+\mathrm{n}\varrho)}},$$

and using $\lambda=\sqrt{\frac{(m+n+(1+m)n\varrho)}{(m+n)(1+\mathrm{n}\varrho)}}$, we get

$$\left|U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G};\ell)-\mathcal{G}(\ell)\right|\leq\frac{3\lambda}{4}\omega_1(\mathcal{G}',\lambda).$$

□

Estimates combining the first and second-order smoothness moduli are more accurate than those applying just the first modulus of continuity.

Theorem 3.5. *If any function $\mathcal{G}\in C_B[0,1]$, then for any value ℓ within the interval $[0,1]$ and $\lambda>0$, it follows*

$$\left|U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G};\ell)-\mathcal{G}(\ell)\right|\leq\frac{3}{2}\omega_1\left(\mathcal{G},\left(\frac{m+n+n\varrho+mn\varrho}{m+n+mn\varrho+nn\varrho}\right)^{\frac{1}{2}}\right).$$

Proof. The well-known property of the first-order smoothness property (first modulus of continuity) is

$$|\mathcal{G}(\imath)-\mathcal{G}(\ell)|\leq\omega_1(\mathcal{G},|\imath-\ell|)\leq\left(1+\lambda^{-1}|\imath-\ell|\right)\omega_1(\mathcal{G},\lambda).$$

By using the previous inequality with the linear positive Lupaş-Durrmeyer type operators, it is as follows

$$\left|U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G};\ell)-\mathcal{G}(\ell)\right|\leq\left(U_{n,\varrho}^{[\frac{m}{n}]}(e_0;\ell)+\frac{1}{\lambda}U_{n,\varrho}^{[\frac{m}{n}]}(|e_1-\ell|;\ell)\right)\omega_1(\mathcal{G},\lambda).$$

For positive linear operators, the Cauchy- Schwarz inequality results in

$$U_{n,\varrho}^{[\frac{m}{n}]}(|e_1-\ell|;\ell)\leq\left(U_{n,\varrho}^{[\frac{m}{n}]}(e_0;\ell)\right)^{\frac{1}{2}}\cdot\left(U_{n,\varrho}^{[\frac{m}{n}]}\left((e_1-\ell)^2;\ell\right)\right)^{\frac{1}{2}}.$$

Understanding that operators of the Lupaş-Durrmeyer type retain constants and are conformable to the conclusions of Lemma 2.2

$$M_{n,\varrho,2}^{[\frac{m}{n}]}(\ell)=U_{n,\varrho}^{[\frac{m}{n}]}\left((e_1-\ell)^2;\ell\right)=\frac{(1-\ell)\ell(m+n+(1+m)n\varrho)}{(m+n)(1+\mathrm{n}\varrho)},$$

we get

$$\left|U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G};\ell)-\mathcal{G}(\ell)\right|\leq\left(1+\lambda^{-1}\sqrt{\frac{(1-\ell)\ell(m+n+(1+m)n\varrho)}{(m+n)(1+\mathrm{n}\varrho)}}\right)\omega_1(\mathcal{G},\lambda).$$

Using the inequality $\sqrt{\ell(1-\ell)}\leq\frac{1}{2}$ into account and choosing $\lambda=\sqrt{\frac{(m+n+(1+m)n\varrho)}{(m+n)(1+\mathrm{n}\varrho)}}$, we get the desired result. □

Theorem 3.6. *Let $\mathcal{G} \in C[0,1]$, then for any $\ell \in [0,1]$ yields*

$$\left|U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G};\ell) - \mathcal{G}(\ell)\right| \leq M\omega_2\left(\mathcal{G}, \tfrac{1}{2}\lambda\right), \text{ with } \lambda = \frac{1}{2}\sqrt{\frac{1+\varrho+m\varrho}{1+n\varrho}},$$

where M is an absolute constant.

Proof. Using Taylor's expansion formula to obtain the following expression for any function $f \in C^2[0,1]$ and for any values of $\imath, \ell \in [0,1]$, we have

$$f(\imath) = f(\ell) + (\imath-\ell)f'(\ell) + \int_\ell^\imath (\imath-u)f''(u)\mathrm{d}u.$$

By using the Lupaş-Durmeyer type operators $U_{n,\varrho}^{[\frac{m}{n}]}$ on both sides of the equation mentioned earlier, we get

$$\begin{aligned} U_{n,\varrho}^{[\frac{m}{n}]}(f;\ell) - f(\ell) &= f'(\ell)U_{n,\varrho}^{[\frac{m}{n}]}(e_1-\ell;\ell) + U_{n,\varrho}^{[\frac{m}{n}]}\left(\int_\ell^\imath (\imath-u)f''(u)\mathrm{d}u;\ell\right) \\ &= U_{n,\varrho}^{[\frac{m}{n}]}\left(\int_\ell^\imath (\imath-u)f''(u)\mathrm{d}u;\ell\right), \end{aligned}$$

applying the results of Lemma 2.2, the result is as follows

$$\left|\int_\ell^\imath (\imath-u)f''(u)\mathrm{d}u\right| \leq (\imath-\ell)^2\|f''\|.$$

Further, keeping in mind the inequality generated at Lemma 2.5, we reach at the following inequality

$$\left|U_{n,\varrho}^{[\frac{m}{n}]}(f;\ell) - f(\ell)\right| \leq \|f''\|U_{n,\varrho}^{[\frac{m}{n}]}\left((e_1-\ell)^2;\ell\right) \leq \tfrac{(m+n+(1+m)n\varrho)}{4(m+n)(1+n\varrho)}\|f''\| \leq \tfrac{\mathcal{D}_\varrho^{[\frac{m}{n}]}}{4(1+n\varrho)}\|f''\| = \lambda^2\|f''\|.$$

For any function $\mathcal{G} \in C[0,1]$ and $f \in C^2[0,1]$, by applying the Proposition 3.3, it follows

$$\begin{aligned} \left|U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G};\ell) - \mathcal{G}(\ell)\right| &\leq \left|U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G}-f;\ell)\right| + \left|U_{n,\varrho}^{[\frac{m}{n}]}(f;\ell) - f(\ell)\right| + |\mathcal{G}(\ell) - f(\ell)| \\ &\leq 2\|\mathcal{G}-f\| + \lambda^2\|f''\| = 2\left(\|\mathcal{G}-f\| + \tfrac{\lambda^2}{2}\|f''\|\right). \end{aligned}$$

