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Multiplicatively hyperbolic type convex functions and some related integral inequalities

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Abstract. In this work, we introduced and explored the concept of multiplicatively hyperbolic type convex functions, examining several of their fundamental algebraic properties. We established Hermite–Hadamard (HH) inequalities for this newly defined class of functions and further derived novel HH type inequalities for both the product and quotient of multiplicatively hyperbolic type convex functions. Furthermore, we obtained new multiplicative integral-based inequalities involving the product and quotient of multiplicatively hyperbolic type convex functions and classical convex functions. In addition, by utilizing the Hölder–İşcan integral inequality, we developed several new HH type integral inequalities specifically for multiplicatively hyperbolic type convex functions. Finally, we conducted a comparative analysis with existing results, demonstrating that our findings offer notable improvements over previously known inequalities. The results of this study have the potential to inspire further research in various scientific disciplines.

1. Introduction and preliminaries

The function $\Theta : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if the following inequality holds for all $\varkappa, y \in I$ and $\omega \in [0,1]$:

$$\Theta(\omega\varkappa + (1-\omega)y) \leq \omega\Theta(\varkappa) + (1-\omega)\Theta(y).$$

The function Θ is said to be concave if $-\Theta$ is convex.

Convexity theory has been instrumental in the development of inequality theory. Numerous prominent findings in the realm of inequalities can be obtained by utilizing the convex properties of functions (see [6, 14, 17, 20, 27, 34, 35]). A widely researched outcome related to convex functions is the Hermite–Hadamard inequality ([11]). This inequality establishes a key relationship between convexity and inequalities, providing significant understanding into the behavior of convex functions.

The HH inequality states that if a function $\Theta : I \to \mathbb{R}$ is convex in I for $\nu_1, \nu_2 \in I$ with $\nu_1 < \nu_2$ and $\Theta \in L[\nu_1, \nu_2]$, then the inequality

$$\Theta\left(\frac{\nu_1+\nu_2}{2}\right) \leq \frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2}\Theta(\varkappa)d\varkappa \leq \frac{\Theta(\nu_1)+\Theta(\nu_2)}{2}, \tag{1}$$

holds.

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Many scholars have shown a keen interest in inequality (1), prompting the introduction of numerous generalizations, extensions, and enhancements in academic works. Those looking for more in-depth information can consult [9, 31] and the references included therein. Furthermore, various refinements of the HH inequality for convex functions have been documented in [7, 18, 21, 23, 26, 28, 29, 32].

Now, we will give some basic definitions and results.

In [36], Toplu et al. introduced the class of hyperbolic type convex functions and derived HH inequalities for this class of functions as follows:

Definition 1.1. *The function $\Theta : I \subset \mathbb{R} \to \mathbb{R}$ is said to be hyperbolic type convex, if*

$$\Theta(\omega\varkappa + (1-\omega)y) \leq \frac{\sinh\omega}{\sinh 1}\Theta(\varkappa) + \frac{\sinh 1 - \sinh\omega}{\sinh 1}\Theta(y),$$

for all $\varkappa, y \in I$ and $\omega \in [0,1]$.

In [1], Ali et al. gave the definition of multiplicatively convex function and HH inequality for this class of functions:

Definition 1.2. *A function $\Theta : I \to (0,\infty)$ is said to be multiplicatively or log convex, if*

$$\Theta(\omega\varkappa + (1-t)\omega) \leq [\Theta(\varkappa)]^{\omega}[\Theta(y)]^{1-\omega},$$

for all $\nu_1, \nu_2 \in I$ and $\omega \in [0,1]$.

Theorem 1.3. *Let Θ be a positive and multiplicatively convex function on interval $[\nu_1, \nu_2]$. Then*

$$\Theta\left(\frac{\nu_1+\nu_2}{2}\right) \leq \left(\int_{\nu_1}^{\nu_2} (\Theta(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}} \leq G\left(\Theta(\nu_1), \Theta(\nu_2)\right),$$

where $G(.,.)$ is the geometric mean.

In [12], İşcan established the Hölder-İşcan integral inequality which is a refinement of the Hölder's integral inequality as follows:

Theorem 1.4. *Let $p > 1$ and $1/p + 1/q = 1$. If f and g are real functions defined on interval $[\nu_1, \nu_2]$ and $|\Theta|^p, |\Upsilon|^q$ are integrable functions on $[\nu_1, \nu_2]$ then*

$$\begin{aligned}\int_{\nu_1}^{\nu_2} |\Theta(\varkappa)\Upsilon(\varkappa)|\,d\varkappa \leq\ & \frac{1}{\nu_2-\nu_1}\left\{\left(\int_{\nu_1}^{\nu_2} (\nu_2-\varkappa)|\Theta(\varkappa)|^p\,d\varkappa\right)^{1/p}\left(\int_{\nu_1}^{\nu_2} (\nu_2-\varkappa)|\Upsilon(\varkappa)|^q\,d\varkappa\right)^{1/q}\right.\\ & \left. + \left(\int_{\nu_1}^{\nu_2} (\varkappa-\nu_1)|\Theta(\varkappa)|^p\,d\varkappa\right)^{1/p}\left(\int_{\nu_1}^{\nu_2} (\varkappa-\nu_1)|\Upsilon(\varkappa)|^q\,d\varkappa\right)^{1/q}\right\}.\end{aligned}$$

In [13], Kadakal et al. gave a refinement of power-mean integral inequality which is called improved power-mean integral inequality as follows:

Theorem 1.5. *Let $q \geq 1$. If f and g are real functions defined on interval $[\nu_1, \nu_2]$ and if $|\Theta|, |\Theta||\Upsilon|^q$ are integrable functions on $[\nu_1, \nu_2]$ then*

$$\begin{aligned}\int_{\nu_1}^{\nu_2} |\Theta(\varkappa)\Upsilon(\varkappa)|\,d\varkappa \leq\ & \frac{1}{\nu_2-\nu_1}\left\{\left(\int_{\nu_1}^{\nu_2} (\nu_2-\varkappa)|\Theta(\varkappa)|\,d\varkappa\right)^{1-1/q}\left(\int_{\nu_1}^{\nu_2} (\nu_2-\varkappa)|\Theta(\varkappa)||\Upsilon(\varkappa)|^q\,d\varkappa\right)^{1/q}\right.\\ & \left. + \left(\int_{\nu_1}^{\nu_2} (\varkappa-\nu_1)|\Theta(\varkappa)|\,d\varkappa\right)^{1-1/q}\left(\int_{\nu_1}^{\nu_2} (\varkappa-\nu_1)|\Theta(\varkappa)||\Upsilon(\varkappa)|^q\,d\varkappa\right)^{1/q}\right\}.\end{aligned}$$

2. Multiplicative calculus

In the 17th century, Isaac Newton and Gottfried Wilhelm Leibniz independently developed the foundations of differential and integral calculus. These revolutionary contributions have since become cornerstones in the fields of analysis and calculus. The core processes of differentiation and integration allow for the manipulation of quantities at an infinitesimally small scale.

In a notable shift from traditional methods, a new form of calculus was introduced between 1967 and 1970 by Grossman and Katz. This innovative system redefined the conventional concepts of addition and subtraction, replacing them with division and multiplication as the primary operations. Known as "multiplicative calculus," this framework is specifically designed for positive functions. Due to its limited applicability, multiplicative calculus has not achieved the same widespread recognition as the calculus developed by Leibniz and Newton. However, it has produced fascinating results in various areas.

The groundwork for multiplicative calculus was first established by [10] in the 1970s, as they revisited and adapted the classical calculus formulated by Newton and Leibniz. Although its application is restricted to positive functions, multiplicative calculus has demonstrated significant potential, as seen in its diverse uses.

For example, in [3], Bashirov et al. presented a key theorem that serves as the foundation of multiplicative calculus. Additionally, Bashirov and Riza pioneered the idea of complex multiplicative calculus in [4]. The study of stochastic multiplicative calculus was explored in [8], where various properties were investigated. For a detailed examination of the applications and scope of this field, readers can consult [2, 5, 15, 16, 22, 24, 25, 30, 33, 37] and the associated references.

2.1. Multiplicative derivatives and integrals

Recall multiplicative derivative which can be found in [3].

