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A C^1-regularity result to the inhomogeneous normalized infinity Laplacian equation

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Abstract. In this paper, we investigate the regularity for the viscosity solution to the Dirichlet problem

$$\begin{cases} -\Delta_\infty^N u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded convex domain and $f(x) \in C(\Omega)$. For $0 < f_{\inf} = \inf_\Omega f \le f \le \sup_\Omega f = f_{\sup} < +\infty$, we first prove the $\frac{1}{2}$-concavity of the viscosity solution by the convex envelope method of Alvarez-Lasry-Lions, and then establish the C^1-regularity based on the upper estimate of semiconcave functions at the singular point. The similar result holds for $-\infty < f_{\inf} \le f \le f_{\sup} < 0$.

1. Introduction

In this paper, we study the regularity of the unique viscosity solution to the Dirichlet problem

$$\begin{cases} -\Delta_\infty^N u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded convex domain and $f(x) \in C(\Omega)$ satisfies $0 < f_{\inf} = \inf_\Omega f \le f \le \sup_\Omega f = f_{\sup} < +\infty$. The normalized infinity Laplacian is given by

$$\Delta_\infty^N u := |Du|^{-2} \left\langle D^2 u Du, Du \right\rangle,$$

which has received significant attention in recent years. The infinity Laplacian $\Delta_\infty^N u$ is singular and highly degenerate, which also has wide applications in mass transportation [13], shape deformation [8] and differential games [20, 25].

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Infinity Laplacian $\Delta_\infty u := \langle D^2 u Du, Du\rangle$ was introduced by Aronsson [3–6] in studying the absolutely minimizing Lipschitz extension. Jensen [16] proved the uniqueness of the viscosity solution to the Dirichlet problem

$$\begin{cases} \Delta_\infty u = 0 & \text{in } \Omega, \\ u = g(x) & \text{on } \partial\Omega, \end{cases}$$

where $g \in C(\partial\Omega)$. For the planar infinity harmonic functions u, Savin [26] and Evans-Savin [14] established the C^1 and $C^{1,\alpha}$-regularity with some $\alpha > 0$, respectively. Furthermore, Koch-Zhang-Zhou [18] established the sharp Sobolev $W^{1,2}_{loc}$-estimate for the gradient Du. For $n \geq 3$, Evans-Smart [15] gained the everywhere differentiability of the infinity harmonic functions.

For the inhomogeneous equation, Lu-Wang [24] obtained the existence and uniqueness of the viscosity solution to the Dirichlet problem

$$\begin{cases} \Delta_\infty u = f(x) & \text{in } \Omega, \\ u = g(x) & \text{on } \partial\Omega, \end{cases}$$

where $f \in C(\Omega)$ with $\inf_\Omega f > 0$ or $\sup_\Omega f < 0$ and $g \in C(\partial\Omega)$. If $f \in C(B_1) \bigcap L^\infty(B_1)$ and $g \in \partial B_1$, Lindgren [21] obtained the linear approximation property. Furthermore, if $f \in C^1(B_1) \bigcap L^\infty(B_1)$ and $g \in \partial B_1$, he also established the everywhere differentiability. If $f \in BV_{loc}(\Omega) \bigcap C(\Omega)$ with $|f| > 0$, Koch-Zhang-Zhou [19] proved $|Du|^\alpha \in W^{1,2}_{loc}(\Omega)$ with $\alpha > 3/2$ and $|Du|^\alpha \in W^{1,p}_{loc}(\Omega)$ with $0 < \alpha \leq 3/2$ with $1 \leq p < 3/(3-\alpha)$ in two-dimension. If $f \in C^{0,1}(\Omega)$, Lu-Miao-Zhou [22] proved the everywhere differentiability of the viscosity solution. In fact, they obtained the regularity of viscosity solutions to the generalized inhomogeneous Aronsson's equation. If $f(x) \equiv 1$ and $g(x) \equiv 0$, Crasta-Fragalà [11] obtained the C^1-regularity in a bounded convex domain satisfying the interior sphere condition.

With the probability methods, Peres-Schramm-Sheeld-Wilson [25] obtained the existence and uniqueness of the viscosity solution to the following Dirichlet problem

$$\begin{cases} \Delta^N_\infty u = f(x) & \text{in } \Omega, \\ u = g(x) & \text{on } \partial\Omega, \end{cases}$$

where $f \in C(\Omega)$ with $\inf_\Omega f > 0$ or $\sup_\Omega f < 0$ and $g \in C(\partial\Omega)$. Lu-Wang [23] and Armstrong-Smart [2] gave another proof by the PDE's methods and the finite difference methods, respectively. If $f(x,u) = u^q$ with $0 \leq q < 1$ and $g(x) = 0$, Juutinen [17] explored the power-concavity property of the positive solution in a bounded convex domain. If $q = 1$, he also proved the log-concavity by a concavity maximum principle. For $f(x) \equiv 1$, Crasta-Fragalà [12] established the C^1-regularity for the unique viscosity solution based on the convex envelope method in a bounded convex domain.

Our main results are stated as follows.

