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Gronwall inequalities for normalized fractional integrals within exponential and Mittag-Leffler kernels with applications

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Abstract. Gronwall inequalities are common tools in studying differential and integral equations analytically. Existence, uniqueness and stability results can be obtained using these inequalities. In this paper, we provide new versions of the Gronwall inequality to the normalized fractional integrals with exponential and Mittag-Leffler kernels. The obtained inequalities are used to establish existence and uniqueness results to the fractional Cauchy problem with the normalized derivative of Mittag-Leffler kernel. Comparison principles are derived based on an estimate of the normalized derivative of a function at its extreme points. These comparison principles are then used to obtain a pre-norm estimates of solutions for related linear fractional differential equations. Two examples are presented to illustrate the efficiency of the obtained results. Further, a numerical example is studied to illustrate the solutions of a non-homogeneous normalized system in the Mittag-Leffler kernel case.

1. Introduction

The normalized fractional derivatives (NFDs) and their related fractional differential equations (FDEs) form a very recent research area. They were considered as normalization to the existing fractional derivatives (FDs) which have smooth derivatives at the starting point and admit geometrical meanings. Unlike the FDs with non-singular kernels, the NFDs don't vanish at the initial point, in general, and therefore the related FDE's admit solutions without the need to impose extra conditions, see [5]. They also satisfy the so-called fundamental theorem of fractional calculus, see [14]. On the other hand, due the fact that the division by the the Atangana-Baleanu (AB) fractional derivative (Caputo-Fabrizio (CF) fractional derivative) starting from a, of $(t-a)$ to get the quantities $AB_\alpha(t)$ and $CF_\alpha(t)$ defined below, we are able to show that the AB and CF fractional integrals vanish at a. This will lead to smooth actions of the integral operators and differential operators on each other, and hence guarantee and verify nontrivial solutions, in both directions, for the

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normalized AB or CF linear equations with constant coefficients. NFDs of Caputo and Riemann-Liouville types were introduced at the first in [17], where the geometric meaning of several concepts of continuum mechanics were investigated with these derivatives. In [18] the power series expansion is developed to obtain the solutions of related linear FDEs with constant coefficients in closed forms. The solutions are given in terms of the Mittag-Leffler functions. Since the NFDs have physical meaning, they have been used to model several viscoelastic and logistic models, see [15, 19]. In this paper we investigate the solution of the fractional initial value problem (FIVP)

$$({}_a nAB^{\alpha}\gamma)(t) = f(t,\gamma), \quad 0<\alpha<1, \ t>a \tag{1}$$

$$\gamma(a) = \gamma_0, \tag{2}$$

where $\gamma_0 \in \mathbb{R}$, $f : [a,b]\times\mathbb{R} \to \mathbb{R}$, is continuous, and ${}_a nAB^{\alpha}$ is the normalized Atangana-Baleanu (AB) derivative of Caputo type. The above initial value problem with the AB-derivative admits a solution only if, $f(a,\gamma(a)) = 0$, see [3, 4, 26], an extra condition which is considered as a drawback of the AB-derivative in particular, and the fractional derivatives with nonsingular kernel, in general. The normalized fractional derivatives overcome this difficulty. In Section 2, we present preliminary results about NFDs. In Section 3, we present Grownwall inequalities of the normalized integral operators of Atangana-Baleanu (AB) and Caputo-Fabrizio (CF) types. We implement these inequalities to establish existence and uniqueness results to the FIVP (1)-(2) with the help of Banach fixed point theory in Section 4. Section 5, is devoted to establish new comparison principles to the fractional differential inequalities with the NFD of AB-type, and their applications. Finally, we close up with some illustrative examples and concluding remarks in Section 6.

2. Preliminaries on normalized fractional derivatives and integrals

The generalized Mittag-Leffler (ML) function is defined as:

$$E^{\sigma}_{\alpha,\mu}(\omega) = \sum_{i=0}^{\infty} \frac{\omega^i (\sigma)_i}{i!\Gamma(\alpha i+\mu)}, \tag{3}$$

where $(\sigma)_i = \sigma(\sigma+1)...(\sigma+i-1)$ is the Pochhammer symbol, $Re(\alpha)>0$, ω,μ,σ are complex numbers, and $(1)_i = i!$. The function $\Gamma(\omega)$ stands for the Gamma special function. The ML functions with two and one parameter are then defined by $E_{\alpha,\mu}(\omega) = E^1_{\alpha,\mu}(\omega)$, and $E_{\alpha}(\omega) = E^1_{\alpha,1}(\omega)$, respectively. For the sake of simplicity, we shall use the following functions derived from ML functions which are known in the literature as the Prabhakar kernels, see [27]. For $\lambda \in \mathbf{R}$, we have

$$\mathcal{E}^{\sigma}_{\alpha,\mu}(\lambda,\omega) = \omega^{\mu-1}E^{\sigma}_{\alpha,\mu}(\lambda\omega^{\alpha}) = \sum_{i=0}^{\infty} \frac{\lambda^i \omega^{i\alpha+\mu-1}(\sigma)_i}{i!\Gamma(\alpha i+\mu)}. \tag{4}$$

For $a \in \mathbf{R}$, $\gamma \in H^1(a,b)$, and $\alpha \in (0,1)$, let ${}_aCF^{\alpha}$ and ${}_aAB^{\alpha}$ are the CF and AB fractional derivatives in the Caputo sense starting from a on the interval $[a,b]$, given by, see [10, 11]

$$({}_aCF^{\alpha}\gamma)(t) = \frac{1}{1-\alpha}\int_a^t Exp(\lambda_{\alpha}(t-s))\gamma'(s)ds, \quad t \ge a \tag{5}$$

$$({}_aAB^{\alpha}\gamma)(t) = \frac{1}{1-\alpha}\int_a^t E_{\alpha}(\lambda_{\alpha}(t-s))\gamma'(s)ds, \quad t \ge a, \tag{6}$$

where $\lambda_{\alpha} = \frac{-\alpha}{1-\alpha} < 0$. Here $H^1(a,b)$ denotes the Sobolev space defined by $H^1(a,b) = \{\gamma \in L^1(a,b) : D\gamma \in L^1(a,b)\}$. Direct calculation will lead to

$$({}_aCF^{\alpha}\,(t-a))(t) = \frac{1}{\alpha}(1-e^{\lambda_{\alpha}(t-a)}) = CF_{\alpha}(t), \tag{7}$$

and

$$({}_aAB^{\alpha}\,(t-a))(t) = \frac{1}{\alpha}\mathcal{E}_{\alpha,2}(\lambda_\alpha, t-a) = AB_\alpha(t), \tag{8}$$

where $\mathcal{E}_{\alpha,2}(\lambda_\alpha, t-a) = \mathcal{E}^1_{\alpha,2}(\lambda_\alpha, t-a)$. Also, it holds that

$$CF_\alpha(a) = AB_\alpha(a) = 0.$$

Definition 2.1. *[16] For $\gamma \in H^1(a,b)$, the normalized CF and AB fractional derivatives of order $\alpha \in (0,1)$ in the Caputo sense are given by*

$$\begin{aligned} ({}_anCF^{\alpha}\gamma)(t) &= \frac{({}_aCF^{\alpha}\gamma)(t)}{CF_\alpha(t)}, \;\; t > a \\ ({}_anAB^{\alpha}\gamma)(t) &= \frac{({}_aAB^{\alpha}\gamma)(t)}{AB_\alpha(t)}, \;\; t > a. \end{aligned} \tag{9}$$