Definition 2.1. *Assume that $\Theta : \mathbb{R} \to \mathbb{R}^+$ is a positive function. The multiplicative derivative of the function Θ is stated as*

$$\frac{d^*\Theta}{d\omega}(\omega) = \Theta^*(\omega) = \lim_{k\to 0}\left(\frac{\Theta(\omega+k)}{\Theta(\omega)}\right)^{\frac{1}{k}}.$$

Recall that the concept of multiplicative integral or * integral is denoted by $\int_{\nu_1}^{\nu_2}(\Theta(\varkappa))^{d\varkappa}$ while the ordinary integral is denoted by $\int_{\nu_1}^{\nu_2}(\Theta(\varkappa))d\varkappa$. This is because the sum of the terms of product is used in the definition of a classical Riemann integral of Θ on $[\nu_1, \nu_2]$, the product of terms raised to certain powers is used in the definition of * integral of Θ on $[\nu_1, \nu_2]$. The following properties of *differentiable exists:

Theorem 2.2. *Let Θ and Υ be *differentiable functions. If α is arbitrary constant, then functions $\alpha\Theta$, $\Theta\Upsilon$, $\Theta+\Upsilon$, Θ/Υ and Θ^{Υ} are *differentiable and*

1. $(\alpha\Theta)^*(\omega) = \alpha\Theta^*(\omega)$,
2. $(\Theta\Upsilon)^*(\omega) = \Theta^*(\omega)\Upsilon^*(\omega)$,
3. $(\Theta+\Upsilon)^*(\omega) = \Theta^*(\omega)^{\frac{\Theta(\omega)}{\Theta(\omega)+\Upsilon(\omega)}}\Upsilon^*(\omega)^{\frac{\Upsilon(\omega)}{\Theta(\omega)+\Upsilon(\omega)}}$,
4. $\left(\frac{\Theta}{\Upsilon}\right)^*(\omega) = \frac{\Theta^*(\omega)}{\Upsilon^*(\omega)}$,
5. $\left(\Theta^{\Upsilon}\right)^*(\omega) = \Theta^*(\omega)^{\Upsilon(\omega)}\Theta(\omega)^{\Upsilon'(\omega)}$.

There is the following relation between Riemann integral and * integral ([3]):

Proposition 2.3. *If Θ is positive and Riemann integrable on $[\nu_1, \nu_2]$, then Θ is * integrable on $[\nu_1, \nu_2]$ and*

$$\int_{\nu_1}^{\nu_2}(\Theta(\varkappa))^{d\varkappa} = e^{\int_{\nu_1}^{\nu_2}(\Theta(\varkappa))d\varkappa}.$$

In [3], Bashirov et al. show that * integral has the following results and notations:

Proposition 2.4. *If Θ is positive and Riemann integrable on $[\nu_1,\nu_2]$, then Θ is multiplicatively integrable on $[\nu_1,\nu_2]$ and*

1. $\int_{\nu_1}^{\nu_2}((\Theta(\varkappa))^p)^{d\varkappa}=\int_{\nu_1}^{\nu_2}\left((\Theta(\varkappa))^{d\varkappa}\right)^p$,
2. $\int_{\nu_1}^{\nu_2}(\Theta(\varkappa)\Upsilon(\varkappa))^{d\varkappa}=\int_{\nu_1}^{\nu_2}(\Theta(\varkappa))^{d\varkappa}\cdot\int_{\nu_1}^{\nu_2}(\Upsilon(\varkappa))^{d\varkappa}$,
3. $\int_{\nu_1}^{\nu_2}\left(\frac{\Theta(\varkappa)}{\Upsilon(\varkappa)}\right)^{d\varkappa}=\frac{\int_{\nu_1}^{\nu_2}(\Theta(\varkappa))^{d\varkappa}}{\int_{\nu_1}^{\nu_2}(\Upsilon(\varkappa))^{d\varkappa}}$,
4. $\int_{\nu_1}^{\nu_2}(\Theta(\varkappa))^{d\varkappa}=\int_{\nu_1}^{\mu}(\Theta(\varkappa))^{d\varkappa}\cdot\int_{\mu}^{\nu_2}(\Theta(\varkappa))^{d\varkappa},\nu_1\le\mu\le\nu_2$,
5. $\int_{\nu_1}^{\nu_1}(\Theta(\varkappa))^{d\varkappa}=1$ *and* $\int_{\nu_1}^{\nu_2}(\Theta(\varkappa))^{d\varkappa}=\left(\int_{\nu_2}^{\nu_1}(\Theta(\varkappa))^{d\varkappa}\right)^{-1}$.

3. Multiplicatively hyperbolic type convexity and some algebraic properties

In this section we give a new definition, which is called multiplicatively hyperbolic type convex function and study some of its basic algebraic properties.

Definition 3.1. *A positive function $\Theta : I\to\mathbb{R}$ is called multiplicatively hyperbolic type convex if for every $\varkappa, y\in I$ and $\omega\in[0,1]$*

$$\Theta(\omega\varkappa+(1-\omega)y)\le(\Theta(\varkappa))^{\frac{\sinh\omega}{\sinh 1}}(\Theta(y))^{\frac{\sinh 1-\sinh\omega}{\sinh 1}}.$$

We will denote by $MHC(I)$ the class of all multiplicatively hyperbolic type convex functions on I.

Theorem 3.2. *Let $\Theta,\Upsilon : [\varkappa,y]\to\mathbb{R}$ be two multiplicatively hyperbolic type convex functions, then $\Theta.\Upsilon$ is multiplicatively hyperbolic type convex function.*

Proof. Let Θ and Υ be multiplicatively hyperbolic type convex functions. Then

$$\begin{aligned}(\Theta.\Upsilon)(\omega\varkappa+(1-\omega)y) &= \Theta(\omega\varkappa+(1-\omega)y).\Upsilon(\omega\varkappa+(1-\omega)y)\\ &\le (\Theta(\varkappa))^{\frac{\sinh\omega}{\sinh 1}}(\Theta(y))^{\frac{\sinh 1-\sinh\omega}{\sinh 1}}.(\Upsilon(\varkappa))^{\frac{\sinh\omega}{\sinh 1}}(\Upsilon(y))^{\frac{\sinh 1-\sinh\omega}{\sinh 1}}\\ &= (\Theta(\varkappa)\Upsilon(\varkappa))^{\frac{\sinh\omega}{\sinh 1}}(\Theta(y)\Upsilon(y))^{\frac{\sinh 1-\sinh\omega}{\sinh 1}}\\ &= ((\Theta.\Upsilon)(\varkappa))^{\frac{\sinh\omega}{\sinh 1}}.((\Theta.\Upsilon)(y))^{\frac{\sinh 1-\sinh\omega}{\sinh 1}}.\end{aligned}$$

□

Theorem 3.3. *Let $\Theta : I\to J$ is convex and $\Upsilon : J\to\mathbb{R}$ is a multiplicatively hyperbolic type convex function and increasing, then $\Upsilon\circ\Theta : I\to\mathbb{R}$ is a multiplicatively hyperbolic type convex function.*

Proof. For $\varkappa, y\in I$ and $\omega\in[0,1]$, one gets

$$\begin{aligned}(\Upsilon\circ\Theta)(\omega\varkappa+(1-\omega)y) &= \Upsilon(\Theta(\omega\varkappa+(1-\omega)y))\\ &\le \Upsilon(\omega\Theta(\varkappa)+(1-\omega)\Theta(y))\\ &\le [\Upsilon(\Theta(\varkappa))]^{\frac{\sinh\omega}{\sinh 1}}.[\Upsilon(\Theta(y))]^{\frac{\sinh 1-\sinh\omega}{\sinh 1}}\\ &= [(\Upsilon\circ\Theta)(\varkappa)]^{\frac{\sinh\omega}{\sinh 1}}.[(\Upsilon\circ\Theta)(y)]^{\frac{\sinh 1-\sinh\omega}{\sinh 1}}.\end{aligned}$$

□

Theorem 3.4. *Let $\Theta_i : [\varkappa,y]\to\mathbb{R}$ be an arbitrary family of multiplicatively hyperbolic type convex functions and let $\Theta(\varkappa)=\sup_i\Theta_i(\varkappa)$. If $J=\{r\in[\varkappa,y] : \Theta(r)<\infty\}\neq\varnothing$, then J is an interval and Θ is a multiplicatively hyperbolic type convex function on J.*

Proof. For all $\varkappa, y \in J$ and $\omega \in [0,1]$, one has

$$\begin{aligned}
\Theta(\omega\varkappa + (1-\omega)y) &= \sup_i \Theta_i(\omega\varkappa + (1-\omega)y) \\
&\leq \sup_i \left[(\Theta_i(\varkappa))^{\frac{\sinh\omega}{\sinh 1}} . (\Theta_i(y))^{\frac{\sinh 1 - \sinh\omega}{\sinh 1}} \right] \\
&\leq \sup_i (\Theta_i(\varkappa))^{\frac{\sinh\omega}{\sinh 1}} . \sup_i (\Theta_i(y))^{\frac{\sinh 1 - \sinh\omega}{\sinh 1}} \\
&= (\Theta(\varkappa))^{\frac{\sinh\omega}{\sinh 1}} . (\Theta(y))^{\frac{\sinh 1 - \sinh\omega}{\sinh 1}} < \infty.
\end{aligned}$$

Thus, Θ is a multiplicatively hyperbolic type convex function on J. This completes the proof. □

4. Hermite-Hadamard inequalities for multiplicatively hyperbolic type convex functions

In this section we derive integral inequalities of HH type for multiplicatively hyperbolic type convex functions and convex functions in the framework of multiplicative calculus.

