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A note on the structure of interpolative metric spaces

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Abstract. In this note we observe that interpolative metric spaces lie between strong b-metric spaces and b-metric spaces. This paper aims to comprehend the connections of these notions and the corresponding structures, namely, interpolative metric spaces, strong b-metric spaces, and b-metric spaces. An example is considered to illustrate our claims.

1. Introduction and Preliminaries

In the history of mathematics, the notion of distance is one of the earliest observed concepts. In modern mathematics, the notion of distance is abstracted under the concept of a metric [19, 20]. It would not be an exaggeration to say that the concept of metric constitutes the fundamental dynamic of modern mathematics. In fact, it has been used not only in mathematics but also to solve problems in other mathematically expressible sciences and disciplines; for example, the concept of metrics has been used extensively in imaging problems and computer science. For this reason, the concept of metrics has attracted the attention of many researchers. This concept has been explored from many different approaches and perspectives, and attempts have been made to generalize, expand, and improve it. In the literature, there are several variants of metrics that can be found. For instance, quasi-metric [36, 42–44, 47, 48, 50], ultra-metric [49], semi-metric [35, 53], bipolar metric [30], modular metric [14, 39], fuzzy metric [45], b-metric [8, 9, 16, 41, 46], strong b-metric [34], partial metric [22, 37], 2-metric [21], D-metric [17], G-metric [38], S-metric [51], A-metric [1], cone metric (Banach-valued metric) [18], b-cone metric [31], TVS-valued metric [23], complex-valued metric [4, 7], C^*-algebra valued metric [5], quaternion-valued metric [2], generalized metric [33], Branciari distance function [13], supra-metric [11, 12], super metric [32], interpolative metric [26, 29], and many others, see e.g. [10, 25, 27]. It should be noted that this list is not complete.

In this paper, we shall restrict ourselves to considering the following three types of metrics: the b-metrics, the strong b-metrics, and the interpolative metrics. We aim to give the relations between these structures.

First, we recall the definitions of the b-metric space, the strong b-metric space, and the interpolative metric space. The notion of b-metric was considered by several researchers, such as Bakhtin [8], Czerwik

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[16], and Berinde [9], among others. It is worth noting that Berinde referred to this abstract space as "quasi-metric"; however, today the concept of quasi-metric is used to describe the structure that emerges by removing the "symmetry condition" from the metric axioms.

Definition 1.1 (*b*-metric). *A b-metric on a (nonempty) set X is a function* $d : X \times X \to [0, +\infty)$ *satisfying the following conditions:*

(b_1) $d(x, y) = 0$ *if and only if* $x = y$, *for all* $x, y \in X$,
(b_2) $d(x, y) = d(y, x)$, *for all* $x, y \in X$,
(b_3) there exists a constant $K \geq 1$ *such that:*

$$d(x, y) \leq K[d(x, z) + d(z, y)],$$

for all $x, y, z \in X$.

Then (X, d) *is called a b-metric space with coefficient K.*

It is a clear generalization and extension of the standard metric. Indeed, *b*-metric coincides with the standard metric for $K = 1$. One of the classical examples of *b*-metric is the following one.

Example 1.2. *Let* $X = C([0, 1], \mathbb{R})$, *the space of continuous real-valued functions on* $[0, 1]$. *Define:*

$$d(f, g) = \left(\int_0^1 |f(t) - g(t)|^2 dt \right)^{\frac{1}{2}}.$$

Then (X, d) *is a b-metric space with* $K = \sqrt{2}$.

It is well known that each *b*-metric d on a set X induces a topology on X where, as in the metric case, a subset A of X is declared open if for each $x \in A$ there is $r > 0$ such that $B(x, r) \subseteq A$, where $B(x, r) = \{y \in X : d(x, y) < r\}$. Although there exist examples of *b*-metrics d with "open" balls $B(x, r)$ that are not open sets for the topology induced by d (see, e.g., [6, 40]), the following equivalence remains valid for any *b*-metric d on a set X: a sequence $(x_n)_{n\in\mathbb{N}}$ in X is convergent to $x \in X$ for the topology induced by d if and only if $d(x, x_n) \to 0$ as $n \to +\infty$.