Using the relation (12) and the infimum on the right-hand side across all of $f \in C^2[0,1]$, we can now obtain

$$\left|U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G};\ell) - \mathcal{G}(\ell)\right| \leq 2K_2\left(\mathcal{G}, \tfrac{1}{2}\lambda^2\right) \leq M\omega_2\left(\mathcal{G}, \tfrac{1}{2}\lambda\right), \text{ with } \lambda = \tfrac{1}{2}\left(\tfrac{\mathcal{D}_\varrho^{[\frac{m}{n}]}}{1+n\varrho}\right)^{\frac{1}{2}},$$

where $\mathcal{D}_\varrho^{[\frac{m}{n}]}$ is taken as $1+\varrho+m\varrho$. □

Theorem 3.7. *Suppose that $\mathcal{G} \in C[0,1]$. Then for any value of $\ell \in [0,1]$ and $\lambda > 0$, the following statement holds*

$$\left|U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G};\ell) - \mathcal{G}(\ell)\right| \leq \frac{9}{8}\omega_2(\mathcal{G},\lambda) \text{ with } \lambda = \sqrt{\frac{(m+n+(1+m)n\varrho)}{(m+n)(1+n\varrho)}}.$$

Proof. Applying Pǎltǎnea result [33] for a linear positive operator $\mathcal{L}$

$$|\mathcal{L}(\mathcal{G};\ell) - \mathcal{G}(\ell)| \leq |\mathcal{L}(e_0;\ell) - 1||\mathcal{G}(\ell)| + \frac{1}{\lambda}|\mathcal{L}(e_1-\ell;\ell)|\omega_1(\mathcal{G},\lambda) + \left(\mathcal{L}(e_0;\ell) + \frac{1}{2\lambda^2}\mathcal{L}\left((e_1-\ell)^2;\ell\right)\right)\omega_2(\mathcal{G},\lambda),$$

we get the estimate for $U_{n,\varrho}^{[\frac{m}{n}]} := \mathcal{L}$

$$\left|U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G};\ell)-\mathcal{G}(\ell)\right| \leq \left|U_{n,\varrho}^{[\frac{m}{n}]}(e_0;\ell)-1\right||\mathcal{G}(\ell)| + \frac{1}{\lambda}\left|U_{n,\varrho}^{[\frac{m}{n}]}(e_1-\ell;\ell)\right|\omega_1(\mathcal{G},\lambda)$$
$$+\left(U_{n,\varrho}^{[\frac{m}{n}]}(e_0;\ell)+\frac{1}{2\lambda^2}U_{n,\varrho}^{[\frac{m}{n}]}\left((e_1-\ell)^2;\ell\right)\right)\omega_2(\mathcal{G},\lambda).$$

Theorem 2.2 and Lemma 2.1 are taken into consideration, and by selecting $\lambda = \sqrt{\frac{(m+n+(1+m)n\varrho)}{(m+n)(1+n\varrho)}}$, we arrive at the desired outcome. □

4. Estimates of the rate of approximation using weighted moduli

In order to prove a global approximation theorem for Lupaş-Durmeyer type operators that takes into account the Ditzian-Totik modulus of smoothness, we find references to certain results from [14]. The Ditzian-Totik smoothness moduli of the first and second orders for any $\mathcal{G} \in C_B[0,1]$ and $\lambda \geq 0$ by

$$\omega_1^{\psi}(\mathcal{G},\lambda) = \sup_{|h|\leq\lambda}\ \sup_{\ell\pm(h/2)\psi(\ell)\in[0,1]}\left|\mathcal{G}\left(\ell+\tfrac{1}{2}h\psi(\ell)\right)-\mathcal{G}\left(\ell-\tfrac{1}{2}h\psi(\ell)\right)\right|,$$

and

$$\omega_2^{\psi}(\mathcal{G},\lambda) = \sup_{|h|\leq\lambda}\ \sup_{\ell\pm h\psi(\ell)\in[0,1]}\left|\mathcal{G}\left(\ell+h\psi(\ell)\right)-2\mathcal{G}(\ell)+\mathcal{G}\left(\ell-h\psi(\ell)\right)\right|, \tag{13}$$

with $\psi(\ell) = \sqrt{\ell(1-\ell)}$, $\ell \in [0,1]$. The second order K-functional can be expressed as follows

$$K_2^{\psi}\left(\mathcal{G},\lambda^2\right) = \inf_{f'\in AC_{\text{loc}}[0,1]}\left(\|\mathcal{G}-f\|+\lambda^2\left\|\psi^2 f''\right\|\right), \tag{14}$$

where $f' \in AC_{\text{loc}}[0,1]$ denotes that f is differentiable and that f' is absolutely continuous on all closed intervals $[a,b] \subset [0,1]$. An inequality between the K-functional (14) and second order modulus of smoothness (13), which is given for a positive constant N, is established in [14] by

$$K_2^{\psi}\left(\mathcal{G},\lambda^2\right) \leq N\omega_2^{\psi}(\mathcal{G},\lambda). \tag{15}$$

Using the information presented, we can provide a proof for the following statement.

Theorem 4.1. *Let $\mathcal{G} \in C[0,1]$, then for any $\ell \in [0,1]$ yields*

$$\left|U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G};\ell)-\mathcal{G}(\ell)\right| \leq N\omega_2^{\psi}(\mathcal{G},\lambda),\ \textit{with}\ \lambda = \left(\frac{\mathcal{D}_{\varrho}^{[\frac{m}{n}]}}{2(1+n\varrho)}\right)^{\frac{1}{2}},$$

where N is an absolute constant.

Proof. Using Taylor's expansion formula, in the proof of Theorem 3.6 we show that for any function $f \in C^2[0,1]$ and for any values $\imath, \ell \in [0,1]$

$$\left|U_{n,\varrho}^{[\frac{m}{n}]}(f;\ell)-f(\ell)\right| \leq U_{n,\varrho}^{[\frac{m}{n}]}\left(\int_{\ell}^{\imath}|\imath-\nu|\cdot|f''(\nu)|\mathrm{d}\nu;\ell\right). \tag{16}$$

Since $\psi^2(\ell)$ is a concave function on $[0,1]$, for $\nu = \lambda\ell+(1-\lambda)\imath$ with $\imath < \nu < \ell$ and $\lambda \in [0,1]$, it follows

$$\frac{|\imath-\nu|}{\psi^2(\nu)} = \frac{|\imath-\lambda t-(1-\lambda)t|}{\psi^2(\lambda\ell+(1-\lambda)t)} \leq \frac{\lambda|\imath-\ell|}{\lambda\psi^2(\ell)+(1-\lambda)\psi^2(\imath)} \leq \frac{|\imath-\ell|}{\psi^2(\ell)}.$$