Theorem 4.1. *Let $\Theta : [\nu_1, \nu_2] \to \mathbb{R}$ be a multiplicatively hyperbolic type convex function on $[\nu_1, \nu_2]$. If $\Theta \in L[\nu_1, \nu_2]$, then*

$$\Theta\left(\frac{\nu_1+\nu_2}{2}\right) \leq \left(\int_{\nu_1}^{\nu_2} (\Theta(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}} \leq (\Theta(\nu_1))^{\frac{\cosh 1 - 1}{\sinh 1}} (\Theta(\nu_2))^{\frac{e-1}{e\sinh 1}}.$$

Proof. Note that

$$\begin{aligned}
\ln\Theta\left(\frac{\nu_1+\nu_2}{2}\right) &= \ln\left(\Theta\left(\frac{\omega\nu_1 + (1-\omega)\nu_2 + \omega\nu_2 + (1-\omega)\nu_1}{2}\right)\right) \\
&= \ln\left(\Theta\left(\frac{\omega\nu_1 + (1-\omega)\nu_2}{2} + \frac{\omega\nu_2 + (1-\omega)\nu_1}{2}\right)\right) \\
&\leq \ln\left[\left(\Theta\left(\frac{\omega\nu_1 + (1-\omega)\nu_2}{2}\right)\right)^{\frac{\sinh\frac{1}{2}}{\sinh 1}} . \left(\Theta\left(\frac{\omega\nu_2 + (1-\omega)\nu_1}{2}\right)\right)^{\frac{\sinh 1 - \sinh\frac{1}{2}}{\sinh 1}}\right] \\
&= \frac{\sinh\frac{1}{2}}{\sinh 1}\ln\Theta\left(\frac{\omega\nu_1 + (1-\omega)\nu_2}{2}\right) + \frac{\sinh 1 - \sinh\frac{1}{2}}{\sinh 1}\ln\Theta\left(\frac{\omega\nu_2 + (1-\omega)\nu_1}{2}\right).
\end{aligned}$$

Integrating the above inequality with respect to ω on $[0,1]$, one has

$$\begin{aligned}
\ln\Theta\left(\frac{\nu_1+\nu_2}{2}\right) &\leq \frac{\sinh\frac{1}{2}}{\sinh 1}\int_0^1 \ln\Theta\left(\frac{\omega\nu_1 + (1-\omega)\nu_2}{2}\right)d\omega + \frac{\sinh 1 - \sinh\frac{1}{2}}{\sinh 1}\int_0^1 \ln\left(\frac{\omega\nu_2 + (1-\omega)\nu_1}{2}\right)d\omega \\
&= \frac{\sinh\frac{1}{2}}{\sinh 1}\frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2} \ln\Theta(\varkappa)\,d\varkappa + \frac{\sinh 1 - \sinh\frac{1}{2}}{\sinh 1}\frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2} \ln\Theta(\varkappa)\,d\varkappa \\
&= \frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2} \ln\Theta(\varkappa)\,d\varkappa.
\end{aligned}$$

Thus,

$$\begin{aligned}
\Theta\left(\frac{\nu_1+\nu_2}{2}\right) &\leq e^{\left(\frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2} \ln\Theta(\varkappa)d\varkappa\right)} \\
&= \left(\int_{\nu_1}^{\nu_2} (\Theta(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}}.
\end{aligned}$$

Hence,

$$\Theta\left(\frac{\nu_1+\nu_2}{2}\right) \leq \left(\int_{\nu_1}^{\nu_2} (\Theta(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}}. \tag{2}$$

Consider the second inequality:

$$\begin{aligned}
\left(\int_{\nu_1}^{\nu_2} (\Theta(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}} &= \left(e^{\left(\int_{\nu_1}^{\nu_2} \ln\Theta(\varkappa)d\varkappa\right)}\right)^{\frac{1}{\nu_2-\nu_1}} \\
&= e^{\frac{1}{\nu_2-\nu_1}\left(\int_{\nu_1}^{\nu_2} \ln\Theta(\varkappa)d\varkappa\right)} \\
&= e^{\left(\int_0^1 \ln\Theta(\omega\nu_1+(1-\omega)\nu_2)d\omega\right)} \\
&\leq e^{\int_0^1 \ln\left[(\Theta(\nu_1))^{\frac{\sinh\omega}{\sinh 1}}.(\Theta(\nu_2))^{\frac{\sinh 1-\sinh\omega}{\sinh 1}}\right]d\omega} \\
&= e^{\int_0^1\left[\frac{\ln\Theta(\nu_1)}{\sinh 1}\sinh\omega+\frac{\ln\Theta(\nu_2)}{\sinh 1}(\sinh 1-\sinh\omega)\right]d\omega} \\
&= e^{\ln\left[(\Theta(\nu_1))^{\frac{\cosh 1-1}{\sinh 1}}(\Theta(\nu_2))^{\frac{e-1}{e\sinh 1}}\right]},
\end{aligned}$$

where

$$\int_0^1 \sinh\omega d\omega = \cosh 1 - 1,$$

$$\int_0^1 (\sinh 1 - \sinh\omega)\, d\omega = \frac{e-1}{e}.$$

Hence,

$$\left(\int_{\nu_1}^{\nu_2} (\Theta(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}} \leq (\Theta(\nu_1))^{\frac{\cosh 1-1}{\sinh 1}} (\Theta(\nu_2))^{\frac{e-1}{e\sinh 1}}. \tag{3}$$

Combining the inequalities (2) and (3), we have

$$\Theta\left(\frac{\nu_1+\nu_2}{2}\right) \leq \left(\int_{\nu_1}^{\nu_2} (\Theta(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}} \leq (\Theta(\nu_1))^{\frac{\cosh 1-1}{\sinh 1}} (\Theta(\nu_2))^{\frac{e-1}{e\sinh 1}}.$$

□

Corollary 4.2. *Let the functions* Θ *and* Υ *be multiplicatively hyperbolic type convex on* $[\nu_1,\nu_2]$. *Then*

$$\Theta\left(\frac{\nu_1+\nu_2}{2}\right)\Upsilon\left(\frac{\nu_1+\nu_2}{2}\right) \leq \left(\int_{\nu_1}^{\nu_2} (\Theta(\varkappa))^{d\varkappa}\int_{\nu_1}^{\nu_2} (\Upsilon(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}} \leq [\Theta(\nu_1)\Upsilon(\nu_1)]^{\frac{\cosh 1-1}{\sinh 1}} [\Theta(\nu_2)\Upsilon(\nu_2)]^{\frac{e-1}{e\sinh 1}}.$$

Corollary 4.3. *Let the functions* Θ *and* Υ *be multiplicatively hyperbolic type convex on* $[\nu_1,\nu_2]$. *Then*

$$\frac{\Theta\left(\frac{\nu_1+\nu_2}{2}\right)}{\Upsilon\left(\frac{\nu_1+\nu_2}{2}\right)} \leq \left(\frac{\int_{\nu_1}^{\nu_2} (\Theta(\varkappa))^{d\varkappa}}{\int_{\nu_1}^{\nu_2} (\Upsilon(\varkappa))^{d\varkappa}}\right)^{\frac{1}{\nu_2-\nu_1}} \leq \left(\frac{\Theta(\nu_1)}{\Upsilon(\nu_1)}\right)^{\frac{\cosh 1-1}{\sinh 1}} \left(\frac{\Theta(\nu_2)}{\Upsilon(\nu_2)}\right)^{\frac{e-1}{e\sinh 1}}.$$

Theorem 4.4. *Let the functions* Θ *and* Υ *be convex and multiplicatively hyperbolic type convex, respectively. Then*

$$\left(\frac{\int_{\nu_1}^{\nu_2} (\Theta(\varkappa))^{d\varkappa}}{\int_{\nu_1}^{\nu_2} (\Upsilon(\varkappa))^{d\varkappa}}\right)^{\frac{1}{\nu_2-\nu_1}} \leq \frac{\left(\frac{(\Theta(\nu_2))^{\Theta(\nu_2)}}{(\Theta(\nu_1))^{\Theta(\nu_1)}}\right)^{\frac{1}{\Theta(\nu_2)-\Theta(\nu_1)}}}{e.(\Upsilon(\nu_1))^{\frac{\cosh 1-1}{\sinh 1}}(\Upsilon(\nu_2))^{\frac{e-1}{e\sinh 1}}}.$$