Theorem 1.1. *Let Ω be a bounded convex domain of $\mathbb{R}^n$. If $f(x) \in C(\Omega)$ satisfies $0 < f_{\inf} \leq f \leq f_{\sup} < +\infty$, then the unique viscosity solution u to Problem (1) is $\frac{1}{2}$-concave in Ω.*

Theorem 1.2. *Let Ω be a bounded convex domain of $\mathbb{R}^n$. If $f(x) \in C(\Omega)$ satisfies $0 < f_{\inf} \leq f \leq f_{\sup} < +\infty$, then the unique viscosity solution u to Problem (1) is of class C^1.*

Theorem 1.1 demonstrates that the solution enjoys the power-concavity with exponent 1/2. Our proof is based on the Alvarez-Lasry-Lions convex envelope technique in [1] and the comparison principle established in [23, 25].

Theorem 1.2 shows the C^1-regularity of the viscosity solution. The key is to combine the solution's local semiconcavity with an upper estimate for semiconcave functions at singular points.

By similar arguments, we can obtain the following symmetric result.

Remark 1.3. *Let Ω be a bounded convex domain of $\mathbb{R}^n$. If the inhomogeneous term $f(x) \in C(\Omega)$ satisfies $-\infty < f_{\inf} \leq f \leq f_{\sup} < 0$, then the unique viscosity solution to Problem (1) is of class C^1.*

The paper is organized as follows. In Section 2, we review the definition of the normalized infinity Laplacian, the concept of viscosity solutions and some related properties. In Section 3, we prove the power-concavity of the solution. In Section 4, we establish the C^1-regularity of the solution based on the power-concavity result.

2. Definitions of viscosity solutions

In this section, we give the definition of the viscosity solutions to the normalized infinity Laplacian equation involving lower terms

$$-\Delta_\infty^N u(x) = g(x, u(x), Du(x)) \quad \text{in } \Omega, \tag{2}$$

where $g : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function.

For a symmetric matrix $A \in \mathbb{R}^{n\times n}_{\text{sym}}$, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote respectively the minimum and the maximum eigenvalue of A, where the set $\mathbb{R}^{n\times n}_{\text{sym}}$ represents the set of all $n \times n$ real symmetric matrices. In the following, if $u, v : \Omega \to \mathbb{R}$ are two functions and $x \in \Omega$, by $u \prec_x v$, we mean that $u(x) = v(x)$ and $u(y) \le v(y)$ for every $y \in \Omega$.

Due to the singularity and high degeneracy of the operator, we adopt the definition in Lu and Wang [23] based on the continuous extension.

Definition 2.1. *For a C^2-function φ defined in a neighborhood of $x \in \mathbb{R}^n$, we define the operators*

$$\Delta_\infty^+\varphi(x) := \begin{cases} |D\varphi(x)|^{-2}\langle D^2\varphi(x)D\varphi(x), D\varphi(x)\rangle & \text{if } D\varphi(x) \neq 0, \\ \lambda_{\max}(D^2\varphi(x)) & \text{if } D\varphi(x) = 0, \end{cases}$$

$$\Delta_\infty^-\varphi(x) := \begin{cases} |D\varphi(x)|^{-2}\langle D^2\varphi(x)D\varphi(x), D\varphi(x)\rangle & \text{if } D\varphi(x) \neq 0, \\ \lambda_{\min}(D^2\varphi(x)) & \text{if } D\varphi(x) = 0. \end{cases}$$

Definition 2.2. *Suppose that $u \in C(\Omega)$ is twice differentiable at $x_0 \in \Omega$. We define the normalized infinity Laplacian of u at x_0 to be the closed interval*

$$\Delta_\infty^N u(x_0) = [\Delta_\infty^- u(x_0), \Delta_\infty^+ u(x_0)],$$

and if $\Delta_\infty^N u(x_0)$ contains only one real number, we do not distinguish $\Delta_\infty^N u(x_0)$ from its single element.

Definition 2.3. *Let Ω be a bounded set and $g : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function.*

An upper semicontinuous function u is called a viscosity sub-solution of (2) in Ω if

$$-\Delta_\infty^+\varphi(x_0) \le g(x_0, u(x_0), Du(x_0)),$$

whenever $u \prec_{x_0} \varphi$ for any $x_0 \in \Omega$ and C^2-test function φ.

Similarly, a lower semicontinuous function u is called a viscosity super-solution of (2) in Ω, if

$$-\Delta_\infty^-\varphi(x_0) \ge g(x_0, u(x_0), Du(x_0)),$$

whenever $\varphi \prec_{x_0} u$ for any $x_0 \in \Omega$ and C^2-test function φ.

A continuous function u is called a viscosity solution of (2) if u is both a viscosity sub-solution and super-solution of Equation (2) in Ω.

Now we recall the concepts of superjets and subjets.

