Filomat 39:35 (2025), 12757–12769
https://doi.org/10.2298/FIL2535757D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Nörlund Orlicz sequence space and their Köthe-Toeplitz duals over n-normed space

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Abstract. In this article, we introduce Nörlund-Orlicz sequence space over n-normed space and established that under certain conditions these space become n-BK space. We investigate some useful algebraic and topological properties of Nörlund-Orlicz sequence space. Additionally, we study the Köthe-Toeplitz duals of Nörlund-Orlicz sequence space.

1. Introduction and Preliminaries

The Theory of sequence space plays an important role in mathematics. We shall write $w, \ell_1, \ell_p, \ell_\infty, c$ and c_0 for the set of all complex, bounded, p-absolutely summable, absolutely summable, convergent and null sequences, respectively. Let $(U, \|.\|)$ be a normed linear space and Γ be a scalar-valued sequence space, then the vector-valued sequence space $\Gamma(U)$ defined by

$$\Gamma(U) = \Big\{(u_r) : u_r \in U \text{ for all } r \in \mathbb{N} \text{ and } \|u\| \in \Gamma\Big\}.$$

The ℓ_p $(1 < p < \infty)$ norm in a Banach space defined by

$$\|u\| = \sum_{r=1}^{\infty} (|u_r|^p)^{\frac{1}{p}}.$$

Peyerimhoff [13] and Mears [10] introduced the concept of Nörlund means. Let (t_r) be a sequence of non-negative real numbers with $t_0 > 0$ and $T_n = \sum_{r=0}^{n} t_r$ for all $n \in \mathbb{N}$. Then the Nörlund mean of the sequence $t = (t_r)$ is denoted by σ_n and defined by

$$\sigma_n = \frac{\sum_{r=0}^{n} t_{n-r} u_r}{T_n}.$$

2020 *Mathematics Subject Classification*. Primary 40A05, 46A20; Secondary 46A45, 46E30.
Keywords. Nörlund sequence space, Orlicz sequence space, difference sequence space, n- normed space, Köthe-Toeplitz dual
Received: 02 April 2025; Accepted: 21 July 2025
Communicated by Miodrag Spalević
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Yeşilkayagil et al. [23] defined the Nörlund sequence space $N\ddot{o}s_p,\ 1 \leq p < \infty$, defined by

$$N\ddot{o}s_p = \left\{ u = (u_r) \in w : \|u\|_p = \left(\sum_{n=1}^{\infty} \frac{1}{T_n} \sum_{r=0}^{n} |t_{n-r} u_r|^p \right)^{\frac{1}{p}} < \infty, 1 \leq p < \infty \right\},$$

and

$$N\ddot{o}s_\infty = \left\{ u = (u_r) \in w : \|u\|_\infty = \sup_n \frac{1}{T_n} \sum_{r=0}^{n} |t_{n-r} u_r| < \infty \right\}.$$

The inclusion $\ell_p \subset N\ddot{o}s_p (1 < p < \infty)$ is strict. Wang [22] defined and investigated the non-absolute Nörlund sequence space N_p as follows.

$$N_p = \left\{ u = (u_r) \in w : \|u\|_p = \left(\sum_{n=1}^{\infty} \left| \frac{1}{T_n} \sum_{r=0}^{n} t_{n-r} u_r \right|^p \right)^{\frac{1}{p}} < \infty, 1 \leq p < \infty \right\},$$

and

$$N_\infty = \left\{ u = (u_r) \in w : \|u\|_\infty = \sup_n \left| \frac{1}{T_n} \sum_{r=0}^{n} t_{n-r} u_r \right| < \infty \right\}.$$

Singh et al. [17] defined Nörlund difference sequence space $N_p(\Delta)$ and $N_\infty(\Delta)$ as

$$N_p(\Delta) = \left\{ u = (u_r) \in w : \|u\|_p = \left(\sum_{n=1}^{\infty} \left| \frac{1}{T_n} \sum_{r=0}^{n} t_{n-r} \Delta u_r \right|^p \right)^{\frac{1}{p}} < \infty, 1 \leq p < \infty \right\},$$

and

$$N_\infty(\Delta) = \left\{ u = (u_r) \in w : \|u\|_\infty = \sup_n \left| \frac{1}{T_n} \sum_{r=0}^{n} t_{n-r} \Delta u_r \right| < \infty \right\}.$$

and prove that for $1 \leq p < \infty$, the inclusions $N_p \subset N_p(\Delta)$ and $N_\infty \subset N_\infty(\Delta)$ are strict. Also, we define the following sequence spaces

$$O_p(\Delta) = \left\{ u = (u_r) \in w : \sum_{n=1}^{\infty} \left(\frac{1}{T_n} \sum_{r=0}^{n} |t_{n-r} \Delta u_r| \right)^p < \infty, 1 \leq p < \infty \right\},$$

and

$$O_\infty(\Delta) = \left\{ u = (u_r) \in w : \sup_{n \geq 1} \frac{1}{T_n} \sum_{r=0}^{n} |t_{n-r} \Delta u_r| < \infty \right\}.$$

The inclusions $O_p(\Delta) \subset N_p(\Delta)$, $N\ddot{o}s_p \subset N_p$ and $N\ddot{o}s_p \subset O_p(\Delta)$ are strict for $1 \leq p < \infty$.

Kizmaz [6] introduced the concept of difference sequence space by studying $\ell_p(\Delta), \ell_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$. Et et al. [3] further generalized the concepts by introducing the spaces $\ell_\infty(\Delta^\ell), c(\Delta^\ell)$ and $c_0(\Delta^\ell)$. For the sequence $u = (u_r), \Delta u = (\Delta u_r) = (u_r - u_{r+1})$, let k, ℓ be non-negative integers, then for $Z = c, c_0$ and ℓ_∞, we have the sequence spaces

$$Z(\Delta_k^\ell) = \{ u = (u_r) \in w : (\Delta_k^\ell u_r) \in Z \},$$

where $\Delta_k^\ell u = (\Delta_k^\ell u_r) = (\Delta_k^{\ell-1} u_r - \Delta_k^{\ell-1} u_{r+1})$ and $\Delta_k^0 = u_r$ for all $r \in \mathbb{N}$. The binomial representation of $\Delta_k^\ell u = (\Delta_k^\ell u_r)$ defined by

$$\Delta_k^\ell u_r = \sum_{m=0}^{\ell} (-1)^m \binom{\ell}{m} u_{r+km}.$$

For more details about the difference sequence space, refer to [18], [14], [2], [15].

Definition 1.1. *A Banach space* $(U, \|\cdot\|)$ *is called a BK-space if it has continuous coordinates, that is, for a complex sequence* $u^n = (u_r^n), u^n \to u$ $(n \to \infty)$ *in then* $u_r^n \to u_r$ $(n \to \infty)$.