Proof. Note that

$$\begin{aligned}
\left(\frac{\int_{\nu_1}^{\nu_2}(\Theta(\varkappa))^{d\varkappa}}{\int_{\nu_1}^{\nu_2}(\Upsilon(\varkappa))^{d\varkappa}}\right)^{\frac{1}{\nu_2-\nu_1}} &= \left(\frac{e^{\left(\int_{\nu_1}^{\nu_2}\ln\Theta(\varkappa)d\varkappa\right)}}{e^{\left(\int_{\nu_1}^{\nu_2}\ln\Upsilon(\varkappa)d\varkappa\right)}}\right)^{\frac{1}{\nu_2-\nu_1}} \\
&= \left(e^{\left(\int_{\nu_1}^{\nu_2}\ln\Theta(\varkappa)d\varkappa-\int_{\nu_1}^{\nu_2}\ln\Upsilon(\varkappa)d\varkappa\right)}\right)^{\frac{1}{\nu_2-\nu_1}} \\
&= e^{\left(\int_0^1\ln\Theta(\nu_2+\omega(\nu_1-\nu_2))d\omega-\int_0^1\ln\Upsilon(\nu_2+\omega(\nu_1-\nu_2))d\omega\right)} \\
&\leq e^{\left(\int_0^1\ln(\Theta(\nu_2)+\omega(\Theta(\nu_1)-\Theta(\nu_2)))d\omega-\int_0^1\ln\left((\Upsilon(\nu_1))^{\frac{\sinh\omega}{\sinh 1}}.(\Upsilon(\nu_2))^{\frac{\sinh 1-\sinh\omega}{\sinh 1}}\right)d\omega\right)} \\
&= e^{\left(\ln\left(\frac{(\Theta(\nu_2))^{\Theta(\nu_2)}}{(\Theta(\nu_1))^{\Theta(\nu_1)}}\right)^{\frac{1}{\Theta(\nu_2)-\Theta(\nu_1)}}-1-\ln\left((\Upsilon(\nu_1))^{\frac{\cosh 1-1}{\sinh 1}}(\Upsilon(\nu_2))^{\frac{e-1}{e\sinh 1}}\right)\right)} \\
&= \frac{\left(\frac{(\Theta(\nu_2))^{\Theta(\nu_2)}}{(\Theta(\nu_1))^{\Theta(\nu_1)}}\right)^{\frac{1}{\Theta(\nu_2)-\Theta(\nu_1)}}}{e.(\Upsilon(\nu_1))^{\frac{\cosh 1-1}{\sinh 1}}(\Upsilon(\nu_2))^{\frac{e-1}{e\sinh 1}}}.
\end{aligned}$$

□

Theorem 4.5. *Let the functions Θ and Υ be multiplicatively hyperbolic type convex and convex, respectively. Then*

$$\left(\frac{\int_{\nu_1}^{\nu_2}(\Theta(\varkappa))^{d\varkappa}}{\int_{\nu_1}^{\nu_2}(\Upsilon(\varkappa))^{d\varkappa}}\right)^{\frac{1}{\nu_2-\nu_1}} \leq \frac{e.(\Theta(\nu_1))^{\frac{\cosh 1-1}{\sinh 1}}(\Theta(\nu_2))^{\frac{e-1}{e\sinh 1}}}{\left(\frac{(\Upsilon(\nu_2))^{\Upsilon(\nu_2)}}{(\Upsilon(\nu_1))^{\Upsilon(\nu_1)}}\right)^{\frac{1}{\Upsilon(\nu_2)-\Upsilon(\nu_1)}}}.$$

Proof. Note that,

$$\begin{aligned}
\left(\frac{\int_{\nu_1}^{\nu_2}(\Theta(\varkappa))^{d\varkappa}}{\int_{\nu_1}^{\nu_2}(\Upsilon(\varkappa))^{d\varkappa}}\right)^{\frac{1}{\nu_2-\nu_1}} &= \left(\frac{e^{\left(\int_{\nu_1}^{\nu_2}\ln\Theta(\varkappa)d\varkappa\right)}}{e^{\left(\int_{\nu_1}^{\nu_2}\ln\Upsilon(\varkappa)d\varkappa\right)}}\right)^{\frac{1}{\nu_2-\nu_1}} \\
&= \left(e^{\left(\int_{\nu_1}^{\nu_2}\ln\Theta(\varkappa)d\varkappa-\int_{\nu_1}^{\nu_2}\ln\Upsilon(\varkappa)d\varkappa\right)}\right)^{\frac{1}{\nu_2-\nu_1}} \\
&= e^{\left(\int_0^1\ln\Theta(\nu_2+\omega(\nu_1-\nu_2))d\omega-\int_0^1\ln\Upsilon(\nu_2+\omega(\nu_1-\nu_2))d\omega\right)} \\
&\leq e^{\left(\int_0^1\ln\left((\Theta(\nu_1))^{\frac{\sinh\omega}{\sinh 1}}.(\Theta(\nu_2))^{\frac{\sinh 1-\sinh\omega}{\sinh 1}}\right)d\omega-\int_0^1\ln(\Upsilon(\nu_2)+\omega(\Upsilon(\nu_1)-\Upsilon(\nu_2)))d\omega\right)} \\
&= e^{\left(\ln\left((\Theta(\nu_1))^{\frac{\cosh 1-1}{\sinh 1}}(\Theta(\nu_2))^{\frac{e-1}{e\sinh 1}}\right)-\ln\left(\frac{(\Upsilon(\nu_2))^{\Upsilon(\nu_2)}}{(\Upsilon(\nu_1))^{\Upsilon(\nu_1)}}\right)^{\frac{1}{\Upsilon(\nu_2)-\Upsilon(\nu_1)}}+1\right)} \\
&= \frac{e.(\Theta(\nu_1))^{\frac{\cosh 1-1}{\sinh 1}}(\Theta(\nu_2))^{\frac{e-1}{e\sinh 1}}}{\left(\frac{(\Upsilon(\nu_2))^{\Upsilon(\nu_2)}}{(\Upsilon(\nu_1))^{\Upsilon(\nu_1)}}\right)^{\frac{1}{\Upsilon(\nu_2)-\Upsilon(\nu_1)}}}.
\end{aligned}$$

□

Theorem 4.6. *Let the functions Θ and Υ be convex and multiplicatively hyperbolic type convex, respectively. Then*

$$\left(\int_{\nu_1}^{\nu_2}(\Theta(\varkappa))^{d\varkappa}\int_{\nu_1}^{\nu_2}(\Upsilon(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}} \leq \frac{\left(\frac{(\Theta(\nu_2))^{\Theta(\nu_2)}}{(\Theta(\nu_1))^{\Theta(\nu_1)}}\right)^{\frac{1}{\Theta(\nu_2)-\Theta(\nu_1)}}(\Upsilon(\nu_1))^{\frac{\cosh 1-1}{\sinh 1}}(\Upsilon(\nu_2))^{\frac{e-1}{e\sinh 1}}}{e}.$$

Proof. Note that

$$
\begin{aligned}
\left(\int_{\nu_1}^{\nu_2} (\Theta(\varkappa))^{d\varkappa} \int_{\nu_1}^{\nu_2} (\Upsilon(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}} &= \left(e^{\int_{\nu_1}^{\nu_2} \ln\Theta(\varkappa)d\varkappa}.e^{\int_{\nu_1}^{\nu_2} \ln\Upsilon(\varkappa)d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}} \\
&= \left(e^{\int_{\nu_1}^{\nu_2} \ln\Theta(\varkappa)d\varkappa+\int_{\nu_1}^{\nu_2} \ln\Upsilon(\varkappa)d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}} \\
&= e^{\left(\int_0^1 \ln\Theta(\nu_2+\omega(\nu_1-\nu_2))d\omega+\int_0^1 \ln\Upsilon(\nu_2+\omega(\nu_1-\nu_2))d\omega\right)} \\
&\leq e^{\left(\int_0^1 \ln(\Theta(\nu_2)+\omega(\Theta(\nu_1)-\Theta(\nu_2)))d\omega-\int_0^1 \ln\left((\Upsilon(\nu_1))^{\frac{\sinh\omega}{\sinh 1}}.(\Upsilon(\nu_2))^{\frac{\sinh 1-\sinh\omega}{\sinh 1}}\right)d\omega\right)} \\
&= e^{\left(\ln\left(\frac{(\Theta(\nu_2))^{\Theta(\nu_2)}}{(\Theta(\nu_1))^{\Theta(\nu_1)}}\right)^{\frac{1}{\Theta(\nu_2)-\Theta(\nu_1)}}-1+\ln\left((\Upsilon(\nu_1))^{\frac{\cosh 1-1}{\sinh 1}}(\Upsilon(\nu_2))^{\frac{e-1}{e\sinh 1}}\right)\right)} \\
&= \frac{\left(\frac{(\Theta(\nu_2))^{\Theta(\nu_2)}}{(\Theta(\nu_1))^{\Theta(\nu_1)}}\right)^{\frac{1}{\Theta(\nu_2)-\Theta(\nu_1)}} (\Upsilon(\nu_1))^{\frac{\cosh 1-1}{\sinh 1}} (\Upsilon(\nu_2))^{\frac{e-1}{e\sinh 1}}}{e}.
\end{aligned}
$$