Definition 2.2. *[16] For $\gamma \in L^1(a,b)$, the normalized CF and AB fractional integral operators are given by*

$$({}^{nCF}_{\;\;a}I^{\alpha}\gamma)(t) = {}^{CF}_{\;\;a}I^{\alpha}[\gamma(t)\,CF_\alpha(t)] = (1-\alpha)CF_\alpha(t)\gamma(t) + \alpha\int_a^t \gamma(s)CF_\alpha(s)ds, \tag{10}$$

and

$$({}^{nAB}_{\;\;a}I^{\alpha}\gamma)(t) = {}^{AB}_{\;\;a}I^{\alpha}[\gamma(t)\,AB_\alpha(t)] = (1-\alpha)AB_\alpha(t)\gamma(t) + \alpha({}_aI^{\alpha}\gamma(.)AB_\alpha(.))(t), \tag{11}$$

where ${}_aI^{\alpha}$, ${}^{CF}_{\;\;a}I^{\alpha}$, and ${}^{AB}_{\;\;a}I^{\alpha}$ denote respectively the Riemann-Liouville, the CF and the AB integral operators given by, [10, 11]

$$({}_aI^{\alpha}\gamma)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\gamma(s)ds, \tag{12}$$

$$({}^{CF}_{\;\;a}I^{\alpha}\gamma)(t) = (1-\alpha)\gamma(t) + \alpha\int_a^t \gamma(s)ds \tag{13}$$

$$({}^{AB}_{\;\;a}I^{\alpha}\gamma)(t) = (1-\alpha)\gamma(t) + \frac{\alpha}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\gamma(s)ds. \tag{14}$$

Proposition 2.3. *For any fractional derivative $\mathbb{D}^{\alpha}_a$ with non-singular kernel $k(t) > 0$, in the Caputo sense, it holds that $(\mathbb{D}^{\alpha}_a\,(t-a))(t)$, is increasing for $t \geq a$, and $0 < \alpha < 1$.*

Proof. We have

$$(\mathbb{D}^{\alpha}_a\gamma)(t) = \int_a^t k(t-s)\gamma'(s)ds,$$

and thus

$$(\mathbb{D}^{\alpha}_a(t-a))(t) = \int_a^t k(t-s)ds.$$

Since $k(t)$ is continuous applying the Leibniz rule for differentiating under integral sign yields

$$\frac{d}{dt}\Big(\mathbb{D}^{\alpha}_a\,(t-a)\Big)(t) = \int_a^t \frac{d}{dt}k(t-s)ds + k(0).$$

Because $\frac{d}{dt}k(t-s) = -\frac{d}{ds}k(t-s)$, we arrive at

$$\begin{aligned} \frac{d}{dt}\Big(\mathbb{D}^{\alpha}_a\,(t-a)\Big)(t) &= -\int_a^t \frac{d}{ds}k(t-s)ds + k(0) \\ &= -k(t-s)|_a^t + k(0) = -k(0) + k(t-a) + k(0) = k(t-a) > 0 \end{aligned} \tag{15}$$

which completes the proof. □

As a direct consequence of the above proposition, we have $CF_\alpha(t)$ and $AB_\alpha(t)$ are increasing for $t \geq a$.

Remark 2.4. *We remark here that $({}_a^{nCF}I^\alpha\gamma)(a) = ({}_a^{nAB}I^\alpha\gamma)(a) = 0$. While $({}_a^{CF}I^\alpha\gamma)(a) = ({}_a^{AB}I^\alpha\gamma)(a) = 0$, only if $\gamma(a) = 0$. This shows that normalized CF and AB fractional linear system with constant coefficients can have other than the trivial solution. This solves the issue of the trivialization in case of CF and AB fractional systems.*

Unlike the fractional derivatives with nonsingular kernels, the normalized fractional derivatives satisfy the fundamental theorem of fractional calculus, mainly

$$
\begin{aligned}
{}_a nCF^\alpha o\, {}_a^{nCF}I^\alpha\gamma(t) &= \gamma(t), \;\; \gamma \in L^1(a,b) \\
{}_a nAB^\alpha o\, {}_a^{nAB}I^\alpha\gamma(t) &= \gamma(t), \;\; \gamma \in L^1(a,b) \\
{}_a^{nCF}I^\alpha o\, {}_a nCF^\alpha\gamma(t) &= \gamma(t) - \gamma(a), \;\; \gamma \in H^1(a,b) \\
{}_a^{nAB}I^\alpha o\, {}_a nAB^\alpha\gamma(t) &= \gamma(t) - \gamma(a), \;\; \gamma \in H^1(a,b) \\
{}_a^{CF}I^\alpha o\, {}_a CF^\alpha\gamma(t) &= \gamma(t) - \gamma(a), \;\; \gamma \in H^1(a,b) \\
{}_a^{AB}I^\alpha o\, {}_a AB^\alpha\gamma(t) &= \gamma(t) - \gamma(a), \;\; \gamma \in H^1(a,b). \\
{}_a CF^\alpha o\, {}_a^{CF}I^\alpha\gamma(t) &= \gamma(t) - \frac{\gamma(a)}{1-\alpha}Exp(\lambda_\alpha, t-a), \;\; \gamma \in L^1(a,b) \\
{}_a AB^\alpha o\, {}_a^{AB}I^\alpha\gamma(t) &= \gamma(t) - \frac{\gamma(a)}{1-\alpha}\mathcal{E}_\alpha(\lambda_\alpha, t-a), \;\; \gamma \in L^1(a,b).
\end{aligned} \tag{16}
$$

For the action of (non-normalized) *CF* and *AB* fractional operators, either integrals or derivatives, on each other, we refer the reader to [1, 2]. The proof the normalized actions follow by definition and the non-nomalized actions. For more properties about the NFDs and their applications, we refer the reader to recent publications [14, 16].