Definition 1.2. *A real valued function* $\|\cdot, \ldots, \cdot\|$ *on* U^n *is called n-norm on U if it satisfy the following four conditions:*
(1) $\|(u_1, u_2, \ldots, u_n)\| = 0$ *if and only if* $u_1, u_2, \ldots, u_n$ *are linear dependent in* U,
(2) $\|(u_1, u_2, \ldots, u_n)\|$ *is invariant under permutation,*
(3) $\|(\gamma u_1, u_2, \ldots, u_n)\| = |\gamma| \, \|(u_1, u_2, \ldots, u_n)\|$ *for any* $\gamma \in \mathbb{R}$ *and*
(4) $\|(u + u', u_2, \ldots, u_n) \leq \|(u, u_2, \ldots, u_n)\| + \|(u', u_2, \ldots, u_n)\|$.
The pair $(U, \|\cdot, \ldots, \cdot\|)$ *is called n-normed space over the field* $\mathbb{R}$.

Remark 1.3. *Let* $(U, \|\cdot, \ldots, \cdot\|)$ *be an n-normed space of dimension* $d \geq n \geq 2$ *and* $\{b_1, b_2, \ldots, b_n\}$ *be linearly independent set in U. Then the following function* $\|\cdot, \ldots, \cdot\|$ *on* U^{n-1} *is defined by*

$$\|(u_1, u_2, \ldots, u_{n-1})\|_\infty = max\{\|(u_1, u_2, \ldots, u_{n-1}, b_i)\| : i = 1, 2, \ldots, n\},$$

is called an $(n-1)$*-norm on U with respect to* $\{b_1, b_2, \ldots, b_n\}$.

Definition 1.4. *A sequence* (u_r) *in an n-normed space* $(U, \|\cdot, \ldots, \cdot\|)$ *is said to converge to some* $T \in U$ *if*

$$\lim_{r \to \infty} \|(u_r - T, f_1, \ldots, f_{n-1})\| = 0,$$

for every $f_1, f_2, \ldots, f_{n-1} \in U$.

Definition 1.5. *A sequence* (u_r) *in an n-normed space* $(U, \|\cdot, \ldots, \cdot\|)$ *is said to be a Cauchy if*

$$\lim_{r,p \to \infty} \|(u_r - u_p, f_1, \ldots, f_{n-1})\| = 0,$$

for every Cauchy sequences $f_1, \ldots, f_{n-1} \in U$.

Remark 1.6. *An n-Banach space is a complete normed-linear space with respect to the n-norm defined on U.*

For more details about sequence space and n-normed spaces, one refer to [11], [4], [5].

Definition 1.7. *A continuous, non-decreasing and convex function* $\mathcal{M} : [0, \infty) \to [0, \infty)$ *is said to be an Orlicz function if it satisfies the following conditions*
(1)$\mathcal{M}(u) = 0$ *for* $u = 0$, $\mathcal{M}(u) > 0$ *for* $u > 0$,
(2)$\mathcal{M}(u) \to \infty$ *as* $u \to \infty$.

Lindenstrauss and Tzafriri [8] used the idea of Orlicz function to define the following sequence space,

$$\ell_{\mathcal{M}} = \left\{u = (u_r) \in w : \sum_{r=1}^{\infty} \mathcal{M}\Big(\frac{|u_r|}{\tau}\Big) < \infty, \text{ for some } \tau > 0\right\}.$$

The Orlicz sequence space $\ell_{\mathcal{M}}$ is a Banach space with the norm defined by

$$\|u\| = inf\left\{\tau > 0 : \sum_{r=1}^{\infty} \mathcal{M}\Big(\frac{|u_r|}{\tau}\Big) \leq 1\right\}.$$

For more details about Orlicz sequence space, refer to [12], [19], [20], [21], [16], [1].
Let $w(n - U)$ denotes U-valued sequence space, where $(U, \|\cdot, \ldots, \cdot\|)$ be an n-normed real linear space, $\mathfrak{M} = (\mathcal{M}_i)$ be a sequence of Orlicz functions and $v = (v_r)$ be a sequence of positive real numbers. Then for every nonzero $f_1, \ldots, f_n \in U$, we define the following sequence spaces

$$\mathcal{N}_p(\mathfrak{M}, v, \Delta_k^\ell, \|\cdot, \ldots, \cdot\|) = \Big\{(u_r) \in w(n-U) : \sum_{i=1}^{\infty} \mathcal{M}_i\Big(\Big\|\frac{1}{T_i}\sum_{r=0}^{i} \frac{v_r t_{n-r}\Delta_k^\ell u_r}{\tau}, f_1, \ldots, f_{n-1}\Big\|\Big)^p < \infty, \text{ for some } \tau > 0\Big\},$$

$$\mathcal{N}_\infty(\mathfrak{M}, v, \Delta_k^\ell, \|\cdot, \ldots, \cdot\|) = \Big\{(u_r) \in w(n-U) : \sup_i \mathcal{M}_i\Big(\Big\|\frac{1}{T_i}\sum_{r=0}^{i} \frac{v_r t_{n-r}\Delta_k^\ell u_r}{\tau}, f_1, \ldots, f_{n-1}\Big\|\Big) < \infty, \text{ for some } \tau > 0\Big\},$$

$$\mathcal{L}_p(\mathfrak{M}, v, \Delta_k^\ell, \|\cdot, \ldots, \cdot\|) = \Big\{(u_r) \in w(n-U) : \sum_{r=1}^{\infty} \mathcal{M}_r\Big(\Big\|\frac{v_r \Delta_k^\ell u_r}{\tau}, f_1, \ldots, f_{n-1}\Big\|\Big)^p < \infty, \text{ for some } \tau > 0\Big\},$$

$$\mathcal{O}_p(\mathfrak{M}, v, \Delta_k^\ell, \|\cdot, \ldots, \cdot\|) = \Big\{(u_r) \in w(n-U) : \sum_{i=1}^{\infty} \mathcal{M}_i\Big(\frac{1}{T_i}\sum_{r=0}^{i} \Big\|\frac{v_r t_{n-r}\Delta_k^\ell u_r}{\tau}, f_1, \ldots, f_{n-1}\Big\|\Big)^p < \infty, \text{ for some } \tau > 0\Big\},$$

and

$$\mathcal{O}_\infty(\mathfrak{M}, v, \Delta_k^\ell, \|\cdot, \ldots, \cdot\|) = \Big\{(u_r) \in w(n-U) : \sup_i \mathcal{M}_i\Big(\frac{1}{T_i}\sum_{r=0}^{i} \Big\|\frac{v_r t_{n-r}\Delta_k^\ell u_r}{\tau}, f_1, \ldots, f_{n-1}\Big\|\Big) < \infty, \text{ for some } \tau > 0\Big\}.$$

Lemma 1.8. *For $1 \le p < \infty$, the Banach spaces*
(*i*) N_p *is normed by*

$$\|u\| = \Big(\sum_{i=1}^{\infty} \Big|\frac{1}{T_i}\sum_{r=0}^{i} t_{n-r}u_r\Big|^p\Big)^{\frac{1}{p}}.$$