Definition 2.4. *Let $u \in C(\Omega)$. The second-order super-jet of u at $x_0 \in \Omega$ is defined to be the set*

$$J_\Omega^{2,+}u(x_0) = \left\{\left(D\varphi(x_0), D^2\varphi(x_0)\right) : \varphi \text{ is } C^2 \text{ and } u \prec_{x_0} \varphi\right\},$$

whose closure is defined to be

$$\overline{J}^{2,+}_{\Omega}u(x_0) = \Big\{(p,X) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}_{sym} : \exists (x_n,p_n,X_n) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{n\times n}_{sym} \text{ such that } (p_n,X_n) \in J^{2,+}_{\Omega}u(x_n) \\ \text{and } (x_n,u(x_n),p_n,X_n) \to (x_0,u(x_0),p,X)\Big\}.$$

The second-order sub-jet of u at $x_0 \in \Omega$ is defined to be the set

$$J^{2,-}_{\Omega}u(x_0) = \Big\{\big(D\varphi(x_0), D^2\varphi(x_0)\big) : \varphi \text{ is } C^2 \text{ and } \varphi \prec_{x_0} u\Big\},$$

whose closure is defined to be

$$\overline{J}^{2,-}_{\Omega}u(x_0) = \Big\{(p,X) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}_{sym} : \exists (x_n,p_n,X_n) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{n\times n}_{sym} \text{such that } (p_n,X_n) \in J^{2,-}_{\Omega}u(x_n) \\ \text{and } (x_n,u(x_n),p_n,X_n) \to (x_0,u(x_0),p,X)\Big\}.$$

In terms of superjets and subjets, we can also give the equivalent definition of viscosity solutions. See for example [10].

Definition 2.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded set and $g : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function.*

For any $x_0 \in \Omega$, an upper semicontinuous function u is called a viscosity sub-solution of (2) *in Ω, if for any $(p,X) \in \overline{J}^{2,+}_{\Omega}u(x_0)$, there holds*

$$\begin{cases} -|p|^{-2}\langle Xp,p\rangle \le g(x_0,u(x_0),p) & \text{if } p \neq 0, \\ -\lambda_{\max}(X) \le g(x_0,u(x_0),0) & \text{if } p = 0. \end{cases}$$

Similarly, a lower semicontinuous function u is called a viscosity super-solution of (2) *in Ω, if for any $(p,X) \in \overline{J}^{2,-}_{\Omega}u(x_0)$, there holds*

$$\begin{cases} -|p|^{-2}\langle Xp,p\rangle \ge g(x_0,u(x_0),p) & \text{if } p \neq 0, \\ -\lambda_{\min}(X) \ge g(x_0,u(x_0),0) & \text{if } p = 0. \end{cases}$$

A continuous function u is called a viscosity solution of (2) *if u is both a viscosity sub-solution and super-solution of Equation* (2) *in Ω.*

Remark 2.6. *The viscosity solution u to Problem* (1) *is strictly positive in Ω. Indeed, u is non-negative by the comparison principle proved in [23, 25]. Assume by contradiction that $u(x_0) = 0$ at some point $x_0 \in \Omega$. Then the function $\varphi \equiv 0$ touches u from below at x_0, and hence u cannot be a viscosity super-solution to the equation $-\Delta^N_\infty u = f(x) > 0$.*

3. Power concavity

In this section, we prove the $\frac{1}{2}$-concavity of the viscosity solution u to Problem (1) by transforming it to $U = -u^{1/2}$ which allows us to resolve critical-point singularities with restricted viscosity solutions. The preservation of convex envelopes is ensured under the interior sphere condition via the comparison principle. One can extend the $\frac{1}{2}$-concavity to any convex domain through outer parallel approximation technique.

The map $u \mapsto U := -u^{1/2}$ establishes a bijective correspondence between positive viscosity sub-solutions and super-solutions of Equation (1) in Ω and a constrained class of negative viscosity super-solutions and sub-solutions of the associated equation

$$-\Delta^N_\infty U = \frac{|DU|^2}{U} + \frac{f}{2U} \quad \text{in } \Omega. \tag{3}$$

By Remark 2.6, the viscosity solution u is strictly positive in Ω, which ensures that $U = -u^{1/2}$ is well-defined and negative.

Lemma 3.1. *Let Ω be a bounded domain and $f(x) \in C(\Omega)$ satisfy $f_{\inf} > 0$. For a strictly positive function $u : \Omega \to (0, +\infty)$, there hold:*

(i) An upper semicontinuous u is a viscosity sub-solution of (1) *if and only if $U := -u^{1/2}$ is a viscosity super-solution to* (3) *in Ω.*

(ii) A lower semicontinuous u is a viscosity super-solution of (1) *if and only if U is a viscosity sub-solution to* (3) *in Ω.*

Proof. We only prove (i) since the proof of (ii) is similar. For any $x \in \Omega$ and $\varphi \in C^2(\Omega)$, let $u \prec_x \varphi$. Since $u > 0$, we have $\varphi(x) = u(x) > 0$. By direct calculations, we obtain

$$u \prec_x \varphi \iff \psi := -\varphi^{1/2} \prec_x U.$$

Then

$$D\psi(x) = -\frac{D\varphi(x)}{2\varphi(x)^{1/2}}$$

and

$$D^2\psi(x) = \frac{1}{4\varphi(x)^{3/2}} D\varphi(x) \otimes D\varphi(x) - \frac{1}{2\varphi(x)^{1/2}} D^2\varphi(x).$$

For the case $D\psi(x) \neq 0$, the result is obvious. For the case $D\psi(x) = 0$, we have $D\varphi(x) = 0$ and $D^2\psi(x) = -\frac{1}{2|\psi(x)|} D^2\varphi(x)$. Particularly, $-\lambda_{\max}(D^2\varphi(x)) \leq f(x)$ implies $-\lambda_{\min}(D^2\psi(x)) \geq -\frac{f(x)}{2U(x)}$. Thus, U is a viscosity super-solution to (3) in Ω. □

To establish the $\frac{1}{2}$-power concavity of the viscosity solution of Problem (1), we need the following condition:

($\boldsymbol{\Omega_{ISC}}$): Ω is a convex domain and satisfies the interior sphere condition.