By applying the inequality derived above to the equation (16) and utilizing Lemma 2.4 , we arrive at the following expression

$$\left|U_{n,\varrho}^{[\frac{m}{n}]}(f;\ell)-f(\ell)\right| \leq U_{n,\varrho}^{[\frac{m}{n}]}\left(\int_{\ell}^{\imath}\frac{|\imath-\nu|}{\psi^2(\nu)}\mathrm{d}\nu;\ell\right)\left\|\psi^2 f''\right\| \leq \frac{1}{\psi^2(\ell)}\left\|\psi^2 f''\right\| U_{n,\varrho}^{[\frac{m}{n}]}\left((e_1-\ell)^2;\ell\right)$$
$$\leq \frac{1}{\psi^2(\ell)}\left\|\psi^2 f''\right\|\frac{\mathcal{D}_{\varrho}^{[\frac{m}{n}]}\ell(1-\ell)}{1+n\varrho} = \left\|\psi^2 f''\right\|\frac{\mathcal{D}_{\varrho}^{[\frac{m}{n}]}}{1+n\varrho}.$$

Using the in equality we derived earlier and Proposition 3.3, we can conclude that for any function $\mathcal{G}\in C[0,1]$ and any function $\mathcal{G}\in AC_{\mathrm{loc}}[0,1]$ the following statement holds

$$\left|U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G};\ell)-\mathcal{G}(\ell)\right| \leq \left|U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G}-f;\ell)\right| + \left|U_{n,\varrho}^{[\frac{m}{n}]}(f;\ell)-f(\ell)\right| + |\mathcal{G}(\ell)-f(\ell)|$$
$$\leq 2\|\mathcal{G}-f\| + \tfrac{\mathcal{D}_{\varrho}^{[\frac{m}{n}]}}{1+n\varrho}\left\|\psi^2 f''\right\| = 2\left(\|\mathcal{G}-f\| + \tfrac{\mathcal{D}_{\varrho}^{[\frac{m}{n}]}}{2(1+n\varrho)}\left\|\psi^2 f''\right\|\right).$$

Take the infimum of the right-hand side of the previous inequality over all functions $\mathcal{G}\in AC_{loc}[0,1]$ and utilize the relation (15), we obtain the following expression

$$\left|U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G};\ell)-\mathcal{G}(\ell)\right| \leq 2K_2\left(\mathcal{G},\lambda^2\right) \leq N\omega_2^{\psi}(\mathcal{G},\lambda), \text{ with } \lambda = \left(\tfrac{\mathcal{D}_{\varrho}^{[\frac{m}{n}]}}{2(1+n\varrho)}\right)^{\frac{1}{2}}.$$

□

5. Rate of convergence

Determine the rate of convergence for differentiable functions whose derivatives have bounded variation on the interval $[0,1]$. The set of differentiable functions $\mathcal{G}$ defined on $[0,1]$, whose derivatives $\mathcal{G}'$ are of bounded variation on $[0,1]$, is denoted as $\mathcal{G}\in DBV[0,1]$ can be represented as

$$\mathcal{G}(\ell) = \int_0^{\ell}\mathcal{G}(\imath)\mathrm{d}\imath + \mathcal{G}(0).$$

The operators $U_{n,\varrho}^{[\frac{m}{a}]}(\mathcal{G})$ can be represented as integrals

$$U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G};\ell) = \int_0^1 \mathcal{K}_{n,\varrho}^{[\frac{m}{n}]}(\ell,\imath)\mathcal{G}(\imath)\mathrm{d}\imath, \tag{17}$$

where the kernel $\mathcal{K}_{n,\varrho}^{[\frac{m}{n}]}$ is given by

$$\mathcal{K}_{n,\varrho}^{[\frac{m}{n}]}(\ell,\imath) = q_{n,0}^{[\frac{m}{n}]}(\ell)\delta(\imath) + q_{n,n}^{[\frac{m}{n}]}(\ell)\delta(1-\imath) + \sum_{\jmath=1}^{n-1} q_{n,\jmath}^{[\frac{m}{n}]}(\ell)\frac{\imath^{\jmath\varrho-1}(1-\imath)^{(n-\jmath)\varrho-1}}{\beta(\jmath\varrho,(n-\jmath)\varrho)},$$

where $\delta(u)$ is the Dirac-delta function.

Lemma 5.1. *Considering a parameter $\frac{m}{n}$ for $m>0$ and $n\in\mathbb{N}$. As $n\to\infty$ then $\frac{m}{n}$ must approaches zero. For a fixed value of $\ell\in(0,1)$, the conclusion holds*

i) $\Theta_{n,\varrho}^{[\frac{m}{n}]}(\ell,y) := \int_0^y \mathcal{K}_{n,\varrho}^{[\frac{m}{n}]}(\ell,\imath)\mathrm{d}\imath \leq \dfrac{\mathcal{D}_{\varrho}^{[\frac{m}{n}]}}{(1+n\varrho)}\dfrac{\ell(1-\ell)}{(\ell-y)^2},\ 0\leq y<\ell;$

ii) $1-\Theta_{n,\varrho}^{[\frac{m}{n}]}(\ell,z) := \int_z^1 \mathcal{K}_{n,\varrho}^{[\frac{m}{n}]}(\ell,\imath)\mathrm{d}\imath \leq \dfrac{\mathcal{D}_{\varrho}^{[\frac{m}{n}]}}{(1+n\varrho)}\dfrac{\ell(1-\ell)}{(z-\ell)^2},\ \ell<z<1.$

Proof. i) Using Lemma 2.4, we get

$$\Theta_{n,\varrho}^{[\frac{m}{n}]}(\ell,y)=\int\limits_{0}^{y}\mathcal{K}_{n,\varrho}^{[\frac{m}{n}]}(\ell,\imath)\mathrm{d}\imath\leq\int_{0}^{y}\left(\frac{\ell-\imath}{\ell-y}\right)^{2}\mathcal{K}_{n,\varrho}^{[\frac{m}{n}]}(\ell,\imath)\mathrm{d}\imath$$

$$=\frac{1}{(\ell-y)^{2}}U_{n,\varrho}^{[\frac{m}{n}]}\left((e_{1}-\ell)^{2};\ell\right)\leq\frac{\mathcal{D}_{\varrho}^{[\frac{m}{n}]}}{(1+n\varrho)}\frac{\ell(1-\ell)}{(\ell-y)^{2}}.$$

ii) The proof doesn't require any further explanation or elaboration, so it has been omitted. □