(*ii*) O_p *is normed by*

$$\|u\| = \Big(\sum_{i=1}^{\infty} \frac{1}{T_i}\sum_{r=0}^{i} \Big|t_{n-r}u_r\Big|^p\Big)^{\frac{1}{p}}.$$

(iii) ℓ_p *is normed by*

$$\|u\| = \Big(\sum_{i=1}^{\infty} |u_i|^p\Big)^{\frac{1}{p}}.$$

(iv) N_∞ *is normed by*

$$\|u\| = \sup_i \left|\frac{1}{T_i}\sum_{r=0}^{i} t_{n-r}u_r\right|.$$

(v) O_∞ *is normed by*

$$\|u\| = \sup_i \frac{1}{T_i}\sum_{r=0}^{i} |t_{n-r}u_r|.$$

Throughout this article, we use the following inequality. Let $p = (p_r)$ be a sequence of positive real number with $0 < p_r \le \sup_r p_r = F$ and let $G = max\{1, 2^{F-1}\}$. Then for the sequences (c_r) and (d_r) in the complex plane, we have

$$|c_r + d_r|^{p_r} \le G(|c_r|^{p_r} + |d_r|^{p_r}). \tag{1}$$

Also, $|c_r|^{p_r} \le max\{1, |c|^F\}$ for all $c \in \mathbb{C}$.

2. Main Results

This section studies Nörlund Orlicz sequence space over n-normed space. Further, we study their completeness and interesting inclusion relations between these spaces.

Theorem 2.1. *Let* $\mathfrak{M} = (\mathcal{M}_i)$ *be a sequence of Orlicz functions and* $v = (v_r)$ *be a sequence of positive real numbers. Then the class of sequence* $Z(\mathfrak{M}, v, \Delta_k^\ell, \|\cdot, \dots, \cdot\|)$*, is linear for* $1 \le p < \infty$ *, where* $Z = \mathcal{N}_p, \mathcal{N}_\infty, \mathcal{O}_p, \mathcal{O}_\infty$ *and* $\mathcal{L}_p$*.*

Proof. Let $u = (u_r), z = (z_r) \in \mathcal{O}_p(\mathfrak{M}, v, \Delta_k^\ell, \|\cdot, \dots, \cdot\|)$ and $\gamma, \delta \in \mathbb{R}$. Then there exist a positive numbers τ_1, τ_2 such that

$$\sum_{i=1}^{\infty} \mathcal{M}_i\Bigg(\frac{1}{T_i}\sum_{r=0}^{i}\left\|\frac{v_r t_{n-r}\Delta_k^\ell u_r}{\tau_1}, f_1, \dots, f_{n-1}\right\|\Bigg)^p < \infty, \text{ for some } \tau_1 > 0$$

and

$$\sum_{i=1}^{\infty} \mathcal{M}_i\Bigg(\frac{1}{T_i}\sum_{r=0}^{i}\left\|\frac{v_r t_{n-r}\Delta_k^\ell z_r}{\tau_2}, f_1, \dots, f_{n-1}\right\|\Bigg)^p < \infty, \text{ for some } \tau_2 > 0$$

Let $\tau_3 = max(2|\gamma|\tau_1, 2|\delta|\tau_2)$. Since $\mathfrak{M} = (\mathcal{M}_i)$ is non-decreasing and convex so by using inequality (1), we have

$$\sum_{i=1}^{\infty}\mathcal{M}_i\left(\frac{1}{T_i}\sum_{r=0}^{i}\left\|\frac{\gamma v_r t_{n-r}\Delta_k^{\ell}u_r+\delta v_r t_{n-r}\Delta_k^{\ell}z_r}{\tau_3},f_1,\dots,f_{n-1}\right\|\right)^p$$
$$\le\sum_{i=1}^{\infty}\mathcal{M}_i\left(\frac{1}{T_i}|\gamma|\sum_{r=0}^{i}\left\|\frac{v_r t_{n-r}\Delta_k^{\ell}u_r}{\tau_3},f_1,\dots,f_{n-1}\right\|\right)^p$$
$$+\sum_{i=1}^{\infty}\mathcal{M}_i\left(\frac{1}{T_i}|\delta|\sum_{r=0}^{i}\left\|\frac{v_r t_{n-r}\Delta_k^{\ell}z_r}{\tau_3},f_1,\dots,f_{n-1}\right\|\right)^p$$
$$\le K\sum_{i=1}^{\infty}\mathcal{M}_i\left(\frac{1}{T_i}\sum_{r=0}^{i}\left\|\frac{v_r t_{n-r}\Delta_k^{\ell}u_r}{\tau_3},f_1,\dots,f_{n-1}\right\|\right)^p$$
$$+K\sum_{i=1}^{\infty}\mathcal{M}_i\left(\frac{1}{T_i}\sum_{r=0}^{i}\left\|\frac{v_r t_{n-r}\Delta_k^{\ell}z_r}{\tau_3},f_1,\dots,f_{n-1}\right\|\right)^p$$
$$<\infty.$$

Thus, $\gamma u+\delta z\in\mathcal{O}_p(\mathfrak{M},v,\Delta_k^{\ell},\|\cdot,\dots,\cdot\|)$. Hence $\mathcal{O}_p(\mathfrak{M},v,\Delta_k^{\ell},\|\cdot,\dots,\cdot\|)$ is a linear space. Using similar arguments, we can show that the space $Z(\mathfrak{M},v,\Delta_k^{\ell},\|\cdot,\dots,\cdot\|)$ is linear for $Z=\mathcal{N}_p,\mathcal{N}_\infty,\mathcal{O}_\infty$ and $\mathcal{L}_p$. □

Theorem 2.2. *Let $\mathfrak{M}=(\mathcal{M}_i)$ be a sequence of Orlicz functions, $v=(v_r)$ be a sequence of positive numbers, and U be an n Banach space. Then for $1\le p<\infty$*
(i) The space $Z(\mathfrak{M},v,\Delta_k^{\ell},\|\cdot,\dots,\cdot\|)$ is an n Banach space, n-normed by $\|u^1,u^2,\dots,u^n\|_{Z(\mathfrak{M},v,\Delta_k^{\ell})}=0$ if $u^1,u^2,u^3,\dots,u^n$ are linearly dependent and $\sum_{r=1}^{\ell}\|u_r,f_1,\dots,f_{n-1}\|+\left(\sum_{i=1}^{\infty}\mathcal{M}_i\frac{1}{T_i}\sum_{r=0}^{i}\left\|\frac{v_r t_{n-r}\Delta_k^{\ell}u_r}{\tau},f_1,\dots,f_{n-1}\right\|^p\right)^{\frac{1}{p}}$ for every $f_1,\dots,f_{n-1}\in U$ if $u^1,u^2,\dots,u^n$ are linearly independent, where $Z=\mathcal{N}_p,\mathcal{O}_p,\mathcal{N}_\infty$ and $\mathcal{O}_\infty$.
(ii) The space $\mathcal{L}_p(\mathfrak{M},v,\Delta_k^{\ell},\|\cdot,\dots,\cdot\|)$ is an n Banach space, n-normed by $\|u^1,u^2,\dots,u^n\|_{\mathcal{L}_p(\mathfrak{M},v,\Delta_k^{\ell})}=0$ if $u^1,u^2,u^3,\dots,u^n$ are linearly dependent and $=\sum_{r=1}^{\ell}\|u_r,f_1,\dots,f_{n-1}\|+\left(\sum_{r=0}^{\infty}\mathcal{M}_r\left\|\frac{v_r\Delta_k^{\ell}u_r}{\tau},f_1,\dots,f_{n-1}\right\|^p\right)^{\frac{1}{p}}$ for every $f_1,\dots,f_{n-1}\in U$ if $u^1,u^2,\dots,u^n$ are linearly independent.