Proposition 2.5. *[14] Let $\alpha, \beta \in \mathbb{C}$ be such that $Re(\alpha) \geq 0$ and $Re(\beta) > 0$. Then, we have*

(1) $\left({}_aI^\alpha(t-a)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(t-a)^{\alpha+\beta-1}, \quad Re(\alpha) > 0.$

(2) $\left({}_a^NI^\alpha(t-a)^{\beta-1}\right)(t) = \frac{\Gamma(\beta-\alpha+1)}{\Gamma(2-\alpha)\Gamma(\beta+1)}(t-a)^\beta, \quad Re(\alpha) > 0$, *where ${}_a^NI^\alpha$ denotes the normalized Riemann-Liouville fractional integral operator.*

Lemma 2.6. *[24] If $Re(\alpha) > 0$ and $Re(\beta) > 0$, then for γ is locally integrable on $[a,b]$, we have*

$$
{}_aI^\alpha({}_aI^\beta\gamma)(t) = {}_aI^\beta({}_aI^\alpha\gamma)(t) = ({}_aI^{\alpha+\beta}\gamma)(t). \tag{17}
$$

Proposition 2.7. *It holds that $(a^{nAB}I^\alpha\, 1)(t) = -\frac{t-a}{\lambda_\alpha} = \frac{t-a}{|\lambda_\alpha|}, \;\; 0 < \alpha < 1$.*

Proof. We have

$$
\begin{aligned}
\alpha\Big({}_aI^\alpha AB_\alpha\Big)(t) &= \Big({}_aI^\alpha\mathcal{E}_{\alpha,2}(\lambda_\alpha, t-a)\Big)(t) \\
&= \sum_{k=0}^{\infty}\frac{(\lambda_\alpha)^k}{\Gamma(\alpha k+2)}({}_aI^\alpha\;(t-a)^{\alpha k+1})(t) \\
&= \sum_{k=0}^{\infty}\frac{(\lambda_\alpha)^k}{\Gamma(\alpha k+2)}\frac{\Gamma(\alpha k+2)}{\Gamma(\alpha k+2+\alpha)}(t-a)^{\alpha k+\alpha+1} \\
&= \frac{1}{\lambda_\alpha}\sum_{k=1}^{\infty}\frac{(\lambda_\alpha)^k}{\Gamma(\alpha k+2)}(t-a)^{\alpha k+1} = \frac{1}{\lambda_\alpha}(\mathcal{E}_{\alpha,2}(\lambda_\alpha, t-a) - (t-a))
\end{aligned} \tag{18}
$$

Thus,

$$
\begin{aligned}
(a^{nAB}I^\alpha\, 1)(t) &= (1-\alpha)AB_\alpha(t) + \alpha\Big({}_aI^\alpha AB_\alpha\Big)(t) \\
&= \frac{1-\alpha}{\alpha}\mathcal{E}_{\alpha,2}(\lambda_\alpha, t-a) + \frac{1}{\lambda_\alpha}(\mathcal{E}_{\alpha,2}(\lambda_\alpha, t-a) - (t-a)) \\
&= -\frac{1}{\lambda_\alpha}(t-a),
\end{aligned} \tag{19}
$$

which completes the proof. Note that because $\lambda_\alpha = -\frac{\alpha}{1-\alpha} < 0$, then $-\lambda_\alpha = |\lambda_\alpha|$. $\square$

Lemma 2.8. *If $a(t)$ is a nondecreasing function on $[a,b]$ and $\gamma(t)$ is locally integrable on $[a,b]$ non-negative, then $\Big({}_{a}^{AB}I^{\alpha}a(.)\gamma(.)\Big)(t) \leq a(t)({}_{a}^{AB}I^{\alpha}\gamma)(t)$.*

Proof. The proof is straight forward. It follows by noticing that $a(s) \leq a(t)$ for all $s \in [a,t]$. $\square$

3. Gronwall inequalities for the normalized fractional integrals

In this section, we prove Gronwall inequality within the normalized fractional derivatives with Mittag-Leffler kernel and exponential kernel. The binomial expansion of the powers of the $CF-$ and $AB-$ fractional integrals will play a crucial role.

Theorem 3.1. *(A normalized Gronwall Inequality) Let $\alpha \in (0,1]$, $AB_\alpha(t)\gamma(t)$, $AB_\alpha(t)l(t)$ be nonnegative locally integrable functions on [a,b) and $r(t)$ be nonnegative and nondecreasing and continuous function defined on $t \in [a,b)$ such that $r(t) \leq K$, where K is a positive constant. If*

$$\gamma(t) \leq l(t) + r(t)\Big({}_{a}^{nAB}I^{\alpha}\gamma\Big)(t), \tag{20}$$

then

$$\gamma(t) \leq l(t) + \sum_{i=1}^{\infty} \frac{(AB_\alpha(t)r(t))^i}{AB_\alpha(t)} \sum_{k=0}^{i} \binom{i}{k} (1-\alpha)^{i-k}\alpha^k \Big({}_{a}I^{k\alpha}AB_\alpha(.)l(.)\Big)(t), \quad t \in [a,b). \tag{21}$$

Proof. Define the linear operator

$$\Lambda\gamma(t) = r(t)\,{}_{a}^{nAB}I^{\alpha}\gamma(t) = r(t)\,{}_{a}^{AB}I^{\alpha}[AB_\alpha(t)\gamma(t)], \ t \in [a,b).$$

From (20) in the assumption, we have $\gamma(t) \leq l(t) + \Lambda\gamma(t)$. If we proceed inductively and use that the operator Λ is linear and nondecreasing on nonnegative functions (since $AB_\alpha(t)$ is nonnegative by Proposition 2.3 and that $AB_\alpha(a) = 0$), then we can prove that $\gamma(t) \leq \sum_{j=0}^{i-1} \Lambda^j l(t) + \Lambda^i\gamma(t)$, for each $i = 1, 2, ...$ We have used the convention that $\Lambda^0\gamma(t) = \gamma(t)$. We claim that

$$\begin{aligned} \Lambda^i\gamma(t) &\leq \frac{\Big(r(t)AB_\alpha(t)\Big)^i}{AB_\alpha(t)} \Big({}_{a}^{AB}I^{\alpha}AB_\alpha(.)\gamma(.)\Big)^i(t) \\ &= \frac{\Big(r(t)AB_\alpha(t)\Big)^i}{AB_\alpha(t)} \sum_{k=0}^{i}\binom{i}{k}(1-\alpha)^{i-k}\alpha^k\Big({}_{a}I^{\alpha k}AB_\alpha(.)\gamma(.)\Big)(t), \end{aligned} \tag{22}$$

and $\Lambda^i\gamma(t) \to 0$ as $i \to \infty$ for $t \in [a,b)$. The power i above in case of the integral type operator means its action i times. It is easy to see that (22) is valid for $i = 1$. Assume that it is true for $i = j$, that is,

$$\Lambda^j\gamma(t) \leq \frac{\Big(r(t)AB_\alpha(t)\Big)^j}{AB_\alpha(t)}\Big({}_{a}^{AB}I^{\alpha}AB_\alpha(.)\gamma(.)\Big)^j(t).$$

