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Estimates of the operators associated with Laguerre polynomials

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Abstract. In this study, we analyze the approximation characteristics of the operators based on Laguerre polynomials. Initially, we present moments and central moments followed by certain results viz. asymptotic formula and difference estimates. Furthermore, we modify these operators to preserve linear functions and we prove that our modified operators have a finer error estimate than the original operators. Also, we demonstrate the convergence of both the operators with graphical illustrations.

1. Introduction

Sucu et al. [18] proposed the subsequent operators for $x \in [0, \infty)$:

$$(S_n^\alpha f)(x) = e^{\frac{-nx}{2}} 2^{-\alpha-1} \sum_{j=0}^{\infty} 2^{-j} L_j^\alpha \left(\frac{-nx}{2}\right) f\left(\frac{j}{n}\right), \quad \alpha > -1, \quad n \in N,$$

where $L_j^\alpha(-x)$ are the modified Laguerre polynomials defined by

$$L_j^\alpha(-x) = \sum_{s=0}^{j} \frac{(\alpha + j)!}{(j-s)!(\alpha+s)!s!} x^s.$$

As the extension of the work on the operator S_n^α, Gupta [10] obtained the new operators as follows:

$$(Q_n^\alpha f)(x) = I_0\left(\frac{2\sqrt{2nx}}{\sqrt{3}}\right) \sum_{j=0}^{\infty} \frac{e^{-nx}}{2^{\alpha+j}} \frac{L_j^\alpha\left(\frac{-1}{3}\right)}{3} f\left(\frac{j}{n}\right), \tag{1}$$

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where $I_0(z)$ represents a modified Bessel's function of first kind i.e.

$$I_0(z) = \sum_{j=0}^{\infty} \frac{(\frac{1}{4}z^2)^j}{(j!)^2}.$$

The approximation properties of various operators have been analysed by researchers in [7], [8], [16] and many more. The examination of several approximation characteristics of the operators defined in (1) is the primary purpose of this manuscript. We derive certain explicit results, such as quantitative asymptotic formulae and difference estimates. Furthermore, we observe here that these operators do not preserve linear functions. Hence, we introduce the modification of Q_n^α and prove that the modified operators have better error estimate. Also, we display the rate of convergence of both the operators using graphs.

2. Auxiliary Results

We apply the subsequent findings in the sequel. In this paper, we indicate that $e_r(m) = m^r, r = 0, 1, 2, ...$

Lemma 2.1. *For the operators (1), the moment generating function is indicated below:*

$$(Q_n^\alpha e^{At})(x) = \frac{1}{(2 - e^{A/n})^\alpha (3 - 2e^{A/n})} exp\left(nx\left(\frac{e^{A/n} - 1}{3 - 2e^{A/n}}\right)\right).$$

Proof. Regarding the computations, see [10]. □

Remark 2.2. *Using Mathematica to extend the right side of Lemma 2.1, we obtain*

$$\begin{aligned}
(Q_n^\alpha e^{At})(x) &= 1 + \frac{A(2+\alpha+nx)}{n} + \frac{A^2\left(10+6\alpha+\alpha^2+9nx+2\alpha nx+n^2x^2\right)}{2n^2} + \frac{A^3}{6n^3}\Big\{74+48\alpha \\
&+12\alpha^2+\alpha^3+97nx+33\alpha nx+3\alpha^2nx+21n^2x^2+3\alpha n^2x^2+n^3x^3\Big\} + \frac{A^4}{24n^4}\Big\{730+490\alpha \\
&+144\alpha^2+20\alpha^3+\alpha^4+1257nx+520\alpha nx+78\alpha^2nx+4\alpha^3nx+403n^2x^2+96\alpha n^2x^2 \\
&+6\alpha^2n^2x^2+38n^3x^3+4\alpha n^3x^3+n^4x^4\Big\} + \frac{A^5}{120n^5}\Big\{9002+6140\alpha+1950\alpha^2+340\alpha^3+30\alpha^4 \\
&+\alpha^5+19201nx+8895\alpha nx+1690\alpha^2nx+150\alpha^3nx+5\alpha^4nx+8145n^2x^2+2495\alpha n^2x^2 \\
&+270\alpha^2n^2x^2+10\alpha^3n^2x^2+1145n^3x^3+210\alpha n^3x^3+10\alpha^2n^3x^3+60n^4x^4+5\alpha n^4x^4+n^5x^5\Big\} \\
&+\frac{A^6}{720n^6}\Big\{133210+91574\alpha+30270\alpha^2+5950\alpha^3+690\alpha^4+42\alpha^5+\alpha^6+338889nx \\
&+168966\alpha nx+36855\alpha^2nx+4280\alpha^3nx+255\alpha^4nx+6\alpha^5nx+178771n^2x^2+63870\alpha n^2x^2 \\
&+9105\alpha^2n^2x^2+600\alpha^3n^2x^2+15\alpha^4n^2x^2+33370n^3x^3+8130\alpha n^3x^3+690\alpha^2n^3x^3+20\alpha^3n^3x^3 \\
&+2615n^4x^4+390\alpha n^4x^4+15\alpha^2n^4x^4+87n^5x^5+6\alpha n^5x^5+n^6x^6\Big\} + O\left(A^7\right).
\end{aligned}$$