Proof. It is easy to show that the spaces $Z(\mathfrak{M},v,\Delta_k^{\ell},\|\cdot,\dots,\cdot\|)$, $\mathcal{L}_p(\mathfrak{M},v,\Delta_k^{\ell},\|\cdot,\dots,\cdot\|)$ are n-normed spaces under the n-norm as defined above. Here, we only prove the completeness for the space $\mathcal{O}_\infty(\mathfrak{M},v,\Delta_k^{\ell},\|\cdot,\dots,\cdot\|)$ and the others can be proved by similar arguments. We consider $(u^h)_{h=1}^{\infty}$ be a Cauchy sequence in $\mathcal{O}_\infty(\mathfrak{M},v,\Delta_k^{\ell},\|\cdot,\dots,\cdot\|)$, where $u^h=(u_i^h)=(u_1^h,u_2^h,\dots)\in\mathcal{O}_\infty(\mathfrak{M},v,\Delta_k^{\ell},\|\cdot,\dots,\cdot\|)$ for each $h\in\mathbb{N}$. Let $\varepsilon>0$. Then there exist a positive integer N such that

$$\|u^h-u^p,g^2,\dots,g^n\|_{\mathcal{O}_\infty(\mathfrak{M},v,\Delta_n^{\ell})}<\varepsilon$$

for all $h,p\ge N$ and for every $g^2,\dots,g^n\in\mathcal{O}_\infty(\mathfrak{M},v,\Delta_k^{\ell},\|\cdot,\dots,\cdot\|)$.
We have

$$\sum_{r=1}^{\ell}\|u_r^h-u_r^p,f_1,\dots,f_{n-1}\|$$
$$+\sup_i\mathcal{M}_i\left(\frac{1}{T_i}\sum_{r=0}^{i}\left\|\frac{v_r t_{n-r}\Delta_k^{\ell}(u_r^h-u_r^p)}{\tau},f_1,\dots,f_{n-1}\right\|\right)<\varepsilon$$

for all $h, p \geq N$ and for every $f_1, \ldots, f_{n-1} \in U$. This shows

$$\sum_{r=1}^{\ell} \|u_r^h - u_r^p, f_1, \ldots, f_{n-1}\| < \varepsilon$$

and

$$\sup_i \mathcal{M}_i\left(\frac{1}{T_i}\sum_{r=0}^{i}\left\|\frac{v_r t_{n-r}\Delta_k^{\ell}(u_r^h - u_r^p)}{\tau}, f_1, \ldots, f_{n-1}\right\|\right) < \varepsilon$$

for all $h, p \geq N$ and for every $f_1, \ldots, f_{n-1} \in U$. Hence, $\|u_r^h - u_r^p, f_1, \ldots, f_{n-1}\| < \varepsilon$ for all $r = 1, 2, \ldots, \ell$ and for every $f_1, \ldots, f_{n-1} \in U$. Therefore, (u_r^h) is a Cauchy sequence for all $r = 1, 2, \ldots, \ell$ in U, an n-Banach space. Hence, (u_r^h) converges in U for all $r = 1, 2, \ldots, \ell$. Let $\lim_{h\to\infty} u_r^h = u_r$ for all $r = 1, 2, \ldots, \ell$.

Further, we have

$$\sup_i \mathcal{M}_i\left(\frac{1}{T_i}\sum_{r=0}^{i}\left\|\frac{v_r t_{n-r}\Delta_k^{\ell}(u_r^h - u_r^p)}{\tau}, f_1, \ldots, f_{n-1}\right\|\right) < \varepsilon,$$

for all $h, p \geq N$ and for every $f_1, \ldots, f_{n-1} \in U$. This implies for every $f_1, f_2, \ldots, f_{n-1} \in U$

$$\mathcal{M}_i \frac{1}{T_i}\sum_{r=0}^{i}\left\|\frac{v_r t_{n-r}\Delta_k^{\ell}(u_r^h - u_r^p)}{\tau}, f_1, \ldots, f_{n-1}\right\| < \varepsilon$$

for all $h, p \geq N$ and $i \in \mathbb{N}$. Thus, $(\Delta_k^{\ell} u_r^h)$ is a Cauchy sequence in $O_\infty(\mathfrak{M}, u, \Delta_k^{\ell}, \|\cdot, \ldots, \cdot\|)$ which is complete. Hence, $(\Delta_k^{\ell} u_r^h)$ converges for each $r \in \mathbb{N}$. Let $\lim_{h\to\infty} \Delta_k^{\ell} u_r^h = z_r$, for each $r \in \mathbb{N}$.

For $r = 1$, we have

$$\lim_{h\to\infty} \Delta_k^{\ell} u_1^h = \lim_{h\to\infty} \sum_{m=0}^{\ell} (-1)^m \binom{\ell}{m} u_{1+km} = z_1, \tag{2}$$

Also,

$$\lim_{h\to\infty} u_r^h = u_r, \tag{3}$$

for $r = 1 + km$, for $m = 1, 2, \ldots, \ell - 1$.

Thus, from equation (2) and (3), we have $\lim_{h\to\infty} u_{1+\ell}^h$ exists. Let $\lim_{h\to\infty} u_{1+\ell}^h = u_{1+\ell}$. Proceeding in this way inductively $\lim_{h\to\infty} u_r^h = u_r$ exists for each $r \in \mathbb{N}$.

Now for every $f_1, \ldots, f_{n-1} \in U$

$$\lim_p \sum_{r=0}^{\ell} \|u_r^h - u_r^p, f_1, \ldots, f_{n-1}\| = \sum_{r=1}^{\ell} \|u_r^h - u_r, f_1, \ldots, f_{n-1}\| < \varepsilon,$$

for all $h \geq N$.