If $i = j+1$, then on the light of that $\Lambda\varphi_1(t) \leq \Lambda\varphi_2(t)$ for $\varphi_1(t) \leq \varphi_2(t)$, that $r(t)$ and $AB_\alpha(t)$ are nondecreasing, and by the help of Lemma 2.8 we have

$$\begin{aligned} \Lambda^{j+1}\gamma(t) &= \Lambda(\Lambda^j\gamma(t)) \leq r(t)\,{}_{a}^{AB}I^{\alpha}\Big[AB_\alpha(t)\frac{\Big(r(t)AB_\alpha(t)\Big)^j}{AB_\alpha(t)}\Big({}_{a}^{AB}I^{\alpha}AB_\alpha(.)\gamma(.)\Big)^j(t)\Big] \\ &\leq \frac{\Big(r(t)AB_\alpha(t)\Big)^{j+1}}{AB_\alpha(t)}\Big({}_{a}^{AB}I^{\alpha}AB_\alpha(.)\gamma(.)\Big)^{j+1}(t). \end{aligned}$$

Hence, the induction step $i = j + 1$ follows. That is (22) is valid for $i = j + 1$. Furthermore, since $AB_\alpha(t)\gamma(t)$ is locally integrable (hence bounded: $AB_\alpha(t)\gamma(t) \leq c$, for some $c > 0$ and for all t) and by using the assumption that $r(t) \leq K$, and Proposition 2.5 with $\beta = 1$, one can figure out that

$$\Lambda^i \gamma(t) \leq \frac{cL^i}{\Gamma(i\alpha + 1)} \to 0, \text{as } i \to \infty,\ c, L > 0.$$

We have used above that $(I^{k\alpha}1)(t) \leq (I^{i\alpha}1)(t)$ for all $0 \leq k \leq i$, $\sum_{k=0}^{i} \binom{i}{k}(1-\alpha)^{i-k}\alpha^k = 1$, and by Proposition 2.3 that $AB_\alpha(t) \leq AB_\alpha(b)$ for all $t \in [a, b)$. Also L is a positive constant depending on $K, (b-a)^\alpha$ and $AB_\alpha(b)$. To complete the proof, we let $i \to \infty$ in

$$\gamma(t) \leq \sum_{j=0}^{i-1} \Lambda^j l(t) + \Lambda^i \gamma(t) = l(t) + \sum_{j=1}^{i-1} \Lambda^j l(t) + \Lambda^i \gamma(t),$$

to reach at $\gamma(t) \leq l(t) + \sum_{j=1}^{\infty} \Lambda^j l(t)$. By the help of the semi group property in Lemma 2.6, and the definition of Λ we get (21). Indeed, we have

$$\begin{aligned} \Lambda^i l(t) &= \left(r(t)\, {}_a^{AB}I^\alpha \Big(AB_\alpha(.) l(.) \Big)(t) \right)^i \qquad (23) \\ &\leq \frac{\Big(r(t) AB_\alpha(t)\Big)^i}{AB_\alpha(t)} \left({}_a^{AB}I^\alpha \Big(AB_\alpha(.) l(.) \Big)(t) \right)^i \\ &= \frac{\Big(r(t) AB_\alpha(t)\Big)^i}{AB_\alpha(t)} \sum_{k=0}^{i} \binom{i}{k} (1-\alpha)^{i-k} \alpha^k \Big({}_aI^{k\alpha} AB_\alpha(.) l(.) \Big)(t), \end{aligned}$$

where above we have used that $r(t)$ and $AB_\alpha(t)$ are nondecreasing by assumption and by Proposition 2.3, respectively. This completes the proof. □

Following similar steps as in the proof of Theorem 3.1, we can state the following Gronwall inequality version in the case of non-normalized case. This version is different from that in Remark 1.1 in [13].

Theorem 3.2. *(An AB-Gronwall Inequality) Let $\alpha \in (0, 1]$, $\gamma(t)$, $l(t)$ be nonnegative locally integrable functions on [a,b) and $r(t)$ be nonnegative and nondecreasing and continuous function defined on $t \in [a, b)$ such that $r(t) \leq K$, where K is a positive constant. If*

$$\gamma(t) \leq l(t) + r(t) \Big({}_a^{AB}I^\alpha \gamma \Big)(t), \qquad (24)$$

then

$$\gamma(t) \leq l(t) + \sum_{i=1}^{\infty} r(t)^i \sum_{k=0}^{i} \binom{i}{k} (1-\alpha)^{i-k} \alpha^k \Big({}_aI^{k\alpha} l(.) \Big)(t), \quad t \in [a, b), \qquad (25)$$

Corollary 3.3. *Under the hypothesis of Theorem 3.1, assume further that $l(t)$ is a nondecreasing function for $t \in [a, b)$, then*

$$\gamma(t) \leq l(t) \Big[1 + \sum_{i=1}^{\infty} \frac{\Big(r(t) AB_\alpha(t)\Big)^i}{AB_\alpha(t)} \sum_{k=0}^{i-1} \binom{i-1}{k} (1-\alpha)^{i-1-k} \alpha^k \frac{(t-a)^{k\alpha+1}}{\Gamma(k\alpha + 2)} \Big]. \qquad (26)$$

Proof. From the proof of (21), using the assumption that $l(t)$ is a nondecreasing function for $t \in [a,b)$, and by noting that ${}_a^{AB}I^{\alpha} AB_\alpha(t) = (t-a)$ we have

$$\begin{aligned}
\gamma(t) &\leq l(t) + \sum_{i=1}^{\infty} \frac{\left(r(t)AB_\alpha(t)\right)^i}{AB_\alpha(t)} \left({}_a^{AB}I^{\alpha}\right)^i [AB_\alpha(t) l(t)] \qquad (27)\\
&\leq l(t)\Big[1 + \sum_{i=1}^{\infty} \frac{\left(r(t)AB_\alpha(t)\right)^i}{AB_\alpha(t)} \left({}_a^{AB}I^{\alpha}\right)^i [AB_\alpha(t)]\Big]\\
&= l(t)\Big[1 + \sum_{i=1}^{\infty} \frac{\left(r(t)AB_\alpha(t)\right)^i}{AB_\alpha(t)} \left({}_a^{AB}I^{\alpha}\right)^{i-1} (t-a)\Big]\\
&= l(t)\Big[1 + \sum_{i=1}^{\infty} \frac{\left(r(t)AB_\alpha(t)\right)^i}{AB_\alpha(t)} \sum_{k=0}^{i-1} \binom{i-1}{k} (1-\alpha)^{i-1-k}\alpha^k \left({}_aI^{k\alpha}\right)(t-a)\Big]\\
&= l(t)\Big[1 + \sum_{i=1}^{\infty} \frac{\left(r(t)AB_\alpha(t)\right)^i}{AB_\alpha(t)} \sum_{k=0}^{i-1} \binom{i-1}{k} (1-\alpha)^{i-1-k}\alpha^k \frac{(t-a)^{k\alpha+1}}{\Gamma(k\alpha+2)}\Big],
\end{aligned}$$

which completes the proof. □

If we also follow the same steps in Corollary 3.3, we can obtain a different version from Theorem 2.1 in [13]. Indeed, we state