First, we show that the convex envelope U_{**} of a restricted super-solution U to Equation (3) is still a restricted super-solution in Ω based on the convex envelope technique introduced by Alvarez, Lasry and Lions [1]. Next, by the comparison principle established in [23, Theorem 3.3], we deduce that if Ω satisfies ($\boldsymbol{\Omega_{ISC}}$), then U is convex, or equivalently, $u^{1/2}$ is concave. Finally, through the approximation of Ω using outer parallel sets, we have that $u^{1/2}$ is concave in any bounded convex domain. Condition ($\boldsymbol{\Omega_{ISC}}$) plays an essential role in the proof of Lemma 3.2 below, which allows us to avoid imposing state constraint boundary conditions on $\partial\Omega$.

Next we prove that, if Ω satisfies ($\boldsymbol{\Omega_{ISC}}$) and U is a viscosity solution to

$$\begin{cases} -\Delta_\infty^N U - \frac{1}{U}\left(|DU|^2 + \frac{f}{2}\right) = 0 & \text{in } \Omega, \\ U = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

then U is convex. We denote by U_{**} the largest convex function below U. First, we establish that, under Condition ($\boldsymbol{\Omega_{ISC}}$), for every $x \in \Omega$, within the characterization

$$U_{**}(x) = \inf\left\{\sum_{i=1}^k \lambda_i U(x_i) : \ x = \sum_{i=1}^k \lambda_i x_i, \ x_i \in \overline{\Omega}, \ \lambda_i > 0, \ \sum_{i=1}^k \lambda_i = 1, \ k \leq n+1\right\},$$

the infimum is attained only at interior points $x_i \in \Omega$.

Lemma 3.2. *Assume that Ω satisfies Condition* ($\boldsymbol{\Omega_{ISC}}$) *and $f(x) \in C(\Omega)$ satisfies $f_{\inf} > 0$. Let u be the solution of Problem* (1)*. Set $U := -u^{1/2}$. For a fixed $x \in \Omega$, let $x_1, \ldots, x_k \in \overline{\Omega}$, $\lambda_1, \ldots, \lambda_k > 0$ with $\sum_{i=1}^k \lambda_i = 1$ be such that*

$$x = \sum_{i=1}^k \lambda_i x_i, \quad U_{**}(x) = \sum_{i=1}^k \lambda_i U(x_i).$$

Then $x_1, \ldots, x_k \in \Omega$.

Proof. For the sake of contradiction, assume that at least one of the points x_i, say x_1, lies on $\partial\Omega$. Let $B_R(y) \subset \Omega$ be a ball such that $\partial B_R(y) \cap \partial\Omega = \{x_1\}$. Since $f_{\inf} > 0$, u satisfies $-\Delta_\infty u \geq 0$ in Ω. By [9], the function

$$\min_{x\in\partial B_r(y)} \frac{u(x) - u(y)}{r}$$

is non-increasing with respect to r. Thus, for any $0 < r < R$, there holds

$$\min_{x\in\partial B_r(y)} \frac{u(x) - u(y)}{|x - y|} \geq \min_{x\in\partial B_R(y)} \frac{u(x) - u(y)}{|x - y|} = -\frac{u(y)}{R}. \tag{5}$$

That is,

$$u(x) \geq u(y)\left(1 - \frac{|x - y|}{R}\right), \quad \forall x \in B_R(y).$$

Hence,

$$U(x) \leq U(y)\left(1 - \frac{|x - y|}{R}\right)^{1/2}, \quad \forall x \in B_R(y). \tag{6}$$

Define $\alpha := \frac{x-x_1}{|x-x_1|}$ (the unit vector in the direction of x in Ω) and $\beta := \frac{y-x_1}{|y-x_1|}$ (the inner normal of $\partial\Omega$ at x_1). Since Ω is convex, for any $t \in [0, 1]$, we have $x_1 + t\alpha \in \overline{\Omega}$ and $\langle\beta, \alpha\rangle > 0$. By the definition of convex envelope, we have $U \geq U_{**}$. Since $U_{**}(x_1) = U(x_1) = 0$, there exists $\mu > 0$ such that

$$U(x_1 + t\alpha) \geq U_{**}(x_1 + t\alpha) = -\mu t \quad \text{for } t \in [0, 1].$$

By (6), we obtain

$$-\mu t \leq U(y)\left(1 - \frac{|R\beta - t\alpha|}{R}\right)^{1/2}, \quad t \in [0, 1].$$

Since

$$|R\beta - t\alpha| = R\left(1 - \frac{t}{R}\langle\beta, \alpha\rangle - o(t)\right),$$

there holds

$$-\mu t \leq U(y)\left(\frac{t\langle\beta, \alpha\rangle}{R} + o(t)\right)^{1/2}, \quad t \to 0^+.$$

Direct calculations yield

$$\mu\sqrt{t} \geq |U(y)|\left(\frac{\langle\beta, \alpha\rangle + o(1)}{R}\right)^{1/2}, \quad t \to 0^+,$$

which leads to a contradiction. We have finished the proof. □

Lemma 3.3. *Assume that Ω satisfies Condition ($\boldsymbol{\Omega_{ISC}}$) and $f(x) \in C(\Omega)$ satisfies $f_{\inf} > 0$. If U is a restricted viscosity super-solution to* (4) *in Ω, then U_{**} is also a restricted viscosity super-solution to the same problem in Ω.*

