Filomat 39:35 (2025), 12725–12756
https://doi.org/10.2298/FIL2535725B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Hybrid harmonic integral inequalities via multiplicative calculus with applications

**Saad Ihsan Butt[a], Muhammad Umar[a], Dawood Khan[b,*]**

[a]*Department of Mathematics, COMSATS University Islamabad, Lahore Campus 54000, Pakistan*
[b]*Department of Mathematics, University of Balochistan Quetta 87300, Pakistan*

**Abstract.** In this study, we employ proportional Caputo-Hybrid (PCH) operators to establish Hermite-Hadamard (HH) type inequalities for multiplicative harmonically convex functions. A key advantage of these fractional operators lies in their flexibility, allowing the recovery of various forms of inequalities. Specifically, traditional HH-type inequalities for multiplicative harmonically convex functions emerge when the parameter $\alpha_o$ is set to 1, while for multiplicatively differentiable harmonic convex functions, they appear when $\alpha_o = 0$. To support our findings, we present graphical illustrations based on concrete examples. Additionally, we explore applications to special functions, leading to novel multiplicative fractional order recurrence relations. A promising avenue for future research involves extending these inequalities to interval calculus, where functions take interval values rather than precise numbers, broadening their applicability to uncertainty analysis, numerical approximations, and fractional differential equations..

## 1. Introduction

Let $I = [\mathfrak{a}_o, \mathfrak{b}_o] \subseteq \Re$ be an closed interval on the real line. A function $\psi : [\mathfrak{a}_o, \mathfrak{b}_o] \longrightarrow \Re$ is said to be convex if it satisfies the fundamental convexity condition:

$$\psi(ß\varkappa_1 + (1 - ß)\varkappa_2) \leq ß\psi(\varkappa_1) + (1 - ß)\psi(\varkappa_2), \tag{1}$$

for all $\varkappa_1, \varkappa_2 \in [\mathfrak{a}_o, \mathfrak{b}_o]$ and $ß \in [0, 1]$ [45].

This inequality encapsulates the essence of convexity by asserting that the function's value at any convex combination of two points does not exceed the corresponding convex combination of the function's values at those points.

A fundamental result in the theory of convexity is the inequality of HH-type, which provides a two-sided bound for the integral mean of a convex function. Specifically, for any convex function $\psi$, defined on the interval $[\mathfrak{a}_o, \mathfrak{b}_o]$, the following inequality holds:

2020 *Mathematics Subject Classification*. Primary 26D15; Secondary 26A51, 26A33, 26D10.
*Keywords*. Multiplicative Calculus; Harmonically Convex Function; Hybrid Operators; Hermite-Hadamard Type Inequalities; Modified Bessel Function.
Received: 31 July 2025; Accepted: 28 September 2025
Communicated by Miodrag Spalević
* Corresponding author: Dawood Khan
*Email addresses:* saadihsanbutt@gmail.com (Saad Ihsan Butt), umarqureshi987@gmail.com (Muhammad Umar), dawooddawood601@gmail.com (Dawood Khan)
ORCID iDs: https://orcid.org/0000-0001-7192-8269 (Saad Ihsan Butt), https://orcid.org/0000-0001-9911-1111 (Muhammad Umar), https://orcid.org/0000-0002-6850-6783 (Dawood Khan)

$$\psi\left(\frac{\mathfrak{a}_o + \mathfrak{b}_o}{2}\right) \leq \frac{1}{\mathfrak{b}_o - \mathfrak{a}_o} \int_{\mathfrak{a}_o}^{\mathfrak{b}_o} \psi(\varkappa) \mathrm{d}(\varkappa) \leq \frac{\psi(\mathfrak{a}_o) + \psi(\mathfrak{b}_o)}{2}. \tag{2}$$

This inequality, independently established by Hermite and Hadamard, serves as a powerful tool in mathematical analysis and has been widely studied in various contexts [51]. It essentially states that, for a convex function, the function's value at the midpoint of an interval is less than or equal to the integral mean, which in turn is bounded above by the arithmetic mean of its values at the endpoints. The midpoint rule, the trapezoidal rule, and Ostrowski-type inequalities are only a few examples of the traditional integral inequalities whose error limits may be improved and refined using the H.H. inequality. Due to its fundamental importance, researchers have extensively generalized and extended inequality (2) to broader function classes and different mathematical settings [12, 20, 47, 56].

Dragomir and Agarwal [20] took a significant step in this approach by creating a series of trapezoidal-type inequalities for convex functions possessing differentiability. Their work paved the way for further work, including the refinement of error estimates for trapezoidal approximations by Pearce and Pečarić [52]. Motivated by these early attempts, Kirmaci and Özdemir [38] extended the study to midpoint-type inequalities for differentiable convex functions, obtaining new results on integral approximation methods.

Other recent works constructed new HH type inequalities and their uses in several mathematical and practical problems. Vivas-Cortez et al. [55] and Mehrez and Agarwal [42] provided new generalizations which added further flexibility in applying HH type results. Additionally, Varošanec [53] created several inequalities with respect to h-convex functions, a generalisation involving many convexity classes such as s-convexity, Godunova-Levin convexity, and P-functions. One of the especially interesting extensions of convexity is the function of exponential trigonometric convexity, applied in functional analysis and optimization problems. Kadakal et al. [30] investigated the class of such functions and discovered new HH type inequalities, adding to the known data on generalized convexity and integral inequalities. These broad applications reflect the fundamental significance of convexity and the HH inequality to mathematical analysis, numerical integration, and applied optimization. The continued evolution of these concepts underscores their significance in theoretical as well as applied contexts.

Convexity is a fundamental concept in applied sciences, optimization, and mathematical analysis. Convexity has been generalized several times with the passage of time to extend its applications in various fields. Harmonic convexity is significant among these generalizations because it covers a broader class of convex functions and has extremely practical applications. The idea of harmonic convex functions was first introduced by Anderson et al. [5] and Í. Íşcan [27], introducing many innovations in theoretical as well as applied mathematics.

Harmonic convexity is particularly relevant in electrical circuit theory, where it provides an essential framework for understanding the resistance of parallel circuits. According to this principle, the total resistance of two resistors connected in parallel is given by the sum of their reciprocals. Specifically, if $s_1$ and $s_2$ are two parallel resistors, the total resistance S is determined by the equation:

$$S = \frac{1}{\frac{1}{s_1} + \frac{1}{s_1}} = \frac{s_1 s_2}{s_1 + s_2}.$$

This expression is exactly half of the harmonic mean, which highlights the deep connection between harmonic convexity and electrical engineering principles.

Beyond circuit theory, harmonic convexity also plays a crucial role in semiconductor physics. In this context, the harmonic mean is used to determine the conductivity effective mass of a semiconductor, which depends on the material's crystallographic properties. The effective mass of charge carriers in a semiconductor is an essential parameter in electronic device modeling, and its harmonic mean formulation facilitates accurate conductivity calculations [21]. A critical aspect of harmonically convex functions is their impact on frequency components in signal processing. These functions often exhibit higher frequency variations that can distort the fundamental waveform, making them less desirable in certain applications.

This characteristic has motivated extensive research into developing mathematical tools for analyzing and mitigating such distortions. An important mathematical application of harmonic convexity is in variational inequalities. Noor et al. [46] showed that the minima of differentiable harmonic convex functions may be efficiently found using harmonic variational inequalities. This result has significant implications in mathematical optimization and numerical analysis, particularly in problems involving harmonic energy distributions. The HH inequality, a fundamental result in convex analysis, has also been extended to the class of harmonic convex functions. Several researchers have investigated various generalizations of HH type integral inequalities within this framework, leading to new insights and refined error estimates. Notably, Í. İşcan and S. Wu [28] explored HH type inequalities for harmonically convex functions using fractional integrals, providing a bridge between harmonic convexity and fractional calculus.

Recent advancements in the study of harmonic convexity have led to significant contributions in mathematical analysis and applied sciences. Gao et al. [26] introduced n-polynomial harmonically exponential type convex functions, establishing new inequalities with applications in applied mathematics. Du and Awan [39] extended this field by utilizing fuzzy integral techniques to develop novel HH type inequalities for harmonically convex functions, with particular relevance in decision-making models under uncertainty. Butt et al. [13] further refined these concepts by formulating generalized fractal Jensen and Jensen-Mercer inequalities, thereby extending classical results in functional analysis. Additionally, Özcan and Butt [48] investigated multiplicatively harmonic convex functions, deriving new inequalities that generalize the classical HH inequality. These contributions collectively highlight the growing importance of harmonic convexity, driving further research and fostering innovative applications in optimization, electrical engineering, semiconductor physics, and numerical integration.

Fractional calculus has engrossed a lot of curiosity from researchers and engineers currently because of its massive potential for resolving challenging questions in an assortment of fields. In contrast to traditional calculus, which compacts with differentiation and integration of integer order, fractional calculus delivers a suppler and wide-ranging mathematical background by encompassing these notions to non-integer orders. This has managed to a prodigious contract of research on numerous varieties of fractional integrals, each of which has its peculiar exceptional mathematical features and usages. These comprise $\psi$-RL ($\psi$-Riemann-Liouville) integrals [40], the generalised fractional integral operators defined in the mean square sense [36], and functional Hadamard fractional integrals [7]. Academics have been predominantly fascinated by these. A superior conception of these fractional integrals' behavior and potential uses in an assortment of scientific and technical arenas has stemmed from their thorough exploration. It must be noted that fractional integral study has been momentously upgraded by a number of central works. One such noteworthy addition was primed in 2016 by Sarikaya et al. [60], who reconnoitered the $(k,s)$-RL operators, accentuating their semigroup features, commutativity, and the creation of a noticeably defined class. The underpinning for later research was documented by Verma and Viswanathan [54], who carried out a thorough analysis of Katugampola fractional integrals, underlining their continuity and potential bounded variation as key characteristics for ensuring mathematical models that are well-posed. Fernandez and Ustaoglu [24] steered another substantial research in this space, proposing a thorough exploration of the tempered fractional integrals, underlining their inimitable properties and theoretical implications.

In recent years, there has been a significant rise in research exploring the boundedness properties of fractional integrals across various functional spaces. A notable contribution in this direction is the comprehensive study by Ledesma et al. [41], which thoroughly investigates the boundedness of tempered fractional integrals within both continuous function spaces and Lebesgue spaces. Their findings not only clarify how these operators behave in different analytical settings but also demonstrate their importance in the broader context of functional analysis. Building on this, Cheng and Luo [17], examined the boundedness of the more generalized $(k,h)$-RL integral operators in the function space $\chi_{\mathrm{h}}^{\mathrm{P}}(0,\infty)$, thereby expanding the theoretical applications of fractional calculus, particularly in the fields of mathematical modeling and applied sciences. Alongside these theoretical developments, there has been a surge in the introduction of new fractional operators that enhance the adaptability of fractional calculus to a wide range of scientific problems. Among the most prominent of these are the Caputo-Fabrizio fractional derivative [16], known for its non-singular exponential kernel; the Atangana-Baleanu derivative [6], which incorporates a Mittag-

Leffler function to model complex memory effects; and the tempered fractional derivative [62], which modifies classical operators by introducing exponential tempering to better represent processes with fading memory. These innovative operators have significantly broadened the practical scope of fractional calculus, enabling its application in areas such as dynamical systems, signal processing, and the study of viscoelastic materials, among others.

The ensuing sources deliver advantageous information for someone who desires to learn further about fractional integral properties: [10, 15, 63, 64]. These papers emphasize fractional integrals' status as indispensable mechanisms in modern mathematical analysis by providing vital means for exploring their mathematical rigor, theoretical substructures, and real life applications.

Among the most notable ultimate purposes of fractional operators is to encompass classical integer-order inequalities to fractional settings, thereby augmentation their applicability in mathematical analysis and applied sciences. Amongst the numerous fractional integral operators, the P.C.H operators have extended noteworthy attention due to their capability to assimilate different fractional calculus tactics. These operators were first familiarized by Sarikaya [61], erection upon previous work by Baleanu et al. [8]. The groundbreaking contribution of Baleanu and his collaborators is the development of a hybrid fractional operator that seamlessly integrates the proportional derivative with the Caputo derivative within a unified theoretical framework. This novel operator is defined as a linear combination of the RL integral operators and the fractional version of Caputo derivative, thereby establishing it as a powerful and versatile tool in the field of fractional calculus. Identifying the potential of this new tactic, Sarikaya employed his own definition of P.C.H operators to develop HH inequalities, encompassing classical outcomes in convex analysis to the fractional dominion. By means of the same conception of P.C.H operators, Sarikaya auxiliary lengthened this structure by acquaint with Simpson's-type inequalities, showcasing its malleability in error approximation and integral estimate. The connotation of these operators was additionally accentuated in Demir's work [18], which further another imperative accumulation to traditional numerical integration techniques by utilizing P.C.H operators to stem Milne-type inequalities. Their theoretical substructures and application were auxiliary concreted when Demir and Tunç later proposed an innovative technique for instituting Simpson's-type inequalities using P.C.H operators in [19].

These developments underscore the growing relevance of P.C.H operators in the study of fractional integral inequalities. Their ability to bridge traditional and fractional calculus methods has led to profound generalizations of classical inequalities, paving the way for novel applications in numerical analysis, optimization, and various branches of applied mathematics.

The theory of multiplicative calculus has garnered significant attention in recent years, particularly after the influential work of Ali et al. [3]. This alternative to classical calculus has proven to be a powerful tool in the study of integral inequalities, leading to the development of numerous multiplicative integer-order inequalities for different function classes. Several important extensions in this domain have been established, including inequalities related to multiplicative preinvex P-convexity [29].

A range of multiplicative inequalities have been proposed by scholars, with notable contributions focusing on H.H-type inequalities in the multiplicative integral setting. For instance, Khan and Budak [35] developed such inequalities for $^*$differentiable functions, while Xie et al. [59] extended these results to $^{**}$differentiable functions. Further research on integer-order inequalities, including those of Ostrowski, Simpson, and Maclaurin types can be found in works such as [4, 43], which provide deeper insights into the structural properties of these inequalities within multiplicative calculus.

Despite the substantial progress made in integer-order multiplicative inequalities, fractional versions, particularly those involving multiplicative fractional integrals remained relatively unexplored for a long time. A significant breakthrough occurred in 2020, when Budak and Özçelik [11] introduced multiplicative R.L fractional integrals and established new HH type inequalities in this framework. This discovery sparked further interest in the mathematical community, leading to subsequent advancements. Fu et al. [25] extended these inequalities using a new class of operators, the multiplicative tempered fractional integrals, which allowed them to explore multiplicative convex functions in greater depth. Building on these results, Peng and Du [49] further generalized HH type inequalities by incorporating differentiable multiplicative $m$-preinvexity and $(s, m)$-preinvexity into the framework of multiplicative tempered fractional integrals.

Peng et al. [50] presented multiplicative fractional integrals with exponential kernels in 2022, a recent milestone in this area. Upper bounds for integral inequalities were obtained using these operators on various mathematical identities. Kashuri et al. [31] also explored this new class of operators in relation to multiplicative Sarikaya fractional integrals, presenting new findings on HH type inequalities. Merad et al. [44] made a significant contribution by establishing symmetric Maclaurin-type inequalities tailored for multiplicatively convex functions within the context of multiplicative fractional calculus. Their work extends classical inequalities by incorporating the structure of multiplicative derivatives, which are particularly suited for modeling growth and decay processes in a multiplicative rather than additive sense. By framing these inequalities symmetrically, the authors enhance their applicability and analytical depth, offering valuable insights for further developments in nonlinear analysis and fractional integral inequalities within multiplicative settings.

Du et al. [22] established multiplicative fractional HH-type inequalities via multiplicative AB-fractional integral operators. In a related study, Ai and Du [2] investigated Newton-type inequality bounds for twice *differentiable functions under multiplicative Katugampola fractional integrals. Furthermore, Du et al. [23] examined Hadamard functional integral operators in the framework of fractional multiplicative calculus, thereby contributing to the growing literature on multiplicative fractional inequalities. Umar et al. [14] gave a landmark contribution to the subject by establishing integral inequalities for harmonically convex functions in the case of P.C.H operators for the first time. This finding created new opportunities to use multiplicative calculus in several types of convex analysis. Beyond these significant advancements, readers seeking the most recent work on multiplicative fractional integrals should consult [1, 9, 33, 34], which includes a wealth of recent papers providing further in-depth understanding of this quickly developing field of study.

Generally, considerable developments in integral inequalities have caused from the improvement of multiplicative calculus and its fractional developments, with implications for applied sciences, numerical integration, and mathematical analysis. These results validate the accumulative connotation of fractional and multiplicative calculus in up-to-date mathematical study and continue to stimulate its improvement.

The preceding analysis highlights the extensive research conducted on integer order and fractional order inequalities, particularly in the framework of HH type inequalities. These inequalities have been widely explored due to their significant applications in various mathematical and applied fields. However, despite the considerable advantages that HH-type inequalities offer, an important research gap remains. Specifically, there has been limited exploration of the fractional version of HH-type inequalities within the setting of P.C.H operators. While P.C.H operators have been successfully employed to establish fundamental inequalities, their potential in the fractional framework, particularly in connection with HH-type inequalities remains largely unexplored. Additionally, another crucial aspect that requires further investigation is the application of these fractional HH-type inequalities to special functions in the context of multiplicative calculus. Special functions, which frequently appear in mathematical physics, engineering, and numerical analysis, could greatly benefit from the application of these inequalities. The multiplicative calculus framework provides a natural and efficient alternative to classical calculus for dealing with growth-based processes, making it a promising direction for future research in this area. Addressing these gaps would not only enhance the theoretical foundation of fractional inequalities but also lead to new applications in fields such as optimization, fractional differential equations, and mathematical modeling. Future research should aim to develop generalized fractional inequalities of HH's type via P.C.H operators, analyze their structural properties, and explore their potential applications to special functions within multiplicative calculus.

The following is how this article is structured:

It begins with an introduction and preliminaries, where fundamental concepts and essential definitions are provided to establish the groundwork for the study. Section 3 presents a detailed investigation of HH's type inequalities for harmonically convex functions within the framework of P.C.H operators. This section explores the derivation and theoretical development of these inequalities. Moving forward, Section 4 focuses on the applications of the derived inequalities. This section demonstrates how the obtained results can be utilized in various mathematical and applied contexts, highlighting their significance in different domains. Finally, Section 5 provides concluding remarks and future directions, summarizing the key findings of the study and outlining potential avenues for further research in this field.

## 2. Preliminaries

This section provides a comprehensive review of fundamental definitions and lemmas that serve as the foundational framework for our findings. These preliminary concepts are essential for understanding the subsequent developments in the study and play a crucial role in establishing the theoretical results presented in later sections.

**Definition 2.1.** *[27] A function $\psi : \mathrm{I} \subseteq \Re \setminus \{0\} \to \Re$ is called harmonic convex, if for all $\lambda_o \in [0,1]$ and $\mathfrak{a}_o, \mathfrak{b}_o \in \mathrm{I}$, the below mentioned condition*

$$\psi\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\lambda_o\mathfrak{a}_o + (1-\lambda_o)\mathfrak{b}_o}\Big) \leq \lambda_o\psi(\mathfrak{b}_o) + (1-\lambda_o)\psi(\mathfrak{a}_o), \tag{3}$$

*holds.*

**Remark 2.2.** *If "≤" in (3) reverses then $\psi$ is termed as harmonic concave function.*

The following is the HH type inequality for harmonic convex functions.

**Theorem 2.3.** *[27] If a function $\psi : \mathrm{I} \subseteq \Re \setminus \{0\} \to \Re$, is harmonically convex on* I, *and $\psi \in \mathrm{L}[\mathfrak{a}_o, \mathfrak{b}_o]$, then*

$$\psi\Big(\frac{2\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{a}_o + \mathfrak{b}_o}\Big) \leq \frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o - \mathfrak{a}_o}\int_{\mathfrak{a}_o}^{\mathfrak{b}_o}\frac{\psi(\mathrm{x})}{\mathrm{x}^2}\mathrm{dx} \leq \frac{\psi(\mathfrak{a}_o) + \psi(\mathfrak{b}_o)}{2}, \tag{4}$$

*holds $\forall \mathfrak{a}_o, \mathfrak{b}_o \in \mathrm{I}$, with $\mathfrak{a}_o < \mathfrak{b}_o$.*

**Lemma 2.4.** *[27] Let $\psi : \mathrm{I} \subseteq \Re \setminus \{0\} \to \Re$ be a differentiable function on $\mathrm{I}^\circ$ for $\mathfrak{a}_o, \mathfrak{b}_o \in \mathrm{I}$ with $\mathfrak{a}_o < \mathfrak{b}_o$. If $\psi' \in \mathrm{L}[\mathfrak{a}_o, \mathfrak{b}_o]$, then*

$$\begin{aligned}&\frac{\psi(\mathfrak{a}_o) + \psi(\mathfrak{b}_o)}{2} - \frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o - \mathfrak{a}_o}\int_{\mathfrak{a}_o}^{\mathfrak{b}_o}\frac{\psi(\mathrm{x})}{\mathrm{x}^2}\mathrm{dx}\\ &= \frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o - \mathfrak{a}_o}\int_0^1\frac{1-2\lambda_o}{(\lambda_o\mathfrak{b}_o + (1-\lambda_o)\mathfrak{a}_o)^2}\psi'\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\lambda_o\mathfrak{b}_o + (1-\lambda_o)\mathfrak{a}_o}\Big)d\lambda_o.\end{aligned} \tag{5}$$

In [27], Işcan proved the following results.