□

5. Some new inequalities for multiplicatively hyperbolic type convexity

Lemma 5.1. *Let $\Theta : I^\circ \subseteq \mathbb{R} \to \mathbb{R}^+$ be a multiplicative differentiable mapping on I°, $\nu_1, \nu_2 \in I^\circ$ with $\nu_1 < \nu_2$. If Θ^* is multiplicative integrable on $[\nu_1, \nu_2]$, then one has*

$$
\frac{\sqrt{\Theta(\nu_1)\Theta(\nu_2)}}{\left(\int_{\nu_1}^{\nu_2} (\Theta(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}}} = \left[\int_0^1 \left([\Theta^*(\omega\nu_1 + (1-\omega)\nu_2)]^{1-2\omega}\right)^{dw}\right]^{\frac{\nu_2-\nu_1}{2}}.
$$

Theorem 5.2. *Let $\Theta : I^\circ \subseteq \mathbb{R} \to \mathbb{R}^+$ be a multiplicative differentiable mapping on I°, $\nu_1, \nu_2 \in I^\circ$ with $\nu_1 < \nu_2$. If Θ is increasing on $[\nu_1, \nu_2]$ and Θ^* is multiplicatively hyperbolic type convex on $[\nu_1, \nu_2]$, then one has*

$$
\left|\frac{\sqrt{\Theta(\nu_1)\Theta(\nu_2)}}{\left(\int_{\nu_1}^{\nu_2} (\Theta(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}}}\right| \leq \left[(\Theta^*(\nu_1))^{\left(\cosh 1-2\sinh 1+4\sinh\frac{1}{2}-1\right)} (\Theta^*(\nu_2))^{\frac{5}{2}\sinh 1-\cosh 1-4\sinh\frac{1}{2}+1}\right]^{\frac{\nu_2-\nu_1}{2\sinh 1}}.
$$

Proof. Using Lemma 5.1, one gets

$$
\begin{aligned}
\left|\frac{\sqrt{\Theta(\nu_1)\Theta(\nu_2)}}{\left(\int_{\nu_1}^{\nu_2} (\Theta(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}}}\right| &\leq \left[\left|\int_0^1 \left([\Theta^*(\omega\nu_1 + (1-\omega)\nu_2)]^{1-2\omega}\right)^{dw}\right|\right]^{\frac{\nu_2-\nu_1}{2}} \\
&\leq e^{\frac{\nu_2-\nu_1}{2}\int_0^1 \left|\ln(\Theta^*(\omega\nu_1+(1-\omega)\nu_2))^{1-2\omega}\right| dw} \\
&= e^{\frac{\nu_2-\nu_1}{2}\int_0^1 |1-2\omega||\ln\Theta^*(\omega\nu_1+(1-\omega)\nu_2)| dw}. \qquad (4)
\end{aligned}
$$

Since the function Θ^* is multiplicatively hyperbolic type convex, one gets

$$
\begin{aligned}
&\int_0^1 |1-2\omega| \ln\Theta^*(\omega\nu_1 + (1-\omega)\nu_2)\, d\omega \\
\leq\ &\int_0^1 |1-2\omega| \ln\left((\Theta^*(\nu_1))^{\frac{\sinh\omega}{\sinh 1}} . (\Theta^*(\nu_2))^{\frac{\sinh 1-\sinh\omega}{\sinh 1}}\right) d\omega \\
=\ &\int_0^1 |1-2\omega| \frac{\sinh\omega}{\sinh 1} \ln\Theta^*(\nu_1)\, d\omega + \int_0^1 |1-2\omega| \frac{\sinh 1-\sinh\omega}{\sinh 1} \ln\Theta^*(\nu_2)\, d\omega
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\ln \Theta^*(\nu_1)}{\sinh 1}\int_0^1 |1-2\omega| \sinh \omega d\omega + \frac{\ln \Theta^*(\nu_2)}{\sinh 1}\int_0^1 |1-2\omega| (\sinh 1 - \sinh \omega)\, d\omega \\
&= \frac{\ln \Theta^*(\nu_1)}{\sinh 1}\left(\cosh 1 - 2\sinh 1 + 4\sinh \frac{1}{2} - 1\right) + \frac{\ln \Theta^*(\nu_2)}{\sinh 1}\left(\frac{5}{2}\sinh 1 - \cosh 1 - 4\sinh\frac{1}{2} + 1\right), \qquad (5)
\end{aligned}
$$

where

$$\int_0^1 |1-2\omega| \sinh \omega d\omega = \cosh 1 - 2\sinh 1 + 4\sinh\frac{1}{2} - 1,$$

$$\int_0^1 |1-2\omega| (\sinh 1 - \sinh \omega)\, d\omega = \frac{5}{2}\sinh 1 - \cosh 1 - 4\sinh\frac{1}{2} + 1.$$

If one substitutes the inequality (5) in (4), one obtains

$$
\begin{aligned}
\left| \frac{\sqrt{\Theta(\nu_1)\Theta(\nu_2)}}{\left(\int_{\nu_1}^{\nu_2} (\Theta(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}}} \right| &\leq e^{\frac{\nu_2-\nu_1}{2}\int_0^1 \left|\ln(\Theta^*(\omega\nu_1+(1-\omega)\nu_2))^{1-2\omega}\right| dw} \\
&\leq e^{\frac{\nu_2-\nu_1}{2}.\left(\left(\cosh 1-2\sinh 1+4\sinh\frac{1}{2}-1\right)\frac{\ln\Theta^*(\nu_1)}{\sinh 1}+\left(\frac{5}{2}\sinh 1-\cosh 1-4\sinh\frac{1}{2}+1\right)\frac{\ln\Theta^*(\nu_2)}{\sinh 1}\right)} \\
&= \left[(\Theta^*(\nu_1))^{\left(\cosh 1-2\sinh 1+4\sinh\frac{1}{2}-1\right)}(\Theta^*(\nu_2))^{\frac{5}{2}\sinh 1-\cosh 1-4\sinh\frac{1}{2}+1}\right]^{\frac{\nu_2-\nu_1}{2\sinh 1}}.
\end{aligned}
$$

□

Theorem 5.3. *Let $\Theta : I^\circ \subset \mathbb{R} \to \mathbb{R}^+$ be a multiplicative differentiable mapping on I°, $\nu_1, \nu_2 \in I^\circ$ with $\nu_1 < \nu_2$. If Θ is increasing on $[\nu_1, \nu_2]$ and $(\ln \Theta^*)^q$ is hyperbolic type convex on $[\nu_1, \nu_2]$, then one has*

$$\left| \frac{\sqrt{\Theta(\nu_1)\Theta(\nu_2)}}{\left(\int_{\nu_1}^{\nu_2} (\Theta(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}}} \right| \leq \left[(\Theta^*(\nu_1))^{(\cosh 1-1)^{\frac{1}{q}}}(\Theta^*(\nu_2))^{(\sinh 1-\cosh 1+1)^{\frac{1}{q}}}\right]^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{\sinh 1}\right)^{\frac{1}{q}}}, \qquad (6)$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using Lemma 5.1 and the Hölder's inequality it follows that

$$
\begin{aligned}
\left| \frac{\sqrt{\Theta(\nu_1)\Theta(\nu_2)}}{\left(\int_{\nu_1}^{\nu_2} (\Theta(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}}} \right| &\leq \left[\left|\int_0^1 \left([\Theta^*(\omega\nu_1 + (1-\omega)\nu_2)]^{1-2\omega}\right)^{dw}\right|\right]^{\frac{\nu_2-\nu_1}{2}} \\
&\leq e^{\frac{\nu_2-\nu_1}{2}\int_0^1 |1-2\omega||\ln\Theta^*(\omega\nu_1+(1-\omega)\nu_2)|dw} \\
&\leq e^{\frac{\nu_2-\nu_1}{2}\left(\int_0^1 |1-2\omega|^p d\omega\right)^{\frac{1}{p}}\left(\int_0^1 (\ln(\Theta^*(\omega\nu_1+(1-\omega)\nu_2)))^q dw\right)^{\frac{1}{q}}}. \qquad (7)
\end{aligned}
$$