Proof. By [1, Lemma 4], we have $U_{**} = 0$ on $\partial\Omega$. To show that U_{**} is still a restricted viscosity super-solution to (4), we only need to verify that U_{**} is a viscosity super-solution of (3) in Ω. In terms of sub-jets, this property can be reformulated as,

$$\forall x \in \Omega,\ \forall (p, A) \in J^{2,-}_{\Omega} U_{**}(x) \implies \begin{cases} -\left\langle A\frac{p}{|p|}, \frac{p}{|p|}\right\rangle \geq \frac{|p|^2}{U_{**}} + \frac{f}{2U_{**}} & \text{if } p \neq 0 \\ -\lambda_{\min}(A) \geq \frac{f}{2U_{**}} & \text{if } p = 0. \end{cases} \tag{7}$$

Let $x \in \Omega$ and $(p, A) \in J_\Omega^{2,-} U_{**}(x)$, with $p \neq 0$ and A positive semidefinite. For every $\epsilon > 0$ small enough, we select points $x_1, \dots, x_k \in \Omega$, positive numbers $\lambda_1, \dots, \lambda_k$ with $\sum_{i=1}^k \lambda_i = 1$, and elements $(p, A_i) \in J_\Omega^{2,-} U(x_i)$, with A_i positive semidefinite. Then, we have

$$\sum_{i=1}^k \lambda_i x_i = x, \quad \sum_{i=1}^k \lambda_i U(x_i) = U_{**}(x), \quad A - \epsilon A^2 \leq \left(\sum_{i=1}^k \lambda_i A_i^{-1} \right)^{-1} =: B.$$

Recall that, we may without loss of generality assume the matrices $A, A_1, \cdots, A_k$ are positive definite, since the case of degenerate matrices can be addressed as in [1, pp. 273]. Since U is a super-solution to (4) in Ω, we have

$$-U(x_i) \leq \frac{1}{\langle A_i p, p \rangle} \left(|p|^4 + \frac{f}{2} |p|^2 \right).$$

Hence, one has

$$-\frac{1}{\sum_{i=1}^k \lambda_i U(x_i)} \left(|p|^4 + \frac{f}{2} |p|^2 \right) \geq \left(\sum_{i=1}^k \lambda_i \frac{1}{\langle A_i p, p \rangle} \right)^{-1}.$$

Then, exploiting the degenerate ellipticity of the operator and concavity of the mapping $Q \mapsto 1/\mathrm{tr}((p \otimes p) Q^{-1})$ (see [1]), we have

$$-\langle (A - \epsilon A^2) p, p \rangle - \frac{1}{U_{**}(x)} \left(|p|^4 + \frac{f(x)}{2} |p|^2 \right) \geq -\langle Bp, p \rangle - \frac{1}{\sum_{i=1}^k \lambda_i U(x_i)} \left(|p|^4 + \frac{f(x)}{2} |p|^2 \right) \geq 0.$$

On the other hand, if $(0, A) \in J_\Omega^{2,-} U_{**}(x)$, it is necessary to demonstrate that

$$\lambda_{\min}(A) \leq -\frac{f(x)}{2U_{**}(x)}.$$

In terms of test functions, this means

$$\psi \prec_x U_{**}, \ D\psi(x) = 0 \implies \lambda_{\min}(D^2 \psi(x)) \leq -\frac{f(x)}{2U_{**}(x)}.$$

Since U_{**} is convex, the conditions $\psi \prec_x U_{**}$ and $D\psi(x) = 0$ imply that x is a minimum point of U_{**}. In particular, $U(x_1) = \cdots = U(x_k) = U_{**}(x)$.

If $k = 1$, we have $U_{**}(x) = U(x)$ and $B = A_1$. Thus, $\lambda_{\min}(A - \epsilon A^2) \leq \lambda_{\min}(B) = \lambda_{\min}(A_1) \leq -f(x)/(2U(x))$.

If $k > 1$, we have that x is not a strict minimum point. Since x is the relative interior of the convex polyhedron with vertices $x_1, \cdots, x_k$, choosing $q := (x_1 - x)/|x_1 - x|$, we have that $U_{**}(x + tq)$ is constant for $|t|$ small enough. Thus, $\psi(x + tq) \leq U_{**}(x) = \psi(x)$ for $|t|$ small enough. Hence,

$$\lambda_{\min}(D^2 \psi(x)) \leq \langle D^2 \psi(x) q, q \rangle \leq 0 < -\frac{f(x)}{2U_{**}(x)}.$$

□

Proof of Theorem 1.1. First, we prove that the unique solution u to (1) is 1/2-concave if Ω satisfies Condition $(\boldsymbol{\Omega_{ISC}})$.

Let $U = -u^{1/2}$. By Lemma 3.1, U is a restricted super-solution to (6) in Ω. Then, U_{**} is a restricted super-solution to (6) in Ω by Lemma 3.3. By Lemma 3.1, the function $v := (U_{**})^2$ is a viscosity sub-solution to (1) in Ω.