Lemma 2.3. *The outcome that follows is valid:*

$$\begin{aligned}
(Q_n^\alpha e_0)(x) &= 1, \\
(Q_n^\alpha e_1)(x) &= \frac{2+\alpha+nx}{n}, \\
(Q_n^\alpha e_2)(x) &= \frac{10+6\alpha+\alpha^2+9nx+2\alpha nx+n^2x^2}{n^2}, \\
(Q_n^\alpha e_3)(x) &= \frac{1}{n^3}\Big\{74+48\alpha+12\alpha^2+\alpha^3+97nx+33\alpha nx+3\alpha^2nx+21n^2x^2+3\alpha n^2x^2+n^3x^3\Big\},
\end{aligned}$$

$$
\begin{aligned}
(Q_n^{\alpha}e_4)(x) &= \frac{1}{n^4}\Big\{730+490\alpha+144\alpha^2+20\alpha^3+\alpha^4+1257nx+520\alpha nx+78\alpha^2nx+4\alpha^3nx\\
&\quad +403n^2x^2+96\alpha n^2x^2+6\alpha^2n^2x^2+38n^3x^3+4\alpha n^3x^3+n^4x^4\Big\},\\
(Q_n^{\alpha}e_5)(x) &= \frac{1}{n^5}\Big\{9002+6140\alpha+1950\alpha^2+340\alpha^3+30\alpha^4+\alpha^5+19201nx+8895\alpha nx\\
&\quad +1690\alpha^2nx+150\alpha^3nx+5\alpha^4nx+8145n^2x^2+2495\alpha n^2x^2+270\alpha^2n^2x^2\\
&\quad +10\alpha^3n^2x^2+1145n^3x^3+210\alpha n^3x^3+10\alpha^2n^3x^3+60n^4x^4+5\alpha n^4x^4+n^5x^5\Big\},\\
(Q_n^{\alpha}e_6)(x) &= \frac{1}{n^6}\Big\{133210+91574\alpha+30270\alpha^2+5950\alpha^3+690\alpha^4+42\alpha^5+\alpha^6+338889nx\\
&\quad +168966\alpha nx+36855\alpha^2nx+4280\alpha^3nx+255\alpha^4nx+6\alpha^5nx+178771n^2x^2\\
&\quad +63870\alpha n^2x^2+9105\alpha^2n^2x^2+600\alpha^3n^2x^2+15\alpha^4n^2x^2+33370n^3x^3+8130\alpha n^3x^3\\
&\quad +690\alpha^2n^3x^3+20\alpha^3n^3x^3+2615n^4x^4+390\alpha n^4x^4+15\alpha^2n^4x^4+87n^5x^5+6\alpha n^5x^5+n^6x^6\Big\}.
\end{aligned}
$$

Proof. Since we are aware that the coefficient of $\frac{A^r}{r!}$ in the moment generating function expansion represents the $r-th$ order moment, we may derive the required outcome from Remark 2.2. □

Lemma 2.4. *We indicate* $\mu_{n,m}^{\alpha}(x) = (Q_n^{\alpha}(e_1 - xe_0)^m)(x)$, *so*

$$
\begin{aligned}
\mu_{n,0}^{\alpha}(x) &= 1,\\
\mu_{n,1}^{\alpha}(x) &= \frac{2+\alpha}{n},\\
\mu_{n,2}^{\alpha}(x) &= \frac{10+6\alpha+\alpha^2+5nx}{n^2},\\
\mu_{n,3}^{\alpha}(x) &= \frac{74+48\alpha+12\alpha^2+\alpha^3+67nx+15\alpha nx}{n^3},\\
\mu_{n,4}^{\alpha}(x) &= \frac{730+20\alpha^3+\alpha^4+961nx+75n^2x^2+6\alpha^2(24+5nx)+\alpha(490+328nx)}{n^4}.
\end{aligned}
$$