As in the metric case, a *b*-metric d on a set X is continuous provided that if $x, y \in X$ and $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are sequences in X such that $d(x, x_n) \to 0$ and $d(y, y_n) \to 0$ as $n \to +\infty$, then, $d(x_n, y_n) \to d(x, y)$ as $n \to +\infty$. Again, the aforementioned examples from [6, 40] show the existence of *b*-metrics that are not continuous.

The concept of a *b*-metric has been extensively investigated by several authors. In this note, we restrict ourselves to the most interesting extension of the *b*-metric, namely, the strong *b*-metric, defined by Kirk and Shahzad [34]. It should be underline that strong *b*-metrics were introduced in order to obtain a *b*-metric structure where "open" balls were open sets and the *b*-metric was a continuous function

Definition 1.3 (Strong *b*-metric). *A strong b-metric on a (nonempty) set X is a function* $d : X \times X \to [0, +\infty)$ *satisfying the following conditions:*

(sb_1) $d(x, y) = 0$ *if and only if* $x = y$, *for all* $x, y \in X$,
(sb_2) $d(x, y) = d(y, x)$, *for all* $x, y \in X$,
(sb_3) there exists a constant $K \geq 1$ *such that:*

$$d(x, y) \leq d(x, z) + Kd(z, y),$$

for all $x, y, z \in X$.

Then (X, d) *is called a strong b-metric space with coefficient K.*

Example 1.4. *Let $X = \{a, b, c\}$, the function $D : X \times X \to [0, \infty)$ be defined by*

$$D(a,a) = D(b,b) = D(c,c) = 0,$$

$$D(a,b) = D(b,a) = 2,$$

$$D(b,c) = D(c,b) = 1,$$

$$D(a,c) = D(c,a) = 6,$$

It is clear that (X, D) is a strong b-metric with $K = 4$.

Very recently, another generalization of the standard metric space, an interpolative metric space, was given in [24, 26, 29] in which the authors propose a new abstract structure for getting a "better" results.

Definition 1.5. *[26, 29] An (α, c)-interpolative metric (or, simply, an interpolative metric if no confusion arises) on a (nonempty) set X is a function $d : X \times X \to [0, +\infty)$ satisfying the following conditions:*

($m1$) *$d(x, y) = 0$, if and only if, $x = y$, for all $x, y \in X$,*
($m2$) *$d(x, y) = d(y, x)$, for all $x, y \in X$,*
($m3$) *there exist constants $\alpha \in (0, 1)$ and $c \geq 0$ such that:*

$$d(x,y) \leq d(x,z) + d(z,y) + c\left[(d(x,z))^{\alpha}\,(d(z,y))^{1-\alpha}\right],$$

for all $x, y, z \in X$.

Then, we call (X, d) an (α, c)-interpolative metric space (or, simply, an interpolative metric space if non confusion arises).

It is evident from [28] that interpolative metric spaces lie in b-metric spaces.

Example 1.6. *Let (X, ρ) be a standard metric space. Define a function $d : X \times X \to [0, +\infty)$ as follows*

$$d(x, y) := \rho(x, y)(\rho(x, y) + A),$$

where $A > 0$. Since ρ is a metric on X, the conditions ($m1$) and ($m2$) are straightforward. For ($m3$), it is enough to consider $c \geq 2$ for any $\alpha \in (0, 1)$. Thus, (X, d) is $(\frac{1}{2}, 2)$-interpolative metric space.
Indeed, we have

$$\begin{aligned}
d(x,y) &= \rho(x,y)(\rho(x,y)+A)\\
&\leq (\rho(x,z)+\rho(z,y))(\rho(x,z)+\rho(z,y)+A)\\
&\leq (\rho(x,z)+\rho(z,y))(\rho(x,z)+\rho(z,y)+A)\\
&\leq [\rho(x,z)(\rho(x,z)+A)+\rho(x,z)\rho(z,y)] + [\rho(z,y)(\rho(z,y)+A)+\rho(z,y)\rho(x,z)]\\
&\leq [\rho(x,z)(\rho(x,z)+A)] + [\rho(z,y)(\rho(z,y)+A)] + 2\rho(x,z)\rho(z,y)\\
&\leq d(x,z)+d(z,y)+2\,(\rho(x,z))^{\frac{1}{2}}\,(\rho(x,z))^{\frac{1}{2}}\,(\rho(z,y))^{\frac{1}{2}}\,(\rho(z,y))^{\frac{1}{2}}\\
&\leq d(x,z)+d(z,y)+2\,(\rho(x,z))^{\frac{1}{2}}\,[\rho(x,z)+A]^{\frac{1}{2}}\,(\rho(z,y))^{\frac{1}{2}}\,[\rho(z,y)+A]^{\frac{1}{2}}\\
&\leq d(x,z)+d(z,y)+2(d(x,z))^{\frac{1}{2}}(d(z,y))^{\frac{1}{2}}
\end{aligned}$$

It is clear that the function d does not form a metric. Note also that the above estimation for the pair $(\frac{1}{2}, 2)$ is very rough, and it can be improved in several ways.