On $\partial\Omega$, we have $u = 0$ and $U = 0$, which imply $U_{**} = 0$. Thus, $v = (U_{**})^2 = 0 = u$ on $\partial\Omega$. Since $-\Delta_\infty^N v \le f = -\Delta_\infty^N u$ in Ω, by the comparison principle in [23, Theorem 3.3], we deduce $v \le u$ in Ω, i.e. $(U_{**})^2 \le U^2$ in Ω. Since $U_{**} \le U$ (by the definition of convex envelope) and $U \le 0$ in Ω, we have $(U_{**})^2 \ge U^2$. Then $U = U_{**}$ is a convex function in Ω. That is, u is 1/2-concave in Ω.

Next, we show that the power-concavity of u remains true if Ω is any bounded convex domain.

For any $\varepsilon \in (0, 1]$, let Ω_ε denote the outer parallel body of Ω defined by

$$\Omega_\varepsilon := \{x \in \mathbb{R}^n : \operatorname{dist}(x, \Omega) < \varepsilon\},$$

and u_ε denote the viscosity solution to

$$\begin{cases} -\Delta_\infty^N u_\varepsilon = f_\varepsilon & \text{in } \Omega', \\ u_\varepsilon = 0 & \text{on } \partial\Omega', \end{cases}$$

where $\Omega \subset \Omega_\varepsilon \subseteq \Omega'$ and $\{f_\varepsilon\}$ is a sequence of continuous functions satisfying $0 < (f_\varepsilon)_{\inf} \le f_\varepsilon \le (f_\varepsilon)_{\sup} < +\infty$, which converges uniformly to f in Ω'. By Theorem 5.3 in [23], as $\varepsilon \to 0^+$, $u_\varepsilon \to u$ uniformly in Ω'.

Since Ω_ε satisfies an interior sphere condition of the radius ε, the function $u_\varepsilon^{1/2}$ is concave in Ω_ε. To show that $u^{1/2}$ is concave in $\overline{\Omega}$, we only need to prove that as $\varepsilon \to 0^+$, $u_\varepsilon \to u$ uniformly on $\partial\Omega$.

For any $y \in \partial\Omega$, take $x_\varepsilon \in \partial\Omega_\varepsilon$ satisfying $|x_\varepsilon - y| = \varepsilon$ and consider the polar quadratic polynomial

$$\eta(x) := \frac{1}{2}\operatorname{diam}(\Omega_\varepsilon)|x - x_\varepsilon| - \frac{(f_\varepsilon)_{\sup}}{2}|x - x_\varepsilon|^2.$$

Since $u_\varepsilon \le \eta$ on $\partial\Omega_\varepsilon$, by the comparison property with the polar quadratic polynomial [23, Theorem 2.2], we have $u_\varepsilon \le \eta$ in Ω_ε. In particular,

$$u_\varepsilon(y) \le \frac{\varepsilon}{2}\left(\operatorname{diam}(\Omega) + 1\right) - \frac{(f_\varepsilon)_{\sup}}{2}\varepsilon^2.$$

Thus, $u_\varepsilon|_{\partial\Omega}$ converges uniformly to 0. □

4. C^1-regularity

In this section, we establish the C^1-regularity of the viscosity solution to Problem (1) based on an upper estimate of semiconcave functions at singular points.

We first recall the definition of semiconcave function. The function $u \colon \Omega \to \mathbb{R}$ is referred to as *semiconcave (with constant C) in* Ω if

$$u(\lambda x + (1 - \lambda)y) \ge \lambda u(x) + (1 - \lambda)u(y) - C\frac{\lambda(1 - \lambda)}{2}|x - y|^2,$$

for all line segments $[x, y] \subset \Omega$ and every $\lambda \in [0, 1]$. Additionally, u is said to be *locally semiconcave in* Ω if it is semiconcave on any compact subset of Ω.

Next, we quote an estimate for locally semiconcave functions in the neighborhood of singular points, which will be used to establish the C^1-regularity.

For a function $u \in C(\Omega)$, let $\Sigma(u)$ denote the singular set of u, i.e., the set of points where u is not differentiable. At each $x_0 \in \Sigma(u)$, the super-differential of u at x_0 is defined by:

$$D^+u(x_0) := \left\{p \in \mathbb{R}^n : \limsup_{x \to x_0} \frac{u(x) - u(x_0) - \langle p, x - x_0\rangle}{|x - x_0|} \le 0\right\},$$

which is a non-empty compact convex set distinct from a singleton. In particular, $D^+u(x_0) \setminus \operatorname{extr} D^+u(x_0)$ is non-empty and contains non-zero elements, where $\operatorname{extr} D^+u(x_0)$ denotes the set of extreme points of the convex set $D^+u(x_0)$.

Lemma 4.1. *[11, Theorem 2] Let $u : \Omega \to \mathbb{R}$ be a locally semiconcave function. Fix $x_0 \in \Sigma(u)$ and let $p \in D^+u(x_0) \setminus \text{extr}D^+u(x_0)$. Assume that there exists $R > 0$ such that $\overline{B}_R(x_0) \subset \Omega$, and C is the semiconcavity constant of u on $\overline{B}_R(x_0)$. Then there exist a constant $K > 0$ and a unit vector $\zeta \in \mathbb{R}^n$ satisfying the following*

$$u(x) \leq u(x_0) + \langle p, x - x_0\rangle - K|\langle \zeta, x - x_0\rangle| + \frac{C}{2}|x - x_0|^2, \quad \forall x \in \overline{B}_R(x_0). \tag{8}$$

In particular, for any $c > 0$, setting $\delta := \min\{K/c, R\}$, we have

$$u(x) \leq u(x_0) + \langle p, x - x_0\rangle - c\langle \zeta, x - x_0\rangle^2 + \frac{C}{2}|x - x_0|^2, \quad \forall x \in \overline{B}_\delta(x_0). \tag{9}$$

Moreover, if $p \neq 0$, the vector ζ can be chosen such that $\langle \zeta, p\rangle \neq 0$.