Using the hyperbolic type convexity of $(\ln \Theta^*)^q$, one obtains

$$
\begin{aligned}
\int_0^1 (\ln(\Theta^*(\omega\nu_1 + (1-\omega)\nu_2)))^q dw &\leq \int_0^1 \left[\frac{\sinh\omega}{\sinh 1}(\ln\Theta^*(\nu_1))^q + \frac{\sinh 1 - \sinh\omega}{\sinh 1}(\ln\Theta^*(\nu_2))^q\right] dw \\
&= \frac{(\cosh 1 - 1)(\ln\Theta^*(\nu_1))^q + (\sinh 1 - \cosh 1 + 1)(\ln\Theta^*(\nu_2))^q}{\sinh 1}, \qquad (8)
\end{aligned}
$$

where

$$\int_0^1 |1-2\omega|^p d\omega = \frac{1}{p+1},$$

$$\int_0^1 \sinh \omega d\omega = \cosh 1 - 1,$$

$$\int_0^1 (\sinh 1 - \sinh \omega)\, d\omega = \sinh 1 - \cosh 1 + 1.$$

By combining (7) and (8), one gets

$$\begin{aligned}
\left| \frac{\sqrt{\Theta(\nu_1)\Theta(\nu_2)}}{\left(\int_{\nu_1}^{\nu_2} (\Theta(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}}} \right| &\leq e^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{\cosh 1-1}{\sinh 1}\right)(\ln \Theta^*(\nu_1))^q + \left(\frac{\sinh 1-\cosh 1+1}{\sinh 1}\right)(\ln \Theta^*(\nu_2))^q\right]^{\frac{1}{q}}} \\
&\leq e^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{\sinh 1}\right)^{\frac{1}{q}}\left[(\cosh 1-1)^{\frac{1}{q}} \ln(\Theta^*(\nu_1)) + (\sinh 1-\cosh 1+1)^{\frac{1}{q}} \ln(\Theta^*(\nu_2))\right]} \\
&= \left[(\Theta^*(\nu_1))^{(\cosh 1-1)^{\frac{1}{q}}} (\Theta^*(\nu_2))^{(\sinh 1-\cosh 1+1)^{\frac{1}{q}}} \right]^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{\sinh 1}\right)^{\frac{1}{q}}}.
\end{aligned}$$

□

Theorem 5.4. *Let $\Theta : I^\circ \subset \mathbb{R} \to \mathbb{R}^+$ be a multiplicative differentiable mapping on I°, $\nu_1, \nu_2 \in I^\circ$ with $\nu_1 < \nu_2$. If Θ is increasing on $[\nu_1, \nu_2]$ and $(\ln \Theta^*)^q$, $q \geq 1$ is hyperbolic type convex on $[\nu_1, \nu_2]$, then*

$$\left| \frac{\sqrt{\Theta(\nu_1)\Theta(\nu_2)}}{\left(\int_{\nu_1}^{\nu_2} (\Theta(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}}} \right| \leq \left[(\Theta^*(\nu_1))^{\left(\frac{-e^2+4e\sqrt{e}-2e-4\sqrt{e}+3}{2e}\right)^{\frac{1}{q}}} (\Theta^*(\nu_2))^{\left(\frac{3e^2-8e\sqrt{e}+4e+8\sqrt{e}-7}{4e}\right)^{\frac{1}{q}}} \right]^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}}. \tag{9}$$

Proof. Assume first that $q > 1$. Using Lemma 5.1, power mean inequality and the hyperbolic type convexity of $(\ln \Theta^*)^q$, one has

$$\begin{aligned}
\left| \frac{\sqrt{\Theta(\nu_1)\Theta(\nu_2)}}{\left(\int_{\nu_1}^{\nu_2} (\Theta(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}}} \right| &\leq \left[\left| \int_0^1 \left([\Theta^*(\omega\nu_1 + (1-\omega)\nu_2)]^{1-2\omega} \right)^{d\omega} \right| \right]^{\frac{\nu_2-\nu_1}{2}} \\
&\leq e^{\frac{\nu_2-\nu_1}{2}\int_0^1 |1-2\omega||\ln \Theta^*(\omega\nu_1+(1-\omega)\nu_2)|d\omega} \\
&\leq e^{\frac{\nu_2-\nu_1}{2}\left(\int_0^1 |1-2\omega|d\omega\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2\omega||\ln(\Theta^*(\omega\nu_1+(1-\omega)\nu_2))|^q d\omega\right)^{\frac{1}{q}}} \\
&\leq e^{\frac{\nu_2-\nu_1}{2}\left(\int_0^1 |1-2\omega|d\omega\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2\omega|\left[\frac{\sinh \omega}{\sinh 1}(\ln \Theta^*(\nu_1))^q + \frac{\sinh 1-\sinh \omega}{\sinh 1}(\ln \Theta^*(\nu_2))^q\right]d\omega\right)^{\frac{1}{q}}} \\
&= e^{\frac{\nu_2-\nu_1}{2}\left(\int_0^1 |1-2\omega|d\omega\right)^{1-\frac{1}{q}}\left((\ln \Theta^*(\nu_1))^q \int_0^1 |1-2\omega|\frac{\sinh \omega}{\sinh 1}d\omega + (\ln \Theta^*(\nu_2))^q \int_0^1 |1-2\omega|\frac{\sinh 1-\sinh \omega}{\sinh 1}d\omega\right)^{\frac{1}{q}}}.
\end{aligned}$$

Since,

$$\int_0^1 |1-2\omega|\, d\omega = \frac{1}{2},$$

$$\int_0^1 |1-2\omega| \sinh \omega d\omega = \frac{-e^2+4e\sqrt{e}-2e-4\sqrt{e}+3}{2e},$$

$$\int_0^1 |1-2\omega| (\sinh 1 - \sinh \omega)\, d\omega = \frac{3e^2-8e\sqrt{e}+4e+8\sqrt{e}-7}{4e},$$

it follows that

$$\left| \frac{\sqrt{\Theta(\nu_1)\Theta(\nu_2)}}{\left(\int_{\nu_1}^{\nu_2} (\Theta(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}}} \right| \leq e^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\frac{-e^2+4e\sqrt{e}-2e-4\sqrt{e}+3}{2e}\right)(\ln \Theta^*(\nu_1))^q + \left(\frac{3e^2-8e\sqrt{e}+4e+8\sqrt{e}-7}{4e}\right)(\ln \Theta^*(\nu_2))^q\right]^{\frac{1}{q}}}$$

$$= e^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\cdot\left(\frac{1}{\sinh 1}\right)^{\frac{1}{q}}\left[\left(\frac{-e^2+4e\sqrt{e}-2e-4\sqrt{e}+3}{2e}\right)^{\frac{1}{q}}\ln(\Theta^*(\nu_1))+\left(\frac{3e^2-8e\sqrt{e}+4e+8\sqrt{e}-7}{4e}\right)^{\frac{1}{q}}\ln(\Theta^*(\nu_2))\right]}$$

$$= \left[(\Theta^*(\nu_1))^{\left(\frac{-e^2+4e\sqrt{e}-2e-4\sqrt{e}+3}{2e}\right)^{\frac{1}{q}}}(\Theta^*(\nu_2))^{\left(\frac{3e^2-8e\sqrt{e}+4e+8\sqrt{e}-7}{4e}\right)^{\frac{1}{q}}}\right]^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}}.$$

For $q = 1$, one uses the estimates from the proof of Theorem 5.2. So, the proof is completed. □

Now, we will prove the Theorem 5.3 by using Hölder-İşcan integral inequality. Then we will show the newly obtained inequality is better approach than the inequality (6).

Theorem 5.5. *Let $\Theta : I^\circ \subset \mathbb{R} \to \mathbb{R}^+$ be a multiplicative differentiable mapping on I°, $\nu_1, \nu_2 \in I^\circ$ with $\nu_1 < \nu_2$. If Θ is increasing on $[\nu_1, \nu_2]$ and $(\ln\Theta^*)^q$ is hyperbolic type convex on $[\nu_1, \nu_2]$, then one has*

$$\left|\frac{\sqrt{\Theta(\nu_1)\Theta(\nu_2)}}{\left(\int_{\nu_1}^{\nu_2}(\Theta(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}}}\right| \le \left[(\Theta^*(\nu_1))^{\left(\left(\frac{e^2-2e-1}{2e}\right)^{\frac{1}{q}}+\left(\frac{1}{e}\right)^{\frac{1}{q}}\right)}(\Theta^*(\nu_2))^{\left(\left(\frac{-e^2+4e+1}{4e}\right)^{\frac{1}{q}}+\left(\frac{e^2-5}{4e}\right)^{\frac{1}{q}}\right)}\right]^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{\sinh 1}\right)^{\frac{1}{q}}}, \tag{10}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 5.1, the Hölder-İşcan integral inequality and hyperbolic type convexity of $(\ln\Theta^*)^q$, it follows that