Corollary 3.4. *Under the hypothesis of Theorem 3.2, assume further that $l(t)$ is a nondecreasing function for $t \in [a,b)$, then*

$$\gamma(t) \leq l(t)\Big[1 + \sum_{i=1}^{\infty} \left(r(t)\right)^i \sum_{k=0}^{i-1} \binom{i-1}{k} (1-\alpha)^{i-1-k}\alpha^k \frac{(t-a)^{k\alpha+1}}{\Gamma(k\alpha+2)}\Big]. \qquad (28)$$

If we make use of (21), and use the identity

$${}_aI^{k\alpha}\mathcal{E}_{\alpha,2}(\lambda, t-a) = \mathcal{E}_{\alpha,k\alpha+2}(\lambda_\alpha, t-a), \qquad (29)$$

then we can state:

Corollary 3.5. *Under the hypothesis of Theorem 3.1, assume further that $l(t)$ is a nondecreasing function for $t \in [a,b)$, then*

$$\begin{aligned}
\gamma(t) &\leq l(t)\Big[1 + \sum_{i=1}^{\infty} \frac{\left(r(t)AB_\alpha(t)\right)^i}{AB_\alpha(t)} \sum_{k=0}^{i} \binom{i}{k} (1-\alpha)^{i-k}\alpha^{k-1} \mathcal{E}_{\alpha,k\alpha+2}(\lambda_\alpha, t-a)\Big] \qquad (30)\\
&\leq l(t)\Big[1 + \sum_{i=1}^{\infty} \frac{\left(r(t)AB_\alpha(t)\right)^i}{\alpha AB_\alpha(t)} \mathcal{E}_{\alpha,i\alpha+2}(\lambda_\alpha, t-a)\Big].
\end{aligned}$$

The second inequality in Corollary 3.5 follows since $AB_\alpha(t)$ is nondecreasing and $AB_\alpha(a) = 0$ and therefore $AB_\alpha(t)$ is nonnegative for $t \geq a$. Hence, ${}_aI^{k\alpha}AB_\alpha(t) \leq {}_aI^{i\alpha}AB_\alpha(t)$ for $k \leq i$. Furthermore, if we use that $AB_\alpha(t)$ is nondecreasing, as proven in Proposition 2.3, then under the assumption of Corollary 3.5, and by making use of Lemma 2.8 with $a(t) = AB_\alpha(t)$ and $\gamma(t) = 1$, we can have the following inequality:

$$\gamma(t) \leq l(t)\Big[1 + \sum_{i=1}^{\infty} \left(r(t)AB_\alpha(t)\right)^i \sum_{k=0}^{i} \binom{i}{k} (1-\alpha)^{i-k}\alpha^k \frac{(t-a)^{\alpha k}}{\Gamma(\alpha k+1)}\Big]$$

Following similar steps as in the proof of Theorem 3.1, for the normalized Caputo-Fabrizio derivatives, we can state the following:

Theorem 3.6. *(A normalized Gronwall Inequality in CF) Let $\alpha \in (0,1]$, $CF_\alpha(t)\gamma(t)$, $CF_\alpha(t)l(t)$ be nonnegative locally integrable functions on [a,b) and $r(t)$ be nonnegative and nondecreasing and continuous function defined on $t \in [a,b)$ such that $r(t) \leq K$, where K is a positive constant. If*

$$\gamma(t) \leq l(t) + r(t)\Big({}_{a}^{nCF}I^\alpha \gamma\Big)(t), \tag{31}$$

then

$$\gamma(t) \leq l(t) + \sum_{i=1}^{\infty} \frac{(CF_\alpha(t) r(t))^i}{CF_\alpha(t)} \sum_{k=0}^{i} \binom{i}{k} (1-\alpha)^{i-k}\alpha^k \Big({}_aI^k CF_\alpha(.)l(.)\Big)(t), \quad t \in [a,b). \tag{32}$$

Corollary 3.7. *Under the hypothesis of Theorem 3.6, assume further that $l(t)$ is a nondecreasing function for $t \in [a,b)$, then*

$$\gamma(t) \leq l(t)\Big[1 + \sum_{i=1}^{\infty} \frac{\Big(r(t)CF_\alpha(t)\Big)^i}{CF_\alpha(t)} \sum_{k=0}^{i-1} \binom{i-1}{k} (1-\alpha)^{i-1-k}\alpha^k \frac{(t-a)^{k+1}}{(k+1)!}\Big]. \tag{33}$$

Proof. The proof is similar to the proof of Corollary 3.3, and by noticing that ${}_{a}^{CF}I^\alpha CF_\alpha(t) = (t-a)$. □

4. Cauchy problem with normalized AB-derivative

As an application of the Grownwall inequalities, we established existence and uniqueness results to the the fractional initial value problem FIVP (1-2).

Proposition 4.1. *For $\gamma \in H^1[a,b]$, $\gamma(t)$ is a solution to the fractional initial value problem (1-2) if and only if $\gamma(t)$ solves the integral equation*

$$\gamma(t) = \gamma_0 + \Big(a^{nAB}I^\alpha f(t,\gamma)\Big)(t). \tag{34}$$

Proof. Applying the operator ${}_{a}^{nAB}I^\alpha$ to both sides of the problem (1-2) and making use of the action:

$${}_{a}^{nAB}I^\alpha o \; {}_anAB^\alpha \gamma(t) = \gamma(t) - \gamma(a), \quad \gamma \in H^1(a,b),$$

we obtain the solution (34). Conversely, applying the operator ${}_anAB^\alpha$ to (34) and making use of the action:

$${}_anAB^\alpha o \; {}_{a}^{nAB}I^\alpha \gamma(t) = \gamma(t),$$

we satisfy the differential equation (1). Moreover, because $AB_\alpha(a) = 0$ implies that $\Big(a^{nAB}I^\alpha f(t,\gamma)\Big)(a) = 0$, and hence the solution representation in (34) satisfies the initial data $\gamma(a) = \gamma_0$. □

In the following we assume that $f(t,\gamma)$ satisfies the Lipschitz condition

$$(H_1) \quad |f(t,\gamma_1) - f(t,\gamma_2)| \leq L|\gamma_1 - \gamma_2|, \;\; L > 0, \;\; \text{for all } t \in [a,b], \; \gamma_1,\gamma_2 \in \mathbb{R}.$$