Again, using the continuity of n-norm, we find that for every $f_1, \ldots, f_{n-1} \in U$

$$\mathcal{M}_i\left(\frac{1}{T_i}\sum_{r=0}^{i}\left\|\frac{v_r t_{n-r}\Delta_k^{\ell} u_r^h}{\tau} - \lim_{p\to\infty}\frac{v_r t_{n-r}\Delta_k^{\ell} u_r^p}{\tau}, f_1, \ldots, f_{n-1}\right\|\right) < \varepsilon,$$

for all $h \geq N$ and $i \in \mathbb{N}$. Hence, for every $f_1, \ldots, f_{n-1} \in U$

$$\sup_i \mathcal{M}_i\left(\frac{1}{T_i}\sum_{r=0}^{i}\left\|\frac{v_r t_{n-r}\Delta_k^{\ell} u_r^h - v_r t_{n-r}\Delta_k^{\ell} u_r}{\tau}, f_1, \ldots, f_{n-1}\right\|\right) < \varepsilon$$

for all $h \geq N$.
Thus, for every $g^2, \ldots, g^n \in \mathcal{O}_\infty(\mathfrak{M}, u, \Delta_k^\ell, \|\cdot, \ldots, \cdot\|)$

$$\|u^h - u, g^2, \ldots, g^n\|_{\mathcal{O}_\infty(\mathfrak{M}, v, \Delta_k^\ell)} < 2\varepsilon$$

for all $h \geq N$. Hence, $(u^h - u) \in \mathcal{O}_\infty(\mathfrak{M}, v, \Delta_k^\ell, \|\cdot, \ldots, \cdot\|)$. Since $\mathcal{O}_\infty(\mathfrak{M}, v, \Delta_k^\ell, \|\cdot, \ldots, \cdot\|)$ is a linear space, thus for all $h \geq N, u = u^h - (u^h - u) \in \mathcal{O}_\infty(\mathfrak{M}, v, \Delta_k^\ell, \|\cdot, \ldots, \cdot\|)$. Hence $\mathcal{O}_\infty(\mathfrak{M}, v, \Delta_k^\ell, \|\cdot, \ldots, \cdot\|)$ is complete and is an n-Banach space. □

Corollary 2.3. *If the base space is a Banach space. Then the spaces $Z(\mathfrak{M}, v, \Delta_k^\ell, \|\cdot, \ldots, \cdot\|)$ is an n-BK space, where $Z = \mathcal{N}_p, \mathcal{O}_p, \mathcal{L}_p, \mathcal{N}_\infty$ and $\mathcal{O}_\infty$.*

Theorem 2.4. *Let $\mathfrak{M} = (\mathcal{M}_i)$ be a sequence of Orlicz functions, $v = (v_r)$ be a sequence of positive numbers. Then $Z(\mathfrak{M}, u, \Delta_k^{\ell-1}, \|\cdot, \ldots, \cdot\|) \subset Z(\mathfrak{M}, u, \Delta_k^\ell, \|\cdot, \ldots, \cdot\|)$, where $Z = \mathcal{N}_p, \mathcal{O}_p, \mathcal{L}_p, \mathcal{N}_\infty$ and $\mathcal{O}_\infty$.*

Proof. Let $u = (u_r) \in \mathcal{N}_p(\mathfrak{M}, v, \Delta_k^{\ell-1}, \|\cdot, \ldots, \cdot\|), 1 \leq p < \infty$. Then for every non zero $f_1, \ldots, f_{n-1} \in U$,

$$\sum_{i=1}^{\infty} \mathcal{M}_i\Big(\Big\|\frac{1}{T_i}\sum_{r=0}^{i}\frac{v_r t_{n-r}\Delta_k^{\ell-1}u_r}{\tau}, f_1, \ldots, f_{n-1}\Big\|\Big)^p < \infty. \tag{4}$$

Now, we have for every nonzero $f_1, \ldots, f_{n-1} \in U$

$$\begin{aligned}
&\mathcal{M}_i\Big(\Big\|\frac{1}{T_i}\sum_{r=0}^{i}\frac{v_r t_{n-r}\Delta_k^{\ell}u_r}{\tau}, f_1, \ldots, f_{n-1}\Big\|\Big)\\
\leq&\mathcal{M}_i\Big(\Big\|\frac{1}{T_i}\sum_{r=0}^{i}\frac{v_r t_{n-r}\Delta_k^{\ell-1}u_r}{\tau}, f_1, \ldots, f_{n-1}\Big\|\Big)\\
+&\mathcal{M}_i\Big(\Big\|\frac{1}{T_i}\sum_{r=0}^{i}\frac{v_r t_{n-r}\Delta_k^{\ell-1}u_{r+1}}{\tau}, f_1, \ldots, f_{n-1}\Big\|\Big)
\end{aligned}$$

It is known that for $1 \leq p < \infty, |c + d|^p \leq 2^p(|c|^p + |d|^p)$. Hence, for $1 \leq p < \infty$,

$$\begin{aligned}
&\mathcal{M}_i\Big(\Big\|\frac{1}{T_i}\sum_{r=0}^{i}\frac{v_r t_{n-r}\Delta_k^{\ell}u_r}{\tau}, f_1, \ldots, f_{n-1}\Big\|\Big)^p\\
\leq&2^p\Big\{\mathcal{M}_i\Big(\Big\|\frac{1}{T_i}\sum_{r=0}^{i}\frac{v_r t_{n-r}\Delta_k^{\ell-1}u_r}{\tau}, f_1, \ldots, f_{n-1}\Big\|\Big)^p\\
+&\mathcal{M}_i\Big(\Big\|\frac{1}{T_i}\sum_{r=0}^{i}\frac{v_r t_{n-r}\Delta_k^{\ell-1}u_{r+1}}{\tau}, f_1, \ldots, f_{n-1}\Big\|\Big)^p\Big\}.
\end{aligned}$$

Then for each positive integer r, we get

$$\begin{aligned}
&\sum_{i=1}^{r}\mathcal{M}_i\Big(\Big\|\frac{1}{T_i}\sum_{r=0}^{i}\frac{v_r t_{n-r}\Delta_k^{\ell}u_r}{\tau}, f_1, \ldots, f_{n-1}\Big\|\Big)^p\\
\leq&2^p\Big\{\sum_{i=1}^{r}\mathcal{M}_i\Big(\Big\|\frac{1}{T_i}\sum_{r=0}^{i}\frac{v_r t_{n-r}\Delta_k^{\ell-1}u_r}{\tau}, f_1, \ldots, f_{n-1}\Big\|\Big)^p\\
+&\sum_{i=1}^{r}\mathcal{M}_i\Big(\Big\|\frac{1}{T_i}\sum_{r=0}^{i}\frac{v_r t_{n-r}\Delta_k^{\ell-1}u_{r+1}}{\tau}, f_1, \ldots, f_{n-1}\Big\|\Big)^p\Big\}
\end{aligned}$$

Taking $r \to \infty$ and using equation (4), we have

$$\sum_{i=1}^{\infty} \mathcal{M}_i\Big(\Big\|\frac{1}{T_i}\sum_{r=0}^{i}\frac{v_r t_{n-r}\Delta_k^{\ell} u_r}{\tau}, f_1, \dots, f_{n-1}\Big\|\Big)^p < \infty.$$

Thus, $\mathcal{N}_p(\mathfrak{M}, v, \Delta_k^{\ell-1}, \|\cdot, \dots, \cdot\|) \subset \mathcal{N}_p(\mathfrak{M}, v, \Delta_k^{\ell}, \|\cdot, \dots, \cdot\|)$ for $1 \le p < \infty$. Using similar arguments we can prove for the spaces $Z = \mathcal{O}_p, \mathcal{L}_p, \mathcal{N}_\infty$ and $\mathcal{O}_\infty$. The inclusion is strict, and it follows from the following example. □