Proof. Lemma 2.3 makes the proof simple to understand. □

Lemma 2.5. *Let* f *be bounded on* $[0,\infty)$ *with* $\|f\| = \sup_{x\in[0,\,\infty)} |f(x)|$, *then*

$$\left|(Q_n^{\alpha}f)(x)\right| \leq \|f\|.$$

3. Asymptotic Formula

The space containing all bounded and uniformly continuous functions on $[0,\infty)$ is denoted by $C_B([0,\infty))$ and

$$C_B^{**}[0,\infty) = \Big\{g \in C_B([0,\infty)) : g', g'' \in C_B([0,\infty))\Big\}.$$

The $K-$ functional is given as

$$K_2(f,\beta) = inf\Big\{\|f-g\| + \beta\|g''\|\Big\},\ \ \beta > 0$$

where $g \in C_B^{**}([0,\infty))$. Then $K_2(f,\beta) \leq C\omega_2(f,\beta^{\frac{1}{2}})$, where C is a positive absolute constant and ω_2 represents the second-order modulus of continuity. Refer to [9, pp. 177, Theorem 2.4] for further details.

Theorem 3.1. *If $f \in C_B([0,\infty))$, then*

$$\left|(Q_n^{\alpha} f)(x) - f(x)\right| \leq C\omega_2(f, \sqrt{\gamma_n(x)}) + \omega\left(f, \frac{2+\alpha}{n}\right),$$

where

$$\gamma_n(x) = \frac{24 + 16\alpha + 3\alpha^2 + 10nx}{2n^2}$$

and ω is modulus of continuity of first order.

Proof. We start by defining the operators $\hat{Q}_n^{\alpha} : C_B([0,\infty)) \to C_B([0,\infty))$ as

$$(\hat{Q}_n^{\alpha} f)(x) = (Q_n^{\alpha} f)(x) - f\left(\frac{2+\alpha+nx}{n}\right) + f(x).$$

These operators obviously preserve linear functions in light of Lemma 2.3. With the help of Taylor's formula, we can express

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-v)g''(v)dv,$$

where $g \in C_B^{**}([0,\infty))$ and $x, t \in [0,\infty)$.
Thus,

$$\begin{aligned} \left|(\hat{Q}_n^{\alpha} g)(x) - g(x)\right| &\leq \left(\hat{Q}_n^{\alpha}\left|\int_x^t (t-v)g''(v)dv\right|\right)(x) \\ &\leq \left(Q_n^{\alpha}\left|\int_x^t (t-v)g''(v)dv\right|\right)(x) + \left|\int_x^{\frac{2+\alpha+nx}{n}} \left(\frac{2+\alpha+nx}{n} - v\right)g''(v)dv\right| \\ &\leq \mu_{n,2}^{\alpha}(x)\|g''\| + \left|\int_x^{\frac{2+\alpha+nx}{n}} \left(\frac{2+\alpha+nx}{n} - v\right)dv\right|\|g''\|. \end{aligned}$$

Now, applying Lemma 2.4 gives

$$\begin{aligned} \left|(\hat{Q}_n^{\alpha} g)(x) - g(x)\right| &\leq \left\{\mu_{n,2}^{\alpha}(x) + \frac{(2+\alpha)^2}{2n^2}\right\}\|g''\| \\ &= \left\{\frac{10 + 6\alpha + \alpha^2 + 5nx}{n^2} + \frac{(2+\alpha)^2}{2n^2}\right\}\|g''\| \\ &:= \gamma_n(x)\|g''\|. \end{aligned}$$

Using Lemma 2.5 and the definition of the operators $\hat{Q}_n^{\alpha}$, we are led to

$$\|(\hat{Q}_n^{\alpha} f)(x)\| \leq \|(Q_n^{\alpha} f)(x)\| + 2\|f\| \leq 3\|f\|.$$