Lemma 4.2. *(Uniqueness result) The fractional initial value problem (1-2) admits at most one solution in $H^1[a,b]$.*

Proof. Let $\gamma_1,\gamma_2 \in H^1[a,b]$ be solutions to (1-2), from Eq. (34) we have

$$\begin{aligned} |\gamma_1 - \gamma_2| &= \Big|\Big(a^{nAB}I^\alpha (f(t,\gamma_1) - f(t,\gamma_2))\Big)(t)\Big| \leq \big(a^{nAB}I^\alpha |f(t,\gamma_1) - f(t,\gamma_2)|\big)(t) \\ &\leq (a^{nAB}I^\alpha L|\gamma_1 - \gamma_2|)(t) = L(a^{nAB}I^\alpha|\gamma_1 - \gamma_2|)(t). \end{aligned} \tag{35}$$

Applying the Gronwall inequality in Theorem 3.2 we arrive at $|\gamma_1 - \gamma_2| \leq 0$, which proves that $\gamma_1 - \gamma_2 = 0$, and completes the proof. □

Lemma 4.3. *Assume that (H_1) holds true and f is t-uniformly bounded in γ. If $L\frac{b-a}{|\lambda_\alpha|} < 1$, then the problem in (1-2) has a unique solution.*

Proof. Let $F = C([a,b],\mathbb{R})$ denotes the Banach space of continuous functions $h(t) : [a,b] \to \mathbb{R}$, with the norm

$$\|h\|_F = \sup_{t\in[a,b]} |h(t)|.$$

We define the operator $\eta : F \to F$, as follows

$$(\eta\gamma)(t) = \gamma_0 + \Big({}^{AB}_{a^n}I^{\alpha} f(t,\gamma)\Big)(t),\ t \in [a,b].$$

We shall prove that η admits a unique fixed point.
At first we prove that η is a bounded operator. We have

$$|(\eta\gamma)(t)| \leq |\gamma_0| + ({}^{AB}_{a^n}I^{\alpha} |f(t,\gamma)(t)|).$$

Because f is t-bounded in γ and there exists $M > 0$, with $|f(t,\gamma)| \leq M,\ t \in [a,b]$, and $\gamma \in H^1[a,b]$. Thus,

$$\begin{aligned} |(\eta\gamma)(t)| &\leq |\gamma_0| + M({}^{AB}_{a^n}I^{\alpha}\, 1)(t) \\ &= |\gamma_0| + M\frac{t-a}{-\lambda_\alpha} \leq |\gamma_0| + M\frac{b-a}{|\lambda_\alpha|} \end{aligned} \tag{36}$$

which proves that η is a bounded well-defined operator. Let $\gamma_1, \gamma_2 \in F$, and $t \in [a,b]$, we have

$$\begin{aligned} |(\eta\gamma_1)(t) - (\eta\gamma_2)(t)| &= |\Big({}^{AB}_{a^n}I^{\alpha} f(t,\gamma_1)\Big)(t) - \Big({}^{AB}_{a^n}I^{\alpha} f(t,\gamma_2)\Big)(t)| \\ &= \Big|\Big({}^{AB}_{a^n}I^{\alpha}\Big(f(t,\gamma_1)(t) - f(t,\gamma_2)\Big)\Big)(t)\Big| \\ &\leq \Big({}^{AB}_{a^n}I\alpha\Big|f(t,\gamma_1)(t) - f(t,\gamma_2)(t)\Big|\Big)(t) \\ &\leq \Big({}^{AB}_{a^n}I^{\alpha}\Big(L|\gamma_1(t) - \gamma_2(t)|\Big)\Big)(t) = L\Big({}^{AB}_{a^n}I^{\alpha}\Big(|\gamma_1(t) - \gamma_2(t)|\Big)\Big)(t). \end{aligned}$$

Thus,

$$\begin{aligned} \|(\eta\gamma_1)(t) - (\eta\gamma_2)(t)\| &\leq L\|\gamma_1(t) - \gamma_2(t)\|\,({}^{AB}_{a^n}I^{\alpha} 1)(t) \\ &= L\|\gamma_1(t) - \gamma_2(t)\|\,(-\frac{t-a}{\lambda_\alpha}) \leq L\|\gamma_1(t) - \gamma_2(t)\|\,\frac{b-a}{|\lambda_\alpha|}. \end{aligned}$$

Because $L\,\frac{b-a}{|\lambda_\alpha|} < 1$, we deduce that η is a contraction mapping and it admits a unique solution γ by the Banach fixed point theorem. □

Lemma 4.4. *(Stability result) Assume that (H_1) holds true, and γ_1, γ_2 are solutions to (1) with $\gamma_1(0) = \hat{\gamma}_1$, and $\gamma_2(0) = \hat{\gamma}_2$. If $L\,\frac{b-a}{|\lambda_\alpha|} < 1$, then it holds that*

$$\|\gamma_1 - \gamma_2\|_F \leq \frac{1}{1 - L\,\frac{b-a}{|\lambda_\alpha|}} |\hat{\gamma}_1 - \hat{\gamma}_2|.$$

Proof. We have

$$\gamma_1 - \gamma_2 = \hat{\gamma}_1 - \hat{\gamma}_2 + \Big({}^{AB}_{a^n}I^{\alpha} f(t,\gamma_1) - f(t,\gamma_2)\Big)(t),$$

and thus,

$$\begin{aligned}|\gamma_1-\gamma_2| &\leq |\hat{\gamma}_1-\hat{\gamma}_2| + \Big(a^{nAB}I^{\alpha}|f(t,\gamma_1)-f(t,\gamma_2)|\Big)(t),\\ &\leq |\hat{\gamma}_1-\hat{\gamma}_2| + \Big(a^{nAB}I^{\alpha}L|\gamma_1-\gamma_2|\Big)(t),\\ &\leq |\hat{\gamma}_1-\hat{\gamma}_2| + L\|\gamma_1-\gamma_2\|_F(a^{nAB}I^{\alpha}\,1)(t) = |\hat{\gamma}_1-\hat{\gamma}_2| + L\|\gamma_1-\gamma_2\|_F\frac{t-a}{|\lambda_\alpha|}\end{aligned}$$

The above inequality yields

$$\|\gamma_1-\gamma_2\|_F \leq |\hat{\gamma}_1-\hat{\gamma}_2| + L|\gamma_1-\gamma_2\|_F\frac{b-a}{|\lambda_\alpha|}.$$

Because $L\,\frac{b-a}{|\lambda_\alpha|} < 1$, we arrive at

$$\|\gamma_1-\gamma_2\|_F \leq \frac{1}{1-L\,\frac{b-a}{|\lambda_\alpha|}}|\hat{\gamma}_1-\hat{\gamma}_2|,$$

which proves the result. □

5. Comparison principles

In the study of fractional differential equations, the comparison principle is a vital tool that enables the establishment of upper and lower bounds for solutions, providing a framework for analyzing the existence, uniqueness, and behavior of solutions. By applying the comparison principle, one can derive rigorous results about the monotonicity, stability, and asymptotic behavior of fractional systems, see [6–9, 20]. In this section, we extend some known comparison principles to the fractional differential inequalities with the normalized derivative of Atangana-Baleanu derivative in the Caputo sense. And then use these comparison principles to obtain a pre-norm estimate of the solution to a related Cauchy problem.