$$\begin{aligned}
&\left|\frac{\sqrt{\Theta(\nu_1)\Theta(\nu_2)}}{\left(\int_{\nu_1}^{\nu_2}(\Theta(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}}}\right| \\
\le\ & e^{\frac{\nu_2-\nu_1}{2}\int_0^1|1-2\omega||\ln\Theta^*(\omega\nu_1+(1-\omega)\nu_2)|d\omega} \\
\le\ & e^{\frac{\nu_2-\nu_1}{2}\left(\int_0^1(1-\omega)|1-2\omega|^p d\omega\right)^{\frac{1}{p}}\left(\int_0^1(1-\omega)|\ln\Theta^*(\omega\nu_1+(1-\omega)\nu_2)|^q d\omega\right)^{\frac{1}{q}}} \\
&\times e^{\frac{\nu_2-\nu_1}{2}\left(\int_0^1\omega|1-2\omega|^p d\omega\right)^{\frac{1}{p}}\left(\int_0^1\omega|\ln\Theta^*(\omega\nu_1+(1-\omega)\nu_2)|^q d\omega\right)^{\frac{1}{q}}} \\
\le\ & e^{\frac{\nu_2-\nu_1}{2}\left(\int_0^1(1-\omega)|1-2\omega|^p d\omega\right)^{\frac{1}{p}}\left(\int_0^1(1-\omega)\left[\frac{\sinh\omega}{\sinh 1}(\ln\Theta^*(\nu_1))^q+\frac{\sinh 1-\sinh\omega}{\sinh 1}(\ln\Theta^*(\nu_2))^q\right]d\omega\right)^{\frac{1}{q}}} \\
&\times e^{\frac{\nu_2-\nu_1}{2}\left(\int_0^1\omega|1-2\omega|^p d\omega\right)^{\frac{1}{p}}\left(\int_0^1\omega\left[\frac{\sinh\omega}{\sinh 1}(\ln\Theta^*(\nu_1))^q+\frac{\sinh 1-\sinh\omega}{\sinh 1}(\ln\Theta^*(\nu_2))^q\right]d\omega\right)^{\frac{1}{q}}} \\
=\ & e^{\frac{\nu_2-\nu_1}{2}\left(\int_0^1(1-\omega)|1-2\omega|^p d\omega\right)^{\frac{1}{p}}\left(\frac{(\ln\Theta^*(\nu_1))^q}{\sinh 1}\int_0^1(1-\omega)\sinh\omega d\omega+\frac{(\ln\Theta^*(\nu_2))^q}{\sinh 1}\int_0^1(1-\omega)(\sinh 1-\sinh\omega)d\omega\right)^{\frac{1}{q}}} \\
&\times e^{\frac{\nu_2-\nu_1}{2}\left(\int_0^1\omega|1-2\omega|^p d\omega\right)^{\frac{1}{p}}\left(\frac{(\ln\Theta^*(\nu_1))^q}{\sinh 1}\int_0^1\omega\sinh\omega d\omega+\frac{(\ln\Theta^*(\nu_2))^q}{\sinh 1}\int_0^1\omega(\sinh 1-\sinh\omega)d\omega\right)^{\frac{1}{q}}} \\
=\ & \left[(\Theta^*(\nu_1))^{\left(\frac{e^2-2e-1}{2e}\right)^{\frac{1}{q}}}(\Theta^*(\nu_2))^{\left(\frac{-e^2+4e+1}{4e}\right)^{\frac{1}{q}}}\right]^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{\sinh 1}\right)^{\frac{1}{q}}} \\
&\times\left[(\Theta^*(\nu_1))^{\left(\frac{1}{e}\right)^{\frac{1}{q}}}(\Theta^*(\nu_2))^{\left(\frac{e^2-5}{4e}\right)^{\frac{1}{q}}}\right]^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{\sinh 1}\right)^{\frac{1}{q}}} \\
=\ & \left[(\Theta^*(\nu_1))^{\left(\left(\frac{e^2-2e-1}{2e}\right)^{\frac{1}{q}}+\left(\frac{1}{e}\right)^{\frac{1}{q}}\right)}(\Theta^*(\nu_2))^{\left(\left(\frac{-e^2+4e+1}{4e}\right)^{\frac{1}{q}}+\left(\frac{e^2-5}{4e}\right)^{\frac{1}{q}}\right)}\right]^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{\sinh 1}\right)^{\frac{1}{q}}},
\end{aligned}$$

where

$$\int_0^1 (1-\omega)\,|1-2\omega|^p\, d\omega = \int_0^1 \omega\,|1-2\omega|^p\, d\omega = \frac{1}{2\,(p+1)},$$
$$\int_0^1 (1-\omega)\sinh\omega d\omega = \frac{e^2-2e-1}{2e},$$
$$\int_0^1 (1-\omega)(\sinh 1-\sinh\omega)\, d\omega = \frac{-e^2+4e+1}{4e},$$
$$\int_0^1 \omega\sinh\omega d\omega = \frac{1}{e},$$
$$\int_0^1 \omega(\sinh 1-\sinh\omega)\, d\omega = \frac{e^2-5}{4e}.$$

□

Remark 5.6. *The inequality (10) gives better result than the inequality (6). Let us show that*

$$\left[(\Theta^*(\nu_1))^{\left(\left(\frac{e^2-2e-1}{2e}\right)^{\frac{1}{q}}+\left(\frac{1}{e}\right)^{\frac{1}{q}}\right)}(\Theta^*(\nu_2))^{\left(\left(\frac{-e^2+4e+1}{4e}\right)^{\frac{1}{q}}+\left(\frac{e^2-5}{4e}\right)^{\frac{1}{q}}\right)}\right]^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{\sinh 1}\right)^{\frac{1}{q}}}$$
$$\leq \left[(\Theta^*(\nu_1))^{(\cosh 1-1)^{\frac{1}{q}}}(\Theta^*(\nu_2))^{(\sinh 1-\cosh 1+1)^{\frac{1}{q}}}\right]^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{\sinh 1}\right)^{\frac{1}{q}}}.$$

Using multiplicatively concavity of the function $\hbar:[0,\infty)\to\mathbb{R}$, $\hbar(x)=x^\tau$, $0<\tau\leq 1$ by sample calculation one gets

$$\left[(\Theta^*(\nu_1))^{\left(\left(\frac{e^2-2e-1}{2e}\right)^{\frac{1}{q}}+\left(\frac{1}{e}\right)^{\frac{1}{q}}\right)}(\Theta^*(\nu_2))^{\left(\left(\frac{-e^2+4e+1}{4e}\right)^{\frac{1}{q}}+\left(\frac{e^2-5}{4e}\right)^{\frac{1}{q}}\right)}\right]^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{\sinh 1}\right)^{\frac{1}{q}}}$$
$$\leq \left[(\Theta^*(\nu_1))^{\left(\frac{1}{2}\cdot\frac{(e-1)^2}{2e}\right)^{\frac{1}{q}}}(\Theta^*(\nu_2))^{\left(\frac{1}{2}\cdot\frac{e-1}{e}\right)^{\frac{1}{q}}}\right]^{2\frac{\nu_2-\nu_1}{2}\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{\sinh 1}\right)^{\frac{1}{q}}}$$
$$= \left[(\Theta^*(\nu_1))^{(\cosh 1-1)^{\frac{1}{q}}}(\Theta^*(\nu_2))^{(\sinh 1-\cosh 1+1)^{\frac{1}{q}}}\right]^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{\sinh 1}\right)^{\frac{1}{q}}},$$

where

$$\cosh 1-1 = \frac{(e-1)^2}{2e},$$
$$\sinh 1-\cosh 1+1 = \frac{e-1}{e}.$$

Theorem 5.7. *Let $\Theta : I^\circ\subset\mathbb{R}\to\mathbb{R}^+$ be a multiplicative differentiable mapping on I°, $\nu_1,\nu_2\in I^\circ$ with $\nu_1<\nu_2$.If Θ is increasing on $[\nu_1,\nu_2]$ and $(\ln\Theta^*)^q$ is hyperbolic type convex on $[\nu_1,\nu_2]$, then one has*

$$\left|\frac{\sqrt{\Theta(\nu_1)\,\Theta(\nu_2)}}{\left(\int_{\nu_1}^{\nu_2}(\Theta(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}}}\right| \leq \left[(\Theta^*(\nu_1))^{\left((\zeta_1)^{\frac{1}{q}}+(\zeta_3)^{\frac{1}{q}}\right)}(\Theta^*(\nu_2))^{\left((\zeta_2)^{\frac{1}{q}}+(\zeta_4)^{\frac{1}{q}}\right)}\right]^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{1}{\sinh 1}\right)^{\frac{1}{q}}}, \tag{11}$$

where

$$\begin{aligned}
\zeta_1 &= \int_0^1 (1-\omega)|1-2\omega|\sinh\omega d\omega = 5\sqrt{e}+\frac{3}{\sqrt{e}}-\frac{3}{2}e-\frac{5}{2e}-5,\\
\zeta_2 &= \int_0^1 (1-\omega)|1-2\omega|(\sinh 1-\sinh\omega)\,d\omega = \frac{13}{8}e+\frac{19}{8e}-5\sqrt{e}-\frac{3}{\sqrt{e}}+5,\\
\zeta_3 &= \int_0^1 \omega|1-2\omega|\sinh\omega d\omega = e+\frac{4}{e}-3\sqrt{e}-\frac{5}{\sqrt{e}}+4,\\
\zeta_4 &= \int_0^1 \omega|1-2\omega|(\sinh 1-\sinh\omega)\,d\omega = 3\sqrt{e}+\frac{5}{\sqrt{e}}-\frac{7}{8}e-\frac{33}{8e}-4.
\end{aligned}$$