In conclusion, we can state that

$$\begin{aligned} \left|(Q_n^{\alpha} f)(x) - f(x)\right| &\leq \left|(\hat{Q}_n^{\alpha} g)(x) - g(x)\right| + \left|(\hat{Q}_n^{\alpha}(f-g))(x) - (f-g)(x)\right| + \left|f\left(\frac{2+\alpha+nx}{n}\right) - f(x)\right| \\ &\leq \gamma_n(x)\|g''\| + 4\|f-g\| + \left|f\left(\frac{2+\alpha+nx}{n}\right) - f(x)\right| \\ &\leq C\Big\{\gamma_n(x)\|g''\| + \|f-g\|\Big\} + \omega\left(f, \frac{2+\alpha}{n}\right). \end{aligned}$$

Using the condition of K-functional and taking infimum over $g \in C_B^{**}([0,\infty))$, we establish the required claim. □

We consider $C^*([0,\infty))$ as the collection of all those g which satisfy $|g(x)| \leq N(1+x^2)$, where N (constant) is independent of x and depends on g. $C^*([0,\infty))$ is a space with the norm

$$\|g\|^* = \sup_{x\in[0,\,\infty)} g(x)(1+x^2)^{-1}.$$

The weighted modulus of continuity [17] be defined as

$$\Omega(g,\zeta) = \sup_{|h|<\zeta, x\in[0,\,\infty)} |g(x+h)-g(x)|\left((1+h^2)(1+x^2)\right)^{-1},$$

where $g \in C([0,\infty)) \cap C^*([0,\infty))$. According to $\Omega(g,\zeta)$, the below inequality is true:

$$|g(t)-g(x)| \leq 2[1+(t-x)^2][1+|t-x|\zeta^{-1}](1+x^2)(1+\zeta^2)\Omega(g,\zeta), \tag{2}$$

where $x, t \in [0,\infty)$.
We now establish the asymptotic result of Voronovskaya type, that has been thoroughly examined by several investigators in [1], [2], [3], [6] etc.

Theorem 3.2. *If f', f'' belong to $\in C([0,\infty)) \cap C^*([0,\infty))$, then*

$$\begin{aligned} &\left|(Q_n^\alpha f)(x) - f(x) - f'(x)\left[\frac{2+\alpha}{n}\right] - \frac{f''(x)}{2}\left[\frac{10+6\alpha+\alpha^2+5nx}{n^2}\right]\right| \\ \leq\;& 8.O(n^{-1})(1+x^2)\Omega(f'', n^{\frac{-1}{2}}). \end{aligned}$$

Proof. Using Taylor's formula,

$$f(u) = f(x) + f'(x)(u-x) + \frac{f''(x)}{2}(u-x)^2 + \xi(u,x)(u-x)^2,$$

where $\theta \in (x,u)$ and $\xi(u,x) = \frac{1}{2}(f''(\theta) - f''(x))$ is a function that is continuous and vanishes at 0.
When we apply the operator Q_n^α to the inequality indicated above, we get

$$\left|(Q_n^\alpha f)(x) - f(x) - f'(x)\left[\frac{2+\alpha}{n}\right] - \frac{f''(x)}{2}\left[\frac{10+6\alpha+\alpha^2+5nx}{n^2}\right]\right| \leq \left(Q_n^\alpha(|\xi(u,x)|(u-x)^2)\right)(x).$$

From Lemma 2.4, we obtain

$$\begin{aligned} \left(Q_n^\alpha(|\xi(u,x)|(u-x)^2)\right)(x) &= 8(1+x^2)\Omega(f'',\zeta)\left[\mu_{n,2}^\alpha(x) + \zeta^{-4}\mu_{n,6}^\alpha(x)\right] \\ &= 8(1+x^2)\Omega(f'',\zeta)\left[O(n^{-1}) + \zeta^{-4}O(n^{-3})\right]. \end{aligned}$$

Suppose we assume $\zeta = n^{\frac{-1}{2}}$, then

$$\left(Q_n^\alpha(|\xi(u,x)|(u-x)^2)\right)(x) \leq 8(1+x^2)\Omega(f'', n^{\frac{-1}{2}})O(n^{-1}),$$

which enable us to get the intended outcome. □

Corollary 3.3. *If $f', f'' \in C([0,\infty)) \cap C^*([0,\infty))$, then*

$$\lim_{n\to\infty} n\left|(Q_n^\alpha f)(x) - f(x)\right| = f'(x)\left[2+\alpha\right] - \frac{5x}{2}f''(x).$$

4. Difference of Operators

Many researchers have been studying the differences of operators in the last few years; current studies may be found in [4], [5], [11], [13], [14] and [15] among other places. In this section, we obtain the quantitative estimates for the differences of the operators Q_n^α with the Baskakov operators and their Kantorovich variant.