Lemma 5.1. *Let a function $\gamma \in H^1[a,b]$ attain a global minimum at $t_0 \in (a,b]$, then it holds that*

$$(\,_anAB^{\alpha}\gamma)(t_0) \leq |\lambda_\alpha|\frac{E_\alpha(\lambda_\alpha(t_0-a)^\alpha)}{\mathcal{E}_{\alpha,2}(\lambda_\alpha,t-a)}\Big(\gamma(t_0)-\gamma(a)\Big) \leq 0. \tag{37}$$

Proof. From Lemma 2.2 in [4], we have

$$(_aAB^{\alpha}\gamma)(t_0) \leq \frac{1}{1-\alpha}E_\alpha(\lambda_\alpha(t_0-a)^\alpha)\Big(\gamma(t_0)-\gamma(a)\Big) \leq 0.$$

The result follows directly as $\mathcal{E}_{\alpha,2}(\lambda_\alpha,t_0-a) > 0$, and

$$(\,_anAB^{\alpha}\gamma)(t_0) = \frac{(_aAB^{\alpha}\gamma)(t_0)}{AB_\alpha(t_0)} = \alpha\frac{(_aAB^{\alpha}\gamma)(t_0)}{\mathcal{E}_{\alpha,2}(\lambda_\alpha,t_0-a)}.$$

□

Lemma 5.2. *Let a function $\gamma \in H^1[a,b] \cap C[a,b]$ satisfy the fractional inequality*

$$(\,_anAB^{\alpha}\gamma)(t) + s(t)\gamma(t) \geq 0, \quad t > a \tag{38}$$

$\gamma(a) \geq 0$, where $s \in C[a,b]$ is a nonnegtive function. Then it holds that

$$\gamma(t) \geq 0,\ t \in [a,b].$$

Proof. Assume the result is untrue, we shall reach a contradiction. Because $\gamma(t) \ngeq 0$, on $[a,b]$ and $\gamma(t)$ is continuous, there exists $t_0 \in [a,b]$, with $\gamma(t_0) < 0$, and γ has a global minimum at t_0. Because $\gamma(a) \geq 0$, then $t_0 \in (a,b]$ with $\gamma(t_0) < \gamma(a)$, and thus

$$|\lambda_\alpha| \frac{E_\alpha(\lambda_\alpha(t_0-a)^\alpha)}{\mathcal{E}_{\alpha,2}(\lambda_\alpha, t_0-a)} \Big(\gamma(t_0) - \gamma(a)\Big) < 0.$$

The above result together with $s(t_0)\gamma(t_0) \leq 0$, contradict the fractional inequality in (39). □

Lemma 5.3. *Let a function* $\gamma \in H^1[a,b] \cap C[a,b]$ *be a solution to*

$$({}_anAB^\alpha \gamma)(t) + s(t)\gamma = g(t), \quad t > a \tag{39}$$

where $s \in C[a,b]$ *is a positive function. Then it holds that*

$$\|\gamma(t)\| = \max_{t\in[a,b]} |\gamma(t)| \leq M = \max_{t\in[a,b]} \{|\frac{g(t)}{s(t)}|, \gamma(a)\},$$

provided that the maximum M exists.

Proof. We have $M \geq |\frac{g(t)}{s(t)}|$, or $Ms(t) \geq |g(t)|$. Let $w_1(t) = M - \gamma,\ t \in [a,b]$. There holds

$$\begin{aligned}
({}_anAB^\alpha w_1)(t) + s(t)w_1(t) &= -({}_anAB^\alpha \gamma)(t) + s(t)(M - \gamma(t)) \\
&= -({}_anAB^\alpha \gamma)(t) - s(t)\gamma(t) + s(t)M = -g(t) + s(t)M \\
&\geq -g(t) + |g(t)| \geq 0
\end{aligned}$$

which together with $w_1(a) = M - \gamma(a) \geq 0$, proves that $w_1(t) = M - \gamma(t) \geq 0,\ t \in [a,b]$, by virtue of the result in Lemma 5.2. Let $w_2(t) = \gamma(t) + M$, there holds

$$\begin{aligned}
({}_anAB^\alpha w_2)(t) + s(t)w_2(t) &= ({}_anAB^\alpha \gamma)(t) + s(t)(\gamma(t) + M) \\
&= ({}_anAB^\alpha \gamma)(t) + s(t)\gamma(t) + s(t)M = g(t) + s(t)M \\
&\geq g(t) + |g(t)| \geq 0,
\end{aligned}$$

which proves that $w_2(t) = \gamma(t) + M \geq 0,\ t \in [a,b]$. We have $\gamma(t) \leq M$, and $-\gamma(t) \leq M$, on $[a,b]$ which imply $|\gamma(t)| \leq M,\ t \in [a,b]$, and completes the proof. □

6. Illustrative examples and discussion

At first we present two examples to illustrate the efficiency of the obtained results.

Example 6.1. *We consider the FIVP*

$$({}_0nAB^\alpha \gamma)(t) = -\frac{1}{2}e^{-t}\gamma(t),\ t \in (0,b],\ \ \gamma(0) = 1. \tag{40}$$

We have

$$|f(t,\gamma_1) - f(t,\gamma_2)| = \frac{1}{2}e^{-t}|\gamma_1 - \gamma_2| \leq \frac{1}{2}|\gamma_1 - \gamma_2|,$$

and thus $f(t,\gamma) = \frac{1}{2}e^{-t}\gamma$, *satisfies the Lipschitz condition with Lipschitz constant* $L = \frac{1}{2}$. *By virtue with result in Lemma 4.3 the problem admits a unique solution provided that* $b < 2|\lambda_\alpha|$. *Also, by the comparison principle in Lemma 5.2 we have*

$$\gamma(t) \geq 0,\ t \in [0,b].$$

Remark 6.2. *Because* $f(0,\gamma(0)) = \frac{1}{2}\gamma(0) = \frac{1}{2} \neq 0$, *then the above problem with the regular Atangana-Baleanu derivative admits no solution, see [4].*

Example 6.3. *As a second example, we consider the nonlinear FIVP*

$$({}_0nAB^{\alpha}\gamma)(t) = \frac{\gamma}{1+\gamma^2},\ t \in (0,b],\ \ \gamma(0) = 1. \tag{41}$$