Proof. Firstly, assume $q>1$. Using Lemma 5.1, improved power-mean integral inequality and hyperbolic type convexity of $(\ln\Theta^*)^q$, it follows that

$$\begin{aligned}
&\left|\frac{\sqrt{\Theta(\nu_1)\Theta(\nu_2)}}{\left(\int_{\nu_1}^{\nu_2}(\Theta(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}}}\right|\\
\leq\ & e^{\frac{\nu_2-\nu_1}{2}\int_0^1|1-2\omega||\ln\Theta^*(\omega\nu_1+(1-\omega)\nu_2)|d\omega}\\
\leq\ & e^{\frac{\nu_2-\nu_1}{2}\left(\int_0^1(1-\omega)|1-2\omega|d\omega\right)^{1-\frac{1}{q}}\left(\int_0^1(1-\omega)|1-2\omega||\ln\Theta^*(\omega\nu_1+(1-\omega)\nu_2)|^q d\omega\right)^{\frac{1}{q}}}\\
&\times e^{\frac{\nu_2-\nu_1}{2}\left(\int_0^1\omega|1-2\omega|d\omega\right)^{1-\frac{1}{q}}\left(\int_0^1\omega|1-2\omega||\ln\Theta^*(\omega\nu_1+(1-\omega)\nu_2)|^q d\omega\right)^{\frac{1}{q}}}\\
\leq\ & e^{\frac{\nu_2-\nu_1}{2}\left(\int_0^1(1-\omega)|1-2\omega|d\omega\right)^{1-\frac{1}{q}}\left(\int_0^1(1-\omega)|1-2\omega|\left[\frac{\sinh\omega}{\sinh 1}(\ln\Theta^*(\nu_1))^q+\frac{\sinh 1-\sinh\omega}{\sinh 1}(\ln\Theta^*(\nu_2))^q\right]d\omega\right)^{\frac{1}{q}}}\\
&\times e^{\frac{\nu_2-\nu_1}{2}\left(\int_0^1\omega|1-2\omega|d\omega\right)^{1-\frac{1}{q}}\left(\int_0^1\omega|1-2\omega|\left[\frac{\sinh\omega}{\sinh 1}(\ln\Theta^*(\nu_1))^q+\frac{\sinh 1-\sinh\omega}{\sinh 1}(\ln\Theta^*(\nu_2))^q\right]d\omega\right)^{\frac{1}{q}}}\\
=\ & e^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{(\ln\Theta^*(\nu_1))^q}{\sinh 1}\int_0^1(1-\omega)|1-2\omega|\sinh\omega d\omega+\frac{(\ln\Theta^*(\nu_2))^q}{\sinh 1}\int_0^1(1-\omega)|1-2\omega|(\sinh 1-\sinh\omega)d\omega\right)^{\frac{1}{q}}}\\
&\times e^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{(\ln\Theta^*(\nu_1))^q}{\sinh 1}\int_0^1\omega|1-2\omega|\sinh\omega d\omega+\frac{(\ln\Theta^*(\nu_2))^q}{\sinh 1}\int_0^1\omega|1-2\omega|(\sinh 1-\sinh\omega)d\omega\right)^{\frac{1}{q}}}\\
=\ & \left[(\Theta^*(\nu_1))^{(\zeta_1)^{\frac{1}{q}}}(\Theta^*(\nu_2))^{(\zeta_2)^{\frac{1}{q}}}\right]^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{1}{\sinh 1}\right)^{\frac{1}{q}}}\\
&\times\left[(\Theta^*(\nu_1))^{\left((\zeta_3)^{\frac{1}{q}}\right)}(\Theta^*(\nu_2))^{\left((\zeta_4)^{\frac{1}{q}}\right)}\right]^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{1}{\sinh 1}\right)^{\frac{1}{q}}}\\
=\ & \left[(\Theta^*(\nu_1))^{\left((\zeta_1)^{\frac{1}{q}}+(\zeta_3)^{\frac{1}{q}}\right)}(\Theta^*(\nu_2))^{\left((\zeta_2)^{\frac{1}{q}}+(\zeta_4)^{\frac{1}{q}}\right)}\right]^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{1}{\sinh 1}\right)^{\frac{1}{q}}},
\end{aligned}$$

where $\int_0^1(1-\omega)|1-2\omega|\,d\omega=\int_0^1\omega|1-2\omega|\,d\omega=\frac{1}{4}$. □

Now, let $q=1$. Then we use the estimates from the proof of Theorem 5.7, which also follow step by step the above estimates. Thus, the proof is completed.

Remark 5.8. *The inequality (11) gives better result than the inequality (9). Let us show that*

$$\left[(\Theta^*(\nu_1))^{\left((\zeta_1)^{\frac{1}{q}}+(\zeta_3)^{\frac{1}{q}}\right)}(\Theta^*(\nu_2))^{\left((\zeta_2)^{\frac{1}{q}}+(\zeta_4)^{\frac{1}{q}}\right)}\right]^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{1}{\sinh 1}\right)^{\frac{1}{q}}}$$

$$\leq \left[(\Theta^*(\nu_1))^{\left(\frac{-e^2+4e\sqrt{e}-2e-4\sqrt{e}+3}{2e}\right)^{\frac{1}{q}}} (\Theta^*(\nu_2))^{\left(\frac{3e^2-8e\sqrt{e}+4e+8\sqrt{e}-7}{4e}\right)^{\frac{1}{q}}} \right]^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}}.$$

Using multiplicatively concavity of the function $\hbar : [0,\infty) \to \mathbb{R}$, $\hbar(x) = x^\tau$, $0 < \tau \leq 1$ bu sample calculation one gets

$$\begin{aligned}
&\left[(\Theta^*(\nu_1))^{\left((\zeta_1)^{\frac{1}{q}}+(\zeta_3)^{\frac{1}{q}}\right)} (\Theta^*(\nu_2))^{\left((\zeta_2)^{\frac{1}{q}}+(\zeta_4)^{\frac{1}{q}}\right)} \right]^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{1}{\sinh 1}\right)^{\frac{1}{q}}} \\
\leq &\left[(\Theta^*(\nu_1))^{\left(\left(\frac{1}{2}.\zeta_1\right)^{\frac{1}{q}}+\left(\frac{1}{2}.\zeta_3\right)^{\frac{1}{q}}\right)} (\Theta^*(\nu_2))^{\left(\left(\frac{1}{2}.\zeta_2\right)^{\frac{1}{q}}+\left(\frac{1}{2}.\zeta_4\right)^{\frac{1}{q}}\right)} \right]^{2\frac{\nu_2-\nu_1}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{1}{\sinh 1}\right)^{\frac{1}{q}}} \\
\leq &\left[(\Theta^*(\nu_1))^{\left(\frac{1}{2}(\zeta_1+\zeta_3)\right)^{\frac{1}{q}}} (\Theta^*(\nu_2))^{\left(\frac{1}{2}(\zeta_2+\zeta_4)\right)^{\frac{1}{q}}} \right]^{2\frac{\nu_2-\nu_1}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{1}{\sinh 1}\right)^{\frac{1}{q}}} \\
= &\left[(\Theta^*(\nu_1))^{\left(\frac{-e^2+4e\sqrt{e}-2e-4\sqrt{e}+3}{2e}\right)^{\frac{1}{q}}} (\Theta^*(\nu_2))^{\left(\frac{3e^2-8e\sqrt{e}+4e+8\sqrt{e}-7}{4e}\right)^{\frac{1}{q}}} \right]^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}}.
\end{aligned}$$

where

$$\zeta_1 + \zeta_3 = \frac{-e^2+4e\sqrt{e}-2e-4\sqrt{e}+3}{2e},$$

$$\zeta_2 + \zeta_4 = \frac{3e^2-8e\sqrt{e}+4e+8\sqrt{e}-7}{4e}.$$

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