The main goal of this note is to demonstrate that an interpolative metric forms a continuous function. Consequently, and taking into account Example 2.1 below, interpolative metric spaces constitute a real generalization of strong b-metric spaces that preserve the properties of openness of open balls and continuity of the function distance.

2. Main Results

We shall start this section with an illustrative example to indicate that an interpolative metric is not a strong b-metric:

Example 2.1. *Let $X = \mathbb{N} \cup \{0\}$ and let ρ be the metric on X given by $\rho(x, x) = 0$ for all $x \in X$, and $\rho(x, y) = x + y$ whenever $x \neq y$.*

Then, the function d on $X \times X$ given by $d(x, y) = \rho(x, y)(\rho(x, y) + 1)$ for all $x, y \in X$, is an interpolative metric on X (see e.g. Example 1.6) which is not a strong b-metric on X. Indeed, for $K \geq 1$ fixed choose $x = 1$, $y \in \mathbb{N}$ with $y > K$ and $z = 0$. Then, we have

$$d(x, y) = (y + 1)(y + 2) > 2K + y(y + 1) = Kd(x, z) + d(z, y).$$

Next we show that every strong b-metric on a set X is an interpolative metric on X.

Proposition 2.2. *Let d is a strong b-metric on a set X with coefficient K. Then d is a $(\frac{1}{2}, K - 1)$-interpolative metric on X.*

Proof. By assumption, we have $d(x, y) \leq Kd(x, z) + d(z, y)$, for all $x, y, z \in X$.

Suppose that $d(x, z) \leq d(z, y)$. Then,

$$\begin{aligned} d(x, y) &\leq d(x, z) + d(z, y) + (K - 1)d(x, z) \\ &\leq d(x, z) + d(z, y) + (K - 1)(d(x, z))^{\frac{1}{2}}(d(z, y))^{\frac{1}{2}}, \end{aligned}$$

and if $d(x, z) > d(z, y)$, we get

$$\begin{aligned} d(x, y) &\leq d(z, y) + d(x, z) + (K - 1)d(z, y) \\ &\leq d(x, z) + d(z, y) + (K - 1)(d(x, z))^{\frac{1}{2}}(d(z, y))^{\frac{1}{2}}. \end{aligned}$$

□

Although the following fact is essentially known we will established it with proof for the sake of completeness.

Proposition 2.3. *Let d be an (α, c)-interpolative metric on a set X. Then, d is a b-metric on X with coefficient $c + 1$.*

Proof. We need to consider two cases: If $d(x, z) \geq d(y, z)$, the relaxed triangle inequality $(m3)$ turns into

$$d(x, y) \leq d(x, z) + d(z, y) + c\left[(d(x, z))^{\alpha}(d(x, z))^{1-\alpha}\right] = (c + 1)d(x, z) + d(z, y).$$

Analogously, if $d(x, z) < d(y, z)$, we get, the relaxed triangle inequality $(m3)$ turns into

$$d(x, y) \leq d(x, z) + d(z, y) + c\left[(d(y, z))^{\alpha}(d(y, z))^{1-\alpha}\right] = (c + 1)d(y, z) + d(x, z).$$

□

We finish this section by showing that for every interpolative metric d on a set X the "open" balls are open sets in the topology induced by d, and that d is continuous.

Proposition 2.4. *Let d be an (α, c)-interpolative metric on a set X. Then, for every $x \in X$ and $r > 0$ the "open" ball $B(x, r)$ is an open set in the topology induced by d.*

Proof. Let $x \in X$, $r > 0$ and $y \in B(x,r)$. Then $d(x,y) = s < r$. Choose $M > 1$ (sufficiently large) verifying $cr < (M^{1-\alpha} - 1)(r-s)$. We show that $B(y,\delta) \subseteq B(x,r)$, where $\delta = \frac{r-s}{M}$.