We have $f(t,\gamma) = \frac{\gamma}{1+\gamma^2} = g(\gamma)$, *and*

$$g'(\gamma) = \frac{1-\gamma^2}{(1+\gamma^2)^2} \le \frac{1}{1+\gamma^2} \le 1.$$

Thus by the mean value theorem $f(t,\gamma) = \frac{\gamma}{1+\gamma^2}$ *satisfies the Lipschitz condition with Lipschitz constant* $L = 1$. *By virtue with result in Lemma 4.3 the problem admits a unique solution provide that* $b < |\lambda_\alpha|$. *Also, by the comparison principle in Lemma 5.2 we have*

$$\gamma(t) \ge 0,\ t \in [0,b].$$

Example 6.4. *We consider the FIVP*

$$({}_0nAB^{\alpha}\gamma)(t) = \lambda\gamma(t) + 1,\ t \in (0,b],\ \ \gamma(a) = \gamma_a. \tag{42}$$

or equivalently,

$$({}_0AB^{\alpha}\gamma)(t) = \lambda AB_\alpha(t)\gamma(t) + 1.AB_\alpha(t),\ t \in (0,b],\ \ \gamma(a) = \gamma_a. \tag{43}$$

We have, the solution of (42) *as described by*

$$\begin{aligned} \gamma(t) &= \gamma(a) + (1-\alpha)\frac{(\lambda+1)}{\alpha}\sum_{i=0}^{\infty}\frac{\lambda_\alpha^i (t-a)^{i\alpha+1}}{\Gamma(\alpha i+2)}(\gamma(t)+1) \\ &+ \frac{\lambda}{\alpha}\sum_{i=0}^{\infty}\frac{\lambda_\alpha^i}{\Gamma(\alpha i+2)}\int_a^t (s-a)^{i\alpha+1}(t-s)^{\alpha-1}\gamma(s)ds + \frac{\lambda}{\alpha}\sum_{i=0}^{\infty}\frac{\lambda_\alpha^i}{\Gamma(\alpha i+2)}\int_a^t (s-a)^{i\alpha+1}(t-s)^{\alpha-1}ds \end{aligned}$$

upon simplification, we get by using $\frac{1}{\Gamma(\alpha)}\int_a^t (s-a)^{i\alpha+1}(t-s)^{\alpha-1}\gamma(s)ds = I_a^\alpha (t-a)^{\alpha i+1}\gamma(t)$

$$\begin{aligned} \gamma(t) &= \gamma(a) + (1-\alpha)\frac{(\lambda+1)}{\alpha}\sum_{i=0}^{\infty}\frac{\lambda_\alpha^i (t-a)^{i\alpha+1}}{\Gamma(\alpha i+2)}(\gamma(t)+1) \\ &+ \lambda\sum_{i=0}^{\infty}\frac{\lambda_\alpha^i}{\Gamma(\alpha i+2)} I_a^\alpha (t-a)^{\alpha i+1}\gamma(t) \\ &+ \frac{\lambda}{\alpha}\sum_{i=0}^{\infty}\frac{\lambda_\alpha^i (t-a)^{i\alpha+1}}{\Gamma(i\alpha+2+\alpha)}. \end{aligned} \tag{44}$$

Let us present the result graphically for different fractional orders, where we take $\lambda = 1,\ a = \gamma_a = 1$. *The graphical illustration is shown in Figure 1.*

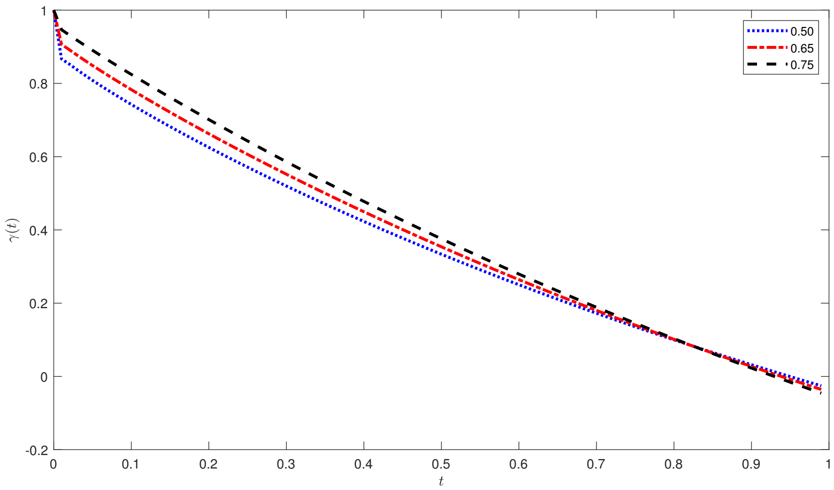

Figure 1: Graphical presentation of expression (43) using the given fractional orders values of α and taking $\lambda = 1,\ a = 1$.

7. Conclusions

Recently, fractional derivatives with nonsingular kernels have been investigated extensively. They show some shortcomings in modeling as the related Cauchy problem admits only the trivial solution, and related fractional differential equations admit solutions by imposing non-necessary extra conditions. Also, they don't satisfy the so-called fundamental theorem of fractional calculus. The normalized fractional derivatives with nonsingular kernel overcome all of the above mentioned shortcomings. We have formulated and proved extended Grownwall inequalities related to the normalized fractional integral operators of Caputo-Fabrizio and Atangana-Baleanu types. We then have established existence and uniqueness results to the fractional Cauchy problem with the normalized fractional derivative of Atangana-Baleanu type via extended Gronwall inequalities, and the Banach fixed point theorem. Some comparison principles were derived to fractional differential inequalities with the normalized AB-derivative. These principles were implemented to estimate the solution of related linear fractional differential equations. We have given two examples to ilustrate the theoretical results, and further a numerical example have been solved to give the solutions of a non-homogeneous normalized system in the Mittag-Leffler kernel case together with an illustrative graph. The results indicate that the normalized derivatives can be implemented in modeling several dynamical systems without further limitations, and they encourage researchers to study related models. During the presentation and investigations of our main results, we have noticed that the fact $AB_\alpha(a) = CF_\alpha(a) = 0$, and that the normalizing factors $AB_\alpha(t)$ and $CF_\alpha(t)$ are nondecreasing positive, played a crucial role.

For possible future contributions in the fields of control theory and stochastic calculus within the normalized Atangana-Baleanu fractional operators we may refer to [12, 28]. The results we have derived for normalized AB-fractional and CF-fractional operators, along with their novel Gronwall inequalities, provide a foundation for exploring and refining various qualitative studies, such as the stability analysis of different fractional differential systems. For further details, we refer the reader to recent works [21–23, 25].

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