**Theorem 2.5.** *Let $\psi : \mathrm{I} \subseteq (0,\infty) \to \Re$ be a differentiable function on $\mathrm{I}^\circ$ for $\mathfrak{a}_o, \mathfrak{b}_o \in \mathrm{I}$ with $\mathfrak{a}_o < \mathfrak{b}_o$ and $\psi' \in \mathrm{L}[\mathfrak{a}_o, \mathfrak{b}_o]$. If $|\psi'|^{\mathrm{q}}$ is possess harmonic convexity on $[\mathfrak{a}_o, \mathfrak{b}_o]$ for $\mathrm{q} \geq 1$, then*

$$\left|\frac{\psi(\mathfrak{a}_o) + \psi(\mathfrak{b}_o)}{2} - \frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o - \mathfrak{a}_o}\int_{\mathfrak{a}_o}^{\mathfrak{b}_o}\frac{\psi(\mathrm{x})}{\mathrm{x}^2}\mathrm{dx}\right| \leq \frac{\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o - \mathfrak{a}_o)}{2}ß_1^{1-\frac{1}{\mathrm{q}}}\Big[ß_2|\psi'(\mathfrak{a}_o)|^{\mathrm{q}} + ß_3|\psi'(\mathfrak{b}_o)|^{\mathrm{q}}\Big]^{\frac{1}{\mathrm{q}}}, \tag{6}$$

*where*

$$ß_1 = \frac{1}{\mathfrak{a}_o\mathfrak{b}_o} - \frac{2}{(\mathfrak{b}_o - \mathfrak{a}_o)^2}\ln\Big(\frac{(\mathfrak{a}_o + \mathfrak{b}_o)^2}{4\mathfrak{a}_o\mathfrak{b}_o}\Big),$$

$$ß_2 = \frac{-1}{\mathfrak{b}_o(\mathfrak{b}_o - \mathfrak{a}_o)} + \frac{3\mathfrak{a}_o + \mathfrak{b}_o}{(\mathfrak{b}_o - \mathfrak{a}_o)^3}\ln\Big(\frac{(\mathfrak{a}_o + \mathfrak{b}_o)^2}{4\mathfrak{a}_o\mathfrak{b}_o}\Big),$$

$$ß_3 = \frac{1}{\mathfrak{a}_o(\mathfrak{b}_o - \mathfrak{a}_o)} - \frac{3\mathfrak{b}_o + \mathfrak{a}_o}{(\mathfrak{b}_o - \mathfrak{a}_o)^3}\ln\Big(\frac{(\mathfrak{a}_o + \mathfrak{b}_o)^2}{4\mathfrak{a}_o\mathfrak{b}_o}\Big) = ß_1 - ß_2.$$

**Theorem 2.6.** *Let $\psi : \mathrm{I} \subseteq (0,\infty) \to \Re$ be a differentiable function on $\mathrm{I}^{\circ}$ for $\mathfrak{a}_o,\ \mathfrak{b}_o \in \mathrm{I}$, with $\mathfrak{a}_o < \mathfrak{b}_o$, and $\psi' \in L[\mathfrak{a}_o, \mathfrak{b}_o]$. If $|\psi'|^{q}$, is harmonic convex function on $[\mathfrak{a}_o, \mathfrak{b}_o]$ for $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\left|\frac{\psi(\mathfrak{a}_o) + \psi(\mathfrak{b}_o)}{2} - \frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o - \mathfrak{a}_o}\int_{\mathfrak{a}_o}^{\mathfrak{b}_o}\frac{\psi(x)}{x^2}dx\right| \le \frac{\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o - \mathfrak{a}_o)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}(\mu_1|\psi'(\mathfrak{a}_o)|^{q} + \mu_2|\psi'\mathfrak{b}_o|^{q})^{\frac{1}{q}}, \tag{7}$$

*where*

$$\mu_1 = \frac{\left[\mathfrak{a}_o{}^{2-2q} + \mathfrak{b}_o{}^{1-2q}[(\mathfrak{b}_o - \mathfrak{a}_o)(1-2q) - \mathfrak{a}_o]\right]}{2(\mathfrak{b}_o - \mathfrak{a}_o)^2(1-q)(1-2q)},$$

$$\mu_2 = \frac{\left[\mathfrak{b}_o{}^{2-2q} - \mathfrak{a}_o{}^{1-2q}[(\mathfrak{b}_o - \mathfrak{a}_o)(1-2q) + \mathfrak{b}_o]\right]}{2(\mathfrak{b}_o - \mathfrak{a}_o)^2(1-q)(1-2q)}.$$

The following examines a few well-known special functions.

[1] Beta function

$$ß(\mathfrak{a}_o, \mathfrak{b}_o) = \frac{\Gamma(\mathfrak{a}_o)\Gamma(\mathfrak{b}_o)}{\Gamma(\mathfrak{a}_o + \mathfrak{b}_o)} = \int_0^1 \lambda_o{}^{\mathfrak{a}_o - 1}(1-\lambda_o)^{\mathfrak{b}_o - 1}d\lambda_o,\ \ \mathfrak{a}_o,\ \mathfrak{b}_o > 0.$$

[2] Hypergeometric function

$${}_2F_1(\mathfrak{a}_o, \mathfrak{b}_o; c; z) = \frac{1}{ß(\mathfrak{b}_o, c - \mathfrak{b}_o)}\int_0^1 \lambda_o{}^{\mathfrak{b}_o - 1}(1-\lambda_o)^{c-\mathfrak{b}_o-1}(1-z\lambda_o)^{-\mathfrak{a}_o}d\lambda_o,\ c > \mathfrak{b}_o > 0,\ |z| < 1.$$

**Lemma 2.7.** *[57] For $0 \le \mathfrak{a}_o < \mathfrak{b}_o$ and $0 < \alpha_o \le 1$, we obtain*

$$|\mathfrak{a}_o{}^{\alpha_o} - \mathfrak{b}_o{}^{\alpha_o}| \le (\mathfrak{b}_o - \mathfrak{a}_o)^{\alpha_o}. \tag{8}$$

We present fundamental concepts related to fractional calculus in the following.

**Definition 2.8.** *[60] The RL fractional operators with $\mathfrak{a}_o \ge 0$ and of order $ß \ge 0$, are given by $\Im^{ß}_{\mathfrak{b}_o{}^-}\psi(\varkappa)$, and $\Im^{ß}_{\mathfrak{a}_o{}^+}\psi(\varkappa)$, and defined as*

$$\Im^{ß}_{\mathfrak{a}_o{}^+}\psi(\varkappa) = \frac{1}{\Gamma(ß)}\int_{\mathfrak{a}_o}^{\varkappa}(\varkappa - y_o)^{ß-1}\psi(y_o)dy_o, \quad (\varkappa > \mathfrak{a}_o), \tag{9}$$

*and*

$$\Im^{ß}_{\mathfrak{b}_o{}^-}\psi(\varkappa) = \frac{1}{\Gamma(ß)}\int_{\varkappa}^{\mathfrak{b}_o}(y_o - \varkappa)^{ß-1}\psi(y_o)dy_o, \quad (\varkappa < \mathfrak{b}_o), \tag{10}$$

*where $\Gamma(ß) = \int_0^\infty y_o{}^{ß-1}e^{-y_o}dy_o$, is defined as gamma function.*

The following is an expression for the fractional form of the HH's type inequality for harmonically convex functions [28]:

**Theorem 2.9.** *Let $\psi : \mathrm{I} \subseteq (0,\infty) \to \Re$ be a harmonic convex function and $\psi \in L[\mathfrak{a}_o, \mathfrak{b}_o]$, where $\mathfrak{a}_o,\ \mathfrak{b}_o \in \mathrm{I}$ and $\Upsilon(\lambda_o) = \frac{1}{\lambda_o}$, with $\mathfrak{a}_o < \mathfrak{b}_o$, then*

$$\psi\left(\frac{2\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{a}_o + \mathfrak{b}_o}\right) \le \frac{\Gamma(\alpha_o + 1)}{2}\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o - \mathfrak{a}_o}\right)^{\alpha_o}\Big[J^{\alpha_o}_{1/\mathfrak{a}_o{}^-}(\psi \circ \Upsilon)(1/\mathfrak{b}_o)$$
$$+ J^{\alpha_o}_{1/\mathfrak{b}_o{}^+}(\psi \circ \Upsilon)(1/\mathfrak{a}_o)\Big] \le \frac{\psi(\mathfrak{a}_o) + \psi(\mathfrak{b}_o)}{2}, \tag{11}$$

*holds $\forall\ \alpha_o > 0$.*

**Definition 2.10.** *[8] Let* $ß > 0$ *and* $ß \notin \{1, 2, ...\}$, $n = [ß] + 1$, $\psi \in AC^n[\mathfrak{a}_o, \mathfrak{b}_o]$, *is the set of all functions having absolute continuity and* n-th *derivatives, then the Caputo fractional derivatives of fractional order* ß *are defined as:*

$$^{C}D^{ß}_{\mathfrak{b}_o^-}\psi(\lambda_o) = \frac{1}{\Gamma(n-ß)}\int_{\varkappa}^{\mathfrak{b}_o}(\lambda_o-\varkappa)^{n-ß-1}\psi^{(n)}(\lambda_o)d\lambda_o,\ \varkappa < \mathfrak{b}_o,$$

*and*

$$^{C}D^{ß}_{\mathfrak{a}_o^+}\psi(\lambda_o) = \frac{1}{\Gamma(n-ß)}\int_{\mathfrak{a}_o}^{\varkappa}(\varkappa-\lambda_o)^{n-ß-1}\psi^{(n)}(\lambda_o)d\lambda_o,\ \varkappa > \mathfrak{a}_o.$$

The Caputo operator is normally employed in aggregation with its derivative alike in fractional calculus. When the order of differentiation is a non-integer value, it displays the fractional derivative of a function pertaining to time. This makes it possible to characterize dynamic systems with memory effects in a more adaptable and generalized approach. The P.C.H operators are non-local mathematical operators that syndicate differentiation and integration constituents. It is a relatively new work in this area. The unique feature of the P.C.H operator is its suave integration of the RL fractional integral and the Caputo derivative into a single linear expression. Its hybrid character creates it more pertinent to a variety of defies, such as numerical analysis, optimization problems, and fractional differential equations.

**Definition 2.11.** *[8] Let* $\psi : I \subset \Re^+ \to \Re$ *be a differentiable function on* $I^\circ$ *and* $\psi, \psi' \in L^1(I)$, *then P.C.H operators are defined as:*

$$^{PC}_{0}D^{ß}_{\lambda_o}\psi(\lambda_o) = \frac{1}{\Gamma(1-ß)}\int_0^{\lambda_o}(K_1(ß,\varkappa)\psi(\varkappa) + K_0(ß,\varkappa)\psi'(\varkappa))(\lambda_o-\varkappa)^{-ß}d\varkappa,$$

*where* $ß \in [0, 1]$ *and* $K_0$ *and* $K_1$ *are functions satisfying*

$$\lim_{ß\to 0^+} K_0(ß,\varkappa) = 0;\ \lim_{ß\to 1} K_0(ß,\varkappa) = 1;\ K_0(ß,\varkappa) \neq 0,\ ß \in (0, 1],$$

$$\lim_{ß\to 0^+} K_1(ß,\varkappa) = 0;\ \lim_{ß\to 1^-} K_1(ß,\varkappa) = 0;\ K_1(ß,\varkappa) \neq 0,\ ß \in [0, 1).$$

**Definition 2.12.** *[61] For the same functions as given in Definition 2.11, the left and right P.C.H operators are defined as:*

$$^{PC}_{\mathfrak{a}_o^+}D^{ß}_{\mathfrak{b}_o}\psi(\mathfrak{b}_o) = \frac{1}{\Gamma(1-ß)}\int_{\mathfrak{a}_o}^{\mathfrak{b}_o}\Big[K_1(ß,\mathfrak{b}_o-\varkappa)\psi(\varkappa) + K_o(ß,\mathfrak{b}_o-\varkappa)\psi'(\varkappa)\Big](\mathfrak{b}_o-\varkappa)^{-ß}d\varkappa,$$

*and*

$$^{PC}_{\mathfrak{b}_o^-}D^{ß}_{\mathfrak{a}_o}\psi(\mathfrak{a}_o) = \frac{1}{\Gamma(1-ß)}\int_{\mathfrak{a}_o}^{\mathfrak{b}_o}\Big[K_1(ß,\varkappa-\mathfrak{a}_o)\psi(\varkappa) + K_o(ß,\varkappa-\mathfrak{a}_o)\psi'(\varkappa)\Big](\varkappa-\mathfrak{a}_o)^{-ß}d\varkappa,$$

*where* $ß \in [0, 1]$ *and* $K_o(ß, \lambda_o) = (1-ß)^2{\lambda_o}^{1-ß}$ *and* $K_1(ß, \lambda_o) = ß^2{\lambda_o}^{ß}$.

### *2.1. Multiplicative Calculus*

This section introduces the fundamental concepts and key features of multiplicative calculus, establishing a solid foundation for further research in this domain.

**Definition 2.13.** *[9] Assume that $\psi : \mathfrak{R} \to \mathfrak{R}$ is a positive mapping, then its multiplicative derivative is designated by $\psi^*$ and is defined as.*

$$\psi^*(\varkappa) = \lim_{\mathrm{h}\to 0}\left(\frac{\psi(\varkappa+\mathrm{h})}{\psi(\varkappa)}\right)^{\frac{1}{\mathrm{h}}}.$$

**Remark 2.14.** *The connection between $\psi^*$ and the standard derivative $\psi'$ is established through the following expression:*

$$\psi^*(\varkappa) = \mathrm{e}^{(\ln\psi(\varkappa))'} = \mathrm{e}^{\frac{\psi'(\varkappa)}{\psi(\varkappa)}}.$$

The multiplicative derivative adheres to the following properties:

**Proposition 2.15.** *[9] Suppose $\mathrm{c} > 0$ and $\psi$ and $\Upsilon$ are multiplicatively differentiable functions then $^*$differentiable functions $c\psi$, $\psi\Upsilon$, $\psi/\Upsilon$, and $\psi + \Upsilon$ are given as*

1. $(c\psi)^*(\varkappa) = \psi^*(\varkappa)$,
2. $(\Upsilon\psi)^*(\varkappa) = \psi^*(\varkappa)\Upsilon^*(\varkappa)$,
3. $(\psi+\Upsilon)^*(\varkappa) = \psi^*(\varkappa)^{\frac{\psi(\varkappa)}{\psi(\varkappa)+\Upsilon(\varkappa)}}\Upsilon^*(\varkappa)^{\frac{\Upsilon(\varkappa)}{\psi(\varkappa)+\Upsilon(\varkappa)}}$,
4. $\left(\frac{\psi}{\Upsilon}\right)^*(\varkappa) = \frac{\psi^*(\varkappa)}{\Upsilon^*(\varkappa)}$,
5. $(\psi^{\Upsilon})^*(\varkappa) = \psi^*(\varkappa)^{\Upsilon(\varkappa)}\psi(\varkappa)^{\Upsilon'(\varkappa)}$.

The multiplicative integral, also known as the $^*$integral, is represented by the notation $\int_{\mathfrak{a}_o}^{\mathfrak{b}_o}(\psi(\varkappa))^{\mathrm{d}\varkappa}$. This notion was introduced by Bashirov et al. in their pioneering work [9], as an alternative to the classical integral calculus, particularly useful in settings where multiplicative changes are more natural than additive ones. In classical Riemann integration, the integral of a function $\psi$ over an interval $[\mathfrak{a}_o, \mathfrak{b}_o]$ is defined by summing product terms of function values and small increments of the variable. However, in multiplicative integration, the process entails exponentiating a product of terms rather than summing them, reflecting an inherent exponential-like accumulation rather than an additive one. This makes the multiplicative integral particularly suitable for applications in exponential growth models, population dynamics, financial mathematics, and other multiplicative systems.

The relationship between the multiplicative integral and the Riemann integral can be expressed as follows [9]:

**Proposition 2.16.** *If $\psi$ is positive and Riemann integrable on $[\mathfrak{a}_o, \mathfrak{b}_o]$, then it is also multiplicatively integrable on the same interval.*

$$\int_{\mathfrak{a}_o}^{\mathfrak{b}_o}(\psi(\varkappa))^{\mathrm{d}\varkappa} = \mathrm{e}^{\int_{\mathfrak{a}_o}^{\mathfrak{b}_o}\ln(\psi(\varkappa))\mathrm{d}\varkappa}.$$

As shown in the work of Bashirov et al. [9], the multiplicative integral possesses several key characteristics and fundamental properties, which include the following:

**Proposition 2.17.** *If $\psi$ is positive and Riemann integrable on $[\mathfrak{a}_o, \mathfrak{b}_o]$, then it is also multiplicatively integrable on that interval. This means that if a function $\psi$ is Riemann integrable, then it is also multiplicatively integrable in the sense of multiplicative calculus. In other words, the existence of the classical Riemann integral ensures that the multiplicative integral of $\psi$ on $[\mathfrak{a}_o, \mathfrak{b}_o]$ is well-defined.*

*If $\psi$ is Riemann integrable on $[\mathfrak{a}_o, \mathfrak{b}_o]$, then it is also *integrable on $[\mathfrak{a}_o, \mathfrak{b}_o]$.*

1. $\int\limits_{\mathfrak{a}_o}^{\mathfrak{b}_o}\left((\psi(\varkappa))^{\mathrm{p}}\right)^{\mathrm{d}\varkappa} = \left(\int\limits_{\mathfrak{a}_o}^{\mathfrak{b}_o}((\psi\varkappa))^{\mathrm{d}\varkappa}\right)^{\mathrm{p}}$,

2. $\int_{\mathfrak{a}_o}^{\mathfrak{b}_o} (\psi(\varkappa)\Upsilon(\varkappa))^{d\varkappa} = \int_{\mathfrak{a}_o}^{\mathfrak{b}_o} (\psi(\varkappa))^{d\varkappa} \cdot \int_{\mathfrak{a}_o}^{\mathfrak{b}_o} (\Upsilon(\varkappa))^{d\varkappa}$,

3. $\int_{\mathfrak{a}_o}^{\mathfrak{b}_o} \left(\frac{\psi(\varkappa)}{\Upsilon(\varkappa)}\right)^{d\varkappa} = \frac{\int_{\mathfrak{a}_o}^{\mathfrak{b}_o} (\psi(\varkappa))^{d\varkappa}}{\int_{\mathfrak{a}_o}^{\mathfrak{b}_o} (\Upsilon(\varkappa))^{d\varkappa}}$,

4. $\int_{\mathfrak{a}_o}^{\mathfrak{b}_o} (\psi(\varkappa))^{d\varkappa} = \int_{\mathfrak{a}_o}^{c} (\psi(\varkappa))^{d\varkappa} \cdot \int_{c}^{\mathfrak{b}_o} (\psi(\varkappa))^{d\varkappa}$, $\mathfrak{a}_o \le c \le \mathfrak{b}_o$,

5. $\int_{\mathfrak{a}_o}^{\mathfrak{a}_o} (\psi(\varkappa))^{d\varkappa} = 1$ *and* $\int_{\mathfrak{a}_o}^{\mathfrak{b}_o} (\psi(\varkappa))^{d\varkappa} = \left(\int_{\mathfrak{b}_o}^{\mathfrak{a}_o} (\psi(\varkappa))^{d\varkappa}\right)^{-1}$.

Alternatively, the subsequent multiplicative RL fractional integrals were put forward by Abdeljawed and Grossman [1].

**Definition 2.18.** *The symbol* $({}_{\mathfrak{a}_o}J_*^{ß}\psi)(\varkappa)$ *is a designation of multiplicative left RL fractional integral of order* $ß \in \mathbb{C}, \mathrm{Re}(ß) > 0$, *with* ß *as an initial point is given by*

$$({}_{\mathfrak{a}_o}J_*^{ß}\psi)(\varkappa) = e^{(J_{\mathfrak{a}_o+}^{ß}(\ln\circ\psi))(\varkappa)},$$

*and what defines the multiplicative right one is*

$$({}_*J_{\mathfrak{b}_o}^{ß}\psi)(\varkappa) = e^{(J_{\mathfrak{b}_o-}^{ß}(\ln\circ\psi))(\varkappa)}.$$

*Here* $J_{\mathfrak{b}_o-}^{ß}$ *and* $J_{\mathfrak{a}_o+}^{ß}$ *describe the right and left RL fractional integral, given by [37]*

$$J_{\mathfrak{b}_o-}^{ß}\psi(\lambda_o) = \frac{1}{\Gamma(ß)}\int_{\varkappa}^{\mathfrak{b}_o} (\lambda_o - \varkappa)^{ß-1}\psi(\lambda_o)d\lambda_o, \ \mathfrak{b}_o > \varkappa,$$

*and*

$$J_{\mathfrak{a}_o+}^{ß}\psi(\lambda_o) = \frac{1}{\Gamma(ß)}\int_{\mathfrak{a}_o}^{\varkappa} (\varkappa - \lambda_o)^{ß-1}\psi(\lambda_o)d\lambda_o, \ \mathfrak{a}_o < \varkappa,$$

*accordingly, where* $\Gamma(ß) = \int_0^\infty e^{-u}u^{ß-1}du$. *Here is* $J_{\mathfrak{b}_o-}^{0}\psi(\varkappa) = \psi(\varkappa) = J_{\mathfrak{a}_o+}^{0}\psi(\varkappa)$.

**Theorem 2.19.** *(Multiplicative integration by parts [9]) Let* $\psi : [\mathfrak{a}_o, \mathfrak{b}_o] \to \mathfrak{R}$ *and* $\Upsilon : [\mathfrak{a}_o, \mathfrak{b}_o] \to \mathfrak{R}$ *possess* $^*$*differentiability and differentiable respectively, so the function* $\psi^{\Upsilon}$ *is* $^*$ *integrable then it implies that*

$$\int_{\mathfrak{a}_o}^{\mathfrak{b}_o} \left(\psi^*(\varkappa)^{\Upsilon(\varkappa)}\right)^{d\varkappa} = \frac{\psi(\mathfrak{b}_o)^{\Upsilon(\mathfrak{b}_o)}}{\psi(\mathfrak{a}_o)^{\Upsilon(\mathfrak{a}_o)}} \cdot \frac{1}{\int_{\mathfrak{a}_o}^{\mathfrak{b}_o} \left(\psi(\varkappa)^{\Upsilon'(\varkappa)}\right)^{d\varkappa}}.$$

**Lemma 2.20.** *[3] Let* $\psi : [\mathfrak{a}_o, \mathfrak{b}_o] \to \mathfrak{R}$ *and* $\Upsilon : [\mathfrak{a}_o, \mathfrak{b}_o] \to \mathfrak{R}$ *be* $^*$*differentiable and differentiable respectively, so* $\psi^{\Upsilon}$ *is* $^*$*integrable then*

$$\int_{\mathfrak{a}_o}^{\mathfrak{b}_o} \left(\psi^*(h(\varkappa))^{h'(\varkappa)\Upsilon(\varkappa)}\right)^{d\varkappa} = \frac{\psi(h(\mathfrak{b}_o))^{\Upsilon(\mathfrak{b}_o)}}{\psi(h(\mathfrak{a}_o))^{\Upsilon(\mathfrak{a}_o)}} \cdot \frac{1}{\int_{\mathfrak{a}_o}^{\mathfrak{b}_o} \left(\psi(h(\varkappa))^{\Upsilon'(\varkappa)}\right)^{d\varkappa}}.$$

In what follows, we provide an overview of the essential terminology and foundational principles of multiplicative calculus, which will underpin the theoretical framework employed throughout this study.

**Proposition 2.21.** *The convexity of* $\log \psi$ *and* $\log \Upsilon$*, implies the convexity of* $\log(\psi\Upsilon)$ *and the convexity of* $\log \psi$ *and the concavity of* $\log \Upsilon$*, implies the convexity of* $\log\left(\frac{\psi}{\Upsilon}\right)$.

**Theorem 2.22.** *Let* $\psi$ *be a multiplicatively convex on* $[\mathfrak{a}_o, \mathfrak{b}_o]$*, then*

$$\psi\left(\frac{\mathfrak{a}_o + \mathfrak{b}_o}{2}\right) \leq \left(\int_{\mathfrak{a}_o}^{\mathfrak{b}_o} (\psi(\varkappa))^{d\varkappa}\right)^{\frac{1}{\mathfrak{b}_o - \mathfrak{a}_o}} \leq G(\psi(\mathfrak{a}_o), \psi(\mathfrak{b}_o)), \tag{12}$$

*holds. Where the symbol* $G(.,.)$ *stands for geometric mean.*

However, the inequality of H.H type for multiplicative RL fractional integrals was demonstrated in [11], which is a noteworthy inequality.