Example 2.5. *Considering $U = \mathbb{R}^3$, a real linear space. Define $\|\cdot,\cdot\| : U \times U \to \mathbb{R}$ by $\|u', v'\| = max\{|u_1'v_2' - u_2'v_1'|, |u_2'v_3' - v_3'u_2'|, |u_3'v_1' - u_1'v_3'|\}$, where $u' = (u_1', u_2', u_3'), v' = (v_1', v_2', v_3') \in \mathbb{R}$. Then $(U, \|\cdot,\cdot\|)$ is a 2-normed linear space. Let $(v_r) = 1, t_{n-r} = 1, (\mathcal{M}_i) = I$, for all $i \in \mathbb{N}, \ell = 2$ and $k = 1$. Let $u' = (u_r') = (r+1, r+1, r+1)$ for all $r \in \mathbb{N}$. Then $\Delta^2(u_r') = (0,0,0)$ for all $r \in \mathbb{N}$. Hence, $(u_r') \in \mathcal{N}_p(\mathfrak{M}, v, \Delta^2, \|\cdot,\cdot\|)$, Thus $\Delta(u_r') = (-1,-1,-1)$ for all $r \in \mathbb{N}$. Hence, $(u_r') \notin \mathcal{N}_p(\mathfrak{M}, v, \Delta, \|\cdot,\cdot\|)$. The inclusion is strict.*

Theorem 2.6. *Let $\mathfrak{M} = (\mathcal{M}_i)$ be a sequence of Orlicz function, $v = (v_r)$ be a sequence of positive numbers. Then*
(i) $\mathcal{O}_p(\mathfrak{M}, v, \Delta_k^{\ell}, \|\cdot, \dots, \cdot\|) \subset \mathcal{N}_p(\mathfrak{M}, v, \Delta_k^{\ell}, \|\cdot, \dots, \cdot\|) \subset \mathcal{N}_\infty(\mathfrak{M}, v, \Delta_k^{\ell}, \|\cdot, \dots, \cdot\|)$ and the inclusions are strict.
(ii) $\mathcal{O}_p(\mathfrak{M}, v, \Delta_k^{\ell}, \|\cdot, \dots, \cdot\|) \subset \mathcal{O}_\infty(\mathfrak{M}, v, \Delta_k^{\ell}, \|\cdot, \dots, \cdot\|) \subset \mathcal{N}_\infty(\mathfrak{M}, v, \Delta_k^{\ell}, \|\cdot, \dots, \cdot\|)$ and the inclusions are strict.

Proof. Clearly, the inclusions follow from the definition. □

Remark 2.7. $\mathcal{L}_p(\mathfrak{M}, v, \Delta_k^{\ell}, \|\cdot, \dots, \cdot\|) \subsetneq \mathcal{O}_p(\mathfrak{M}, v, \Delta_k^{\ell}, \|\cdot, \dots, \cdot\|)$.

Example 2.8. *Let $p = 1$ and 2-norm $\|\cdot,\cdot\|$ on $U = \mathbb{R}^3$ in example 2.5. Let $\ell = 2, k = 1, (v_r) = 1$ and $(\mathcal{M}_i) = I$. Consider the sequence $(u_r') = \{(2,2,2), (0,0,0), (0,0,0), \dots\}$. Then $\Delta^2 u_r' = (2,2,2)$ for $r = 1$ and $\Delta^2 u_r' = (0,0,0)$ for all $r > 1$. Then $(u_r') \in \mathcal{L}(\mathfrak{M}, v, \Delta^2, \|\cdot,\cdot\|)$ but $(u_r') \notin \mathcal{O}(\mathfrak{M}, v, \Delta^2, \|\cdot,\cdot\|)$.*

Theorem 2.9. *For $1 \le p < q$, we have*
(i) $\mathcal{O}_q(\mathfrak{M}, v, \Delta_k^{\ell}, \|\cdot, \dots, \cdot\|) \supset \mathcal{N}_q(\mathfrak{M}, v, \Delta_k^{\ell}, \|\cdot, \dots, \cdot\|)$;
(ii) $\mathcal{L}_q(\mathfrak{M}, v, \Delta_k^{\ell}, \|\cdot, \dots, \cdot\|) \supset \mathcal{L}_p(\mathfrak{M}, v, \Delta_k^{\ell}, \|\cdot, \dots, \cdot\|)$;
(iii) $\mathcal{O}_q(\mathfrak{M}, v, \Delta_k^{\ell}, \|\cdot, \dots, \cdot\|) \supset \mathcal{O}_p(\mathfrak{M}, v, \Delta_k^{\ell}, \|\cdot, \dots, \cdot\|)$.

Proof. Let $u \in \mathcal{O}_p(\mathfrak{M}, v, \Delta_k^{\ell}, \|\cdot, \dots, \cdot\|)$. Then there exist $\tau > 0$ such that

$$\sum_{i=1}^{\infty} \mathcal{M}_i\Big(\Big\|\frac{1}{T_i}\sum_{r=0}^{i}\frac{v_r t_{n-r}\Delta_k^{\ell} u_r}{\tau}, f_1, \dots, f_{n-1}\Big\|\Big)^p < \infty.$$

This implies

$$\sum_{i=1}^{\infty} \mathcal{M}_i\Big(\Big\|\frac{1}{T_i}\sum_{r=0}^{i}\frac{v_r t_{n-r}\Delta_k^{\ell} u_r}{\tau}, f_1, \dots, f_{n-1}\Big\|\Big)^p < 1.$$

Since $(\mathcal{M}_i)$ is non decreasing for sufficiently large values of i, thus we get

$$\begin{aligned}
&\sum_{i=1}^{\infty} \mathcal{M}_i\Big(\Big\|\frac{1}{T_i}\sum_{r=0}^{i}\frac{v_r t_{n-r}\Delta_k^{\ell} u_r}{\tau}, f_1, \dots, f_{n-1}\Big\|\Big)^q \\
\le &\sum_{i=1}^{\infty} \mathcal{M}_i\Big(\Big\|\frac{1}{T_i}\sum_{r=0}^{i}\frac{v_r t_{n-r}\Delta_k^{\ell} u_r}{\tau}, f_1, \dots, f_{n-1}\Big\|\Big)^p \\
&< \infty.
\end{aligned}$$

Thus, $u \in \mathcal{O}_q(\mathfrak{M}, v, \Delta_k^{\ell}, \|\cdot, \dots, \cdot\|)$. Using similar arguments we can establish the inclusions (i) and (ii). □