Indeed, if $z \in B(y,\delta)$, we have

$$\begin{aligned} d(x,y) &\leq d(x,z) + d(/z,y) + cd(x,z)d(z,y) \\ &< s + \frac{r-s}{M} + c(r^{\alpha}\frac{(r-s)^{1-\alpha}}{M^{1-\alpha}}) \\ &< s + \frac{r-s}{M^{1-\alpha}} + \frac{(M^{1-\alpha}-1)(r-s)}{M^{1-\alpha}} \\ &= s + r - s = r. \end{aligned}$$

□

Lemma 2.5. *(See.g., [15, Remark 2.5]). Let d be a b-metric on a set X and let $x \in X$. Then, $B(x,r)$ is open for all $r > 0$ if and only if the function $d(x,\cdot)$ is upper semicontinuous.*

Proposition 2.6. *Every interpolative metric on a set X is a continuous function.*

Proof. Let d be an (α,c)-interpolative metric on a set X. Let $x,y \in X$ and $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences in X such that $d(x,x_n) \to 0$ and $d(y,y_n) \to 0$ as $n \to \infty$.

Fix $\varepsilon \in (0,1)$. We show that there is $n_\varepsilon \in \mathbb{N}$ such that $\left|d(x,y) - d(x_n,y_n)\right| < \varepsilon$ for all $n \geq n_\varepsilon$.

Choose a $\delta > 0$ verifying $\max\{\delta^{\alpha},\delta^{1-\alpha}\} < \varepsilon/4$, and $c\delta^{\alpha}((d(x,y)+1)) < \varepsilon/4$.

There exists $n_\varepsilon \in \mathbb{N}$ satisfying the following relations for each $n \geq n_\varepsilon$, $d(x,x_n) < \delta$, $d(y,y_n) < \delta$, and by Proposition 2.4 and Lemma 2.5, $d(x,y_n) < d(x,y) + \delta$ and $d(y,x_n) < d(y,x) + \delta$.

Then,

$$\begin{aligned} d(x_n,y_n) &\leq d(x_n,x) + d(x,y_n) + c[(d(x_n,x))^{\alpha}(d(x,y_n))^{1-\alpha}] \\ &< 2\delta + d(x,y) + c\delta^{\alpha}(d(x,y)+\delta)^{1-\alpha} \\ &< \varepsilon + d(x,y), \end{aligned}$$

for all $n \geq n_\varepsilon$. We also have

$$\begin{aligned} d(y,x_n) &\leq d(y,y_n) + d(y_n,x_n) + c[(d(y,y_n))^{\alpha}(d(y_n,x_n))^{1-\alpha}] \\ &< d(y_n,x_n) + \delta + c[\delta^{\alpha}(\varepsilon + d(x,y))^{1-\alpha}] \\ &< d(y_n,x_n) + \delta + \varepsilon/4, \end{aligned}$$

for all $n \geq n_\varepsilon$. Consequently,

$$\begin{aligned} d(x,y) &\leq d(x,x_n) + d(x_n,y) + c[(d(x,x_n))^{\alpha}(d(x_n,y))^{1-\alpha}] \\ &< \delta + d(x_n,y_n) + \delta + \varepsilon/4 + c[\delta^{\alpha}(d(x,y)+\delta)^{1-\alpha}] \\ &< \varepsilon + d(x_n,y_n), \end{aligned}$$

for all $n \geq n_\varepsilon$. This concludes the proof. □

3. Further remarks

We conclude this note by emphasizing the need to be cautious and careful when attempting to generalize or extend the concept of interpolative metric.

Specifically, if we define a symmetric b-distance on a set X as a function $d : X \times X \to [0,+\infty)$ satisfying the following two conditions:

(sbd_1) $d(x,y) = d(y,x)$, for all $x,y \in X$,

(sbd_2) there exists a constant $K \geq 1$ such that $d(x,y) \leq K[d(x,z) + d(z,y)]$, for all $x,y,z \in X$,

then, it is tempting and natural to try to define and address the following notion:

A (K,α,c)-interpolative symmetric b-distance (or simply an interpolative symmetric b-distance if no confusion arises) on a set X is a function $d : X \times X \to [0,+\infty)$ satisfying the following two conditions:

(isbd_1) $d(x,y) = d(y,x)$, for all $x,y \in X$,

(isbd_2) there exist constants $K \geq 1$, $\alpha \in (0,1)$ and $c \geq 0$ such that

$$d(x,y) \leq K[d(x,z) + d(z,y)] + c[(d(x,z))^{\alpha}(d(z,y))^{1-\alpha}],$$

for all $x,y,z \in X$.