Now we are ready to give the C^1-regularity.

Theorem 4.2. *Let Ω be a bounded convex domain of $\mathbb{R}^n$ and $f(x,t,p) \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ satisfy $0 < f_{\inf} \leq f \leq f_{\sup} < +\infty$. Suppose that $u \in C(\Omega)$ is a viscosity solution to $-\Delta_\infty^N u = f(x,u,Du)$ in Ω. If u enjoys local semiconcavity in Ω, then u is everywhere differentiable in Ω (thereby belonging to the C^1-class).*

Proof. Let $u \in C(\Omega)$ be a locally semiconcave viscosity solution to $-\Delta_\infty^N u = f(x,u,Du)$ in Ω. For the sake of contradiction, suppose that the singular set $\Sigma(u)$ is non-empty. Let $x_0 \in \Sigma(u)$. Take $p \in D^+u(x_0) \setminus \text{extr}D^+u(\Sigma(u))$ with $p \neq 0$. By Theorem 4.1, there exists a unit vector $\zeta \in \mathbb{R}^n$ satisfying $\langle \zeta, p\rangle \neq 0$, such that for any $c > 0$, there holds

$$u(x) \leq u(x_0) + \langle p, x - x_0\rangle - c\langle \zeta, x - x_0\rangle^2 + \frac{C}{2}|x - x_0|^2, \quad \forall x \in B_\delta(x_0),$$

where δ depending on c and C is the local semiconcavity constant of u in $B_\delta(x_0)$. Define

$$\varphi(x) := u(x_0) + \langle p, x - x_0\rangle - c\langle \zeta, x - x_0\rangle^2 + \frac{C}{2}|x - x_0|^2, \quad x \in B_\delta(x_0).$$

Obviously, $u \prec_{x_0} \varphi$. Since $D\varphi(x_0) = p \neq 0$ and u is a viscosity subsolution to $-\Delta_\infty^N u = f(x,u,Du)$ in Ω, we derive

$$-\Delta_\infty^+ \varphi(x_0) \leq f(x_0, u(x_0), D\varphi(x_0)) = f(x_0, u(x_0), p).$$

By direct computations, we have

$$\Delta_\infty^+ \varphi(x_0) = \frac{1}{|p|^2}\langle D^2\varphi(x_0)p, p\rangle = -2c\frac{\langle \zeta, p\rangle^2}{|p|^2} + C.$$

Clearly, choosing $c > \frac{1}{2}\frac{|p|^2}{\langle \zeta,p\rangle^2}(f_{\sup} + C)$ large enough, we get $-\Delta_\infty^+\varphi(x_0) > f(x_0, u(x_0), p)$, which contradicts the definition of the viscosity subsolution. That is, u has no singular points and is everywhere differentiable in Ω. By [7, Proposition 3.3.4], we have $u \in C^1(\Omega)$. □

Now we are ready to give the proof of Theorem 1.2.

Proof of Theorem 1.2. By Theorem 4.2, to prove $u \in C^1(\Omega)$, it is sufficient to prove that u is locally semiconcave in Ω.

Let K be an arbitrary compact convex domain of Ω and M denote the Lipschitz constant of $v := u^{1/2}$ on K. For any $x, y \in K$ and $\lambda \in [0,1]$, by Theorem 1.1, there holds

$$\begin{aligned}
&u(\lambda x + (1-\lambda)y) - \lambda u(x) - (1-\lambda)u(y) + M^2\lambda(1-\lambda)|x-y|^2 \\
&\geq \left|\lambda v(x) + (1-\lambda)v(y)\right|^2 - \lambda v(x)^2 - (1-\lambda)v(y)^2 + M^2\lambda(1-\lambda)|x-y|^2 \\
&= \lambda(1-\lambda)\left[M^2|x-y|^2 - \left|v(x) - v(y)\right|^2\right] \geq 0.
\end{aligned}$$

Thus, u is semiconcave in K with the semiconcavity constant of $C = 2M^2$. □

Remark 4.3. *Let Ω satisfy Condition ($\mathbf{\Omega_{ISC}}$). If u is the viscosity solution to Problem* (1)*, then u is locally semiconcave in Ω by the proof of Theorem 1.2. Indeed, recall that the function $U := -v = -u^{1/2}$ satisfies* (6)*. Let $y \in \Omega$ and $x_1 \in \partial\Omega \cap B_R(y)$. Choosing $x = x_1 + \lambda\nu$ with $U(x_1) = 0$ and $0 < \lambda < R$, we have*

$$\lim_{\lambda\to 0^+} \frac{U(x_1+\lambda\nu)-U(x_1)}{\lambda} \le \lim_{\lambda\to 0^+} \frac{U(y)}{\lambda}\left(\frac{\lambda}{R}\right)^{1/2} = -\infty.$$

That is, the normal derivative of U with respect to the external normal tends to $+\infty$ on $\partial\Omega$. Hence, $M \to +\infty$ as $K \to \Omega$.