It is defined that the Baskakov operators are

$$(V_n f)(x) = \sum_{j=0}^{\infty} v_{n,j}(x) F_{n,j}(f), \tag{3}$$

where $v_{n,j}(x) = \binom{n+j-1}{j}\frac{x^j}{(1+x)^{n+j}}$ and $F_{n,j}(f) = f(\frac{j}{n})$.

Remark 4.1. *We have*

$$b^{F_{n,j}} = F_{n,j}(e_1) = \frac{j}{n},$$
$$\mu_2^{F_{n,j}} = F_{n,j}(e_1 - b^{F_{n,j}} e_0)^2 = 0.$$

Below are few moments of Baskakov operators:

$$(V_n e_0)(x) = 1, (V_n e_1)(x) = x, (V_n e_2)(x) = \frac{(n+1)x^2 + x}{n}$$

and

$$(V_n(e_1 - x)^2)(x) = \frac{x(1+x)}{n}.$$

For $x \in [0, \infty)$, the Baskakov-Kantorovich operators (refer to [19]) are expressed as follows:

$$(K_n f)(x) = \sum_{j=0}^{\infty} v_{n,j}(x) J_{n,j}(f). \tag{4}$$

Here

$$J_{n,j}(f) = n \int_{\frac{j}{n}}^{\frac{j+1}{n}} f(t)dt$$

and $v_{n,j}$ is the Baskakov basis function defined in (3).

Also, the operators defined in (1) can be expressed as

$$(Q_n^\alpha f)(x) = \sum_{j=0}^{\infty} q_{n,j}^\alpha(x) F_{n,j}(f),$$

where

$$q_{n,j}^\alpha(x) = \left(\frac{2\sqrt{2nx}}{\sqrt{3}}\right) \sum_{i=0}^{\infty} \frac{(1/4z^2)^i}{(i!)^2} \frac{e^{-nx}}{2^{\alpha+j}} \frac{1}{3} \sum_{s=0}^{j} \frac{(\alpha+j)!}{(j-s)!(\alpha+s)!s!} \left(\frac{1}{3}\right)^s$$

and

$$F_{n,j}(f) = f\left(\frac{j}{n}\right).$$

We apply the following result for the difference of positive linear operators with different bases, as given by Gupta [11] and Gupta-Acu [12]:

Theorem A. If $U_n := \sum_{j=0}^{\infty} u_{n,j}(x)L_{n,j}(f)$ and $W_n := \sum_{j=0}^{\infty} w_{n,j}(x)M_{n,j}(f)$, then

$$|((U_n - W_n)f)(x)| \leq \frac{C(x)}{2}\|f''\| + 2\omega(f,\gamma_1) + 2\omega(f,\gamma_2), n \in N,$$

where $C(x) = \sum_{j=0}^{\infty} u_{n,j}(x)\mu_2^{L_{n,j}} + \sum_{j=0}^{\infty} w_{n,j}(x)\mu_2^{M_{n,j}}$,

$$\gamma_1^2 = \sum_{j=0}^{\infty} u_{n,j}(x)\left(L_{n,j}(e_1) - x\right)^2$$

and

$$\gamma_2^2 = \sum_{j=0}^{\infty} w_{n,j}(x)\left(M_{n,j}(e_1) - x\right)^2$$

with

$$\mu_r^{H_{n,j}} = \sum_{i=0}^{r} \binom{r}{i}(-1)^i H_{n,j}(e_{r-i})(H_{n,j}(e_1))^i.$$

Remark 4.2. *If* $L_{n,j}^2(e_1) \leq L_{n,j}(e_1^2)$ *and* $M_{n,j}^2(e_1) \leq M_{n,j}(e_1^2)$ *then Theorem A takes the form*

$$|((U_n - W_n)f)(x)| \leq 2\omega(f,\nu_1) + 2\omega(f,\nu_2), n \in N,$$

where

$$\nu_1^2(x) = (L_n(e_1 - x)^2)(x)$$

and

$$\nu_2^2(x) = (M_n(e_1 - x)^2)(x).$$

With the help of Remark 4.2, the difference of Baskakov operators V_n and the operators Q_n^α is derived as:

Theorem 4.3. *If* $f \in C_B([0,\infty))$, $x \in [0,\infty)$ *and* $n \in N$, *then*

$$|((V_n - Q_n^\alpha)f)(x)| \leq 2\omega\left(f, \sqrt{\frac{x(x+1)}{n}}\right) + 2\omega\left(f, \sqrt{\frac{10 + 6\alpha + \alpha^2 + 5nx}{n^2}}\right).$$

Now, we derive the estimate of the difference between Baskakov-Kantorovich operators $K_n f$ and the operators Q_n^α. In light of Theorem A, it follows:

Theorem 4.4. *If* $f \in C_B([0,\infty))$, *then for Baskakov-Kantorovich operators specified as*

$(K_n f)(x) = \sum_{j=0}^{\infty} v_{n,j}(x)J_{n,j}(f)$

and for the operators specified as

$(Q_n^\alpha f)(x) = \sum_{j=0}^{\infty} q_{n,j}^\alpha(x)F_{n,j}(f)$, *we get*

$$|((K_n - Q_n^\alpha)f)(x)| \leq \frac{1}{24n^2}\|f''\| + 2\omega\left(f, \sqrt{\frac{1}{4n^2} + \frac{x(n+1)}{n}}\right) + 2\omega\left(f, \sqrt{\frac{\alpha^2 + 6\alpha + 5nx + 10}{n^2}}\right).$$

Proof. We have

$$b^{J_{n,j}} = J_{n,j}(e_1) = \frac{2j+1}{2n}.$$

Also,

$$\begin{aligned}\mu_2^{J_{n,j}} &= J_{n,j}(e_2) - 2[J_{n,j}(e_1)]^2 + [J_{n,j}(e_1)]^2 \\ &= \frac{n}{3}\left(\left(\frac{j+1}{n}\right)^3 - \left(\frac{j}{n}\right)^3\right) - \left(\frac{2j+1}{2n}\right)^2 = \frac{1}{12n^2}.\end{aligned}$$

Next, we have $b^{F_{n,j}}$ is $F_{n,j}(e_1)$ whose value is $\frac{j}{n}$, and $\mu_r^{F_{n,j}} = 0, r \in N$. Using values mentioned above, we obtain

$$\begin{aligned}C(x) &= \sum_{j=0}^{\infty} v_{n,j}(x)\mu_2^{J_{n,j}} + \sum_{j=0}^{\infty} q_{n,j}^{\alpha}(x)\mu_2^{F_{n,j}} \\ &= \sum_{j=0}^{\infty} v_{n,j}(x)\frac{1}{12n^2} \\ &= \frac{1}{12n^2}.\end{aligned}$$

Further, we have

$$\begin{aligned}\gamma_1^2 &= \sum_{j=0}^{\infty} v_{n,j}(x)\left(J_{n,j}(e_1) - x\right)^2 \\ &= \sum_{j=0}^{\infty} v_{n,j}(x)\left(\frac{2j+1}{2n} - x\right)^2 \\ &= \frac{1}{4n^2} + \frac{x(n+1)}{n}\end{aligned}$$

and

$$\begin{aligned}\gamma_2^2 &= \sum_{j=0}^{\infty} q_{n,j}^{\alpha}(x)\left(F_{n,j}(e_1) - x\right)^2 \\ &= \sum_{j=0}^{\infty} q_{n,j}^{\alpha}(x)\left(\frac{j}{n} - x\right)^2 \\ &= \frac{\alpha^2 + 6\alpha + 5nx + 10}{n^2}.\end{aligned}$$

Hence the result follows.