Obviously, every symmetric b-distance, with coefficient K, is a $(K,\alpha,0)$-interpolative symmetric b-distance. We also have the following converse (compare Proposition 2.3).

Proposition 3.1. *Let d be a (K,α,c)-interpolative symmetric b-distance on a set X. Then, d is a symmetric b-distance on X with coefficient $K + c$.*

Proof. For each $x,y,z \in X$ we have

$$\begin{aligned} d(x,y) &\leq K[d(x,z) + d(z,y)] + c[(d(x,z))^{\alpha}(d(z,y))^{1-\alpha}] \\ &\leq K[d(x,z) + d(z,y)] + c[d(x,z) + d(z,y)] \\ &= (K+c)[d(x,z) + d(z,y)]. \end{aligned}$$

Hence, d is a symmetric b-distance on X with coefficient $K + c$. □

From the preceding result it follows that the class of interpolative symmetric b-distances on a set X coincides with the class of symmetric b-distances on X.

Particular cases:

1. If d is a (K,α,c)-interpolative symmetric b-distance on a set X verifying that

$d(x,y) = 0$ if and only if $x = y$,

we will refer to d as a (K,α,c)-interpolative b-metric on X. Then, it follows from Proposition 3.1 that d is a b-metric on X with coefficient $K + c$.

Therefore, the class of interpolative b-metrics coincides with the class of b-metrics on any set X.

2. In [52] it was introduced the notion of a partial b-metric as a simultaneous generalization of the notion of a b-metric and Matthew's notion of a partial metric [37].

Let us recall that a partial b-metric on a set X is a symmetric b-distance on X verifying the following three conditions:

(pm_1) $d(x,y) = d(x,x) = d(y,y)$ if and only if $x = y$, for all $x,y \in X$,

(pm_2) $d(x,x) \leq d(x,y)$, for all $x,y \in X$,

(pm_3) there is a constant $K \geq 1$ such that $d(x,y) + d(z,z) \leq K[d(x,z) + d(z,y)]$, for all $x,y,z \in X$.

A partial b-metric space is a pair (X,d) such that X is a set and d is a partial b-metric on X.

If one defines a (K,α,c)-interpolative partial b-metric on a set X as a (K,α,c)-interpolative symmetric b-distance d on X that satisfies conditions (pm_1), (pm_2) and (pm_3), we deduce from Proposition 3.1 that d is a partial b-metric on X, with coefficient $K + c$.

Therefore, the class of interpolative partial b-metrics coincides with the class of partial b-metrics on any set X.

3. In [3] it was introduced the notion of a b-metric like as a simultaneous generalization of the notion of a b-metric and the notion of a metric-like.

Let us recall that a b-metric-like on a set X is a symmetric b-distance d on X verifying

(bml_0) $d(x,y) = 0 \Rightarrow x = y$.

A b-metric-like space is a pair (X,d) such that X is a set and d is a b-metric-like on X.

Obviously, every partial b-metric, and hence every b-metric, on a set X is a b-metric-like on X.

If one defines a (K,α,c)-interpolative b-metric-like on a set X as a (K,α,c)-interpolative symmetric b-distance d on X that satisfies condition (bml_0), we deduce from Proposition 3.1 that d is a b-metric-like on X, with coefficient $K + c$.

Therefore, the class of interpolative b-metrics-like coincides with the class of b-metrics-like on any set X.

4. Conclusion

The notion of metric is a powerful and important tool not only for mathematics but also for several quantitative sciences. Regarding its importance and potential in solving several problems in distinct disciplines, it has been improved and generalized in different ways. One of the most interesting examples of this trend was the notion of b-metric. On the other hand, b-metric has a very significant weakness: Not all b-metric are continuous. This gap was removed in the notion of strong b-metric. In this note we indicate that there is a proper class between b-metric and strong b-metric. In conclusion, we have the following inclusions:

$$\text{metric} \subsetneq \text{ strong } b\text{-metric} \subsetneq \text{ interpolative metric} \subsetneq b\text{-metric}$$

Keeping the huge application potential of the distance notions in mind, such a continuous interpolative metric will play a crucial role in applied sciences.

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