3. Köthe-Toeplitz duals of Nörlund Orlicz sequence space

Köthe and Toeplitz [7] introduced the idea of dual sequence space. Then Maddox [9] generalized this notion to U-valued sequence classes where U is an n Banach space. The α and β-duals of a (complex-valued) sequence space E, denoted by E^α and E^β respectively, and defined by

$$E^\alpha = \Big\{(c_r) \in w : \sum_{r=1}^{\infty} |c_r u_r| < \infty \text{ for all } u = (u_r) \in w\Big\},$$

$$E^\beta = \Big\{(c_r) \in w : \sum_{r=1}^{\infty} c_r u_r \text{ converges for all } u = (u_r) \in w\Big\}.$$

Definition 3.1. *A real valued n-functional defined on $B_1 \times \cdots \times B_n$, where $B_1, \ldots, B_n$ are linear manifolds of a linear n-normed space. Let H be an n-functional defined on a domain $B_1 \times \cdots \times B_n$. Then H is called a linear n-functional whenever for all ${}^1b_1, {}^1b_2, \ldots, {}^1b_n \in B_1, {}^2b_1, {}^2b_2, \ldots, {}^2b_n \in B_2$ and ${}^nb_1, {}^nb_2, \ldots, {}^nb_n \in B_n$ we have*
(i) $H({}^1b_1, {}^1b_2, \ldots, {}^1b_n, {}^2b_1, {}^2b_2, \ldots, {}^2b_n, \ldots, {}^nb_1, {}^nb_2, \ldots, {}^nb_n) = \sum_{1\le i_1,\, i_2,\, \ldots,\, i_n \le n} H({}^1b_{i_i}, {}^1b_{i_2}, \ldots, {}^1b_{i_n})$
(ii) $H(\beta_1 b_1, \ldots, \beta_n b_n) = \beta_1, \ldots, \beta_n\, H(b_1, b_2, \ldots, b_n)$, for all $\beta_1, \beta_2, \ldots, \beta_n \in \mathbb{R}$.

Remark 3.2. *Let H be an n-functional with domain $D(H)$. Then H is called bounded if there is a real constant $R \ge 0$ such that $|H(b_1, \ldots, b_n)| \le R\|b_1, \ldots, b_n\|$ for all $(b_1, \ldots, b_n) \in D(H)$. If H is bounded, we define the norm $\|H\| = glb\{R : H(b_1, \ldots, b_n) \le R\|b_1, \ldots, b_n\|$ for all $(b_1, \ldots, b_n) \in D(H)\}$. If H is not bounded, we define $\|H\| = +\infty$.*

Proposition 3.3. *A linear n-functional H is continuous if and only if it is bounded.*

Proposition 3.4. *Let A^* be the set of bounded linear n-functionals with domain $A_1 \times \ldots \times A_n$. Then A^* is an n-Banach space up to linear dependence.*

For any $n(> 1)$-normed space E, the continuous dual of E denote by E^*.

Definition 3.5. *Let E be an n-normed linear space, normed by $\|\cdot, \ldots, \cdot\|_E$, the Köthe-Toeplitz dual of the sequence space $Z(E)$ whose base space is E, defined as*

$$[Z(E)]^\alpha = \{(v_r) : v_r \in E^*, r \in \mathbb{N} \text{ and } (\|u_r, g_2, \ldots, g_n\|_E \|v_r, f_2, \ldots, f_n\|_{E^*}) \in \ell_1\}$$

for every $f_2, \ldots, f_n \in E^*$, $g_2, \ldots, g_n \in E$, $(u_r) \in Z(E)$. Clearly for $\phi \in U^\alpha$, if $U \subset V$, then $V^\alpha \subset U^\alpha$.

Remark 3.6. *We consider the set*

$$S\mathcal{N}_p(\mathfrak{M}, v, \Delta_k^\ell, \|\cdot, \ldots, \cdot\|) = \{u = (u_r) : u \in \mathcal{N}_p(\mathfrak{M}, v, \Delta_k^\ell, \|\cdot, \ldots, \cdot\|), u_1 = \ldots = u_\ell = 0\}.$$

Then for $1 \le p < \infty$, $S\mathcal{N}_p(\mathfrak{M}, v, \Delta_k^\ell, \|\cdot, \ldots, \cdot\|)$ is a subspace of $\mathcal{N}_p(\mathfrak{M}, v, \Delta_k^\ell, \|\cdot, \ldots, \cdot\|)$. The other subspaces are also determined in the same way.

Theorem 3.7. *$u \in S\mathcal{N}_\infty(\Delta^\ell)$ implies $\sup_r r^{-\ell}\|u_r, f_2, \ldots, f_n\| < \infty$.*

Proof. Since, $u \in S\mathcal{N}_\infty(\Delta^\ell)$, thus

$$\sup_i \left\| \frac{1}{T_i} \sum_{r=0}^{i} \Delta^\ell u_r t_{n-r} \right\| < \infty.$$

Let,

$$\sup_r \left\| \frac{1}{T_i} \sum_{r=0}^{i} \Delta^\ell u_r t_{n-r} \right\| < \infty$$
$$\Longrightarrow \sup_r \left\| \frac{1}{T_i} \Delta^{\ell-1} u_r t_{n-r} - \frac{1}{T_i} \Delta^{\ell-1} u_{r+1} t_{n-(r+1)} \right\| < \infty.$$

Now,

$$\begin{aligned}\left\|\frac{1}{T_i}\Delta^{\ell-1}u_1t_{n-1}-\frac{1}{T_i}\Delta^{\ell-1}u_{r+1}t_{n-(r+1)}\right\| &=\left\|\frac{1}{T_i}\sum_{p=1}^{r}\left(\Delta^{\ell-1}u_pt_{n-p}-\Delta^{\ell-1}u_{p+1}t_{n-(p+1)}\right)\right\|\\ &\leq\sum_{p=1}^{r}\left\|\frac{1}{T_i}\Delta^{\ell-1}u_pt_{n-p}-\frac{1}{T_i}\Delta^{\ell-1}u_{p+1}t_{n-(p+1)}\right\|\\ &=O(r).\end{aligned}$$

This implies

$$\sup_r r^{-1}\left\|\Delta^{\ell-1}u_r,f_2,\dots,f_n\right\|<\infty,$$

Similarly,

$$\sup_r r^{-2}\left\|\Delta^{\ell-2}u_r,f_2,\dots,f_n\right\|<\infty,$$

continuing this process, we have

$$\sup_r r^{-k}\left\|\Delta^{\ell-i}u_r,f_2,\dots,f_n\right\|<\infty,$$

for $k=1,2,\dots,\ell$.
For $k=\ell$, we get the desired result. □

Lemma 3.8. *$u\in S\mathcal{N}_\infty(\mathfrak{M},v,\Delta_k^\ell,\|\cdot,\dots,\cdot\|)$ implies $\sup_r r^{-\ell}\|u_r,f_2,\dots,f_n\|<\infty$ for every $f_2,\dots,f_n\in U$.*