Theorem 5.2. *Assuming that $\mathcal{G}\in DBV[0,1]$, and that the $\frac{m}{n}\to 0$ as $n\to\infty$. Then, for every $\ell\in(0,1)$, and $n\to\infty$, we have*

$$\left|U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G};\ell)-\mathcal{G}(\ell)\right|\leq\sqrt{\frac{\mathcal{D}_{\varrho}^{[\frac{m}{n}]}\ell(1-\ell)}{(1+n\varrho)}}\frac{|\mathcal{G}'(\ell+)-\mathcal{G}'(\ell-)|}{2}+\frac{\mathcal{D}_{\varrho}^{[\frac{m}{n}]}(1-\ell)}{(1+n\varrho)}\sum_{\jmath=1}^{[\sqrt{n}]}\bigvee_{\ell-(\ell/\jmath)}^{\ell}(\mathcal{G}'_{\ell})$$

$$+\frac{\ell}{\sqrt{n}}\bigvee_{\ell-(\ell/\sqrt{n})}^{\ell}(\mathcal{G}'_{\ell})+\frac{\mathcal{D}_{\varrho}^{[\frac{m}{n}]}\ell}{(1+n\varrho)}\sum_{\jmath=1}^{[\sqrt{n}]}\bigvee_{\ell}^{\ell+((1-\ell)/\jmath)}(\mathcal{G}'_{\ell})+\frac{(1-\ell)}{\sqrt{n}}\bigvee_{\ell}^{\ell+((1-\ell)/\sqrt{n})}(\mathcal{G}'_{\ell}),$$

where $\bigvee_{a}^{b}(\mathcal{G}'_{\ell})$ denotes the total variation of $\mathcal{G}'_{\ell}$ on $[a,b]$ and $\mathcal{G}'_{\ell}$ is defined by

$$\mathcal{G}'_{\ell}(\imath)=\begin{cases}\mathcal{G}'(\imath)-\mathcal{G}'(\ell-), & 0\leq\imath<\ell\\ 0, & \imath=\ell\\ \mathcal{G}'(\imath)-\mathcal{G}'(\ell+), & \ell<\imath<1.\end{cases}\tag{18}$$

Proof. The Lupaş-Durmeyer type operators maintain constants and by utilizing equation (17), this applies to every value of $\ell\in(0,1)$, we have

$$U_{n,\varrho}^{[\frac{m}{n}]}(f;\ell)-\mathcal{G}(\ell)=\int_{0}^{1}\mathcal{K}_{n,\varrho}^{[\frac{m}{n}]}(\ell,\imath)\left(\mathcal{G}(\imath)-\mathcal{G}(\ell)\right)\mathrm{d}\imath=\int_{0}^{1}\mathcal{K}_{n,\varrho}^{[\frac{m}{n}]}(\ell,\imath)\int_{\ell}^{\imath}\mathcal{G}'(u)\mathrm{d}u\,\mathrm{d}\imath.\tag{19}$$

For any function $\mathcal{G}\in DBV[0,1]$, using equation (18), we get

$$\mathcal{G}'(u)=\mathcal{G}'_{\ell}(u)+\frac{\mathcal{G}'(\ell+)+\mathcal{G}'(\ell-)}{2}+\frac{\mathcal{G}'(\ell+)-\mathcal{G}'(\ell-)}{2}\mathrm{sgn}(u-\ell)\tag{20}$$

$$+\lambda_{\ell}(u)\left(\mathcal{G}'(u)-\frac{\mathcal{G}'(\ell+)+\mathcal{G}'(\ell-)}{2}\right),$$

where

$$\lambda_{\ell}(u)=\begin{cases}1, & u=\ell\\ 0, & u\neq\ell.\end{cases}$$

Obviously,

$$\int_{0}^{1}\left(\int_{\ell}^{\imath}\left(\mathcal{G}'(u)-\frac{\mathcal{G}'(\ell+)+\mathcal{G}'(\ell-)}{2}\right)\lambda_{\ell}(u)\mathrm{d}u\right)\mathcal{K}_{n,\varrho}^{[\frac{m}{n}]}(\ell,\imath)\mathrm{d}\imath=0,$$

and

$$\int_{0}^{1}\left(\int_{\ell}^{\imath}\frac{\mathcal{G}'(\ell+)+\mathcal{G}'(\ell-)}{2}\mathrm{d}u\right)\mathcal{K}_{n,\varrho}^{[\frac{m}{n}]}(\ell,\imath)\mathrm{d}\imath$$

$$=\frac{\mathcal{G}'(\ell+)+\mathcal{G}'(\ell-)}{2}\int_{0}^{1}(t-\ell)\mathcal{K}_{n,\varrho}^{[\frac{m}{n}]}(\ell,\imath)\mathrm{d}\imath=\frac{\mathcal{G}'(\ell+)+\mathcal{G}'(\ell-)}{2}\cdot U_{n,\varrho}^{[\frac{m}{n}]}(e_{1}-\ell;\ell)=0.$$

The following is the result of using the Chauchy-Schwarz's inequality for linear positive operators:

$$\left|\int_0^1 \mathcal{K}_{n,\varrho}^{[\frac{m}{n}]}(\ell,\imath)\Big(\int_\ell^\imath \frac{\mathcal{G}'(\ell+)-\mathcal{G}'(\ell-)}{2}\operatorname{sgn}(u-\ell)\mathrm{d}u\Big)\mathrm{d}\imath\right| \le \frac{|\mathcal{G}'(\ell+)-\mathcal{G}'(\ell-)|}{2}\int_0^1 |\imath-\ell|\mathcal{K}_{n,\varrho}^{[\frac{m}{n}]}(\ell,\imath)\mathrm{d}\imath$$
$$\le \frac{|\mathcal{G}'(\ell+)-\mathcal{G}'(\ell-)|}{2}U_{n,\varrho}^{[\frac{m}{n}]}(|\imath-\ell|;\ell) \le \frac{|\mathcal{G}'(\ell+)-\mathcal{G}'(\ell-)|}{2}\left(U_{n,\varrho}^{[\frac{m}{n}]}((\imath-\ell)^2;\ell)\right)^{1/2}.$$

By applying Lemma 2.2 or Lemma 2.4 together with equation (19) and (20), a new result can be

$$\left|U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G};\ell)-\mathcal{G}(\ell)\right|$$
$$\le \frac{|\mathcal{G}'(\ell+)-\mathcal{G}'(\ell-)|}{2}\sqrt{\frac{\mathcal{D}_\varrho^{[\frac{m}{n}]}\ell(1-\ell)}{(1+n\varrho)}} + \left|\int_0^\ell\left(\int_\ell^\imath \mathcal{G}'_\ell(u)\mathrm{d}u\right)\mathcal{K}_{n,\varrho}^{[\frac{m}{n}]}(\ell,\imath)\mathrm{d}\imath + \int_\ell^1\left(\int_\ell^\imath \mathcal{G}'_\ell(u)\mathrm{d}u\right)\mathcal{K}_{n,\varrho}^{[\frac{m}{n}]}(\ell,\imath)\mathrm{d}\imath\right|. \tag{21}$$