5. Modified Operators Preserving Linear Functions

We can see that linear functions are not preserved by the operators defined in (1). We were inspired to investigate this further, and in this section, we suggest modifying these operators. The modified form of (1), for $f \in C([0,\infty))$, can be defined as follows if we take $r_n(x) = \frac{nx-(2+\alpha)}{n}$, into consideration:

$$(\widetilde{Q}_n^{\alpha} f)(x) = I_0\left(\frac{2\sqrt{2nr_n(x)}}{\sqrt{3}}\right)\sum_{j=0}^{\infty} \frac{e^{-nr_n(x)}}{2^{\alpha+j}} \frac{L_j^{\alpha}\left(\frac{-1}{3}\right)}{3} f\left(\frac{j}{n}\right). \tag{5}$$

Out of Lemmas 2.3 and 2.4, the following lemmas follow:

Lemma 5.1. *The following is true for the operators defined in (5):*

$$\begin{aligned}
(\widetilde{Q}_n^{\alpha}e_0)(x) &= 1,\\
(\widetilde{Q}_n^{\alpha}e_1)(x) &= x,\\
(\widetilde{Q}_n^{\alpha}e_2)(x) &= \frac{-3\alpha+n^2x^2+5nx-4}{n^2},\\
(\widetilde{Q}_n^{\alpha}e_3)(x) &= \frac{-\alpha(9nx+31)+n^3x^3+15n^2x^2+25nx-44}{n^3},\\
(\widetilde{Q}_n^{\alpha}e_4)(x) &= \frac{27\alpha^2-\alpha\left(18n^2x^2+214nx+267\right)+n^4x^4+30n^3x^3+199n^2x^2+69nx-460}{n^4},\\
(\widetilde{Q}_n^{\alpha}e_5)(x) &= \frac{1}{n^5}\Big(15\alpha^2(9nx+62)-\alpha\left(30n^3x^3+760n^2x^2+3995nx+1791\right)+n^5x^5+50n^4x^4\\
&\quad +705n^3x^3+2635n^2x^2-1479nx-5052\Big).
\end{aligned}$$

Lemma 5.2. *For the operators defined in (5), there hold:*

$$\begin{aligned}
(\widetilde{Q}_n^{\alpha}(e_1-xe_0))(x) &= 0,\\
(\widetilde{Q}_n^{\alpha}(e_1-xe_0)^2)(x) &= \frac{-3\alpha+5nx-4}{n^2}.
\end{aligned}$$

Theorem 5.3. *For $f \in C[0,\infty]$, we can write*

$$\lim_{n\to\infty}(\widetilde{Q}_n^{(1/n)}f)(x) = f(x).$$

Proof. As a consequence of Lemma 5.1, it is clear that

$$\lim_{n\to\infty}(\widetilde{Q}_n^{\alpha}e_0)(x) = 1,$$

$$\lim_{n\to\infty}(\widetilde{Q}_n^{\alpha}e_1)(x) = x$$

and

$$\lim_{n\to\infty}(\widetilde{Q}_n^{\alpha}e_2)(x) = x^2.$$

Hence, on applying Korovkin theorem, we may conclude the result. □

Theorem 5.4. *For $f \in C([0,\infty))$, it implies*

$$|(\widetilde{Q}_n^{\alpha}f)(x) - f(x)| \le 2\omega\left(f,\eta\right),$$

where $\eta = \sqrt{(\widetilde{Q}_n^{\alpha}(e_1-xe_0)^2)(x)}$.

Proof. For $\eta > 0$, the modulus of continuity has the below mentioned property:

$$|f(y)-f(x)| \le \omega(f,\eta)\left(1+\eta^{-2}(y-x)^2\right). \tag{6}$$

Thus,

$$\begin{aligned}
|(\widetilde{Q}_n^{\alpha}f)(x)-f(x)| &\le (\widetilde{Q}_n^{\alpha}|f(y)-f(x)|, x)\\
&\le \omega(f,\eta)\left(1+\eta^{-2}(\widetilde{Q}_n^{\alpha}(e_1-xe_0)^2)(x)\right).
\end{aligned}$$

Lemma 5.2, provides

$$\lim_{n\to\infty}(\widetilde{Q}_n^{\alpha}(e_1-xe_0)^2)(x) = 0.$$

Hence, if we select $\eta = \sqrt{(\widetilde{Q}_n^{\alpha}(e_1-xe_0)^2)(x)}$, the desired result follows. □

On the same lines, we have

Theorem 5.5. *If $f \in C([0,\infty))$, then*

$$|(Q_n^\alpha f)(x) - f(x)| \leq 2\omega(f,\gamma),$$

where $\gamma = \sqrt{\frac{10+6\alpha+\alpha^2+5nx}{n^2}}$.