**Theorem 2.23.** *Let* $\psi$ *be a multiplicatively convex on* $[\mathfrak{a}_o, \mathfrak{b}_o]$*, then we obtain the below mentioned H.H inequality involving end points of the interval for multiplicative fractional integrals of RL*

$$\psi\left(\frac{\mathfrak{a}_o + \mathfrak{b}_o}{2}\right) \leq \left[ {}_{\mathfrak{a}_o}\mathrm{I}_*^{ß} \psi(\mathfrak{b}_o) ._* \mathrm{I}_{\mathfrak{b}_o}^{ß} \psi(\mathfrak{a}_o)\right]^{\frac{\Gamma(ß+1)}{2(\mathfrak{b}_o - \mathfrak{a}_o)^{ß}}} \leq G(\psi(\mathfrak{a}_o), \psi(\mathfrak{b}_o)), \tag{13}$$

*where the notation* $G(.,.)$ *stands for geometric mean.*

**Theorem 2.24.** *[11] Let the positive function* $\psi$ *possess multiplicatively convexity on* $[\mathfrak{a}_o, \mathfrak{b}_o]$*, then we obtain the subsequent H.H inequality midpoints of the interval for multiplicative fractional integrals of RL:*

$$\psi\left(\frac{\mathfrak{a}_o + \mathfrak{b}_o}{2}\right) \leq \left[ {}_{\frac{\mathfrak{a}_o + \mathfrak{b}_o}{2}}\mathrm{I}_*^{ß} \psi(\mathfrak{b}_o) ._* \mathrm{I}_{\frac{\mathfrak{a}_o + \mathfrak{b}_o}{2}}^{ß} \psi(\mathfrak{a}_o)\right]^{\frac{2^{ß-1}\Gamma(ß+1)}{(\mathfrak{b}_o - \mathfrak{a}_o)^{ß}}} \leq G(\psi(\mathfrak{a}_o), \psi(\mathfrak{b}_o)), \tag{14}$$

*where the notation* $G(.,.)$ *stands for geometric mean.*

**Definition 2.25.** *Let the function* $\psi : \mathrm{I} \subseteq \mathfrak{R} \setminus \{0\} \to \mathfrak{R}$ *be a multiplicatively harmonic convex, then*

$$\psi\left(\frac{\mathfrak{a}_o \mathfrak{b}_o}{\lambda_o \mathfrak{a}_o + (1 - \lambda_o)\mathfrak{b}_o}\right) \leq [\psi(\mathfrak{b}_o)]^{\lambda_o} [\psi(\mathfrak{a}_o)]^{1-\lambda_o}, \tag{15}$$

*holds* $\forall\ \mathfrak{a}_o, \mathfrak{b}_o \in \mathrm{I}$ *and* $\lambda_o \in [0, 1]$.

### *2.2. Multiplicative P.C.H-fractional integrals*

The behaviour of RL-fractional integral operators is similar to that of *P.C.H*-fractional integral operators. The inspiration and logical basis for developing and presenting the idea of consecutive multiplicative *P.C.H*-fractional integrals are provided by this resemblance. Stated differently, it is both natural and illuminating to extend the concept to multiplicative calculus by defining a consecutive or repeated application of the *P.C.H*-fractional integral in a multiplicative framework because the two types of fractional integrals have similar qualities.

**Definition 2.26.** *The multiplicative left P.C.H operator of order* $ß \in \mathbb{C}$ *designated by* $({}_{\mathfrak{a}_o}\mathrm{D}_*^{ß}\psi)(\varkappa)$ *with* $\mathrm{Re}(ß) > 0$ *by assuming* ß *as an initial point is given by*

$$\begin{aligned}
({}_{\mathfrak{a}_o}\mathrm{D}_*^{ß}\psi)(\mathfrak{a}_o) &= \exp\left\{ {}_{\mathfrak{b}_o^-}^{PC}\mathrm{D}_{\mathfrak{a}_o}^{ß} (\ln \circ \psi)(\mathfrak{a}_o) \right\} \\
&= \exp\left\{ \frac{1}{\Gamma(1 - ß)} \int_{\mathfrak{a}_o}^{\mathfrak{b}_o} \left[ \mathrm{K}_1(ß, \varkappa - \mathfrak{a}_o)(ln \circ \psi)(\varkappa) + \mathrm{K}_0(ß, \varkappa - \mathfrak{a}_o)(ln \circ \psi^*)(\varkappa) \right] (\varkappa - \mathfrak{a}_o)^{-ß} d\varkappa \right\},
\end{aligned}$$

*and what defines the multiplicative right one is*

$$({}_*\mathrm{D}^{ß}_{\mathfrak{b}_o}\psi)(\mathfrak{b}_o) = \exp\Big\{{}^{\mathrm{PC}}_{\mathfrak{a}_o^+}\mathrm{D}^{ß}_{\mathfrak{b}_o}(\ln\circ\psi)(\mathfrak{b}_o)\Big\}$$
$$= \exp\Big\{\frac{1}{\Gamma(1-ß)}\int_{\mathfrak{a}_o}^{\mathfrak{b}_o}\Big[\mathrm{K}_1(ß,\mathfrak{b}_o-\varkappa)(ln\circ\psi)(\varkappa)+\mathrm{K}_0(ß,\mathfrak{b}_o-\varkappa)(ln\circ\psi^*)(\varkappa)\Big](\mathfrak{b}_o-\varkappa)^{-ß}d\varkappa\Big\},$$

*where the expression* $\Gamma(ß) = \int_0^\infty \varkappa^{ß-1}\exp\{-\varkappa\}d\varkappa$, $ß \in [0,1]$, $\mathrm{K}_0(ß,\lambda_o) = (1-ß)^2\lambda_o{}^{1-ß}$ *and* $\mathrm{K}_1(ß,\lambda_o) = ß^2\lambda_o{}^{ß}$.

Several scholars have overextended their studies afar integer-order integrals to investigate fractional order inequalities of the HH type because of the various applicability of fractional integrals and HH type inequalities. Using fractional integral approaches, an increasing number of HH type inequalities for diverse classes of functions have been initiated in current years. Emerging HH type inequalities for harmonically convex functions in the framework of *P.C.H* fractional integral operators is the painstaking objective of this work. Furthermore, it protracts the theoretic foundations of fractional calculus and its applications by means of an identity settled for fractional integrals to paradigm novel integral inequalities.

## 3. Main Result

The first goal of our paper is to get the HH's inequalities for the *P.C.H* operators.

**Theorem 3.1.** *Let* $\psi : \mathrm{I} \subset \mathfrak{R}^+ \to \mathfrak{R}$ *be a differentiable function on* $\mathrm{I}^\circ$*, where* $\mathrm{I}^\circ$ *is the interior of* $\mathrm{I}$*, where* $\mathfrak{a}_o, \mathfrak{b}_o \in \mathrm{I}^\circ$ *with* $\mathfrak{a}_o < \mathfrak{b}_o$ *and the functions* $\psi, \psi^*$ *are multiplicatively harmonic convex on* $\mathrm{I}$ *and* $\Upsilon(\lambda_o) = \frac{1}{\lambda_o}$*, then*

$$\psi\Big(\frac{2\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{a}_o+\mathfrak{b}_o}\Big)^{\alpha_o{}^2\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)^{\alpha_o}}\cdot\psi^*\Big(\frac{2\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{a}_o+\mathfrak{b}_o}\Big)^{\frac{(1-\alpha_o)}{2}\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)^{1-\alpha_o}}$$
$$\le\Bigg[\Big({}_{\frac{1}{\mathfrak{a}_o}}\mathrm{D}^{\alpha_o}_{*}(\psi\circ\Upsilon)\Big(\frac{1}{\mathfrak{b}_o}\Big)\Big)\Big({}_*\mathrm{D}^{\alpha_o}_{\frac{1}{\mathfrak{b}_o}}(\psi\circ\Upsilon)\Big(\frac{1}{\mathfrak{a}_o}\Big)\Big)\Bigg]^{\frac{\Gamma(1-\alpha_o)}{2}\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)^{1-\alpha_o}}$$
$$\le\Big[G(\psi(\mathfrak{a}_o),\psi(\mathfrak{b}_o))\Big]^{\alpha_o{}^2\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)^{\alpha_o}}\Big[G(\psi^*(\mathfrak{a}_o),\psi^*(\mathfrak{b}_o))\Big]^{(1-\alpha_o)\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)^{1-\alpha_o}}, \tag{16}$$

*holds with* $\alpha_o > 0$.

*Proof. Since the functions* $\psi$ *and* $\psi^*$ *possess multiplicative harmonic convexity on* $[\mathfrak{a}_o,\mathfrak{b}_o]$, $\forall$ $\mathrm{x},\mathrm{y}\in[\mathfrak{a}_o,\mathfrak{b}_o]$ *(with* $\lambda_o = 1/2$ *in the inequality (15)), we attain*

$$\psi\Big(\frac{2\mathrm{xy}}{\mathrm{x}+\mathrm{y}}\Big)\le\psi(\mathrm{x})^{\frac{1}{2}}\psi(\mathrm{y})^{\frac{1}{2}}.$$

*Choosing* $\mathrm{x} = \frac{\mathfrak{a}_o\mathfrak{b}_o}{\lambda_o\mathfrak{b}_o+(1-\lambda_o)\mathfrak{a}_o}$, $\mathrm{y} = \frac{\mathfrak{a}_o\mathfrak{b}_o}{\lambda_o\mathfrak{a}_o+(1-\lambda_o)\mathfrak{b}_o}$, *we get*

$$\psi\left(\frac{2\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{a}_o+\mathfrak{b}_o}\right)\le\psi\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\lambda_o\mathfrak{b}_o+(1-\lambda_o)\mathfrak{a}_o}\right)^{\frac{1}{2}}\cdot\psi\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\lambda_o\mathfrak{a}_o+(1-\lambda_o)\mathfrak{b}_o}\right)^{\frac{1}{2}}$$

$$\ln\psi\left(\frac{2\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{a}_o+\mathfrak{b}_o}\right)\le\frac{1}{2}\ln\psi\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\lambda_o\mathfrak{b}_o+(1-\lambda_o)\mathfrak{a}_o}\right)+\frac{1}{2}\ln\psi\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\lambda_o\mathfrak{a}_o+(1-\lambda_o)\mathfrak{b}_o}\right). \tag{17}$$

*Similarly for* $\psi^*$,

$$\psi^*\left(\frac{2\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{a}_o+\mathfrak{b}_o}\right)\leq\psi^*\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\lambda_o\mathfrak{b}_o+(1-\lambda_o)\mathfrak{a}_o}\right)^{\frac{1}{2}}\cdot\psi^*\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\lambda_o\mathfrak{a}_o+(1-\lambda_o)\mathfrak{b}_o}\right)^{\frac{1}{2}}$$

$$\ln\psi^*\left(\frac{2\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{a}_o+\mathfrak{b}_o}\right)\leq\frac{1}{2}\ln\psi^*\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\lambda_o\mathfrak{b}_o+(1-\lambda_o)\mathfrak{a}_o}\right)+\frac{1}{2}\ln\psi^*\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\lambda_o\mathfrak{a}_o+(1-\lambda_o)\mathfrak{b}_o}\right). \tag{18}$$

*Multiplying* (17) *by* $\alpha_o{}^2\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)^{\alpha_o}$ *and* (18) *by* $(1-\alpha_o)^2\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)^{1-\alpha_o}\lambda_o{}^{1-2\alpha_o}$*, respectively, we have*

$$\alpha_o{}^2\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)^{\alpha_o}\ln\psi\left(\frac{2\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{a}_o+\mathfrak{b}_o}\right)\leq\frac{1}{2}\left[\alpha_o{}^2\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)^{\alpha_o}\ln\psi\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\lambda_o\mathfrak{b}_o+(1-\lambda_o)\mathfrak{a}_o}\right)\right.$$
$$\left.+\alpha_o{}^2\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)^{\alpha_o}\ln\psi\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\lambda_o\mathfrak{a}_o+(1-\lambda_o)\mathfrak{b}_o}\right)\right],$$

*and*

$$(1-\alpha_o)^2\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)^{1-\alpha_o}\lambda_o{}^{1-2\alpha_o}\ln\psi^*\left(\frac{2\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{a}_o+\mathfrak{b}_o}\right)$$
$$\leq\frac{1}{2}\left[(1-\alpha_o)^2\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)^{1-\alpha_o}\lambda_o{}^{1-2\alpha_o}\ln\psi^*\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\lambda_o\mathfrak{b}_o+(1-\lambda_o)\mathfrak{a}_o}\right)\right.$$
$$\left.+(1-\alpha_o)^2\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)^{1-\alpha_o}\lambda_o{}^{1-2\alpha_o}\ln\psi^*\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\lambda_o\mathfrak{a}_o+(1-\lambda_o)\mathfrak{b}_o}\right)\right].$$

*By summing these two expressions term by term and subsequently integrating the resulting equation with respect to* $\lambda_o$*, over the interval* $[0,1]$*, we obtain:*

$$\alpha_o{}^2\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)^{\alpha_o}\ln\psi\left(\frac{2\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{a}_o+\mathfrak{b}_o}\right)+(1-\alpha_o)^2\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)^{1-\alpha_o}\ln\psi^*\left(\frac{2\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{a}_o+\mathfrak{b}_o}\right)\int_0^1\lambda_o{}^{1-2\alpha_o}d\lambda_o$$
$$\leq\frac{1}{2}\int_0^1\left[\alpha_o{}^2\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)^{\alpha_o}\lambda_o{}^{\alpha_o}\ln\psi\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\lambda_o\mathfrak{b}_o+(1-\lambda_o)\mathfrak{a}_o}\right)\right.$$
$$\left.+(1-\alpha_o)^2\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)^{1-\alpha_o}\lambda_o{}^{1-\alpha_o}\ln\psi^*\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\lambda_o\mathfrak{b}_o+(1-\lambda_o)\mathfrak{a}_o}\right)\right]\lambda_o{}^{-\alpha_o}d\lambda_o$$
$$+\frac{1}{2}\int_0^1\left[\alpha_o{}^2\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)^{\alpha_o}\lambda_o{}^{\alpha_o}\ln\psi\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\lambda_o\mathfrak{a}_o+(1-\lambda_o)\mathfrak{b}_o}\right)\right.$$
$$\left.+(1-\alpha_o)^2\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)^{1-\alpha_o}\lambda_o{}^{1-\alpha_o}\ln\psi^*\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\lambda_o\mathfrak{a}_o+(1-\lambda_o)\mathfrak{b}_o}\right)\right]\lambda_o{}^{-\alpha_o}d\lambda_o.$$

*Using the change of variable, we obtain that*

$$\alpha_o{}^2\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)^{\alpha_o}\ln\psi\left(\frac{2\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{a}_o+\mathfrak{b}_o}\right)+\frac{1}{2}(1-\alpha_o)\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)^{1-\alpha_o}\ln\psi^*\left(\frac{2\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{a}_o+\mathfrak{b}_o}\right)$$
$$\leq\frac{1}{2}\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)\int_{\frac{1}{\mathfrak{b}_o}}^{\frac{1}{\mathfrak{a}_o}}\left[\alpha_o{}^2\left(\tau-\frac{1}{\mathfrak{b}_o}\right)^{\alpha_o}\ln\psi\left(\frac{1}{\tau}\right)+(1-\alpha_o)^2\left(\tau-\frac{1}{\mathfrak{b}_o}\right)^{1-\alpha_o}\ln\psi^*\left(\frac{1}{\tau}\right)\right]\left(\tau-\frac{1}{\mathfrak{b}_o}\right)^{-\alpha_o}d\tau$$
$$+\frac{1}{2}\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)\int_{\frac{1}{\mathfrak{b}_o}}^{\frac{1}{\mathfrak{a}_o}}\left[\alpha_o{}^2\left(\frac{1}{\mathfrak{a}_o}-\tau\right)^{\alpha_o}\ln\psi\left(\frac{1}{\tau}\right)+(1-\alpha_o)^2\left(\frac{1}{\mathfrak{a}_o}-\tau\right)^{1-\alpha_o}\ln\psi^*\left(\frac{1}{\tau}\right)\right]\left(\frac{1}{\mathfrak{a}_o}-\tau\right)^{-\alpha_o}d\tau$$

$$= \frac{\Gamma(1-\alpha_o)}{2}\Big(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\Big)\Big[{}^{PC}_{\frac{1}{\mathfrak{a}_o}^-}D^{\alpha_o}_{\frac{1}{\mathfrak{b}_o}}\ln(\psi\circ\Upsilon)\Big(\frac{1}{\mathfrak{b}_o}\Big)+{}^{PC}_{\frac{1}{\mathfrak{b}_o}^+}D^{\alpha_o}_{\frac{1}{\mathfrak{a}_o}}\ln(\psi\circ\Upsilon)\Big(\frac{1}{\mathfrak{a}_o}\Big)\Big]$$
$$\times\psi\Big(\frac{2\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{a}_o+\mathfrak{b}_o}\Big)^{{\alpha_o}^2\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)^{\alpha_o}}\cdot\psi^*\Big(\frac{2\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{a}_o+\mathfrak{b}_o}\Big)^{\frac{(1-\alpha_o)}{2}\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)^{1-\alpha_o}}$$
$$\leq\Big[\Big({}_{\frac{1}{\mathfrak{b}_o}}D^{\alpha_o}_*(\psi\circ\Upsilon)\Big(\frac{1}{\mathfrak{b}_o}\Big)\Big)\Big({}_*D^{\alpha_o}_{\frac{1}{\mathfrak{a}_o}}(\psi\circ\Upsilon)\Big(\frac{1}{\mathfrak{a}_o}\Big)\Big)\Big]^{\frac{\Gamma(1-\alpha_o)}{2}\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)}.$$

*Thus, we successfully derive the first part of inequality (16).*

*To proceed with the second part of the proof, let* $\psi$ *and* $\psi^*$ *be two harmonically convex functions. Then, for all* $\lambda_o\in[0,1]$*, the following holds. By applying the definition of harmonic convexity, we obtain:*

$$\ln\psi\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\lambda_o\mathfrak{a}_o+(1-\lambda_o)\mathfrak{b}_o}\Big)+\ln\psi\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\lambda_o\mathfrak{b}_o+(1-\lambda_o)\mathfrak{a}_o}\Big)\leq\ln\psi(\mathfrak{a}_o)+\ln\psi(\mathfrak{b}_o)$$

*and*

$$\ln\psi^*\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\lambda_o\mathfrak{a}_o+(1-\lambda_o)\mathfrak{b}_o}\Big)+\ln\psi^*\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\lambda_o\mathfrak{b}_o+(1-\lambda_o)\mathfrak{a}_o}\Big)]\leq\ln\psi^*(\mathfrak{a}_o)+\ln\psi^*(\mathfrak{b}_o).$$

*Multiplying the result by* ${\alpha_o}^2\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)^{\alpha_o}$ *and* $(1-\alpha_o)^2\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)^{1-\alpha_o}{\lambda_o}^{1-2\alpha_o}$ *respectively, we have*

$${\alpha_o}^2\Big(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\Big)^{\alpha_o}\ln\psi\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\lambda_o\mathfrak{b}_o+(1-\lambda_o)\mathfrak{a}_o}\Big)$$
$$+{\alpha_o}^2\Big(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\Big)^{\alpha_o}\ln\psi\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\lambda_o\mathfrak{a}_o+(1-\lambda_o)\mathfrak{b}_o}\Big)$$
$$\leq{\alpha_o}^2\Big(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\Big)^{\alpha_o}[\ln\psi(\mathfrak{a}_o)+\ln\psi(\mathfrak{b}_o)]$$

*and*

$$\frac{1}{2}\Big[(1-\alpha_o)^2\Big(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\Big)^{1-\alpha_o}{\lambda_o}^{1-2\alpha_o}\ln\psi^*\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\lambda_o\mathfrak{b}_o+(1-\lambda_o)\mathfrak{a}_o}\Big)$$
$$+(1-\alpha_o)^2\Big(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\Big)^{1-\alpha_o}{\lambda_o}^{1-2\alpha_o}\ln\psi^*\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\lambda_o\mathfrak{a}_o+(1-\lambda_o)\mathfrak{b}_o}\Big)\Big]$$
$$\leq\frac{1}{2}(1-\alpha_o)^2\Big(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\Big)^{1-\alpha_o}{\lambda_o}^{1-2\alpha_o}[\ln\psi^*(\mathfrak{a}_o)+\ln\psi^*(\mathfrak{b}_o)].$$

*By summing these two expressions side by side and integrating the resulting equation with respect to* $\lambda_o$ *over the interval* $[0,1]$*, we obtain:*

$$\frac{1}{2}\int_0^1\Big[{\alpha_o}^2\Big(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\Big)^{\alpha_o}{\lambda_o}^{\alpha_o}\ln\psi\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\lambda_o\mathfrak{b}_o+(1-\lambda_o)\mathfrak{a}_o}\Big)$$
$$+(1-\alpha_o)^2\Big(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\Big)^{1-\alpha_o}{\lambda_o}^{1-\alpha_o}\ln\psi^*\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\lambda_o\mathfrak{b}_o+(1-\lambda_o)\mathfrak{a}_o}\Big)\Big]{\lambda_o}^{-\alpha_o}d\lambda_o$$
$$+\frac{1}{2}\int_0^1\Big[{\alpha_o}^2\Big(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\Big)^{\alpha_o}{\lambda_o}^{\alpha_o}\ln\psi\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\lambda_o\mathfrak{a}_o+(1-\lambda_o)\mathfrak{b}_o}\Big)$$
$$+(1-\alpha_o)^2\Big(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\Big)^{1-\alpha_o}{\lambda_o}^{1-\alpha_o}\ln\psi^*\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\lambda_o\mathfrak{a}_o+(1-\lambda_o)\mathfrak{b}_o}\Big)\Big]{\lambda_o}^{-\alpha_o}d\lambda_o$$

$$\leq \alpha_o{}^2\Big(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\Big)^{\alpha_o}\Big[\frac{\ln\psi(\mathfrak{a}_o)+\ln\psi(\mathfrak{b}_o)}{2}\Big]+(1-\alpha_o)^2\Big(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\Big)^{1-\alpha_o}\Big[\frac{\ln\psi^*(\mathfrak{a}_o)+\ln\psi^*(\mathfrak{b}_o)}{2}\Big]\int_0^1\lambda_o{}^{1-2\alpha_o}d\lambda_o.$$

*Through the substitution of the variable, we achieve*

$$\begin{aligned}&\frac{\Gamma(1-\alpha_o)}{2}\Big(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\Big)^{1-\alpha_o}\Big[{}^{PC}_{\frac{1}{\mathfrak{a}_o}-}D^{\alpha_o}_{\frac{1}{\mathfrak{b}_o}}(\ln\psi\circ\Upsilon)\Big(\frac{1}{\mathfrak{b}_o}\Big)+{}^{PC}_{\frac{1}{\mathfrak{b}_o}+}D^{\alpha_o}_{\frac{1}{\mathfrak{a}_o}}\ln(\psi\circ\Upsilon)\Big(\frac{1}{\mathfrak{a}_o}\Big)\Big]\\&\leq \alpha_o{}^2\Big(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\Big)^{\alpha_o}\Big[\frac{\ln\psi(\mathfrak{a}_o)+\ln\psi(\mathfrak{b}_o)}{2}\Big]+(1-\alpha_o)\Big(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\Big)^{1-\alpha_o}\Big[\frac{\ln\psi^*(\mathfrak{a}_o)+\ln\psi^*(\mathfrak{b}_o)}{4}\Big]\\&\times\Big[\Big({}_{\frac{1}{\mathfrak{b}_o}}D^{\alpha_o}_{*}(\psi\circ\Upsilon)\Big(\frac{1}{\mathfrak{b}_o}\Big)\Big)\Big({}_{*}D^{\alpha_o}_{\frac{1}{\mathfrak{a}_o}}(\psi\circ\Upsilon)\Big(\frac{1}{\mathfrak{a}_o}\Big)\Big)\Big]^{\frac{\Gamma(1-\alpha_o)}{2}\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)}\\&\leq\Big[G(\psi(\mathfrak{a}_o),\psi(\mathfrak{b}_o))\Big]^{\alpha_o{}^2\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)^{\alpha_o}}\Big[G(\psi^*(\mathfrak{a}_o),\psi^*(\mathfrak{b}_o))\Big]^{(1-\alpha_o)\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)^{1-\alpha_o}}.\end{aligned}$$

*Hence, we obtain the second part of (16).* □

**Example 3.2.** *The Figure 1 explains the validity of Theorem 3.1 for $\psi(x_o) = exp\{x_o^4\}$. Let $L_t, M_t$ and $R_t$, represent the left, middle and right terms of Theorem 3.1.*

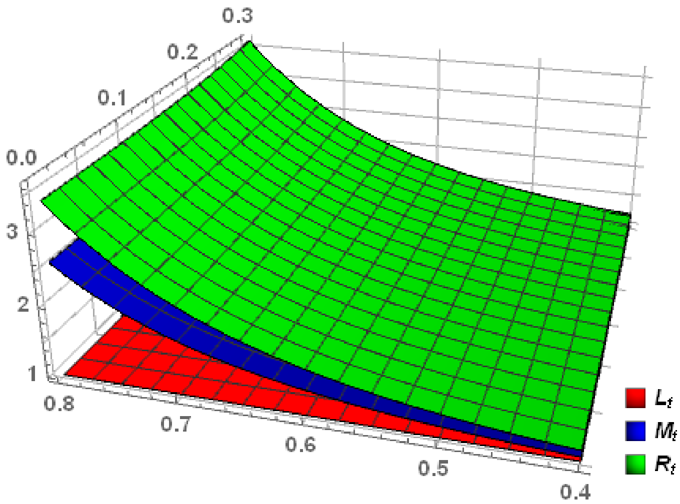


Figure 1: A visual representation that demonstrates that Theorem 3.1 is true for $\mathfrak{a}_o \in [0, 0.3]$, $\mathfrak{b}_o \in [0.4, 0.8]$ and $\alpha_o = 0.5$.