Let us focus on

$$\mathcal{M}_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G}'_\ell,\ell) = \int_0^\ell\left(\int_\ell^\imath \mathcal{G}'_\ell(u)\mathrm{d}u\right)\mathcal{K}_{n,\varrho}^{[\frac{m}{n}]}(\ell,\imath)\mathrm{d}\imath,$$

$$\mathcal{F}_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G}'_\ell,\ell) = \int_\ell^1\left(\int_\ell^\imath \mathcal{G}'_\ell(u)\mathrm{d}u\right)\mathcal{K}_{n,\varrho}^{[\frac{m}{n}]}(\ell,\imath)\mathrm{d}\imath.$$

In order to finalize the proof, it is enough to estimate $\mathcal{F}_{n,\varrho}^{[\frac{m}{n}]}$ and $\mathcal{M}_{n,\varrho}^{[\frac{m}{n}]}$. Since $\int_e^g d_\imath\Theta_{n,\varrho}^{[\mathrm{m}]}(\ell,\imath) \le 1$ for all $[0,l]\subset[0,1]$. Utilizing the integration formula by parts and applying Lemma 5.1 with $y=\ell-\frac{\ell}{\sqrt{n}}$, to express the following

$$\left|\mathcal{M}_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G}'_\ell,\ell)\right| = \left|\int_0^\ell\left(\int_\ell^\imath \mathcal{G}'_\ell(u)\mathrm{d}u\right)\mathrm{d}_\imath\Theta_{n,\varrho}^{[\frac{m}{n}]}(\ell,\imath)\right| = \left|\int_0^\ell \Theta_{n,\varrho}^{[\frac{m}{n}]}(\ell,\imath)\mathcal{G}'_\ell(\imath)\mathrm{d}\imath\right|$$
$$\le \left(\int_0^y + \int_y^\ell\right)|\mathcal{G}'_\ell(\imath)||\Theta_{n,\varrho}^{[\frac{m}{n}]}(\ell,\imath)|\mathrm{d}\imath$$
$$\le \frac{\mathcal{D}_\varrho^{[\frac{m}{n}]}\ell(1-\ell)}{(1+n\varrho)}\int_0^y \bigvee_\imath^\ell(\mathcal{G}'_\ell)(\ell-\imath)^{-2}\mathrm{d}\imath + \int_y^\ell \bigvee_\imath^\ell(\mathcal{G}'_\ell)\mathrm{d}\imath$$
$$\le \frac{\mathcal{D}_\varrho^{[\frac{m}{n}]}\ell(1-\ell)}{(1+n\varrho)}\int_0^y \bigvee_\imath^\ell(\mathcal{G}'_\ell)(l-\imath)^{-2}\mathrm{d}\imath + \frac{\ell}{\sqrt{n}}\bigvee_{\ell-(\ell\sqrt{n})}^\ell(\mathcal{G}'_\ell).$$

Now put $u=\ell/(\ell-\imath)$, we find

$$\frac{\mathcal{D}_\varrho^{[\frac{m}{n}]}\ell(1-\ell)}{(1+n\varrho)}\int_0^{\ell-(\ell/\sqrt{n})}(\ell-\imath)^{-2}\bigvee_\imath^\ell(\mathcal{G}'_\ell)\mathrm{d}\imath = \frac{\mathcal{D}_\varrho^{[\frac{m}{n}]}(1-\ell)}{(1+n\varrho)}\int_1^{\sqrt{n}}\bigvee_{\ell-(\ell/u)}^\ell(\mathcal{G}'_\ell)\mathrm{d}u$$
$$\le \frac{\mathcal{D}_\varrho^{[\frac{m}{n}]}(1-\ell)}{(1+n\varrho)}\sum_{\jmath=1}^{[\sqrt{n}]}\int_\jmath^{\jmath+1}\bigvee_{\ell-(\ell/\jmath)}^\ell(\mathcal{G}'_\ell)\mathrm{d}u \le \frac{\mathcal{D}_\varrho^{[\frac{m}{n}]}(1-\ell)}{(1+n\varrho)}\sum_{\jmath=1}^{[\sqrt{n}]}\bigvee_{\ell-(\ell/\jmath)}^\ell(\mathcal{G}'_\ell).$$

Thus

$$\left|\mathcal{M}_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G}'_\ell,\ell)\right| \le \frac{\mathcal{D}_\varrho^{[\frac{m}{n}]}(1-\ell)}{(1+n\varrho)}\sum_{\jmath=1}^{[\sqrt{n}]}\bigvee_{\ell-(\ell/\jmath)}^\ell(\mathcal{G}'_\ell) + \frac{\ell}{\sqrt{n}}\bigvee_{\ell-(\ell/\sqrt{n})}^\ell(\mathcal{G}'_\ell). \tag{22}$$

By utilizing the integration formula by parts and implementing Lemma 5.1 with $z = \ell + \frac{(1-\ell)}{\sqrt{n}}$, the result is as follows:

$$
\begin{aligned}
&\left|\mathcal{F}_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G}'_{\ell},\ell)\right| \\
&= \left|\int_{\ell}^{1}\left(\int_{\ell}^{\imath} f'_{\ell}(u)\mathrm{d}u\right)\mathcal{K}_{n,\varrho}^{[\frac{m}{n}]}(\ell,\imath)\mathrm{d}\imath\right| \\
&= \left|\int_{\ell}^{z}\left(\int_{\ell}^{\imath} f'_{\ell}(u)\mathrm{d}u\right)\mathrm{d}_{\imath}\left(1-\Theta_{n,\varrho}^{[\frac{m}{n}]}(\ell,\imath)\right)+\int_{z}^{1}\left(\int_{\ell}^{\imath} f'_{\ell}(u)\mathrm{d}u\right)\mathrm{d}_{\imath}\left(1-\Theta_{n,\varrho}^{[\frac{m}{n}]}(\ell,\imath)\right)\right| \\
&= \left|\left[\int_{\ell}^{\imath} f'_{\ell}(u)\left(1-\Theta_{n,\varrho}^{[\frac{m}{n}]}(\ell,\imath)\right)\mathrm{d}u\right]_{\ell}^{z}-\int_{\ell}^{z} f'_{\ell}(\imath)\left(1-\Theta_{n,\varrho}^{[\frac{m}{n}]}(\ell,\imath)\right)\mathrm{d}\imath+\int_{z}^{1}\left(\int_{\ell}^{\imath} f'_{\ell}(u)\mathrm{d}u\right)\mathrm{d}_{\imath}\left(1-\Theta_{n,\varrho}^{[\frac{m}{n}]}(\ell,\imath)\right)\right| \\
&= \left|\int_{\ell}^{z} f'_{\ell}(u)\mathrm{d}u\left(1-\Theta_{n,\varrho}^{[\frac{m}{n}]}(\ell,z)\right)-\int_{\ell}^{z} f'_{\ell}(\imath)\left(1-\Theta_{n,\varrho}^{[\frac{m}{n}]}(\ell,\imath)\right)\mathrm{d}\imath+\left[\int_{\ell}^{\imath} f'_{\ell}(u)\mathrm{d}u\left(1-\Theta_{n,\varrho}^{[\frac{m}{n}]}(\ell,\imath)\right)\right]_{z}^{1}\right. \\
&\quad \left.-\int_{z}^{1} f'_{\ell}(\imath)\left(1-\Theta_{n,\varrho}^{[\frac{m}{n}]}(\ell,\imath)\right)\mathrm{d}\imath\right| \\
&= \left|\int_{\ell}^{z} f'_{\ell}(\imath)\left(1-\Theta_{n,\varrho}^{[\frac{m}{n}]}(\ell,\imath)\right)\mathrm{d}\imath+\int_{z}^{1} f'_{\ell}(\imath)\left(1-\Theta_{n,\varrho}^{[\frac{m}{n}]}(\ell,\imath)\right)\mathrm{d}\imath\right| \\
&\leq \frac{\mathcal{D}_{\varrho}^{[\frac{m}{n}]}\ell(1-\ell)}{(1+n\varrho)}\int_{z}^{1}\bigvee_{\ell}^{\imath}(\mathcal{G}'_{\ell})(\imath-\ell)^{-2}\mathrm{d}\imath+\int_{\ell}^{z}\bigvee_{\ell}^{\imath}(\mathcal{G}'_{\ell})\mathrm{d}\imath \\
&= \frac{\mathcal{D}_{\varrho}^{[\frac{m}{n}]}\ell(1-\ell)}{(1+n\varrho)}\int_{\ell+((1-\ell)/\sqrt{n})}^{1}\bigvee_{\ell}^{\imath}(\mathcal{G}'_{\ell})(\imath-\ell)^{-2}\mathrm{d}\imath+\frac{(1-\ell)}{\sqrt{n}}\bigvee_{\ell}^{\ell+((1-\ell)/\sqrt{n})}(\mathcal{G}'_{\ell}).
\end{aligned}
$$

By the substitution of $v = (1-\ell)/(\imath-\ell)$, we get

$$
\begin{aligned}
\left|\mathcal{F}_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G}'_{\ell},\ell)\right| &\leq \frac{\mathcal{D}_{\varrho}^{[\frac{m}{n}]}\ell(1-\ell)}{(1+n\varrho)}\int_{1}^{\sqrt{n}}\bigvee_{\ell}^{\ell+((1-\ell)/v)}(\mathcal{G}'_{\ell})(1-\ell)^{-1}\mathrm{d}v+\frac{(1-\ell)}{\sqrt{n}}\bigvee_{\ell}^{\ell+((1-\ell)/\sqrt{n})}(\mathcal{G}'_{\ell}) \\
&\leq \frac{\mathcal{D}_{\varrho}^{[\frac{m}{n}]}\ell}{(1+n\varrho)}\sum_{\jmath=1}^{[\sqrt{n}]}\int_{\jmath}^{\jmath+1}\bigvee_{\ell}^{\ell+((1-\ell)/v)}(\mathcal{G}'_{\ell})\mathrm{d}v+\frac{(1-\ell)}{\sqrt{n}}\bigvee_{\ell}^{\ell+((1-\ell)/\sqrt{n})}(\mathcal{G}'_{\ell}) \\
&= \frac{\mathcal{D}_{\varrho}^{[\frac{m}{n}]}\ell}{(1+n\varrho)}\sum_{\jmath=1}^{[\sqrt{n}]}\bigvee_{\ell}^{\ell+((1-\ell)/\jmath)}(\mathcal{G}'_{\ell})+\frac{(1-\ell)}{\sqrt{n}}\bigvee_{\ell}^{\ell+((1-\ell))/\sqrt{n}}(\mathcal{G}'_{\ell}).
\end{aligned}
\tag{23}
$$

Collecting the estimates (21)-(23), we get the required result. □

6. Chebyshev-Grüss theorem

Theorem 6.1. *Using the Chebyshev-Grüss inequality[4], [Theorem 6,[11]], a uniform inequality hold for* $U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G};\ell) : C[0,1] \to C[0,1]$ *that is:*

$$
\left.U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G}\theta)-U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G})U_{n,\varrho}^{[\frac{m}{n}]}(\theta)\right|\leq\frac{1}{4}\tilde{\omega}\left(\mathcal{G};2\sqrt{\frac{(m+n+(1+m)n\varrho)}{2(m+n)(1+n\varrho)}}\right)\cdot\tilde{\omega}\left(\theta;2\sqrt{\frac{(m+n+(1+m)n\varrho)}{2(m+n)(1+n\varrho)}}\right).
$$

7. Grüss-Voronovskaya theorems

Theorem 7.1. *Let* $g, \theta \in C^2[0,1]$*. Then following equality hold*

$$
\lim_{n\to\infty} n\left[U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G}\theta)-U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G})U_{n,\varrho}^{[\frac{m}{n}]}(\theta)\right] = -YS\mathcal{G}'\theta', \quad y \in [0,1].
$$

Proof. *Let* $Y = \ell(1-\ell)$ *and* $S = \frac{(m+n+(1+m)n\varrho)}{(m+n)(1+n\varrho)}$, *we have*

$$\begin{aligned} n\left[U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G}\theta) - U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G})U_{n,\varrho}^{[\frac{m}{n}]}(\theta)\right] &= n\Big\{U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G}\theta) - (\mathcal{G}\theta) - \left(\frac{YS}{2n}(\mathcal{G}''\theta + 2\mathcal{G}'\theta' + \mathcal{G}\theta'')\right) \\ &- \theta\left[U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G}) - \mathcal{G} - \left(\frac{YS}{2n}\mathcal{G}''\right)\right] - U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G})\left[U_{n,\varrho}^{[\frac{m}{n}]}(\theta) - \theta - \left(\frac{YS}{2n}\theta''\right)\right] \\ &- \frac{YS}{n}\mathcal{G}'\theta' - \left(\frac{YS}{2n}\theta''\right)\left[\mathcal{G} - U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G})\right]\Big\}, \end{aligned}$$

applying the Theorem 3.1 and 3.2, we get

$$\lim_{n\to\infty} n\left[U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G}\theta) - U_{n,\varrho}^{[\frac{m}{n}]}(\mathcal{G})U_{n,\varrho}^{[\frac{m}{n}]}(\theta)\right] = -YS\mathcal{G}'\theta'.$$