Proof. Using Theorem(3.7), we easily prove Lemma (3.8). □

Theorem 3.9. *Let $v=(v_r)$ be a sequence of positive numbers, $\mathfrak{M}=(\mathcal{M}_i)$ be a sequence of Orlicz functions. Then the Köthe-Toeplitz duals of the space $S\mathcal{N}_p(\mathfrak{M},v,\Delta_k^\ell,\|\cdot,\dots,\cdot\|)$ is $\mathcal{U}$, that is, $[S\mathcal{N}_\infty(\mathfrak{M},v,\Delta_k^\ell,\|\cdot,\dots,\cdot\|)]^\alpha=\mathcal{U}$, where*

$$\mathcal{U}=\left\{c=(c_r):\sum_{r=1}^{\infty}r^\ell\|c_r,f_2,\dots,f_n\|_{U^*}<\infty,\ \textit{for every } f_2,\dots,f_n\in U\right\}.$$

Proof. If $c\in\mathcal{U}$, then

$$\begin{aligned}&\sum_{r=1}^{\infty}\|c_r,f_2,\dots,f_n\|_{U^*}\|u_r,g_2,\dots,g_n\|_U\\ &=\sum_{r=1}^{\infty}r^\ell\|c_r,f_2,\dots,f_n\|_{U^*}(r^{-\ell}\|u_r,g_2,\dots,g_n\|_U)\\ &<\infty,\end{aligned}$$

for each $u\in S\mathcal{N}_\infty(\mathfrak{M},v,\Delta_k^\ell,\|\cdot,\dots,\cdot\|)$ by Lemma (3.8). Hence, $u\in[S\mathcal{N}_\infty(\mathfrak{M},v,\Delta_k^\ell,\|\cdot,\dots,\cdot\|)]^\alpha$.
Again, let $c\in[S\mathcal{N}_\infty(\mathfrak{M},v,\Delta_k^\ell,\|\cdot,\dots,\cdot\|)]^\alpha$. Then

$$\sum_{r=1}^{\infty}\|c_r,f_2,\dots,f_n\|_{U^*}\|u_r,g_2,\dots,g_n\|_U<\infty,$$

for each $u\in S\mathcal{N}_\infty(\mathfrak{M},v,\Delta_k^\ell,\|\cdot,\dots,\cdot\|)$.
Define the sequence $u=(u_r)$ by

$$u_r = \begin{cases} 0, & r \leq \ell, \\ r^\ell, r > \ell \end{cases}$$

and choose $g_2, \ldots, g_n \in U$ such that

$$\|r^\ell, g_2, \ldots, g_n\|_U = r^\ell \|1, g_2, \ldots, g_n\|_U = \begin{cases} 0, & r \leq \ell \\ r^\ell, r > \ell. \end{cases}$$

Thus, we have $f_2, \ldots, f_n \in U^*$

$$\begin{aligned} \sum_{r=1}^{\infty} r^\ell \|c_r, f_2, \ldots, f_n\|_{U^*} &= \sum_{r=1}^{\infty} \|r^\ell, g_2, \ldots, g_n\|_U \|c_r, f_2, \ldots, f_n\|_{U^*} \\ &= \sum_{r=1}^{\ell} \|r^\ell, g_2, \ldots, g_n\|_U \|c_r, f_2, \ldots, f_n\|_{U^*} \\ &+ \sum_{r=1}^{\infty} \|r^\ell, g_2, \ldots, g_n\|_U \|c_r, f_2, \ldots, f_n\|_{U^*} \\ &< \infty. \end{aligned}$$

This implies $c \in \mathcal{U}$. □

Theorem 3.10. *Let $v = (v_r)$ be a sequence of positive real numbers, $\mathfrak{M} = (\mathcal{M}_i)$ be a sequence of Orlicz functions Then $[S\mathcal{N}_\infty(\mathfrak{M}, v, \Delta_k^\ell, \|\cdot, \ldots, \cdot\|)]^\alpha = [O_\infty(\mathfrak{M}, v, \Delta_k^\ell, \|\cdot, \ldots, \cdot\|)]^\alpha$.*

Proof. Since $S\mathcal{N}_\infty(\mathfrak{M}, v, \Delta_k^\ell \|\cdot, \ldots, \cdot\|) \subset O_\infty(\mathfrak{M}, v, \Delta_k^\ell, \|\cdot, \ldots, \cdot\|)$, we have $[O_\infty(\mathfrak{M}, v, \Delta_k^\ell \|\cdot, \ldots, \cdot\|)]^\alpha \subset [S\mathcal{N}_\infty(\mathfrak{M}, v, \Delta_k^\ell, \|\cdot, \ldots, \cdot\|)]^\alpha$. Let $c \in [S\mathcal{N}_\infty(\mathfrak{M}, v, \Delta_k^\ell, \|\cdot, \ldots, \cdot\|)]^\alpha$ and $u \in O_\infty(\mathfrak{M}, v, \Delta_k^\ell, \|\cdot, \ldots, \cdot\|)$. Consider the sequence $u = (u_r)$ defined by

$$u_r = \begin{cases} u_r, r \leq \ell, \\ u'_r, r > \ell, \end{cases}$$

where, $u' = (u'_r) \in S\mathcal{N}_\infty(\mathfrak{M}, v, \Delta_k^\ell, \|\cdot, \ldots, \cdot\|)$. Then we write

$$\begin{aligned} \sum_{r=1}^{\infty} \|c_r, f_2, \ldots, f_n\|_{U^*} \|u_r, g_2, \ldots, g_n\|_U &= \sum_{r=1}^{\ell} \|c_r, f_2, \ldots, f_n\|_{U^*} \|u_r, g_2, \ldots, g_n\|_U \\ &+ \sum_{r=1}^{\infty} \|c_r, f_2, \ldots, f_n\|_{U^*} \|u'_r, g_2, \ldots, g_n\|_U \\ &< \infty. \end{aligned}$$

This implies $c \in [O_\infty(\mathfrak{M}, v, \Delta_k^\ell \cdot, \ldots, \cdot\|)]^\alpha$. □

Corollary 3.11. *Let $v = (v_r)$ be a sequence of positive real numbers, $\mathfrak{M} = (\mathcal{M}_i)$ be a sequence of Orlicz functions. Then $[S\mathcal{N}_\infty \mathfrak{M}, v, \Delta_k^\ell, \|\cdot, \ldots, \cdot\|)]^\alpha = [\mathcal{N}_\infty(\mathfrak{M}, v, \Delta_k^\ell, \|\cdot, \ldots, \cdot\|)]^\alpha$.*

4. Conclusion

In this article, we obtained many useful topological and algebraic properties of Nörlund Orlicz sequence space and established their Köthe-Toeplitz duals. These results will be helpful to study Euler, Hölder, Hausdorff, and other means in the setting of Orlicz sequence space. Additionally, these investigations generalize the concept of summability theory in the setting of Orlicz sequence space.

Declaration

Conflict of interest: The author declares that the article is free from conflict of interest.
Contribution of author: All authors have equal contributions for the preparation of the article.
Funding: The article is not financially supported by any agency.

Acknowledgments

The first author is grateful to the Indian government's Ministry of Education for the Institute Fellowship (GATE), which financially supported my research efforts. We want to express our sincere thanks to the reviewers for the kind remarks, which improve the presentation of the paper.

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