.

Lemma below is required to bolster our other main conclusions:

**Lemma 3.3.** *Let $\psi : I \subset \mathfrak{R}^+ \to \mathfrak{R}$ be a differentiable function on $I^\circ$. Let $\mathfrak{a}_o, \mathfrak{b}_o \in I^\circ$, with $\mathfrak{a}_o < \mathfrak{b}_o$ and $\psi^*$, $\psi^{**} \in L[\mathfrak{a}_o, \mathfrak{b}_o]$, $\Upsilon(\lambda_o) = \frac{1}{\lambda_o}$, then the following identity holds,*

$$I_1\times I_2\times I_3=\frac{\left[\psi(\mathfrak{a}_o)\psi(\mathfrak{b}_o)\right]^{\alpha_o{}^2\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{1-2\alpha_o}}\cdot\left[\psi^*(\mathfrak{a}_o)\psi^*(\mathfrak{b}_o)\right]^{\frac{(1-\alpha_o)}{2}}}{\left[\left({}_{\frac{1}{\mathfrak{a}_o}}D^{\alpha_o}_{*}(\psi\circ\Upsilon)(\frac{1}{\mathfrak{a}_o})\right)\cdot\left({}_{*}D^{\alpha_o}_{\frac{1}{\mathfrak{b}_o}}(\psi\circ\Upsilon)(\frac{1}{\mathfrak{b}_o})\right)\right]^{\Gamma(1-\alpha_o)\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{2-2\alpha_o}}},\tag{19}$$

*where*

$$\mathrm{I}_1 = \left( \int_0^1 \left( \psi^* \left( \frac{\mathfrak{a}_o \mathfrak{b}_o}{\mathfrak{b}_o \lambda_o + (1-\lambda_o)\mathfrak{a}_o} \right)^{1-2\lambda_o} \right)^{d\lambda_o} \right)^{\frac{\alpha_o{}^2 \mathfrak{a}_o \mathfrak{b}_o (\mathfrak{b}_o - \mathfrak{a}_o)}{(\mathfrak{b}_o\lambda_o + (1-\lambda_o)\mathfrak{a}_o)^2} \left( \frac{\mathfrak{a}_o \mathfrak{b}_o}{\mathfrak{b}_o - \mathfrak{a}_o} \right)^{1-2\alpha_o}},$$

$$\mathrm{I}_2 = \left( \int_0^1 \left( \psi^{**} \left( \frac{\mathfrak{a}_o \mathfrak{b}_o}{\mathfrak{b}_o \lambda_o + (1-\lambda_o)\mathfrak{a}_o} \right)^{\lambda_o{}^{2-2\alpha_o}} \right)^{d\lambda_o} \right)^{\frac{(\alpha_o - 1)\mathfrak{a}_o \mathfrak{b}_o (\mathfrak{b}_o - \mathfrak{a}_o)}{2(\mathfrak{b}_o\lambda_o + (1-\lambda_o)\mathfrak{a}_o)^2}},$$

*and*

$$\mathrm{I}_3 = \left( \int_0^1 \left( \psi^{**} \left( \frac{\mathfrak{a}_o \mathfrak{b}_o}{\mathfrak{b}_o \lambda_o + (1-\lambda_o)\mathfrak{a}_o} \right)^{(1-\lambda_o)^{2-2\alpha_o}} \right)^{d\lambda_o} \right)^{\frac{(1-\alpha_o)\mathfrak{a}_o \mathfrak{b}_o (\mathfrak{b}_o - \mathfrak{a}_o)}{2(\mathfrak{b}_o\lambda_o + (1-\lambda_o)\mathfrak{a}_o)^2}}.$$

*Proof.* *Using integration by parts for multiplicative integrals from* $\mathrm{I}_1$, *we have*

$$\mathrm{I}_1 = \left( \int_0^1 \left( \psi^* \left( \frac{\mathfrak{a}_o \mathfrak{b}_o}{\mathfrak{b}_o \lambda_o + (1-\lambda_o)\mathfrak{a}_o} \right)^{1-2\lambda_o} \right)^{d\lambda_o} \right)^{\frac{\alpha_o{}^2 \mathfrak{a}_o \mathfrak{b}_o (\mathfrak{b}_o - \mathfrak{a}_o)}{(\mathfrak{b}_o\lambda_o + (1-\lambda_o)\mathfrak{a}_o)^2} \left( \frac{\mathfrak{a}_o \mathfrak{b}_o}{\mathfrak{b}_o - \mathfrak{a}_o} \right)^{1-2\alpha_o}}$$

$$= \int_0^1 \left( \psi^* \left( \frac{\mathfrak{a}_o \mathfrak{b}_o}{\mathfrak{b}_o \lambda_o + (1-\lambda_o)\mathfrak{a}_o} \right)^{\frac{\mathfrak{a}_o \mathfrak{b}_o (\mathfrak{b}_o - \mathfrak{a}_o)}{(\mathfrak{b}_o\lambda_o + (1-\lambda_o)\mathfrak{a}_o)^2} \alpha_o{}^2 \left( \frac{\mathfrak{a}_o \mathfrak{b}_o}{\mathfrak{b}_o - \mathfrak{a}_o} \right)^{1-2\alpha_o} (1-2\lambda_o)} \right)^{d\lambda_o}$$

$$= \frac{[\psi(\mathfrak{a}_o)]^{\alpha_o{}^2 \left( \frac{\mathfrak{a}_o \mathfrak{b}_o}{\mathfrak{b}_o - \mathfrak{a}_o} \right)^{1-2\alpha_o}}}{[\psi(\mathfrak{b}_o)]^{-\alpha_o{}^2 \left( \frac{\mathfrak{a}_o \mathfrak{b}_o}{\mathfrak{b}_o - \mathfrak{a}_o} \right)^{1-2\alpha_o}}} \cdot \frac{1}{\int_0^1 \left( \psi \left( \frac{\mathfrak{a}_o \mathfrak{b}_o}{\mathfrak{b}_o \lambda_o + (1-\lambda_o)\mathfrak{a}_o} \right)^{2\alpha_o{}^2 \left( \frac{\mathfrak{a}_o \mathfrak{b}_o}{\mathfrak{b}_o - \mathfrak{a}_o} \right)^{1-2\alpha_o}} \right)^{d\lambda_o}}.$$

*It implies that*

$$\mathrm{I}_1 = [\psi(\mathfrak{a}_o)\, \psi(\mathfrak{b}_o)]^{\alpha_o{}^2 \left( \frac{\mathfrak{a}_o \mathfrak{b}_o}{\mathfrak{b}_o - \mathfrak{a}_o} \right)^{1-2\alpha_o}} \cdot \frac{1}{exp \left\{ 2\alpha_o{}^2 \left( \frac{\mathfrak{a}_o \mathfrak{b}_o}{\mathfrak{b}_o - \mathfrak{a}_o} \right)^{2-2\alpha_o} \int_{\frac{1}{\mathfrak{b}_o}}^{\frac{1}{\mathfrak{a}_o}} \ln(\psi\left(\frac{1}{\varkappa}\right)) d\varkappa \right\}}. \tag{20}$$

*Similarly for* $\mathrm{I}_2$, *we acquire*

$$\mathrm{I}_2 = \left( \int_0^1 \left( \psi^{**} \left( \frac{\mathfrak{a}_o \mathfrak{b}_o}{\mathfrak{b}_o \lambda_o + (1-\lambda_o)\mathfrak{a}_o} \right)^{\lambda_o{}^{2-2\alpha_o}} \right)^{d\lambda_o} \right)^{\frac{(\alpha_o - 1)\mathfrak{a}_o \mathfrak{b}_o (\mathfrak{b}_o - \mathfrak{a}_o)}{2(\mathfrak{b}_o\lambda_o + (1-\lambda_o)\mathfrak{a}_o)^2}}$$

$$= \int_0^1 \left( \psi^{**} \left( \frac{\mathfrak{a}_o \mathfrak{b}_o}{\mathfrak{b}_o \lambda_o + (1-\lambda_o)\mathfrak{a}_o} \right)^{\frac{-\mathfrak{a}_o \mathfrak{b}_o (\mathfrak{b}_o - \mathfrak{a}_o)}{(\mathfrak{b}_o\lambda_o + (1-\lambda_o)\mathfrak{a}_o)^2} \frac{(1-\alpha_o)\lambda_o{}^{2-2\alpha_o}}{2}} \right)^{d\lambda_o}$$

$$= [\psi^*(\mathfrak{a}_o)]^{\frac{(1-\alpha_o)}{2}} \cdot \frac{1}{\int_0^1 \left( \psi^* \left( \frac{\mathfrak{a}_o \mathfrak{b}_o}{\mathfrak{b}_o \lambda_o + (1-\lambda_o)\mathfrak{a}_o} \right)^{(1-\alpha_o)^2 \lambda_o{}^{1-2\alpha_o}} \right)^{d\lambda_o}}.$$

*It implies that*

$$\mathrm{I}_2 = [\psi^*(\mathfrak{a}_o)]^{\frac{(1-\alpha_o)}{2}} \cdot \frac{1}{exp\left\{(1-\alpha_o)^2\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{2-2\alpha_o}\int\limits_{\frac{1}{\mathfrak{b}_o}}^{\frac{1}{\mathfrak{a}_o}} \ln(\psi^*\left(\frac{1}{\varkappa}\right))\left(\varkappa-\frac{1}{\mathfrak{b}_o}\right)^{1-2\alpha_o} d\varkappa\right\}}. \tag{21}$$

*By moving in the same fashion*

$$\begin{aligned}
\mathrm{I}_3 &= \left(\int\limits_0^1\left(\psi^{**}\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o\lambda_o+(1-\lambda_o)\mathfrak{a}_o}\right)^{(1-\lambda_o)^{2-2\alpha_o}}\right)^{d\lambda_o}\right)^{\frac{(1-\alpha_o)\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{2(\mathfrak{b}_o\lambda_o+(1-\lambda_o)\mathfrak{a}_o)^2}} \\
&= \int\limits_0^1\left(\psi^{**}\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o\lambda_o+(1-\lambda_o)\mathfrak{a}_o}\right)^{\frac{-\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{(\mathfrak{a}_o\lambda_o+(1-\lambda_o)\mathfrak{b}_o)^2}\ \frac{(\alpha_o-1)}{2}(1-\lambda_o)^{2-2\alpha_o}}\right)^{d\lambda_o} \\
&= [\psi^*(\mathfrak{b}_o)]^{\frac{(1-\alpha_o)}{2}} \cdot \frac{1}{\int\limits_0^1\left(\psi^*\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o\lambda_o+(1-\lambda_o)\mathfrak{a}_o}\right)^{(1-\alpha_o)^2(1-\lambda_o)^{1-2\alpha_o}}\right)^{d\lambda_o}}.
\end{aligned}$$

*It implies tha*

$$\mathrm{I}_3 = [\psi^*(\mathfrak{b}_o)]^{\frac{(1-\alpha_o)}{2}} \cdot \frac{1}{exp\left\{(1-\alpha_o)^2\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{2-2\alpha_o}\int\limits_{\frac{1}{\mathfrak{b}_o}}^{\frac{1}{\mathfrak{a}_o}} \ln(\psi^*\left(\frac{1}{\varkappa}\right))\left(\frac{1}{\mathfrak{a}_o}-\varkappa\right)^{1-2\alpha_o} d\varkappa\right\}}. \tag{22}$$

*Using* (20), (21) *and* (22), *we get*

$$\begin{aligned}
\mathrm{I}_1\times\mathrm{I}_2\times\mathrm{I}_3 &= [\psi(\mathfrak{a}_o)\,\psi(\mathfrak{b}_o)]^{\alpha_o{}^2\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{1-2\alpha_o}} \cdot [\psi^*(\mathfrak{a}_o)\,\psi^*(\mathfrak{b}_o)]^{\frac{(1-\alpha_o)}{2}} \\
&\times \frac{1}{exp\left\{\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{2-2\alpha_o}\int\limits_{\frac{1}{\mathfrak{b}_o}}^{\frac{1}{\mathfrak{a}_o}}\left[\alpha_o{}^2\ln(\psi\left(\frac{1}{\varkappa}\right))\left(\varkappa-\frac{1}{\mathfrak{b}_o}\right)^{\alpha_o}+(1-\alpha_o)^2\ln(\psi^*\left(\frac{1}{\varkappa}\right))\left(\varkappa-\frac{1}{\mathfrak{b}_o}\right)^{1-\alpha_o}\right]\left(\varkappa-\frac{1}{\mathfrak{b}_o}\right)^{-\alpha_o}d\varkappa\right\}} \\
&\times \frac{1}{exp\left\{\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{2-2\alpha_o}\int\limits_{\frac{1}{\mathfrak{b}_o}}^{\frac{1}{\mathfrak{a}_o}}\left[\alpha_o{}^2\ln(\psi\left(\frac{1}{\varkappa}\right))\left(\frac{1}{\mathfrak{a}_o}-\varkappa\right)^{\alpha_o}+(1-\alpha_o)^2\ln(\psi^*\left(\frac{1}{\varkappa}\right))\left(\frac{1}{\mathfrak{a}_o}-\varkappa\right)^{1-\alpha_o}\right]\left(\frac{1}{\mathfrak{a}_o}-\varkappa\right)^{-\alpha_o}d\varkappa\right\}} \\
&= \frac{[\psi(\mathfrak{a}_o)\,\psi(\mathfrak{b}_o)]^{\alpha_o{}^2\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{1-2\alpha_o}} \cdot [\psi^*(\mathfrak{a}_o)\,\psi^*(\mathfrak{b}_o)]^{\frac{(1-\alpha_o)}{2}}}{\exp\left\{\Gamma(1-\alpha_o)\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{2-2\alpha_o}\left[\left({}^{PC}_{\frac{1}{\mathfrak{a}_o}^-}\mathrm{D}^{\alpha_o}_{\frac{1}{\mathfrak{b}_o}}\ln(\psi\circ\Upsilon)(\tfrac{1}{\mathfrak{b}_o})\right)+\left({}^{PC}_{\frac{1}{\mathfrak{b}_o}^+}\mathrm{D}^{\alpha_o}_{\frac{1}{\mathfrak{a}_o}}\ln(\psi\circ\Upsilon)(\tfrac{1}{\mathfrak{a}_o})\right)\right]\right\}}.
\end{aligned}$$

*It implies that*

$$\mathrm{I}_1\times\mathrm{I}_2\times\mathrm{I}_3 = \frac{[\psi(\mathfrak{a}_o)\,\psi(\mathfrak{b}_o)]^{\alpha_o{}^2\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{1-2\alpha_o}} \cdot [\psi^*(\mathfrak{a}_o)\,\psi^*(\mathfrak{b}_o)]^{\frac{(1-\alpha_o)}{2}}}{\left[\left({}_{\frac{1}{\mathfrak{b}_o}}\mathrm{D}^{\alpha_o}_{*}(\psi\circ\Upsilon)(\tfrac{1}{\mathfrak{b}_o})\right)\cdot\left({}_{*}\mathrm{D}^{\alpha_o}_{\frac{1}{\mathfrak{a}_o}}(\psi\circ\Upsilon)(\tfrac{1}{\mathfrak{a}_o})\right)\right]^{\Gamma(1-\alpha_o)\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{2-2\alpha_o}}}. \tag{23}$$

*Which is the desire inequality (19).* □

The resulting integral inequality may be derived using Lemma 3.3.

**Theorem 3.4.** *Let* $\psi : \mathrm{I} \subset \mathfrak{R}^+ \to \mathfrak{R}$ *be a differentiable function on* $\mathrm{I}^\circ$, *with* $\mathfrak{a}_o, \mathfrak{b}_o \in \mathrm{I}^\circ$ *where* $\mathfrak{a}_o < \mathfrak{b}_o$. *If* $|\psi^*|^{\mathfrak{q}}, |\psi^{**}|^{\mathfrak{q}}$ *are multilplicative harmonically convex on* $[\mathfrak{a}_o, \mathfrak{b}_o]$ *for some fixed* $\mathfrak{q} \geq 1$ *and* $\Upsilon(\lambda_o) = \frac{1}{\lambda_o}$, *then*

$$\left| \frac{\left[\psi(\mathfrak{a}_o)\,\psi(\mathfrak{b}_o)\right]^{\alpha_o{}^2\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{1-2\alpha_o}} \cdot \left[\psi^*(\mathfrak{a}_o)\,\psi^*(\mathfrak{b}_o)\right]^{\frac{(1-\alpha_o)}{2}}}{\left[\left({}_{\frac{1}{\mathfrak{a}_o}}\mathrm{D}_*^{\alpha_o}(\psi\circ\Upsilon)(\tfrac{1}{\mathfrak{a}_o})\right)\cdot\left({}_*\mathrm{D}_{\frac{1}{\mathfrak{b}_o}}^{\alpha_o}(\psi\circ\Upsilon)(\tfrac{1}{\mathfrak{b}_o})\right)\right]^{\Gamma(1-\alpha_o)\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{2-2\alpha_o}}} \right|$$
$$\leq \left[(\psi^*(\mathfrak{a}_o))^{(\text{ß}_2)^{\frac{1}{\mathfrak{q}}}} \cdot (\psi^*(\mathfrak{b}_o))^{(\text{ß}_3)^{\frac{1}{\mathfrak{q}}}}\right]^{\alpha_o{}^2\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{1-2\alpha_o}\text{ß}_1{}^{1-1/\mathfrak{q}}}$$
$$\times \left[(\psi^{**}(\mathfrak{a}_o))^{(\mathrm{C}_2(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o))^{\frac{1}{\mathfrak{q}}}} \cdot (\psi^{**}(\mathfrak{b}_o))^{(\mathrm{C}_3(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o))^{\frac{1}{\mathfrak{q}}}}\right]^{(1-\alpha_o)\frac{\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{2}\mathrm{C}_1{}^{1-1/\mathfrak{q}}(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o)}, \tag{24}$$

*holds, where* ß$_1$, ß$_2$, ß$_3$ *are defined in Theorem 2.5, and*

$$\mathrm{C}_1(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o) = \frac{\mathfrak{a}_o{}^{-2}}{3-2\alpha_o}\left[{}_2\mathrm{F}_1\left(2,1;4-2\alpha_o;1-\frac{\mathfrak{b}_o}{\mathfrak{a}_o}\right) + {}_2\mathrm{F}_1\left(2,3-2\alpha_o;4-2\alpha_o;1-\frac{\mathfrak{a}_o}{\mathfrak{b}_o}\right)\right]$$

$$\mathrm{C}_2(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o) = \frac{\mathfrak{a}_o{}^{-2}}{4-2\alpha_o}\left[{}_2\mathrm{F}_1\left(2,2;5-2\alpha_o;1-\frac{\mathfrak{b}_o}{\mathfrak{a}_o}\right) + {}_2\mathrm{F}_1\left(2,4-2\alpha_o;5-2\alpha_o;1-\frac{\mathfrak{a}_o}{\mathfrak{b}_o}\right)\right],$$

$$\mathrm{C}_3(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o) = \frac{\mathfrak{a}_o{}^{-2}}{3-2\alpha_o}\left[{}_2\mathrm{F}_1\left(2,1;5-2\alpha_o;1-\frac{\mathfrak{b}_o}{\mathfrak{a}_o}\right) + {}_2\mathrm{F}_1\left(2,3-2\alpha_o;5-2\alpha_o;1-\frac{\mathfrak{a}_o}{\mathfrak{b}_o}\right)\right],$$