□

8. Numerical Examples

Example 8.1. *Let's consider the function* $\mathcal{G} = \ell^2\sin(2\pi\ell)$*(blue), where* $\varrho = 10$, $m = 0.1$, *and n takes on the value of 20,40,60 and 80. To calculate the convergence of operator* $U_{n,\varrho}^{\frac{m}{n}}$*. The resulting value of* $U_{20,10}^{\frac{0.1}{20}}$*(green),* $U_{40,10}^{\frac{0.1}{40}}$*(magenta),* $U_{60,10}^{\frac{0.1}{60}}$*(red),* $U_{80,10}^{\frac{0.1}{80}}$*(black) plot on a graph to observe the convergence Fig. 1. As the value of n increases, the plots of the operator become progressively closer to the function graph. This trend is shown by the convergence of the operator's plots towards the function graph.*

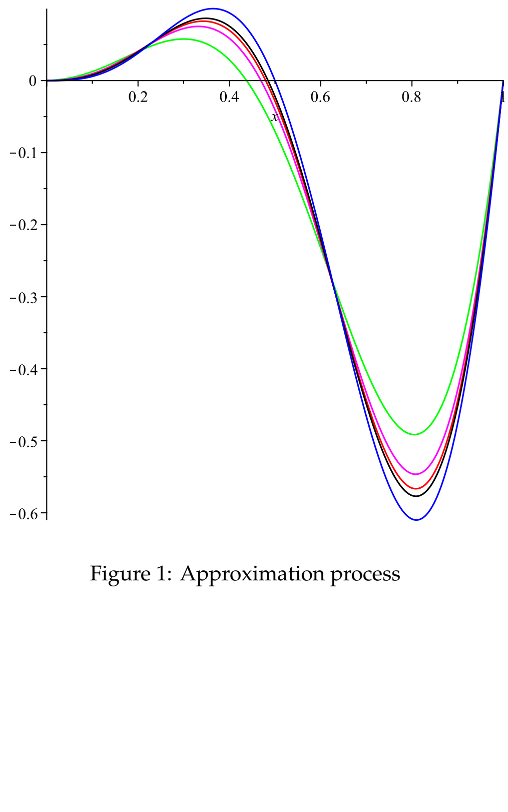

Figure 1: Approximation process

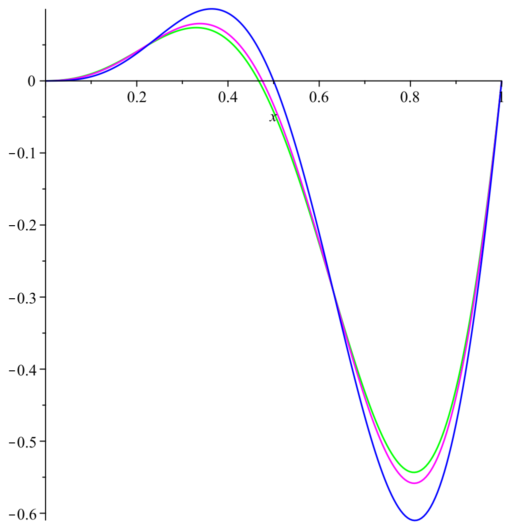

Figure 2: Convergence of $U^{\frac{m}{n}}_{n,\varrho}(\mathcal{G};\ell)$ and $U^{\alpha}_{n,\rho}(\mathcal{G};\ell)$ to the function $\mathcal{G}(\ell) = \ell^2\sin(2\pi\ell)$

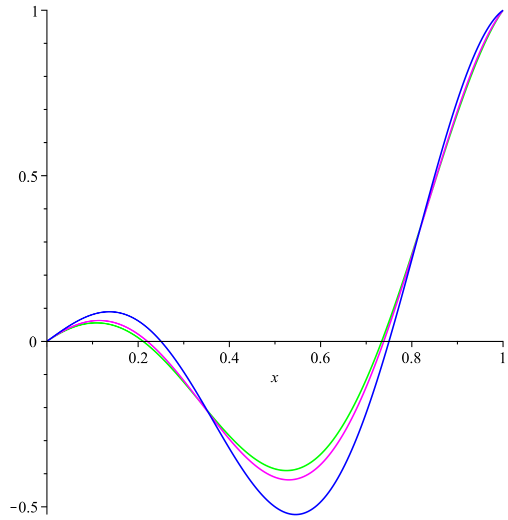

Figure 3: Convergence of $U^{\frac{m}{n}}_{n,\varrho}(\mathcal{G};\ell)$ and $U^{\alpha}_{n,\rho}(\mathcal{G};\ell)$ to the function $\mathcal{G}(\ell) = \ell\cos(2\pi\ell)$

Example 8.2. *We analyze the convergence of the operator $U^{\frac{m}{n}}_{n,\varrho}(\mathcal{G};\ell)$(magenta) and Kajla Stancu-Durrmeyer type operators [23] $U^{\alpha}_{n,\rho}(\mathcal{G};\ell)$(green) , we need to evaluate how well they approximate the function $\mathcal{G}(\ell) = \ell^2\sin(2\pi\ell)$(blue) for the give value of $\varrho = \rho = 10$, $m = 0.1$, $\alpha = \frac{1}{2n}$ and $n = 50$. Since it was observed that $U^{\frac{m}{n}}_{n,\varrho}(\mathcal{G};\ell)$ give a better approximation to $\mathcal{G}(\ell)$ than $U^{\alpha}_{n,\rho}(\mathcal{G};\ell)$ in Fig. 2. While in Fig. 3, consider the function $\mathcal{G}(\ell) = \ell\cos(2\pi\ell)$(blue) over the interval [0,1]. We want to approximate $\mathcal{G}(\ell)$ using the operators $U^{\frac{m}{n}}_{n,\varrho}(\mathcal{G};\ell)$(magenta) and $U^{\alpha}_{n,\rho}(\mathcal{G};\ell)$(green) with $n = 30$ and using the same parametric values as above. Both operators have a similar shape, but $U^{\frac{m}{n}}_{n,\varrho}(\mathcal{G};\ell)$(magenta) is more closer to $\mathcal{G}$(blue).*

Example 8.3. *Consider the function $\mathcal{G}(\ell) = \ell^2\sin(3\pi\ell)$ (blue), with parameters $\varrho = \rho = 10$, $m = 0.1$, $n = 20$, and $\alpha = 0.1$. In Fig. 4, we compare the operator $U^{\frac{m}{n}}_{n,\varrho}$ (green) with:*

- *Stancu operators [36] (red),*
- *Lupaş operators [27] (magenta),*
- *Kajla operators [21] (cyan), where $\tau(x) = \frac{1}{1+x}$,*
- *Kajla-Stancu-Durrmeyer type operators [23] (yellow).*

The graph illustrates their convergence toward $\mathcal{G}(\ell)$.