Remark 5.6. *The error estimate for our modified operators in Theorem 5.4 is finer than the one taken into account for the original operators Q_n^α because, if we assume, for $\alpha > -1$,*

$$\begin{aligned} \eta &\leq \gamma \\ &\Leftrightarrow \frac{-4-3\alpha+5nx}{n^2} \leq \frac{10+6\alpha+\alpha^2+5nx}{n^2} \\ &\Leftrightarrow \frac{-(14+9\alpha+\alpha^2)}{n^2} \leq 0 \end{aligned}$$

which is valid for all x and for any $n \in \mathbf{N}$. We find that our modified operator provides a better approximation as a result.

6. Graphical Representation

The next two graphs are provided below to illustrate the convergence of the operators stated by (1) and modified operators stated by (5):

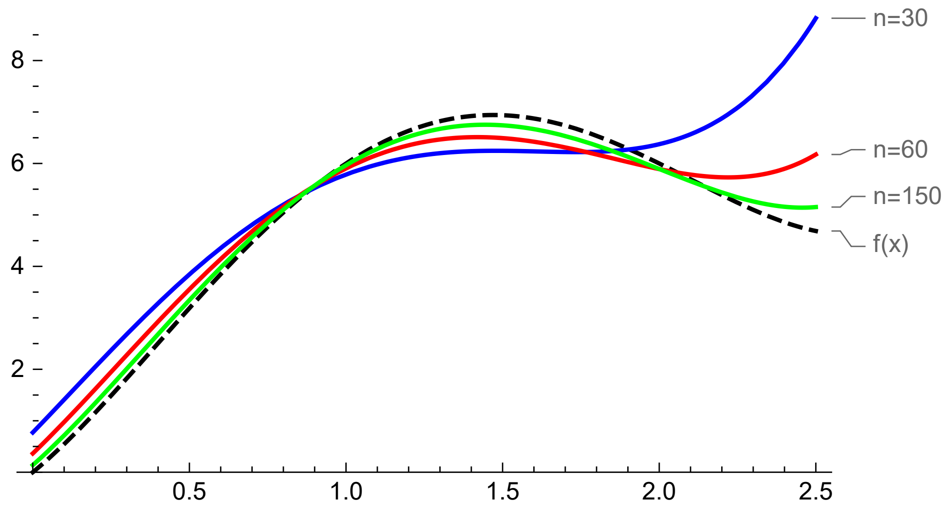

Figure 1: Convergence of $(Q_n^\alpha f)(x)$ for $f(x) = x^4 - 5x^3 + 5x^2 + 5x$ (black) is demonstrated for $\alpha = 2$ and $n = 30, 60$ and 150.

□

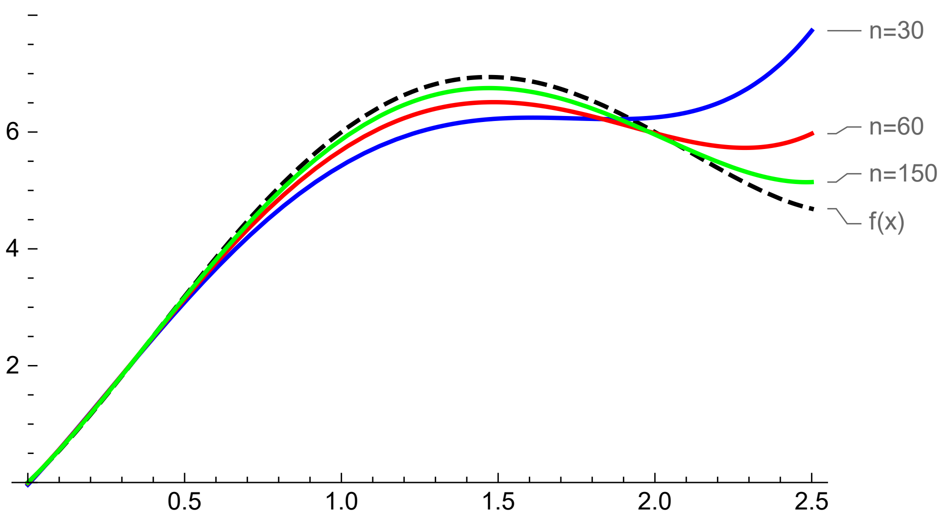

Figure 2: Convergence of $(\widetilde{Q}_n^{\alpha} f)(x)$ for $f(x) = x^4 - 5x^3 + 5x^2 + 5x$ (black) is demonstrated for $\alpha = 2$ and $n = 30, 60$ and 150.

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