*Proof.* *Let* $\mathrm{U}_{\lambda_o} = \lambda_o\mathfrak{b}_o + (1-\lambda_o)\mathfrak{a}_o$. *Using the modulus property, the power mean inequality, and the multiplicative harmonic convexity of* $(\ln\psi^*)^{\mathfrak{q}}$ *and* $(\ln\psi^{**})^{\mathfrak{q}}$, *we obtain the following result from Lemma 3.3:*

$$\left| \frac{\left[\psi(\mathfrak{a}_o)\,\psi(\mathfrak{b}_o)\right]^{\alpha_o{}^2\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{1-2\alpha_o}} \cdot \left[\psi^*(\mathfrak{a}_o)\,\psi^*(\mathfrak{b}_o)\right]^{\frac{(1-\alpha_o)}{2}}}{\left[\left({}_{\frac{1}{\mathfrak{a}_o}}\mathrm{D}_*^{\alpha_o}(\psi\circ\Upsilon)(\tfrac{1}{\mathfrak{a}_o})\right)\cdot\left({}_*\mathrm{D}_{\frac{1}{\mathfrak{b}_o}}^{\alpha_o}(\psi\circ\Upsilon)(\tfrac{1}{\mathfrak{b}_o})\right)\right]^{\Gamma(1-\alpha_o)\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{2-2\alpha_o}}} \right|$$
$$\leq \exp\left\{\alpha_o{}^2\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{1-2\alpha_o}\left(\int_0^1 \frac{|1-2\lambda_o|}{\mathrm{A}_{\lambda_o}^2}\mathrm{d}\lambda_o\right)^{1-1/\mathfrak{q}}\right.$$
$$\times\left(\int_0^1 \frac{|1-2\lambda_o|[\lambda_o(\ln\psi^*(\mathfrak{a}_o))^{\mathfrak{q}} + (1-\lambda_o)(\ln\psi^*(\mathfrak{b}_o))^{\mathfrak{q}}]}{\mathrm{A}_{\lambda_o}^2}\mathrm{d}\lambda_o\right)^{1/\mathfrak{q}}$$
$$+ (1-\alpha_o)\frac{\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{2}\left(\int_0^1 \frac{|(1-\lambda_o)^{2-2\alpha_o} - \lambda_o{}^{2-2\alpha_o}|}{\mathrm{A}_{\lambda_o}^2}\mathrm{d}\lambda_o\right)^{1-1/\mathfrak{q}}$$
$$\left.\times\left(\int_0^1 \frac{|(1-\lambda_o)^{2-2\alpha_o} - \lambda_o{}^{2-2\alpha_o}|[\lambda_o(\ln\psi^{**}(\mathfrak{a}_o))^{\mathfrak{q}} + (1-\lambda_o)(\ln\psi^{**}(\mathfrak{b}_o))^{\mathfrak{q}}]}{\mathrm{A}_{\lambda_o}^2}\mathrm{d}\lambda_o\right)^{1/\mathfrak{q}}\right\}$$
$$\leq \exp\left\{\alpha_o{}^2\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{1-2\alpha_o}\left(\int_0^1 \frac{|1-2\lambda_o|}{\mathrm{A}_{\lambda_o}^2}\mathrm{d}\lambda_o\right)^{1-1/\mathfrak{q}}\right.$$

$$\times\Big(\int_0^1 \frac{|1-2\lambda_o|[\lambda_o(\ln\psi^*(\mathfrak{a}_o))^{\mathfrak{q}}+(1-\lambda_o)(\ln\psi^*(\mathfrak{b}_o))^{\mathfrak{q}}]}{\mathrm{A}^2_{\lambda_o}}d\lambda_o\Big)^{1/\mathfrak{q}}$$

$$+(1-\alpha_o)\frac{\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{2}\Big(\int_0^1 \frac{[(1-\lambda_o)^{2-2\alpha_o}+\lambda_o{}^{2-2\alpha_o}]}{\mathrm{A}^2_{\lambda_o}}d\lambda_o\Big)^{1-1/\mathfrak{q}}$$

$$\times\Big(\int_0^1 \frac{[(1-\lambda_o)^{2-2\alpha_o}+\lambda_o{}^{2-2\alpha_o}][\lambda_o(\ln\psi^{**}(\mathfrak{a}_o))^{\mathfrak{q}}+(1-\lambda_o)(\ln\psi^{**}(\mathfrak{b}_o))^{\mathfrak{q}}]}{\mathrm{A}^2_{\lambda_o}}d\lambda_o\Big)^{1/\mathfrak{q}}\Big\}$$

$$\leq \exp\Big\{\alpha_o{}^2\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\Big)^{1-2\alpha_o}ß_1{}^{1-1/\mathfrak{q}}\Big[ß_2(\ln\psi^*(\mathfrak{a}_o))^{\mathfrak{q}}+ß_3(\ln\psi^*(\mathfrak{b}_o))^{\mathfrak{q}}\Big]^{1/\mathfrak{q}}$$

$$+(1-\alpha_o)\frac{\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{2}\mathrm{C}_1{}^{1-1/\mathfrak{q}}(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o)\Big[\mathrm{C}_2(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o)(\ln\psi^{**}(\mathfrak{a}_o))^{\mathfrak{q}}+\mathrm{C}_3(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o)(\ln\psi^{**}(\mathfrak{b}_o))^{\mathfrak{q}}\Big]^{1/\mathfrak{q}}\Big\}$$

$$\leq \exp\Big\{\alpha_o{}^2\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\Big)^{1-2\alpha_o}ß_1{}^{1-1/\mathfrak{q}}$$

$$\times\Big[\Big((ß_2)^{\frac{1}{\mathfrak{q}}}\ln\psi^*(\mathfrak{a}_o)\Big)^{\mathfrak{q}}+\Big((ß_3)^{\frac{1}{\mathfrak{q}}}\ln\psi^*(\mathfrak{b}_o)\Big)^{\mathfrak{q}}\Big]^{1/\mathfrak{q}}$$

$$+(1-\alpha_o)\frac{\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{2}\mathrm{C}_1{}^{1-1/\mathfrak{q}}(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o)$$

$$\times\Big[\Big((\mathrm{C}_2(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o))^{\frac{1}{\mathfrak{q}}}\ln\psi^{**}(\mathfrak{a}_o)\Big)^{\mathfrak{q}}+\Big((\mathrm{C}_3(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o))^{\frac{1}{\mathfrak{q}}}\ln\psi^{**}(\mathfrak{b}_o)\Big)^{\mathfrak{q}}\Big]^{1/\mathfrak{q}}\Big\}.$$

*By the use of $A^q+B^q\leq(A+B)^q$ for $A\geq0, B\geq0$ with $q\geq1$, we have that*

$$\left|\frac{[\psi(\mathfrak{a}_o)\,\psi(\mathfrak{b}_o)]^{\alpha_o{}^2\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{1-2\alpha_o}}\cdot[\psi^*(\mathfrak{a}_o)\,\psi^*(\mathfrak{b}_o)]^{\frac{(1-\alpha_o)}{2}}}{\left[\left({}_{\frac{1}{\mathfrak{a}_o}}\mathrm{D}^{\alpha_o}_*(\psi\circ\Upsilon)(\tfrac{1}{\mathfrak{a}_o})\right)\cdot\left({}_*\mathrm{D}^{\alpha_o}_{\frac{1}{\mathfrak{b}_o}}(\psi\circ\Upsilon)(\tfrac{1}{\mathfrak{b}_o})\right)\right]^{\Gamma(1-\alpha_o)\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{2-2\alpha_o}}}\right|$$

$$\leq \exp\Big\{\alpha_o{}^2\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\Big)^{1-2\alpha_o}ß_1{}^{1-1/\mathfrak{q}}\Big[(ß_2)^{\frac{1}{\mathfrak{q}}}\ln\psi^*(\mathfrak{a}_o)+(ß_3)^{\frac{1}{\mathfrak{q}}}\ln\psi^*(\mathfrak{b}_o)\Big]$$

$$+(1-\alpha_o)\frac{\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{2}\mathrm{C}_1{}^{1-1/\mathfrak{q}}(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o)\Big[(\mathrm{C}_2(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o))^{\frac{1}{\mathfrak{q}}}\ln\psi^{**}(\mathfrak{a}_o)+(\mathrm{C}_3(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o))^{\frac{1}{\mathfrak{q}}}\ln\psi^{**}(\mathfrak{b}_o)\Big]\Big\}$$

$$=\Big[(\psi^*(\mathfrak{a}_o))^{(ß_2)^{\frac{1}{\mathfrak{q}}}}\cdot(\psi^*(\mathfrak{b}_o))^{(ß_3)^{\frac{1}{\mathfrak{q}}}}\Big]^{\alpha_o{}^2\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{1-2\alpha_o}ß_1{}^{1-1/\mathfrak{q}}}$$

$$\times\Big[(\psi^{**}(\mathfrak{a}_o))^{(\mathrm{C}_2(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o))^{\frac{1}{\mathfrak{q}}}}\cdot(\psi^{**}(\mathfrak{b}_o))^{(\mathrm{C}_3(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o))^{\frac{1}{\mathfrak{q}}}}\Big]^{(1-\alpha_o)\frac{\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{2}\mathrm{C}_1{}^{1-1/\mathfrak{q}}(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o)}. \quad (25)$$

*Calculating $ß_1$, $ß_2$, $ß_3$, $\mathrm{C}_1(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o)$, $\mathrm{C}_2(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o)$, and $\mathrm{C}_3(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o)$, we have*

$$ß_1=\int_0^1\frac{|1-2\lambda_o|}{\mathrm{A}^2_{\lambda_o}}d\lambda_o=\frac{1}{\mathfrak{a}_o\mathfrak{b}_o}-\frac{2}{(\mathfrak{b}_o-\mathfrak{a}_o)^2}\ln\Big(\frac{(\mathfrak{a}_o+\mathfrak{b}_o)^2}{4\mathfrak{a}_o\mathfrak{b}_o}\Big),$$

$$ß_2=\int_0^1\frac{|1-2\lambda_o|\lambda_o}{\mathrm{A}^2_{\lambda_o}}d\lambda_o=\frac{-1}{\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}+\frac{3\mathfrak{a}_o+\mathfrak{b}_o}{(\mathfrak{b}_o-\mathfrak{a}_o)^3}\ln\Big(\frac{(\mathfrak{a}_o+\mathfrak{b}_o)^2}{4\mathfrak{a}_o\mathfrak{b}_o}\Big),$$

$$ß_3=\int_0^1\frac{|1-2\lambda_o|(1-\lambda_o)}{\mathrm{A}^2_{\lambda_o}}d\lambda_o=\frac{1}{\mathfrak{a}_o(\mathfrak{b}_o-\mathfrak{a}_o)}-\frac{3\mathfrak{b}_o+\mathfrak{a}_o}{(\mathfrak{b}_o-\mathfrak{a}_o)^3}\ln\Big(\frac{(\mathfrak{a}_o+\mathfrak{b}_o)^2}{4\mathfrak{a}_o\mathfrak{b}_o}\Big)=ß_1-ß_2,$$

$$C_1(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o)=\frac{[(1-\lambda_o)^{2-2\alpha_o}+\lambda_o{}^{2-2\alpha_o}]}{A^2_{\lambda_o}}d\lambda_o$$
$$=\frac{\mathfrak{a}_o{}^{-2}}{3-2\alpha_o}\left[{}_2F_1\left(2,1;4-2\alpha_o;1-\frac{\mathfrak{b}_o}{\mathfrak{a}_o}\right)+{}_2F_1\left(2,3-2\alpha_o;4-2\alpha_o;1-\frac{\mathfrak{b}_o}{\mathfrak{a}_o}\right)\right]$$

$$C_2(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o)=\frac{[(1-\lambda_o)^{2-2\alpha_o}+\lambda_o{}^{2-2\alpha_o}]\lambda_o}{A^2_{\lambda_o}}d\lambda_o$$
$$=\frac{\mathfrak{a}_o{}^{-2}}{4-2\alpha_o}\left[{}_2F_1\left(2,2;5-2\alpha_o;1-\frac{\mathfrak{b}_o}{\mathfrak{a}_o}\right)+{}_2F_1\left(2,4-2\alpha_o;5-2\alpha_o;1-\frac{\mathfrak{b}_o}{\mathfrak{a}_o}\right)\right]$$

$$C_3(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o)=\frac{[(1-\lambda_o)^{2-2\alpha_o}+\lambda_o{}^{2-2\alpha_o}](1-\lambda_o)}{A^2_{\lambda_o}}d\lambda_o$$
$$=\frac{\mathfrak{a}_o{}^{-2}}{3-2\alpha_o}\left[{}_2F_1\left(2,1;5-2\alpha_o;1-\frac{\mathfrak{b}_o}{\mathfrak{a}_o}\right)+{}_2F_1\left(2,3-2\alpha_o;5-2\alpha_o;1-\frac{\mathfrak{b}_o}{\mathfrak{a}_o}\right)\right].$$

*Hence the proof.* □

***Remark 3.5.*** *If $\alpha_o=1$ is taken in Theorem 3.4, then one attains*

$$\left|\frac{(\psi(\mathfrak{a}_o)\psi(\mathfrak{b}_o))^{\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}}}{\left(\int_{\frac{1}{\mathfrak{b}_o}}^{\frac{1}{\mathfrak{a}_o}}\left(\psi(\frac{1}{x_o})\right)^{dx_o}\right)^2}\right|\le\left[(\psi^*(\mathfrak{a}_o))^{\frac{\beta_2}{q}}\cdot(\psi^*(\mathfrak{b}_o))^{\frac{\beta_3}{q}}\right]^{(\mathfrak{b}_o-\mathfrak{a}_o)^2[\beta_1]^{1-\frac{1}{q}}}.$$

***Example 3.6.*** *The following graph describes the viability of Theorem 3.4 for $\psi(x_o)=\exp\{x_o^4\}$. Let $L_t, M_t$ and $R_t$, represent the left, middle and right terms of Theorem 3.4.*

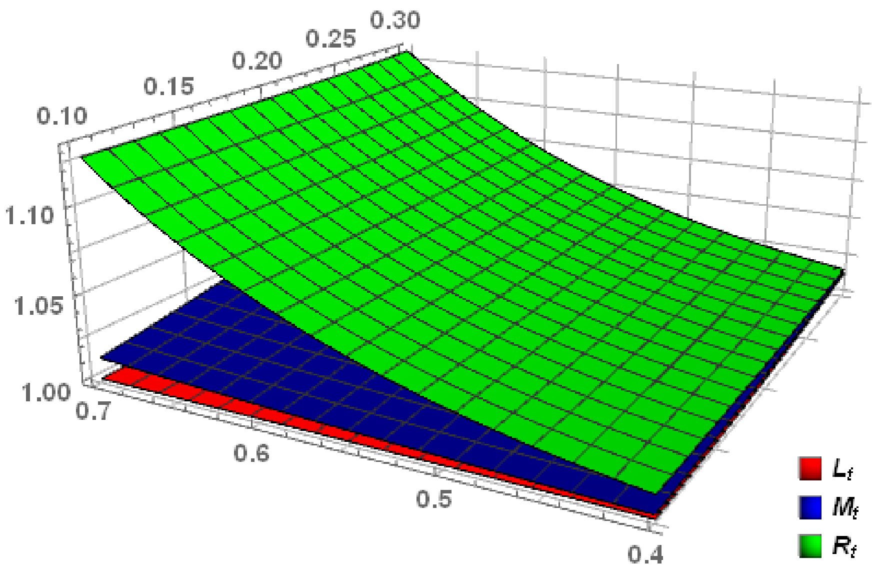


Figure 2: A visual representation that demonstrates that Theorem 3.4 is true for $\mathfrak{a}_o\in[0.1,0.3]$, $\mathfrak{b}_o\in[0.4,0.7]$ and $\alpha_o=0.5$.

.

By applying Lemma 2.7 and Lemma 3.3, we establish an additional conclusion for multiplicative harmonically convex functions within the range $0 \leq \alpha_o \leq 1$.

**Theorem 3.7.** *Let $\psi : \mathrm{I} \subset \mathfrak{R}^+ \to \mathfrak{R}$ be a differentiable function on $\mathrm{I}^\circ$, with $\mathfrak{a}_o, \mathfrak{b}_o \in \mathrm{I}^\circ$ where $\mathfrak{a}_o < \mathfrak{b}_o$. If $(\psi^*)^{\mathrm{q}}, (\psi^{**})^{\mathrm{q}}$ are multilplicative harmonically convex on $[\mathfrak{a}_o, \mathfrak{b}_o]$ for some fixed $\mathrm{q} \geq 1$ and $\Upsilon(\lambda_o) = \frac{1}{\lambda_o}$, then*

$$\left| \frac{[\psi(\mathfrak{a}_o)\psi(\mathfrak{b}_o)]^{\alpha_o^2\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{1-2\alpha_o}} \cdot [\psi^*(\mathfrak{a}_o)\psi^*(\mathfrak{b}_o)]^{\frac{(1-\alpha_o)}{2}}}{\left[\left({}_{\frac{1}{\mathfrak{a}_o}}\mathrm{D}_*^{\alpha_o}(\psi\circ\Upsilon)(\frac{1}{\mathfrak{a}_o})\right)\cdot\left({}_*\mathrm{D}_{\frac{1}{\mathfrak{b}_o}}^{\alpha_o}(\psi\circ\Upsilon)(\frac{1}{\mathfrak{b}_o})\right)\right]^{\Gamma(1-\alpha_o)\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{2-2\alpha_o}}} \right|$$
$$\leq \left[(\psi^*(\mathfrak{a}_o))^{(\text{ß}_2)^{\frac{1}{\mathrm{q}}}} \cdot (\psi^*(\mathfrak{b}_o))^{(\text{ß}_3)^{\frac{1}{\mathrm{q}}}}\right]^{\alpha_o^2\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{1-2\alpha_o}\text{ß}_1{}^{1-1/\mathrm{q}}}$$
$$\times \left[(\psi^{**}(\mathfrak{a}_o))^{(\mathrm{K}_2(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o))^{\frac{1}{\mathrm{q}}}} \cdot (\psi^{**}(\mathfrak{b}_o))^{(\mathrm{K}_3(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o))^{\frac{1}{\mathrm{q}}}}\right]^{(1-\alpha_o)\frac{\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{2}\mathrm{K}_1{}^{1-1/\mathrm{q}}(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o)}, \tag{26}$$

*holds, where*

$$\mathrm{K}_1(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o) = \frac{\mathfrak{a}_o{}^{-2}}{3-2\alpha_o}\Big[{}_2\mathrm{F}_1\Big(2, 3-2\alpha_o; 4-2\alpha_o; 1-\frac{\mathfrak{b}_o}{\mathfrak{a}_o}\Big) - {}_2\mathrm{F}_1\Big(2, 1; 4-2\alpha_o; 1-\frac{\mathfrak{b}_o}{\mathfrak{a}_o}\Big)$$
$$+ {}_2\mathrm{F}_1\Big(2, 1; 4-2\alpha_o; \frac{1}{2}\Big(1-\frac{\mathfrak{b}_o}{\mathfrak{a}_o}\Big)\Big)\Big],$$

$$\mathrm{K}_2(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o) = \frac{\mathfrak{a}_o{}^{-2}}{4-2\alpha_o}\Big[{}_2\mathrm{F}_1\Big(2, 3-2\alpha_o; 5-2\alpha_o; 1-\frac{\mathfrak{b}_o}{\mathfrak{a}_o}\Big) - \frac{1}{3-2\alpha_o}{}_2\mathrm{F}_1\Big(2, 2; 5-2\alpha_o; 1-\frac{\mathfrak{b}_o}{\mathfrak{a}_o}\Big)$$
$$+ \frac{1}{3-2\alpha_o}{}_2\mathrm{F}_1\Big(2, 2; 5-2\alpha_o; \frac{1}{2}\Big(1-\frac{\mathfrak{b}_o}{\mathfrak{a}_o}\Big)\Big)\Big],$$

*and*

$$\mathrm{K}_3(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o) = \frac{\mathfrak{a}_o{}^{-2}}{4-2\alpha_o}\Big[\frac{1}{3-2\alpha_o}{}_2\mathrm{F}_1\Big(2, 3-2\alpha_o; 5-2\alpha_o; 1-\frac{\mathfrak{b}_o}{\mathfrak{a}_o}\Big) - {}_2\mathrm{F}_1\Big(2, 1; 5-2\alpha_o; 1-\frac{\mathfrak{b}_o}{\mathfrak{a}_o}\Big)$$
$$+ {}_2\mathrm{F}_1\Big(2, 1; 5-2\alpha_o; \frac{1}{2}\Big(1-\frac{\mathfrak{b}_o}{\mathfrak{a}_o}\Big)\Big)\Big].$$

*Proof.* *Assume that $\mathrm{U}_{\lambda_o} = \lambda_o\mathfrak{b}_o + (1-\lambda_o)\mathfrak{a}_o$. By utilizing the modulus property, the power mean inequality, and the multiplicative harmonic convexity of $(\psi^*)^{\mathrm{q}}$ and $(\psi^{**})^{\mathrm{q}}$ in conjunction with Lemma (3.3), we derive the following result:*

$$\left| \frac{[\psi(\mathfrak{a}_o)\psi(\mathfrak{b}_o)]^{\alpha_o^2\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{1-2\alpha_o}} \cdot [\psi^*(\mathfrak{a}_o)\psi^*(\mathfrak{b}_o)]^{\frac{(1-\alpha_o)}{2}}}{\left[\left({}_{\frac{1}{\mathfrak{a}_o}}\mathrm{D}_*^{\alpha_o}(\psi\circ\Upsilon)(\frac{1}{\mathfrak{a}_o})\right)\cdot\left({}_*\mathrm{D}_{\frac{1}{\mathfrak{b}_o}}^{\alpha_o}(\psi\circ\Upsilon)(\frac{1}{\mathfrak{b}_o})\right)\right]^{\Gamma(1-\alpha_o)\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{2-2\alpha_o}}} \right|$$
$$\leq \exp\Big\{\alpha_o{}^2\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\Big)^{1-2\alpha_o}\Big(\int_0^1 \frac{|2\lambda_o-1|}{\mathrm{U}^2_{\lambda_o}}d\lambda_o\Big)^{1-1/\mathrm{q}}$$
$$\times\Big(\int_0^1 \frac{|2\lambda_o-1|[\lambda_o|\ln\psi^*(\mathfrak{a}_o)|^{\mathrm{q}} + (1-\lambda_o)|\ln\psi^*(\mathfrak{b}_o)|^{\mathrm{q}}]}{\mathrm{U}^2_{\lambda_o}}d\lambda_o\Big)^{1/\mathrm{q}}$$

$$+ (1-\alpha_o)\frac{\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{2}\Big(\int_0^1 \frac{|(1-\lambda_o)^{2-2\alpha_o}-\lambda_o{}^{2-2\alpha_o}|}{\mathrm{U}^2_{\lambda_o}}\mathrm{d}\lambda_o\Big)^{1-1/\mathrm{q}}$$

$$\times\Big(\int_0^1 \frac{|(1-\lambda_o)^{2-2\alpha_o}-\lambda_o{}^{2-2\alpha_o}|[\lambda_o|\ln\psi^{**}(\mathfrak{a}_o)|^{\mathrm{q}}+(1-\lambda_o)|\ln\psi^{**}(\mathfrak{b}_o)|^{\mathrm{q}}]}{\mathrm{U}^2_{\lambda_o}}\mathrm{d}\lambda_o\Big)^{1/\mathrm{q}}\Big\}$$

$$\leq \exp\Big\{\alpha_o{}^2\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\Big)^{1-2\alpha_o}\text{ß}_1{}^{1-1/\mathrm{q}}\Big[\text{ß}_2|\ln\psi^{*}(\mathfrak{a}_o)|^{\mathrm{q}}+\text{ß}_3|\ln\psi^{*}(\mathfrak{b}_o)|^{\mathrm{q}}\Big]^{1/\mathrm{q}}$$

$$+ (1-\alpha_o)\frac{\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{2}\mathrm{K}_1{}^{1-1/\mathrm{q}}(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o)\Big[\mathrm{K}_2(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o)|\psi^{**}(\mathfrak{a}_o)|^{\mathrm{q}}+\mathrm{K}_3(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o)|\psi^{**}(\mathfrak{b}_o)|^{\mathrm{q}}\Big]^{1/\mathrm{q}}\Big\}$$

$$\leq \exp\Big\{\alpha_o{}^2\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\Big)^{1-2\alpha_o}\text{ß}_1{}^{1-1/\mathrm{q}}\Big[\Big((\text{ß}_2)^{\frac{1}{\mathrm{q}}}\ln\psi^{*}(\mathfrak{a}_o)\Big)^{\mathrm{q}}+\Big((\text{ß}_3)^{\frac{1}{\mathrm{q}}}\ln\psi^{*}(\mathfrak{b}_o)\Big)^{\mathrm{q}}\Big]^{1/\mathrm{q}}$$

$$+ (1-\alpha_o)\frac{\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{2}\mathrm{K}_1{}^{1-1/\mathrm{q}}(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o)\Big[\Big((\mathrm{K}_2(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o))^{\frac{1}{\mathrm{q}}}\ln\psi^{**}(\mathfrak{a}_o)\Big)^{\mathrm{q}}$$

$$+\Big((\mathrm{K}_3(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o))^{\frac{1}{\mathrm{q}}}\ln\psi^{**}(\mathfrak{b}_o)\Big)^{\mathrm{q}}\Big]^{1/\mathrm{q}}\Big\}.$$