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References

[1] O. Alvarez, J.M. Lasry, P. L. Lions, *Convex viscosity solutions and state constraints* (English, with English and French summaries), J. Math. Pures Appl. **76** (1997), no. 3, 265–288.

[2] S. N. Armstrong, C. K. Smart, *A finite difference approach to the infinity Laplace equation and tug-of-war games*, Trans. Amer. Math. Soc. **364** (2012), no. 2, 595–636.

[3] G. Aronsson, *Minimization problems for the functional* $\sup_x F(x, f(x), f'(x))$, Ark. Mat. **6** (1965), 33–53.

[4] G. Aronsson, *Minimization problems for the functional* $\sup_x F(x, f(x), f'(x))$*). II*, Ark. Mat. **6** (1966), 409–431.

[5] G. Aronsson, *Extension of functions satisfying Lipschitz conditions*, Ark. Mat. **6** (1967), 551–561.

[6] G. Aronsson, *Minimization problems for the functional* $\sup_x F(x, f(x), f'(x))$ *III*, Ark. Mat. **7** (1969), 509–512.

[7] P. Cannarsa, C. Sinestrari, *Semiconcave functions, Hamilton-Jacobi equations and optimal control*, Progress in Nonlinear Differential Equations and their Applications, 58, Birkhäuser Boston, Inc., Boston, MA, 2004.

[8] G. Cong, M. Esser, B. Parvin, G. Bebis, *Shape metamorphism using p-Laplace equation*, Proceedings of the 17th International Conference on Pattern Recognition **4** (2004), 15–18.

[9] M. G. Crandall, L. C. Evans, R. F. Gariepy, *Optimal Lipschitz extensions and the infinity Laplacian*, Calc. Var. Partial Differential Equations **13** (2001), no. 2, 123–139.

[10] M. G. Crandall, H. Ishii, P. L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N. S.) **27** (1992), 1–67.

[11] G. Crasta, I. Fragalà, *On the Dirichlet and Serrin problems for the inhomogeneous infinity Laplacian in convex domains: regularity and geometric results*, Arch. Ration. Mech. Anal. **218** (2015), no. 3, 1577–1607.

[12] G. Crasta, I. Fragalà, *A* C^1 *regularity result for the inhomogeneous normalized infinity Laplacian*, Proc. Amer. Math. Soc. **144** (2016), no. 6, 2547–2558.

[13] L. C. Evans, W. Gangbo, *Differential equations methods for the Monge-Kantorovich mass transfer problem*, Mem. Amer. Math. Soc. **137** (1999), no. 653, viii+66.

[14] L. C. Evans, O. Savin, $C^{1,\alpha}$ *regularity for infinity harmonic functions in two dimensions*, Calc. Var. Partial Differential Equations **32** (2008), no. 3, 325–347.

[15] L. C. Evans, C. K. Smart, *Everywhere differentiability of infinity harmonic functions*, Calc. Var. Partial Differential Equations **42** (2011), no. 1-2, 289–299.

[16] R. Jensen, *Uniqueness of Lipschitz extensions: Minimizing the sup norm of the gradient*, Arch. Ration. Mech. Anal. **123** (1993), no. 1, 51–74.

[17] P. Juutinen, *Concavity maximum principle for viscosity solutions of singular equations*, NoDEA Nonlinear Differential Equations Appl. **17** (2010), no. 5, 601–618.

[18] H. Koch, Y. Zhang, Y. Zhou, *An asymptotic sharp Sobolev regularity for planar infinity harmonic functions*, J. Math. Pures Appl. **132** (2019), no.9, 457–482.

[19] H. Koch, Y. Zhang, Y. Zhou, *Some sharp Sobolev regularity for inhomogeneous infinity Laplace equation in plane*, J. Math. Pures Appl. **132** (2019), no.9, 483–521.

[20] R. V. Kohn, S. Serfaty, *A deterministic-control-based approach to motion by curvature*, Comm. Pure Appl. Math. **59** (2006), no. 3, 344–407.

[21] E. Lindgren, *On the regularity of solutions of the inhomogeneous infinity Laplace equation*, Proc. Amer. Math. Soc. **142** (2014), no. 1, 277–288.

[22] G. Lu, Q. Miao, Y. Zhou, *Viscosity solutions to inhomogeneous Aronsson's equations involving Hamiltonians* $\langle A(x)p, p\rangle$, Calc. Var. Partial Differential Equations **58** (2019), no. 1, 1–37.

[23] G. Lu, P. Wang, *A PDE perspective of the normalized infinity Laplacian*, Comm. Partial Differential Equations **33** (2008), no. 10–12, 1788-1817.

[24] G. Lu, P. Wang, *Inhomogeneous infinity Laplace equation*, Advances in Math. **217** (2008), no. 4, 1838–1868.

[25] Y. Peres, O. Schramm, S. Sheffield, D. B. Wilson, *Tug-of-war and the infinity Laplacian*, J. Amer. Math. Soc. **22** (2009), no. 1, 167–210.

[26] O. Savin, C^1 *regularity for infinity harmonic functions in two dimensions*, Arch. Ration. Mech. Anal. **176** (2005), no. 3, 351–361.