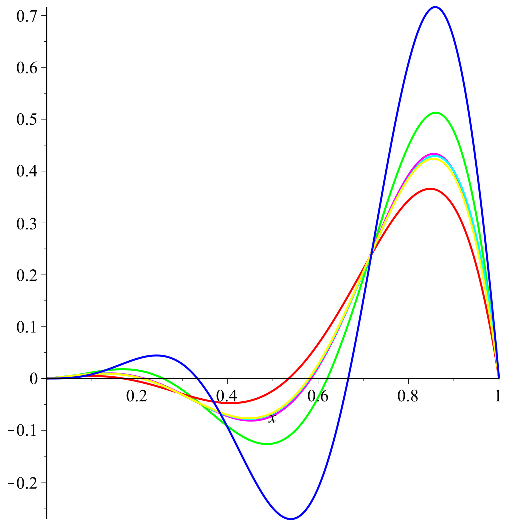

Figure 4: Comparison process.

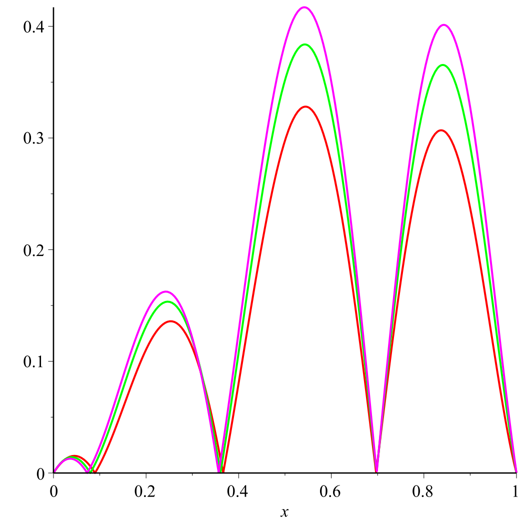

Figure 5: Error Estimation for $m = 0.1$(red), $m = 0.5$(green) and $m = 0.8$(magenta).

Example 8.4. *The error of approximation of the operators* $U_{n,\varrho}^{\frac{m}{n}}(\mathcal{G};\ell)$ *for the function* $\mathcal{G}(\ell) = \ell^2\sin(3\pi\ell)$ *for the parameter values of* $\varrho = 10$, $m = 0.1$ $n = 30$*(green),* $n = 40$*(magenta) and* $n = 50$*(red) display in Fig 6. The error of approximation is defined as* $E_{n,\varrho}^{\frac{m}{n}}(\mathcal{G};\ell) = |U_{n,\varrho}^{\frac{m}{n}}(\mathcal{G};\ell) - \mathcal{G}(\ell)|$.

In Fig. 5 error of approximation of the operators $U_{n,\varrho}^{\frac{m}{n}}(\mathcal{G};\ell)$ *for the function* $\mathcal{G}(\ell) = \ell\sin(3\pi\ell)$ *using the parameter values of* $\varrho = 10$, $n = 15$ *with different value of m.*

In Fig. 7. the error of approximation of the operators for these parametric values given above and $\mathcal{G}(\ell) = \ell^2\cos(3\pi\ell)$. *The operator's performance is indicated by the curve on the graph, with a lower curve indicating a better performance and a higher curve indicating a poor performance. By increasing the value of n we can minimize the error between the operator's approximation and the certain function.*

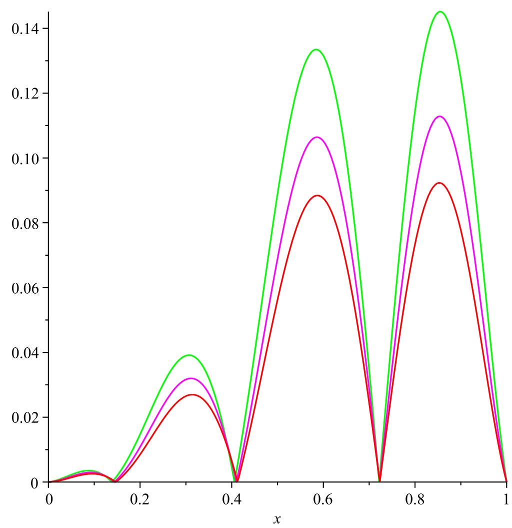

Figure 6: Error Estimation for $\mathcal{G}(\ell) = \ell^2\sin(3\pi\ell)$

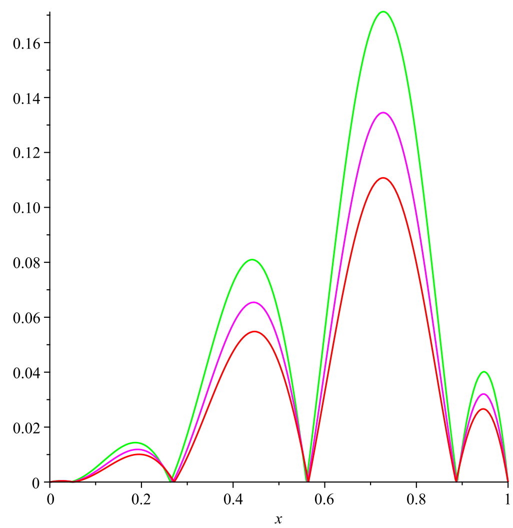

Figure 7: Error Estimation for $\mathcal{G}(\ell) = \ell^2\cos(3\pi\ell)$

9. Conclusion

- The new operators $U_{n,\varrho}^{[\frac{m}{n}]}$ defined on the interval $[0,1]$ converge uniformly to continuous functions.
- These operators have two parameters, m and ϱ, which provide a more generalized framework. By choosing suitable values of m and ϱ, various types of operators can be derived.
- Numerical example demonstrate that decreasing the value of m results in a reduction of approximation error, highlighting the effectiveness of the proposed operators.
- In comparison graphs, our operators show the best approximation results compared to other operators, proving their superiority.

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