*By the use of $A^q + B^q \leq (A + B)^q$ for $A \geq 0, B \geq 0$ with $q \geq 1$, we have that*

$$\left|\frac{[\psi(\mathfrak{a}_o)\,\psi(\mathfrak{b}_o)]^{\alpha_o{}^2\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{1-2\alpha_o}}\cdot[\psi^{*}(\mathfrak{a}_o)\,\psi^{*}(\mathfrak{b}_o)]^{\frac{(1-\alpha_o)}{2}}}{\Big[\Big({}_{\frac{1}{\mathfrak{a}_o}}\mathrm{D}^{\alpha_o}_{*}(\psi\circ\Upsilon)(\frac{1}{\mathfrak{a}_o})\Big)\cdot\Big({}_{*}\mathrm{D}^{\alpha_o}_{\frac{1}{\mathfrak{b}_o}}(\psi\circ\Upsilon)(\frac{1}{\mathfrak{b}_o})\Big)\Big]^{\Gamma(1-\alpha_o)\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{2-2\alpha_o}}}\right|$$

$$\leq \exp\Big\{\alpha_o{}^2\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\Big)^{1-2\alpha_o}\text{ß}_1{}^{1-1/\mathrm{q}}\Big[(\text{ß}_2)^{\frac{1}{\mathrm{q}}}\ln\psi^{*}(\mathfrak{a}_o)+(\text{ß}_3)^{\frac{1}{\mathrm{q}}}\ln\psi^{*}(\mathfrak{b}_o)\Big]$$

$$+ (1-\alpha_o)\frac{\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{2}\mathrm{K}_1{}^{1-1/\mathrm{q}}(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o)\Big[(\mathrm{K}_2(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o))^{\frac{1}{\mathrm{q}}}\ln\psi^{**}(\mathfrak{a}_o)+(\mathrm{K}_3(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o))^{\frac{1}{\mathrm{q}}}\ln\psi^{**}(\mathfrak{b}_o)\Big]\Big\}$$

$$= \Big[(\psi^{*}(\mathfrak{a}_o))^{(\text{ß}_2)^{\frac{1}{\mathrm{q}}}}\cdot(\psi^{*}(\mathfrak{b}_o))^{(\text{ß}_3)^{\frac{1}{\mathrm{q}}}}\Big]^{\alpha_o{}^2\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{1-2\alpha_o}\text{ß}_1{}^{1-1/\mathrm{q}}}$$

$$\times\Big[(\psi^{**}(\mathfrak{a}_o))^{(\mathrm{K}_2(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o))^{\frac{1}{\mathrm{q}}}}\cdot(\psi^{**}(\mathfrak{b}_o))^{(\mathrm{K}_3(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o))^{\frac{1}{\mathrm{q}}}}\Big]^{(1-\alpha_o)\frac{\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{2}\mathrm{K}_1{}^{1-1/\mathrm{q}}(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o)}. \quad (27)$$

*Now calculating* $\mathrm{K}_1, \mathrm{K}_2$ *and* $\mathrm{K}_3$ *, by Lemma (2.7), we have*

$$\mathrm{K}_1(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o)=\int_0^1\frac{(1-\lambda_o)^{2-2\alpha_o}-\lambda_o{}^{2-2\alpha_o}}{\mathrm{U}^2_{\lambda_o}}\mathrm{d}\lambda_o$$

$$=\int_0^{\frac{1}{2}}\frac{(1-\lambda_o)^{2-2\alpha_o}-\lambda_o{}^{2-2\alpha_o}}{\mathrm{U}^2_{\lambda_o}}\mathrm{d}\lambda_o+\int_{\frac{1}{2}}^1\frac{\lambda_o{}^{2-2\alpha_o}-(1-\lambda_o)^{2-2\alpha_o}}{\mathrm{U}^2_{\lambda_o}}\mathrm{d}\lambda_o$$

$$=\int_0^1\frac{\lambda_o{}^{2-2\alpha_o}-(1-\lambda_o)^{2-2\alpha_o}}{\mathrm{U}^2_{\lambda_o}}\mathrm{d}\lambda_o+2\int_0^{\frac{1}{2}}\frac{(1-\lambda_o)^{2-2\alpha_o}-\lambda_o{}^{2-2\alpha_o}}{\mathrm{U}^2_{\lambda_o}}\mathrm{d}\lambda_o$$

$$\leq\int_0^1\lambda_o{}^{2-2\alpha_o}\mathrm{U}^{-2}_{\lambda_o}\mathrm{d}\lambda_o-\int_0^1(1-\lambda_o)^{2-2\alpha_o}\mathrm{U}^{-2}_{\lambda_o}\mathrm{d}\lambda_o+2\int_0^{\frac{1}{2}}(1-2\lambda_o)^{2-2\alpha_o}\mathrm{U}^{-2}_{\lambda_o}\mathrm{d}\lambda_o$$

$$=\int_0^1\lambda_o{}^{2-2\alpha_o}\mathrm{U}^{-2}_{\lambda_o}\mathrm{d}\lambda_o-\int_0^1(1-\lambda_o)^{2-2\alpha_o}\mathrm{U}^{-2}_{\lambda_o}\mathrm{d}\lambda_o+\int_0^1(1-\mathrm{u})^{2-2\alpha_o}\mathrm{U}^{-2}\Big(1-\mathrm{u}\frac{1}{2}\Big(1-\frac{\mathfrak{b}_o}{\mathfrak{a}_o}\Big)\Big)^{-2}\mathrm{d}\mathrm{u}$$

$$\mathrm{K}_1(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o) \le \frac{\mathfrak{a}_o{}^{-2}}{3-2\alpha_o}\Big[{}_2\mathrm{F}_1\Big(2,3-2\alpha_o;4-2\alpha_o;1-\frac{\mathfrak{b}_o}{\mathfrak{a}_o}\Big) - {}_2\mathrm{F}_1\Big(2,1;4-2\alpha_o;1-\frac{\mathfrak{b}_o}{\mathfrak{a}_o}\Big) + {}_2\mathrm{F}_1\Big(2,1;4-2\alpha_o;\frac{1}{2}\Big(1-\frac{\mathfrak{b}_o}{\mathfrak{a}_o}\Big)\Big)\Big]. \tag{28}$$

*similarly we get*

$$\mathrm{K}_2(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o) \le \int_0^1 \frac{(1-\lambda_o)^{2-2\alpha_o} - \lambda_o{}^{2-2\alpha_o}}{\mathrm{U}_{\lambda_o}^2}\lambda_o d\lambda_o$$

$$\le \int_0^1 \lambda_o{}^{3-2\alpha_o}\mathrm{U}_{\lambda_o}^{-2}d\lambda_o - \int_0^1 (1-\lambda_o)^{2-2\alpha_o\lambda_o}\mathrm{U}_{\lambda_o}^{-2}d\lambda_o + 2\int_0^{\frac{1}{2}}(1-2\lambda_o)^{2-2\alpha_o}\lambda_o\mathrm{U}_{\lambda_o}^{-2}d\lambda_o$$

$$= \frac{\mathfrak{a}_o{}^{-2}}{4-2\alpha_o}\Big[{}_2\mathrm{F}_1\Big(2,3-2\alpha_o;5-2\alpha_o;1-\frac{\mathfrak{b}_o}{\mathfrak{a}_o}\Big) - \frac{1}{3-2\alpha_o}{}_2\mathrm{F}_1\Big(2,2;5-2\alpha_o;1-\frac{\mathfrak{b}_o}{\mathfrak{a}_o}\Big) + \frac{1}{3-2\alpha_o}{}_2\mathrm{F}_1\Big(2,2;5-2\alpha_o;\frac{1}{2}\Big(1-\frac{\mathfrak{b}_o}{\mathfrak{a}_o}\Big)\Big)\Big], \tag{29}$$

$$\mathrm{K}_3(\alpha_o;\mathfrak{a}_o,\mathfrak{b}_o) = \int_0^1 \frac{(1-\lambda_o)^{2-2\alpha_o} - \lambda_o{}^{2-2\alpha_o}}{\mathrm{U}_{\lambda_o}^2}(1-\lambda_o)d\lambda_o$$

$$\le \int_0^1 \lambda_o{}^{2-2\alpha_o}(1-\lambda_o)\mathrm{U}_{\lambda_o}^{-2}d\lambda_o - \int_0^1 (1-\lambda_o)^{3-2\alpha_o\lambda_o}\mathrm{U}_{\lambda_o}^{-2}d\lambda_o + 2\int_0^{\frac{1}{2}}(1-2\lambda_o)^{2-2\alpha_o}(1-\lambda_o)\mathrm{U}_{\lambda_o}^{-2}d\lambda_o$$

$$= \frac{\mathfrak{a}_o{}^{-2}}{4-2\alpha_o}\Big[\frac{1}{3-2\alpha_o}{}_2\mathrm{F}_1\Big(2,3-2\alpha_o;5-2\alpha_o;1-\frac{\mathfrak{b}_o}{\mathfrak{a}_o}\Big) - {}_2\mathrm{F}_1\Big(2,1;5-2\alpha_o;1-\frac{\mathfrak{b}_o}{\mathfrak{a}_o}\Big) + {}_2\mathrm{F}_1\Big(2,1;5-2\alpha_o;\frac{1}{2}\Big(1-\frac{\mathfrak{a}_o}{\mathfrak{b}_o}\Big)\Big)\Big].$$

*Hence the proof.* □

**Theorem 3.8.** *Let* $\psi : \mathrm{I} \subset \mathfrak{R}^+ \to \mathfrak{R}$ *be a differentiable function on* $\mathrm{I}^\circ$*, the interior of the interval* $\mathrm{I}$*, where* $\mathfrak{a}_o, \mathfrak{b}_o \in \mathrm{I}^\circ$ *with* $\mathfrak{a}_o < \mathfrak{b}_o$*. If* $(\psi^*)^{\mathrm{q}}, (\psi^{**})^{\mathrm{q}}$ *are multilplicatively harmonic convex on* $[\mathfrak{a}_o, \mathfrak{b}_o]$ *for some fixed* $\mathrm{q} > 1$ *and* $\frac{1}{\mathrm{p}} + \frac{1}{\mathrm{q}} = 1$ *and* $\Upsilon(\lambda_o) = \frac{1}{\lambda_o}$*, then*

$$\left|\frac{[\psi(\mathfrak{a}_o)\psi(\mathfrak{b}_o)]^{\alpha_o{}^2\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{1-2\alpha_o}} \cdot [\psi^*(\mathfrak{a}_o)\psi^*(\mathfrak{b}_o)]^{\frac{(1-\alpha_o)}{2}}}{\Big[\Big({}_{\frac{1}{\mathfrak{a}_o}}\mathrm{D}_*^{\alpha_o}(\psi\circ\Upsilon)(\frac{1}{\mathfrak{a}_o})\Big)\cdot\Big({}_*\mathrm{D}_{\frac{1}{\mathfrak{b}_o}}^{\alpha_o}(\psi\circ\Upsilon)(\frac{1}{\mathfrak{b}_o})\Big)\Big]^{\Gamma(1-\alpha_o)\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{2-2\alpha_o}}}\right|$$

$$\le \Big[(\psi^*(\mathfrak{a}_o))^{\mu_1{}^{\frac{1}{\mathrm{q}}}}\cdot(\psi^*(\mathfrak{b}_o))^{\mu_2{}^{\frac{1}{\mathrm{q}}}}\Big]^{\alpha_o{}^2\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{1-2\alpha_o}\left(\frac{1}{\mathrm{p}+1}\right)^{1/\mathrm{p}}}$$

$$\times\Big[\psi^{**}(\mathfrak{a}_o)\cdot\psi^{**}(\mathfrak{b}_o)\Big]^{(1-\alpha_o)\frac{\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{4}(\mathrm{K}_4^{1/\mathrm{p}}+\mathrm{K}_5^{1/\mathrm{p}})}, \tag{30}$$

*holds, where*

$$\mathrm{K}_4 = \frac{\mathfrak{a}_o{}^{-2\mathrm{p}}}{2\mathrm{p}(1-\alpha_o)-1}{}_2\mathrm{F}_1\Big(2\mathrm{p},1;2\mathrm{p}(1-\alpha_o)+2;1-\frac{\mathfrak{b}_o}{\mathfrak{a}_o}\Big),$$

*and*

$$K_5 = \frac{\mathfrak{a}_o^{-2p}}{2p(1-\alpha_o)-1}\,{}_2F_1\Big(2p, 2p(1-\alpha_o)+1; 2p(1-\alpha_o)+2; 1-\frac{\mathfrak{b}_o}{\mathfrak{a}_o}\Big).$$

*with* $\mu_1$ *and* $\mu_2$ *are defined in Theorem 2.6.*

*Proof.* *Let* $U_{\lambda_o} = \lambda_o\mathfrak{b}_o + (1-\lambda_o)\mathfrak{a}_o$. *Lemma (3.3) and (2.7) can be determined by using the Hölder inequality and the multiplicative harmonically convexity of* $(\psi^*)^q$ *and* $(\psi^{**})^q$.

$$\left|\frac{[\psi(\mathfrak{a}_o)\psi(\mathfrak{b}_o)]^{\alpha_o^2\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{1-2\alpha_o}}\cdot[\psi^*(\mathfrak{a}_o)\psi^*(\mathfrak{b}_o)]^{\frac{(1-\alpha_o)}{2}}}{\left[\left({}_{\frac{1}{\mathfrak{a}_o}}D_*^{\alpha_o}(\psi\circ\Upsilon)(\frac{1}{\mathfrak{a}_o})\right)\cdot\left({}_*D_{\frac{1}{\mathfrak{b}_o}}^{\alpha_o}(\psi\circ\Upsilon)(\frac{1}{\mathfrak{b}_o})\right)\right]^{\Gamma(1-\alpha_o)\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{2-2\alpha_o}}}\right|$$
$$\leq \exp\Bigg\{\alpha_o^2\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\Big)^{1-2\alpha_o}\Big(\int_0^1|1-2\lambda_o|^p d\lambda_o\Big)^{\frac{1}{p}}\times\Big(\int_0^1\frac{1}{U_{\lambda_o}^{2q}}\Big(\ln\psi^*\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{U_{\lambda_o}^{2q}}\Big)\Big)^q d\lambda_o\Big)^{\frac{1}{q}}$$
$$+(1-\alpha_o)\frac{\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{2}\Bigg[\Big(\int_0^1\frac{(1-\lambda_o)^{2-2\alpha_o p}}{U_{\lambda_o}^{2p}}d\lambda_o\Big)^{1/p}\Big(\int_0^1\Big(\ln\psi^{**}\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{U_{\lambda_o}^2}\Big)\Big)^q d\lambda_o\Big)^{1/q}$$
$$+\Big[\Big(\int_0^1\frac{\lambda_o^{2-2\alpha_o p}}{U_{\lambda_o}^{2p}}d\lambda_o\Big)^{1/p}\Big(\int_0^1\Big(\ln\psi^{**}\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{U_{\lambda_o}^2}\Big)\Big)^q d\lambda_o\Big)^{1/q}\Big]\Bigg\}$$
$$\leq \exp\Bigg\{\alpha_o^2\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\Big)^{1-2\alpha_o}\Big(\frac{1}{p+1}\Big)^{1/p}\times\Big(\int_0^1\frac{\lambda_o(\ln\psi^{**}(\mathfrak{a}_o))^q+(1-\lambda_o)(\ln\psi^{**}(\mathfrak{b}_o))^q}{U_{\lambda_o}^{2q}}d\lambda_o\Big)^{1/q}$$
$$+(1-\alpha_o)\frac{\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{2}(K_4^{1/p}+K_5^{1/p})\Big(\int_0^1\lambda_o(\ln\psi^{**}(\mathfrak{a}_o))^q+(1-\lambda_o)(\ln\psi^{**}(\mathfrak{b}_o))^q\Big)^{1/q}\Bigg\}$$
$$\leq \exp\Bigg\{\alpha_o^2\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\Big)^{1-2\alpha_o}\Big(\frac{1}{p+1}\Big)^{1/p}\Big(\mu_1(\ln\psi^*(\mathfrak{a}_o))^q+\mu_2(\ln\psi^*(\mathfrak{b}_o))^q\Big)^{\frac{1}{q}}$$
$$+(1-\alpha_o)\frac{\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{2}(K_4^{1/p}+K_5^{1/p})\Big(\frac{(\ln\psi^{**}(\mathfrak{a}_o))^q+(\ln\psi^{**}(\mathfrak{b}_o))^q}{2}\Big)^{1/q}\Bigg\}$$
$$\leq \exp\Bigg\{\alpha_o^2\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\Big)^{1-2\alpha_o}\Big(\frac{1}{p+1}\Big)^{1/p}\Big(\Big(\mu_1^{\frac{1}{q}}\ln\psi^*(\mathfrak{a}_o)\Big)^q+\Big(\mu_2^{\frac{1}{q}}\ln\psi^*(\mathfrak{b}_o)\Big)^q\Big)^{\frac{1}{q}}$$
$$+(1-\alpha_o)\frac{\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{2}(K_4^{1/p}+K_5^{1/p})\Bigg(\Bigg(\Big(\frac{1}{2}\Big)^{\frac{1}{q}}\ln(\psi^{**}(\mathfrak{a}_o))\Bigg)^q+\Bigg(\Big(\frac{1}{2}\Big)^{\frac{1}{q}}\ln(\psi^{**}(\mathfrak{b}_o))\Bigg)^q\Bigg)^{1/q}\Bigg\}.$$

*By the use of* $A^q + B^q \leq (A+B)^q$ *for* $A \geq 0, B \geq 0$ *and* $\left(\frac{1}{2}\right)^{\frac{1}{q}} \leq \frac{1}{2}$ *with* $q \geq 1$*, we have that*

$$\left|\frac{[\psi(\mathfrak{a}_o)\psi(\mathfrak{b}_o)]^{\alpha_o^2\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{1-2\alpha_o}}\cdot[\psi^*(\mathfrak{a}_o)\psi^*(\mathfrak{b}_o)]^{\frac{(1-\alpha_o)}{2}}}{\left[\left({}_{\frac{1}{\mathfrak{a}_o}}D_*^{\alpha_o}(\psi\circ\Upsilon)(\frac{1}{\mathfrak{a}_o})\right)\cdot\left({}_*D_{\frac{1}{\mathfrak{b}_o}}^{\alpha_o}(\psi\circ\Upsilon)(\frac{1}{\mathfrak{b}_o})\right)\right]^{\Gamma(1-\alpha_o)\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{2-2\alpha_o}}}\right|$$
$$\leq \exp\Bigg\{\alpha_o^2\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\Big)^{1-2\alpha_o}\Big(\frac{1}{p+1}\Big)^{1/p}\Big[\mu_1^{\frac{1}{q}}\ln\psi^*(\mathfrak{a}_o)+\mu_2^{\frac{1}{q}}\ln\psi^*(\mathfrak{b}_o)\Big]$$
$$+(1-\alpha_o)\frac{\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{2}(K_4^{1/p}+K_5^{1/p})\Big[\Big(\frac{1}{2}\Big)^{\frac{1}{q}}\ln(\psi^{**}(\mathfrak{a}_o))+\Big(\frac{1}{2}\Big)^{\frac{1}{q}}\ln(\psi^{**}(\mathfrak{b}_o))\Big]\Bigg\}$$

$$= \left[ (\psi^*(\mathfrak{a}_o))^{\mu_1^{\frac{1}{q}}} \cdot (\psi^*(\mathfrak{b}_o))^{\mu_2^{\frac{1}{q}}} \right]^{\alpha_o{}^2 \mathfrak{a}_o \mathfrak{b}_o (\mathfrak{b}_o - \mathfrak{a}_o) \left( \frac{\mathfrak{a}_o \mathfrak{b}_o}{\mathfrak{b}_o - \mathfrak{a}_o} \right)^{1-2\alpha_o} \left( \frac{1}{p+1} \right)^{1/p}}$$
$$\times \left[ \psi^{**}(\mathfrak{a}_o) \cdot \psi^{**}(\mathfrak{b}_o) \right]^{(1-\alpha_o) \frac{\mathfrak{a}_o \mathfrak{b}_o (\mathfrak{b}_o - \mathfrak{a}_o)}{4} (K_4^{1/p} + K_5^{1/p})}.$$

*Calculating $\mu_1, \mu_2, K_4$ and $K_5$, we have*

$$\mu_1 = \int_0^1 \frac{\lambda_o}{U_{\lambda_o}^{2q}} d\lambda_o = \frac{[\mathfrak{a}_o{}^{2-2q} + \mathfrak{b}_o{}^{1-2q}[(\mathfrak{b}_o - \mathfrak{a}_o)(1-2q) - \mathfrak{a}_o]]}{2(\mathfrak{b}_o - \mathfrak{a}_o)^2 (1-q)(1-2q)},$$

$$\mu_2 = \int_0^1 \frac{1-\lambda_o}{U_{\lambda_o}^{2q}} d\lambda_o = \frac{[\mathfrak{b}_o{}^{2-2q} + \mathfrak{a}_o{}^{1-2q}[(\mathfrak{b}_o - \mathfrak{a}_o)(1-2q) + \mathfrak{b}_o]]}{2(\mathfrak{b}_o - \mathfrak{a}_o)^2 (1-q)(1-2q)},$$

$$K_4 = \left( \int_0^1 \frac{(1-\lambda_o)^{2-2\alpha_o p}}{U_{\lambda_o}^{2p}} d\lambda_o \right) = \frac{\mathfrak{a}_o{}^{-2p}}{2p(1-\alpha_o) - 1} {}_2F_1\left( 2p, 1; 2p(1-\alpha_o) + 2; 1 - \frac{\mathfrak{b}_o}{\mathfrak{a}_o} \right),$$

*and*

$$K_5 = \left( \int_0^1 \frac{\lambda_o{}^{2-2\alpha_o p}}{U_{\lambda_o}^{2p}} d\lambda_o \right) = \frac{\mathfrak{a}_o{}^{-2p}}{2p(1-\alpha_o) - 1} {}_2F_1\left( 2p, 2p(1-\alpha_o) + 1; 2p(1-\alpha_o) + 2; 1 - \frac{\mathfrak{b}_o}{\mathfrak{a}_o} \right).$$

*Hence the proof.* □

**Theorem 3.9.** *Let $\psi : I \subset \Re^+ \to \Re$ be a differentiable function on $I^\circ$, with $\mathfrak{a}_o, \mathfrak{b}_o \in I^\circ$ where $\mathfrak{a}_o < \mathfrak{b}_o$. If $(\psi^*)^q, (\psi^{**})^q$ are multilplicative harmonically convex on $[\mathfrak{a}_o, \mathfrak{b}_o]$ for some fixed $q > 1$ and $\Upsilon(\lambda_o) = \frac{1}{\lambda_o}$, then*

$$\left| \frac{[\psi(\mathfrak{a}_o)\psi(\mathfrak{b}_o)]^{\alpha_o{}^2 \left( \frac{\mathfrak{a}_o \mathfrak{b}_o}{\mathfrak{b}_o - \mathfrak{a}_o} \right)^{1-2\alpha_o}} \cdot [\psi^*(\mathfrak{a}_o)\psi^*(\mathfrak{b}_o)]^{\frac{(1-\alpha_o)}{2}}}{\left[ \left( {}_{\frac{1}{\mathfrak{a}_o}}D_*^{\alpha_o} (\psi \circ \Upsilon)(\frac{1}{\mathfrak{a}_o}) \right) \cdot \left( {}_*D_{\frac{1}{\mathfrak{b}_o}}^{\alpha_o} (\psi \circ \Upsilon)(\frac{1}{\mathfrak{b}_o}) \right) \right]^{\Gamma(1-\alpha_o) \left( \frac{\mathfrak{a}_o \mathfrak{b}_o}{\mathfrak{b}_o - \mathfrak{a}_o} \right)^{2-2\alpha_o}}} \right|$$
$$\leq \left[ (\psi^*(\mathfrak{a}_o))^{\mu_1^{\frac{1}{q}}} \cdot (\psi^*(\mathfrak{b}_o))^{\mu_2^{\frac{1}{q}}} \right]^{\alpha_o{}^2 \mathfrak{a}_o \mathfrak{b}_o (\mathfrak{b}_o - \mathfrak{a}_o) \left( \frac{\mathfrak{a}_o \mathfrak{b}_o}{\mathfrak{b}_o - \mathfrak{a}_o} \right)^{1-2\alpha_o} \left( \frac{1}{p+1} \right)^{1/p}}$$
$$\times \left[ (\psi^{**}(\mathfrak{a}_o))^{K_{10}^{1/p}} \cdot (\psi^{**}(\mathfrak{b}_o))^{K_{11}^{1/p}} \right]^{(1-\alpha_o) \frac{\mathfrak{a}_o \mathfrak{b}_o (\mathfrak{b}_o - \mathfrak{a}_o)}{4} (K_9^{1/p})}, \tag{31}$$

*holds, where $\mu_1$ and $\mu_2$ are defined in Theorem 2.6.*

*Proof.* *Let $U_{\lambda_o} = \lambda_o \mathfrak{b}_o + (1-\lambda_o)\mathfrak{a}_o$. Lemma (3.3) and (2.7) may be determined by using the Hölder inequality and the multiplicative harmonically convexity of $(\psi^*)^q$ and $(\psi^{**})^q$.*

$$\left| \frac{[\psi(\mathfrak{a}_o)\psi(\mathfrak{b}_o)]^{\alpha_o{}^2 \left( \frac{\mathfrak{a}_o \mathfrak{b}_o}{\mathfrak{b}_o - \mathfrak{a}_o} \right)^{1-2\alpha_o}} \cdot [\psi^*(\mathfrak{a}_o)\psi^*(\mathfrak{b}_o)]^{\frac{(1-\alpha_o)}{2}}}{\left[ \left( {}_{\frac{1}{\mathfrak{a}_o}}D_*^{\alpha_o} (\psi \circ \Upsilon)(\frac{1}{\mathfrak{a}_o}) \right) \cdot \left( {}_*D_{\frac{1}{\mathfrak{b}_o}}^{\alpha_o} (\psi \circ \Upsilon)(\frac{1}{\mathfrak{b}_o}) \right) \right]^{\Gamma(1-\alpha_o) \left( \frac{\mathfrak{a}_o \mathfrak{b}_o}{\mathfrak{b}_o - \mathfrak{a}_o} \right)^{2-2\alpha_o}}} \right|$$

$$\leq \exp\Bigg\{{\alpha_o}^2 \mathfrak{a}_o \mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\Big)^{1-2\alpha_o}\Big(\int_0^1 |1-2\lambda_o|^{\mathrm{p}}d\lambda_o\Big)^{\frac{1}{\mathrm{p}}} \times\Big(\int_0^1 \frac{1}{\mathrm{U}^{2\mathrm{q}}_{\lambda_o}}\Big(\ln\psi^*\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathrm{U}^{2\mathrm{q}}_{\lambda_o}}\Big)\Big)^{\mathrm{q}}d\lambda_o\Big)^{\frac{1}{\mathrm{q}}}$$

$$+(1-\alpha_o)\frac{\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{2}\Big(\int_0^1 |(1-\lambda_o)^{2-2\alpha_o}-{\lambda_o}^{2-2\alpha_o}|^{\mathrm{p}}d\lambda_o\Big)^{1/\mathrm{p}}\Big(\int_0^1\Big(\ln\psi^{**}\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathrm{U}^2_{\lambda_o}}\Big)\Big)^{\mathrm{q}}d\lambda_o\Big)^{1/\mathrm{q}}\Bigg\}$$

$$\leq \exp\Bigg\{{\alpha_o}^2 \mathfrak{a}_o \mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\Big)^{1-2\alpha_o}\Big(\frac{1}{\mathrm{p}+1}\Big)^{1/\mathrm{p}}\Big(\int_0^1 \frac{\lambda_o(\ln\psi^{**}(\mathfrak{a}_o))^{\mathrm{q}}+(1-\lambda_o)(\ln\psi^{**}(\mathfrak{b}_o))^{\mathrm{q}}}{\mathrm{U}^{2\mathrm{q}}_{\lambda_o}}d\lambda_o\Big)^{1/\mathrm{q}}$$

$$+(1-\alpha_o)\frac{\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{2}\Big(\int_0^1 |1-2\lambda_o|^{(2-2\alpha_o)\mathrm{p}}d\lambda_o\Big)^{\frac{1}{\mathrm{p}}}\Big(\int_0^1 \frac{\lambda_o(\ln\psi^{**}(\mathfrak{a}_o))^{\mathrm{q}}+(1-\lambda_o)(\ln\psi^{**}(\mathfrak{b}_o))^{\mathrm{q}}}{\mathrm{U}^{2\mathrm{q}}_{\lambda_o}}d\lambda_o\Big)^{1/\mathrm{q}}\Bigg\}$$

$$\leq \exp\Bigg\{{\alpha_o}^2 \mathfrak{a}_o \mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\Big)^{1-2\alpha_o}\Big(\frac{1}{\mathrm{p}+1}\Big)^{1/\mathrm{p}}\Big(\mu_1(\ln\psi^*(\mathfrak{a}_o))^{\mathrm{q}}+\mu_2(\ln\psi^*(\mathfrak{b}_o))^{\mathrm{q}}\Big)^{\frac{1}{\mathrm{q}}}$$

$$+(1-\alpha_o)\frac{\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{2}\mathrm{K}_9^{1/\mathrm{p}}\Big(\mathrm{K}_{10}(\ln\psi^{**}(\mathfrak{a}_o))^{\mathrm{q}}+\mathrm{K}_{11}(\ln\psi^{**}(\mathfrak{b}_o))^{\mathrm{q}}\Big)^{1/\mathrm{q}}\Bigg\}$$

$$\leq \exp\Bigg\{{\alpha_o}^2 \mathfrak{a}_o \mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\Big)^{1-2\alpha_o}\Big(\frac{1}{\mathrm{p}+1}\Big)^{1/\mathrm{p}}\Big(\Big({\mu_1}^{\frac{1}{\mathrm{q}}}\ln\psi^*(\mathfrak{a}_o)\Big)^{\mathrm{q}}+\Big({\mu_2}^{\frac{1}{\mathrm{q}}}\ln\psi^*(\mathfrak{b}_o)\Big)^{\mathrm{q}}\Big)^{\frac{1}{\mathrm{q}}}$$

$$+(1-\alpha_o)\frac{\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{2}\mathrm{K}_9^{1/\mathrm{p}}\Big(\Big((\mathrm{K}_{10})^{\frac{1}{\mathrm{q}}}\ln(\psi^{**}(\mathfrak{a}_o))\Big)^{\mathrm{q}}+\Big((\mathrm{K}_{11})^{\frac{1}{\mathrm{q}}}\ln(\psi^{**}(\mathfrak{b}_o))\Big)^{\mathrm{q}}\Big)^{1/\mathrm{q}}\Bigg\}.$$

*By the use of $A^q+B^q\leq(A+B)^q$ for $A\geq 0, B\geq 0$ with $q\geq 1$, we have that*

$$\left|\frac{[\psi(\mathfrak{a}_o)\,\psi(\mathfrak{b}_o)]^{{\alpha_o}^2\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{1-2\alpha_o}}\cdot[\psi^*(\mathfrak{a}_o)\,\psi^*(\mathfrak{b}_o)]^{\frac{(1-\alpha_o)}{2}}}{\Big[\Big({}_{\frac{1}{\mathfrak{a}_o}}\mathrm{D}^{\alpha_o}_{*}(\psi\circ\Upsilon)(\tfrac{1}{\mathfrak{a}_o})\Big)\cdot\Big({}_{*}\mathrm{D}^{\alpha_o}_{\frac{1}{\mathfrak{b}_o}}(\psi\circ\Upsilon)(\tfrac{1}{\mathfrak{b}_o})\Big)\Big]^{\Gamma(1-\alpha_o)\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{2-2\alpha_o}}}\right|$$

$$\leq \exp\Bigg\{{\alpha_o}^2 \mathfrak{a}_o \mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)\Big(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\Big)^{1-2\alpha_o}\Big(\frac{1}{\mathrm{p}+1}\Big)^{1/\mathrm{p}}\Big[{\mu_1}^{\frac{1}{\mathrm{q}}}\ln\psi^*(\mathfrak{a}_o)+{\mu_2}^{\frac{1}{\mathrm{q}}}\ln\psi^*(\mathfrak{b}_o)\Big]$$

$$+(1-\alpha_o)\frac{\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{2}\mathrm{K}_9^{1/\mathrm{p}}\Big[(\mathrm{K}_{10})^{\frac{1}{\mathrm{q}}}\ln(\psi^{**}(\mathfrak{a}_o))+(\mathrm{K}_{11})^{\frac{1}{\mathrm{q}}}\ln(\psi^{**}(\mathfrak{b}_o))\Big]\Bigg\}$$

$$=\Big[(\psi^*(\mathfrak{a}_o))^{{\mu_1}^{\frac{1}{\mathrm{q}}}}\cdot(\psi^*(\mathfrak{b}_o))^{{\mu_2}^{\frac{1}{\mathrm{q}}}}\Big]^{{\alpha_o}^2\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{1-2\alpha_o}\left(\frac{1}{\mathrm{p}+1}\right)^{1/\mathrm{p}}}$$

$$\times\Big[(\psi^{**}(\mathfrak{a}_o))^{\mathrm{K}_{10}^{1/\mathrm{p}}}\cdot(\psi^{**}(\mathfrak{b}_o))^{\mathrm{K}_{11}^{1/\mathrm{p}}}\Big]^{(1-\alpha_o)\frac{\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{4}(\mathrm{K}_9^{1/\mathrm{p}})}.$$

*where*

$$\int_0^1 |1-2\lambda_o|^{(2-2\alpha_o)\mathrm{p}}d\lambda_o=\frac{1}{(2-2\alpha_o)\mathrm{p}+1},$$

$$\int_0^1 \lambda_o\mathrm{U}^{-2\mathrm{q}}_{\lambda_o}d\lambda_o={\mathfrak{a}_o}^{-2\mathrm{q}}\int_0^1\lambda_o\Big(1-\lambda_o\Big(1-\frac{\mathfrak{b}_o}{\mathfrak{a}_o}\Big)\Big)^{-2\mathrm{q}}d\lambda_o=\frac{1}{2{\mathfrak{a}_o}^{2\mathrm{q}}}{}_2\psi_1\Big(2\mathrm{q},2;3;1-\frac{\mathfrak{a}_o}{\mathfrak{b}_o}\Big),$$

*and*

$$\int_0^1 (1-\lambda_o)\mathrm{U}^{-2\mathrm{q}}_{\lambda_o}d\lambda_o=\frac{1}{2{\mathfrak{a}_o}^{2\mathrm{q}}}{}_2\psi_1\Big(2\mathrm{q},1;3;1-\frac{\mathfrak{a}_o}{\mathfrak{b}_o}\Big).$$

*Hence the proof.* □

**Theorem 3.10.** *Let $\psi : \mathrm{I} \subset \mathfrak{R}^+ \to \mathfrak{R}$ be a differentiable on $\mathrm{I}^\circ$, with $\mathfrak{a}_o, \mathfrak{b}_o \in \mathrm{I}^\circ$ where $\mathfrak{a}_o < \mathfrak{b}_o$. If $(\psi^*)^q, (\psi^{**})^q$, are multilplicatively harmonic convex on $[\mathfrak{a}_o, \mathfrak{b}_o]$ for some fixed $q > 1$ and $\Upsilon(\lambda_o) = \frac{1}{\lambda_o}$, then*

$$\left|\frac{[\psi(\mathfrak{a}_o)\psi(\mathfrak{b}_o)]^{\alpha_o^2\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{1-2\alpha_o}} \cdot [\psi^*(\mathfrak{a}_o)\psi^*(\mathfrak{b}_o)]^{\frac{(1-\alpha_o)}{2}}}{\left[\left({}_{\frac{1}{\mathfrak{a}_o}}D_*^{\alpha_o}(\psi\circ\Upsilon)(\frac{1}{\mathfrak{a}_o})\right)\cdot\left({}_*D^{\alpha_o}_{\frac{1}{\mathfrak{b}_o}}(\psi\circ\Upsilon)(\frac{1}{\mathfrak{b}_o})\right)\right]^{\Gamma(1-\alpha_o)\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{2-2\alpha_o}}}\right|$$
$$\leq \left(\psi^*(\mathfrak{b}_o)\cdot\psi^*(\mathfrak{a}_o)\right)^{\frac{\alpha_o^2\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{2}\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{1-2\alpha_o}\left(\frac{1}{q+1}\right)^{1/q}\left(\frac{L_{2p-2}^{2-2/p}(\mathfrak{a}_o,\mathfrak{b}_o)}{(ab)^{2p-1}}\right)^{1/p}}$$
$$\times\left(\psi^{**}(\mathfrak{b}_o)\cdot\psi^{**}(\mathfrak{b}_o)\right)^{(1-\alpha_o)\frac{\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{2}\left(\frac{1}{(2-2\alpha_o)q+1}\right)^{1/q}\left(\frac{L_{2p-2}^{2-2/p}(\mathfrak{a}_o,\mathfrak{b}_o)}{(ab)^{2p-1}}\right)^{1/p}},$$

*holds, where $L_{2p-2}(a,b) = \left(\frac{\mathfrak{b}_o{}^{2p-1}-\mathfrak{a}_o{}^{2p-1}}{(2p-1)(\mathfrak{b}_o-\mathfrak{a}_o)}\right)^{1/(2p-2)}$ is $2p-2$–Logarithmic mean.*

*Proof.* *Suppose that $U_{\lambda_o} = \lambda_o\mathfrak{b}_o + (1-t)\mathfrak{a}_o$. From 3.3 and 2.7 $(\psi^{**})^q$, we derive the following, assuming the multiplicative harmonic convexity of $(\psi^*)^q$ and the Hölder inequality*

$$\left|\frac{[\psi(\mathfrak{a}_o)\psi(\mathfrak{b}_o)]^{\alpha_o^2\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{1-2\alpha_o}} \cdot [\psi^*(\mathfrak{a}_o)\psi^*(\mathfrak{b}_o)]^{\frac{(1-\alpha_o)}{2}}}{\left[\left({}_{\frac{1}{\mathfrak{a}_o}}D_*^{\alpha_o}(\psi\circ\Upsilon)(\frac{1}{\mathfrak{a}_o})\right)\cdot\left({}_*D^{\alpha_o}_{\frac{1}{\mathfrak{b}_o}}(\psi\circ\Upsilon)(\frac{1}{\mathfrak{b}_o})\right)\right]^{\Gamma(1-\alpha_o)\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{2-2\alpha_o}}}\right|$$
$$\leq \exp\left\{\alpha_o^2\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{1-2\alpha_o}\left(\int_0^1\frac{1}{U_{\lambda_o}^{2p}}d\lambda_o\right)^{1/p}\left(\int_0^1|1-2\lambda_o|^q\left(\ln\psi^*\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{U_{\lambda_o}}\right)\right)^q d\lambda_o\right)^{1/q}\right.$$
$$\left.+(1-\alpha_o)\frac{\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{2}\left(\int_0^1\frac{1}{U_{\lambda_o}^{2p}}d\lambda_o\right)^{1/p}\left(\int_0^1|1-2\lambda_o|^{(2-2\alpha_o)q}\left(\ln\psi^{**}\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{U_{\lambda_o}}\right)\right)^q d\lambda_o\right)^{1/q}\right\}$$
$$\leq \exp\left\{\alpha_o^2\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{1-2\alpha_o}\left(\int_0^1\frac{1}{U_{\lambda_o}^{2p}}d\lambda_o\right)^{1/p}\right.$$
$$\left(\int_0^1|1-2\lambda_o|^q[\lambda_o(\ln\psi^*(\mathfrak{b}_o))^q+(1-\lambda_o)(\ln\psi^*(\mathfrak{a}_o))^q]d\lambda_o\right)^{1/q}$$
$$\left.+(1-\alpha_o)\frac{\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{2}\left(\int_0^1\frac{1}{U_{\lambda_o}^{2p}}d\lambda_o\right)^{1/p}\left(\int_0^1|1-2\lambda_o|^{(2-2\alpha_o)q}[\lambda_o(\ln\psi^{**}(\mathfrak{b}_o))^q+(1-\lambda_o)(\ln\psi^{**}(\mathfrak{a}_o))^q]d\lambda_o\right)^{1/q}\right\}$$
$$=\exp\left\{\alpha_o^2\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{1-2\alpha_o}\left(\frac{L_{2p-2}^{2-2/p}(\mathfrak{a}_o,\mathfrak{b}_o)}{(ab)^{2p-1}}\right)^{1/p}\left(\frac{1}{2(q+1)}(\ln\psi^*(\mathfrak{b}_o))^q+\frac{1}{2(q+1)}(\ln\psi^*(\mathfrak{a}_o))^q d\lambda_o\right)^{1/q}\right.$$
$$\left.+(1-\alpha_o)\frac{\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{2}\left(\frac{L_{2p-2}^{2-2/p}(\mathfrak{a}_o,\mathfrak{b}_o)}{(ab)^{2p-1}}\right)^{1/p}\left(\frac{1}{(2-2\alpha_o)q+1}\right)^{1/q}\left(\frac{(\ln\psi^{**}(\mathfrak{b}_o))^q+(\ln\psi^{**}(\mathfrak{b}_o))^q}{2}\right)^{1/q}\right\}$$

$$\left|\frac{[\psi(\mathfrak{a}_o)\psi(\mathfrak{b}_o)]^{\alpha_o^2\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{1-2\alpha_o}} \cdot [\psi^*(\mathfrak{a}_o)\psi^*(\mathfrak{b}_o)]^{\frac{(1-\alpha_o)}{2}}}{\left[\left({}_{\frac{1}{\mathfrak{a}_o}}D_*^{\alpha_o}(\psi\circ\Upsilon)(\frac{1}{\mathfrak{a}_o})\right)\cdot\left({}_*D^{\alpha_o}_{\frac{1}{\mathfrak{b}_o}}(\psi\circ\Upsilon)(\frac{1}{\mathfrak{b}_o})\right)\right]^{\Gamma(1-\alpha_o)\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{2-2\alpha_o}}}\right|$$

$$\leq \left(\psi^{*}(\mathfrak{b}_o)\cdot\psi^{*}(\mathfrak{a}_o)\right)^{\frac{\alpha_o{}^2\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{2}\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{1-2\alpha_o}\left(\frac{1}{q+1}\right)^{1/q}\left(\frac{L_{2p-2}^{2-2/p}(\mathfrak{a}_o,\mathfrak{b}_o)}{(ab)^{2p-1}}\right)^{1/p}}$$

$$\times\left(\psi^{**}(\mathfrak{b}_o)\cdot\psi^{**}(\mathfrak{b}_o)\right)^{(1-\alpha_o)\frac{\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{2}\left(\frac{1}{(2-2\alpha_o)q+1}\right)^{1/q}\left(\frac{L_{2p-2}^{2-2/p}(\mathfrak{a}_o,\mathfrak{b}_o)}{(ab)^{2p-1}}\right)^{1/p}}.$$

*Where*

$$\int_0^1 \frac{1}{U_{\lambda_o}^{2p}}d\lambda_o = \mathfrak{a}_o{}^{-2p}\int_0^1\left(1-\lambda_o\left(1-\frac{\mathfrak{b}_o}{\mathfrak{a}_o}\right)\right)^{-2p}d\lambda_o = \mathfrak{a}_o{}^{-2p}\cdot{}_2\psi_1\left(2p,1;2,1-\frac{\mathfrak{b}_o}{\mathfrak{a}_o}\right) = \frac{L_{2p-2}^{2-2/p}(\mathfrak{a}_o,\mathfrak{b}_o)}{(\mathfrak{a}_o\mathfrak{b}_o)^{2p-1}},$$

$$\int_0^1 |1-2\lambda_o|^q\lambda_o d\lambda_o = \int_0^{1/2}(1-2\lambda_o)^q\lambda_o d\lambda_o + \int_{1/2}^1(2\lambda_o-1)^q\lambda_o d\lambda_o = \frac{1}{2(q+1)},$$

$$\int_0^1 |1-2\lambda_o|^q(1-\lambda_o)d\lambda_o = \frac{1}{2(q+1)},$$

$$\int_0^1 |1-2\lambda_o|^{(2-2\alpha_o)q}\lambda_o d\lambda_o = \int_0^{1/2}(1-2\lambda_o)^{(2-2\alpha_o)q}\lambda_o d\lambda_o + \int_{1/2}^1(2\lambda_o-1)^{(2-2\alpha_o)q}\lambda_o d\lambda_o = \frac{1}{2((2-2\alpha_o)q+1)},$$

*and*

$$\int_0^1 |1-2\lambda_o|^{(2-2\alpha_o)q}(1-\lambda_o)d\lambda_o = \frac{1}{2((2-2\alpha_o)q+1)}.$$

□

*The proof ends at this stage.*

**Example 3.11.** *The following graph explains the veracity of Theorem 3.10 for* $\psi(x_o) = \exp\{x_o^4\}$.

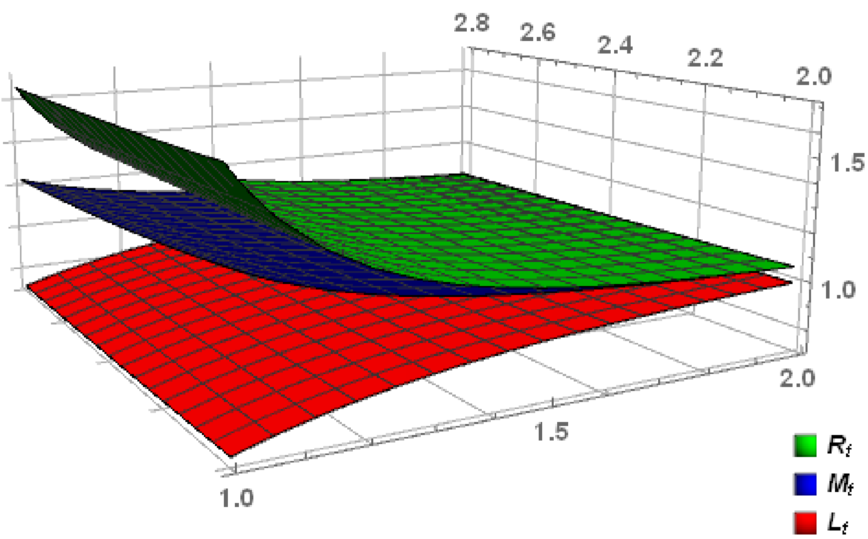


Figure 3: Graphical explanation that confirms the veracity of Theorem 3.10 for $\mathfrak{a}_o \in [1, 1.99]$, $\mathfrak{b}_o \in [2, 2.8]$, $q = 3$ and $\alpha_o = 0.5$.

## 4. Applications to Bessel function

Here, we present a new function expressed through the modified first-kind Bessel function. Exploiting the function, new fractional recurrence equations for the first kind modified Bessel function that cannot be realized using standard analytic methods are formulated.

We consider here the modified first-kind Bessel function $\mathfrak{J}_{\mathfrak{p}}$ [58]. The function is fundamental in many areas of mathematical physics and engineering. Applying its characteristics under the scope of fractional calculus, we find recurrence relations which allow us to further understand its applications and behavior.

$$\mathfrak{J}_{\mathfrak{p}}(\varkappa) = \sum_{\mathfrak{n}=0}^{\infty} \frac{(\frac{\varkappa}{2})^{\mathfrak{p}+2\mathfrak{n}}}{\mathfrak{n}!\Gamma(\mathfrak{p}+\mathfrak{n}+1)}, \qquad \textit{where} \quad \varkappa \in \mathfrak{R}. \tag{32}$$

The authors of [15] and [32] introduced a novel function, denoted as $\mathfrak{J}_{\mathfrak{s}}(\varkappa)$, which is defined over the domain $[0,\infty)$ and maps to the range $[0,\infty)$. This function is formulated in terms of the modified Bessel function of the first kind and is specifically defined for $\mathfrak{p} \geq 1$ and $\mathfrak{p} \in Z_+$. The explicit definition of this function is given by

$$\mathfrak{J}_{\mathfrak{s}}(\varkappa) = \varkappa^{\mathfrak{p}}\mathfrak{J}_{\mathfrak{p}}(\varkappa). \tag{33}$$

$$\mathfrak{J}_{\mathfrak{s}}^{'}(\varkappa) = \varkappa^{\mathfrak{p}}\mathfrak{J}_{\mathfrak{p}-1}(\varkappa). \tag{34}$$

$$\mathfrak{J}_{\mathfrak{s}}^{''}(\varkappa) = \varkappa^{\mathfrak{p}-1}\mathfrak{J}_{\mathfrak{p}-1}(\varkappa) + \varkappa^{\mathfrak{p}}\mathfrak{J}_{\mathfrak{p}-2}(\varkappa). \tag{35}$$

As $\mathfrak{J}_{\mathfrak{s}}^{''}(\varkappa) > 0$, $\forall\ \varkappa > 0$ and $\mathfrak{p} \geq 1$. $\Rightarrow \mathfrak{J}_{\mathfrak{s}}(\varkappa)$ is convex on $[0,\infty[$. Since the function $\mathfrak{J}_{\mathfrak{s}}(\varkappa)$ is increasing too, therefore we can term it as harmonic convex function, it implies that $\exp\{\mathfrak{J}_{\mathfrak{s}}(\varkappa)\}$, is multiplicative harmonically convex function.

**Proposition 4.1.** *Let* $\mathfrak{p} \geq 1$, *and* $\mathfrak{a}_o, \mathfrak{b}_o \in [0,\infty[$ *such that* $0 < \mathfrak{a}_o < \mathfrak{b}_o$, *then*

$$\begin{aligned}
& exp\Bigg\{\left(\frac{2\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{a}_o+\mathfrak{b}_o}\right)^{\frac{(1-\alpha_o)(\mathfrak{b}_o-\mathfrak{a}_o)\mathfrak{p}}{2\mathfrak{a}_o\mathfrak{b}_o}\left[{\alpha_o}^2-\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{\alpha_o}\right]} \mathfrak{J}_{\mathfrak{p}}\left(\frac{2\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{a}_o+\mathfrak{b}_o}\right)^{{\alpha_o}^2\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)^{\alpha_o} - \frac{(1-\alpha_o)}{2}\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)^{1-\alpha_o}} \\
& \mathfrak{J}_{\mathfrak{p}-1}\left(\frac{2\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{a}_o+\mathfrak{b}_o}\right)^{\frac{(1-\alpha_o)}{2}\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)^{1-\alpha_o}} \Bigg\} \\
& \leq \left[exp\left\{\left(\frac{1}{\mathfrak{a}_o\mathfrak{b}_o}\right)^{\mathfrak{p}}\right\}\left({}_{\frac{1}{\mathfrak{a}_o}}D_*^{\alpha_o}(\mathfrak{J}_{\mathfrak{p}}\circ\Upsilon)\left(\frac{1}{\mathfrak{b}_o}\right)\right)\left({}_*D_{\frac{1}{\mathfrak{b}_o}}^{\alpha_o}(\mathfrak{J}_{\mathfrak{p}}\circ\Upsilon)\left(\frac{1}{\mathfrak{a}_o}\right)\right)\right]^{\frac{\Gamma(1-\alpha_o)}{2}\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)^{1-\alpha_o}} \\
& \leq exp\left\{(\mathfrak{a}_o\mathfrak{b}_o)^{\frac{{\alpha_o}^2(1-\alpha_o)(\mathfrak{b}_o-\mathfrak{a}_o)\mathfrak{p}}{\mathfrak{a}_o\mathfrak{b}_o}}\Big[G(\mathfrak{J}_{\mathfrak{p}}(\mathfrak{a}_o),\mathfrak{J}_{\mathfrak{p}}(\mathfrak{b}_o))\Big]^{{\alpha_o}^2\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)^{\alpha_o}}\Big[G(\mathfrak{J}_{\mathfrak{p}-1}(\mathfrak{a}_o),\mathfrak{J}_{\mathfrak{p}-1}(\mathfrak{b}_o))\Big]^{(1-\alpha_o)\left(\frac{\mathfrak{b}_o-\mathfrak{a}_o}{\mathfrak{a}_o\mathfrak{b}_o}\right)^{1-\alpha_o}}\right\},
\end{aligned} \tag{36}$$

*holds.*

*Proof. The result* (36) *is obtained by employing Theorem 3.1 for* $ß > 0$ *and changing* $\psi(\varkappa)$ *by* $exp\{\mathfrak{J}_{\mathfrak{s}}(\varkappa)\}$ *where*

$$\mathfrak{J}_{\mathfrak{s}}(\varkappa) = \varkappa^{\mathfrak{p}}\mathfrak{J}_{\mathfrak{p}}(\varkappa).$$

*Hence the proof.* □

**Proposition 4.2.** *Let* $\mathfrak{p} \geq 1$, *and* $\mathfrak{a}_o, \mathfrak{b}_o \in [0, \infty[$ *such that* $0 \leq \mathfrak{a}_o < \mathfrak{b}_o$, *then*

$$\left| \frac{exp\left\{\alpha_o{}^2\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{1-2\alpha_o}\left[\mathfrak{a}_o{}^{\mathfrak{p}}\mathfrak{J}_{\mathfrak{p}}(\mathfrak{a}_o)+\mathfrak{b}_o{}^{\mathfrak{p}}\mathfrak{J}_{\mathfrak{p}}(\mathfrak{b}_o)+\left(\frac{1-\alpha_o}{2}\left[\frac{\mathfrak{J}_{\mathfrak{p}-1}(\mathfrak{a}_o)}{\mathfrak{J}_{\mathfrak{p}}(\mathfrak{a}_o)}\right]\right)\right]+\left(\frac{1-\alpha_o}{2}\left[\frac{\mathfrak{J}_{\mathfrak{p}-1}(\mathfrak{a}_o)}{\mathfrak{J}_{\mathfrak{p}}(\mathfrak{a}_o)}\right]\right)\right\}}{\left[\left({}_{\frac{1}{\mathfrak{a}_o}}\mathrm{D}_*^{\alpha_o}(\mathfrak{J}_{\mathfrak{s}}\circ\Upsilon)\left(\frac{1}{\mathfrak{a}_o}\right)\right)\cdot\left({}_*\mathrm{D}_{\frac{1}{\mathfrak{b}_o}}^{\alpha_o}(\mathfrak{J}_{\mathfrak{s}}\circ\Upsilon)\left(\frac{1}{\mathfrak{b}_o}\right)\right)\right]^{\Gamma(1-\alpha_o)\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{2-2\alpha_o}}} \right|$$

$$\leq exp\left\{\frac{\alpha_o{}^2\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{2}\left(\frac{\mathfrak{a}_o\mathfrak{b}_o}{\mathfrak{b}_o-\mathfrak{a}_o}\right)^{1-2\alpha_o}\left(\frac{1}{\mathfrak{q}+1}\right)^{1/\mathfrak{q}}\left(\frac{L_{2\mathfrak{p}-2}^{2-2/\mathfrak{p}}(\mathfrak{a}_o,\mathfrak{b}_o)}{(\mathfrak{a}_o\mathfrak{b}_o)^{2\mathfrak{p}-1}}\right)^{1/\mathfrak{p}}\left[\frac{\mathfrak{J}_{\mathfrak{p}-1}(\mathfrak{a}_o)}{\mathfrak{J}_{\mathfrak{p}}(\mathfrak{a}_o)}+\frac{\mathfrak{J}_{\mathfrak{p}-1}(\mathfrak{b}_o)}{\mathfrak{J}_{\mathfrak{p}}(\mathfrak{a}_o)}\right]\right\}$$

$$\times exp\left\{\frac{(1-\alpha_o)\mathfrak{a}_o\mathfrak{b}_o(\mathfrak{b}_o-\mathfrak{a}_o)}{2((2-2\alpha_o)\mathfrak{q}+1)^{1/\mathfrak{q}}}\left(\frac{L_{2\mathfrak{p}-2}^{2-2/\mathfrak{p}}(\mathfrak{a}_o,\mathfrak{b}_o)}{(\mathfrak{a}_o\mathfrak{b}_o)^{2\mathfrak{p}-1}}\right)^{1/\mathfrak{p}}\left[\frac{\mathfrak{b}_o{}^{\mathfrak{p}-1}\mathfrak{J}_{\mathfrak{p}-1}(\mathfrak{b}_o)+\mathfrak{b}_o{}^{\mathfrak{p}}\mathfrak{J}_{\mathfrak{p}-2}(\mathfrak{b}_o)-\mathfrak{b}_o{}^{2\mathfrak{p}}\mathfrak{J}_{\mathfrak{p}-1}^2(\mathfrak{b}_o)}{\mathfrak{b}_o{}^{2\mathfrak{p}}\mathfrak{J}_{\mathfrak{p}}^2(\mathfrak{b}_o)}\right.\right.$$

$$\left.\left.+\frac{\mathfrak{a}_o{}^{\mathfrak{p}-1}\mathfrak{J}_{\mathfrak{p}-1}(\mathfrak{a}_o)+\mathfrak{a}_o{}^{\mathfrak{p}}\mathfrak{J}_{\mathfrak{p}-2}(\mathfrak{a}_o)-\mathfrak{a}_o{}^{2\mathfrak{p}}\mathfrak{J}_{\mathfrak{p}-1}^2(\mathfrak{a}_o)}{\mathfrak{a}_o{}^{2\mathfrak{p}}\mathfrak{J}_{\mathfrak{p}}^2(\mathfrak{a}_o)}\right]\right\}, \quad (37)$$

*holds.*

*Proof.* *The result* (37) *is obtained by employing Theorem 3.1 for* $\text{ß} > 0$ *and changing* $\psi(\varkappa)$ *by* $\mathfrak{J}_{\mathfrak{s}}(\varkappa)$ *where* $\Upsilon(\lambda_o) = \frac{1}{\lambda_o}$

$$\mathfrak{J}_{\mathfrak{s}}(\varkappa) = \varkappa^{\mathfrak{p}}\mathfrak{J}_{\mathfrak{p}}(\varkappa).$$

*Hence the proof.* □

## 5. Conclusion

In this study, we successfully employed P.C.H operators to establish HH type inequalities for multiplicative harmonically convex functions. Our findings highlight the adaptability of these fractional operators, which allow the retrieval of various forms of HH-type inequalities based on different choices of the parameter $\alpha_o$. Specifically, when $\alpha_o = 1$, the inequalities correspond to the first derivative, while for $\alpha_o = 0$, they apply to the second derivative. This flexibility underscores the significance of P.C.H operators in generalizing and extending classical integral inequalities within the multiplicative calculus framework.

Furthermore, we supported our theoretical results with graphical representations, demonstrating their validity through concrete examples. We also explored the implications of our derived inequalities in the context of special functions, leading to the formulation of new multiplicative fractional-order recurrence relations. These results add to the expanding corpus of work on integral inequalities, fractional calculus, and its uses in numerical techniques, optimization, and uncertainty analysis.

## References


[1] T. Abdeljawad and M. Grossman, *On geometric fractional calculus*, J. Semigroup Theory Appl., 2016(2016) ID 2, 14 p.

[2] D. Y. Ai and T. S. Du, *A study on Newton-type inequalities bounds for twice *differentiable functions under multiplicative Katugampola fractional integrals*, Fractals, 33(5) (2025), 2550032.

[3] M. A. Ali, M. Abbas, Z. Zhang, I. B. Sial and R. Arif, *On integral inequalities for product and quotient of two multiplicatively convex functions*, Asian Res. J. Math. 12(3)(2019), 1-11.

[4] M. A. Ali, H. Budak, M. Z. Sarikaya and Z. Y. Zhang, *Ostrowski and Simpson type inequalities for multiplicative integrals*, Proyecciones, 40(3)(2021), 743-763

[5] G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen, *Generalized convexity and inequalities*, Journal of Mathematical Analysis and Applications. 15 (2007), 1294–308.

[6] A. Atangana and D. Baleanu, *New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model*,Therm. Sci., 20(2) (2016), 763-769.

[7] K. Balachandran, M. Matar, N. Annapoorani and D. Prabu, *Hadamard functional fractional integrals and derivatives and fractional differential equations*, Filomat, 38(3) (2024), 779-792.

[8] D. Balenau, A. Ferrandez and A. Akgül, *On a fractional operator combining proportional and classical differintegrals*, Mathematics, 8(3) (2020), 13.
[9] A. E. Bashirov, E. M. Kurpinar and A. Özyapici, *Multiplicative calculus and its applications*, J. Math. Anal. Appl., 337(1)(2008), 36-48.
[10] B. Benaissa and H. Budak, *General $(k, p)$-Riemann-Liouville fractional integrals*, Filomat, 38(8) (2024), 2579-2586.
[11] H. Budak and K. Özçelik, *On Hermite-Hadamard type inequalities for multiplicative fractional integrals*, Miskolc Math. Notes., 21(1)(2020), 91-99.
[12] S. I. Butt, *Generalized Jensen-Hermite-Hadamard Mercer type inequalities for generalized strongly convex functions on fractal sets*, Turkish Journal of Scince, 8(2) (2024),51-63.
[13] S. I. Butt, P. Agarwal, S. Yousaf and J. L. G. Guirao, *Generalized fractal Jensen and Jensen-Mercer inequalities for harmonic convex function with applications*, J. Inequal. Appl., 2022 (1), 1–18.
[14] S. I. Butt, M. Umar, D. Khan, Y. Seol and S. Tipurić-Spužević, *Hermite-Hadamard-type inequalities for harmonically convex functions via proportional Caputo-hybrid operators with applications*, Fractal Fract., 9(2)(2025), 77.
[15] S. I. Butt, D. Khan, *Integral inequalities of h-superquadratic functions and their fractional perspective with applications*, Math. Meth. Appl. Sci., (2024), 1-30.
[16] M. Caputo and M. Fabrizio, *A new definition of fractional derivative without singular kernel*, Prog. Fract. Differ. Appl., 1(2) (2015), 73-85.
[17] Q. J. Cheng and C. Y. Luo, *Analytical properties, fractal dimensions and related inequalities of $(k, h)$-Riemann- Liouville fractional integrals*, J. Comput. Appl. Math., 450 (2024), 115999.
[18] İ. Demir, *A new approach of Milne-type inequalities based on proportional Caputo-hybrid operator*, J. Adv. Appl. Comput. Math., 10 (2023), 102-119.
[19] İ. Demir and T. Tunç, *A new approach to Simpson-type inequality with proportional Caputo-hybrid operator*, Math. Meth. Appl. Sci., 2024, 1-14.
[20] S. S. Dragomir and R. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl.Math.Lett, 11(5) (1998), 91-95.
[21] S. Dimitrijev, *Effective mass in semiconductors*, Bart J. Van Zeghbroeck, (1997).
[22] T. S. Du, Y. Long and J. G. Liao, *Multiplicative fractional HH-type inequalities via multiplicative AB-fractional integral operators*, J. Comput. Appl. Math., 474 (2026), 116970.
[23] T. S. Du, Z. Y. Zhou and Z. R. Tan, *Hadamard functional integral operators within fractional multiplicative calculus*, Chaos Solitons Fractals, 199 (2025), 116710.
[24] A. Fernandez and C. Ustaoğlu, *On some analytic properties of tempered fractional calculus*, J. Comput. Appl. Math., 366 (2020), 112400.
[25] H. Fu, Y. Peng and T. S. Du, *Some inequalities for multiplicative tempered fractional integrals involving the $\lambda$-incomplete gamma functions*, AIMS Math., 6(7)(2021), 7456-7478.
[26] W. Gao, A. Kashuri, S. I. Butt, M. Tariq, A. Aslam and M. Nadeem, *New inequalities via n-polynomial harmonically exponential type convex functions*, AIMS Mathematics, 5(6) (2020), 6856–6873.
[27] Í. Íşcan, *Hermite-Hadamard inequalities for harmonically convex functions*, Hacettepe Journal of Mathematics and Statistics, 43(6) (2014), 935–942.
[28] Í. Íşcan and S. Wu, *Hermite-Hadamard type inequalities for harmonically convex functions via fractional integrals*, Applied Mathematics and Computation, 238(1) (2014), 237-244.
[29] H. Kadakal and M. Kadakal, *Multiplicatively preinvex P-functions*, J. Sci. Arts., 23(1)(2023), 21-32.
[30] M. Kadakal, İ. İşcan, P. Agarwal and M. Jleli, *Exponential trigonometric convex functions and Hermite-Hadamard type inequalities*, Math. Slovaca, 71(1) (2021), 43-56.
[31] A. Kashuri, S. K. Sahoo, M. Aljuaid, M. Tariq and M. D. L. Sen, *Some new Hermite-Hadamard type inequalities pertaining to generalized multiplicative fractional integrals*, Symmetry, 15(2023) ID 868, 14 p.
[32] D. Khan and S. I. Butt, *Superquadraticity and its fractional perspective via center-radius cr-order relation*, Chaos, Solitons and Fractals, 182 (2024), 114821.
[33] D. Khan, S. I. Butt and Y. Seol, *Analysis on multiplicatively $(P, m)$-superquadratic functions and related fractional inequalities with applications*, Fractals, (2024), 2450129.
[34] D. Khan, S. I. Butt and Y. Seol, *Properties and integral inequalities of P-superquadratic functions via multiplicative calculus with applications*, Bound. Value Probl., 2024(1)(2024), 1-40.
[35] S. Khan and H. Budak, *On midpoint and trapezoid type inequalities for multiplicative integrals*, Mathematica, 64(87)(2022), 95-108.
[36] T. U. Khan and M. A. Khan, *New generalized mean square stochastic fractional operators with applications*, Chaos Solitons and Fractals, 142 (2021), 110452.
[37] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, In North Holland Mathematics Studies, 204, Elsevier Science, Publishers BV, Amsterdam, (2006).
[38] U. S. Kirmaci and M. E. Özdemir, *On some inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula*, Appl. Math. Comput, 153(2) (2004), 361-368.
[39] M. A. Latif and T. S. Du, *Hermite-Hadamard type inequalities for harmonically-convex functions using fuzzy integrals*, Filomat, 36(12) (2022), 4099–4110.
[40] C. E. T. Ledesma and J. V. D. C. Sousa, *Fractional integration by parts and Sobolev-type inequalities for $\psi$- fractional operators*, Math. Meth. Appl. Sci., 45(16) (2022), 9945-9966.
[41] C. E. T. Ledesma, H. A. C. Gutierrez, J. P. A. Rodriguez and W. Z. Vera, *Some boundedness results for Riemann-Liouville tempered fractional integrals*, Fract. Calc. Appl. Anal., 27 (2024), 818-847.
[42] K. Mehrez and P. Agarwal, *New Hermite-Hadamard type integral inequalities for convex functions and their applications*, J. Comput. Appl. Math, 350 (2019), 274-285.
[43] B. Meftah, *Maclaurin type inequalities for multiplicatively convex functions*, Proc. Amer. Math. Soc., 151(5)(2023), 2115-2125.

[44] M. Merad, B. Meftah, A. Moumen and M. Bouye, *Fractional Maclaurin-type inequalities for multiplicatively convex functions*, Fractal Fract., 7(2023) ID 879, 13 p.
[45] C. P. Niculescu and L. E. Persson, *Old and new on the Hermite-Hadamard inequality*, Real Anal. Exchange, 29(2) (2004), 663-686.
[46] M. A. Noor and K. I. Noor, *Harmonic variational inequalities*, Appl. Math. Inform. Sci., 10(5) (2016), 1811–114.
[47] J. E. Nápoles Valdés, F. Rabossi and A. D. Samaniego, *Convex functions: Ariadne's thread or Charlotte's Spiderweb?*, Advanced Mathematical Models and Applications, 5(2) (2020).
[48] S. Özcan and S. I. Butt, *Hermite-Hadamard type inequalities for multiplicatively harmonic convex functions*. Journal of Inequalities and Applications, 2023 (2023), 1–18.
[49] Y. Peng and T. S. Du, *Hermite-Hadamard-type inequalities for* $^*$*differentiable multiplicative m-preinvexity and* $(s, m)$*-preinvexity via the multiplicative tempered fractional integrals*, J. Math. Inequal., 17(3)(2023), 1179-1201.
[50] Y. Peng, H. Fu and T. S. Du, *Estimations of bounds on the multiplicative fractional integral inequalities having exponential kernels*, Commun. Math. Stat. 12(02) (2024), 187-211.
[51] J. Pečarić and Y. L. Tong, *Convex functions, partial orderings and statistical applications*, Academic Press, New York, 1992.
[52] C. E. Pearce and J. Pečarić, *Inequalities for differentiable mappings with application to special means and quadrature formulae*, Appl. Math. Lett, 13(2) (2000), 51-55.
[53] S. Varošanec, *On h-convexity*, J. Math. Anal. Appl, 326(1) (2007), 303-311.
[54] S. Verma and P. Viswanathan, *A note on Katugampola fractional calculus and fractal dimensions*, Appl. Math. Comput., 339 (2018), 220-230.
[55] M. Vivas-Cortez, M. A. Ali, H. Budak, H. Kalsoom and P. Agarwal, *Some new Hermite-Hadamard and related inequalities for convex functions via (p, q)-integral*, Entropy, 23(7) (2021), 828.
[56] M. Vivas-Cortez, P. Kérus and J. E. Nápoles Valdés, *Some generalized Hermite-Hadamard-Fejér inequality for convex functions*, Advances in Difference Equations, 2021 (2021), 1-11.
[57] J. Wang, C. Zhu and Y. Zhou, *New generalized Hermite-Hadamard type inequalities and applications to special means*, J. Inequal. Appl., 2013(325) (2013), 15.
[58] G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge University Press, (1944).
[59] J. Q. Xie, M. A. Ali and T. Sitthiwirattham, *Some new midpoint and trapezoidal type inequalities in multi plicative calculus with applications*, Filomat, 37(20)(2023), 6665-6675.
[60] M. Z. Sarikaya, Z. Dahmani, M. E. Kiris and F. Ahmad, $(k, s)$*-Riemann-Liouville fractional integral and applications*, Hacet. J. Math. Stat., 45(1) (2016), 77-89.
[61] M. Z. Sarikaya, *On Hermite-Hadamard type inequalities for proportional Caputo-hybrid operator*, Konuralp. J. Math., 11(1) (2023), 31-39.
[62] F. Sabzikar, M. M. Meerschaert, and J. Chen, *Tempered fractional calculus*, J. Comput. Phys., 293 (2015), 14-28.
[63] W. Sun, *On generalization of some inequalities for generalized harmonically convex functions via local fractional integrals*, Quaest. Math., 42(9) (2019), 1159–1183.
[64] W. Sun, *Some new inequalities for generalized h-convex functions involving local fractional integral operators with Mittag-Leffler kernel*, Math. Meth. Appl. Sci., 44(6) (2021), 4985–4998.