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Characteristic and lower characteristic of bounded linear operators

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Abstract. In this research paper, we generalize the concept of characteristic and lower characteristic of linear operators on linear relations, examining some properties and elaborating a connection with upper semi-Fredholm relations.

1. Introduction

Among the importance of studying Fredholm and upper semi-Fredholm relations is to characterize spectrum and essential spectrum (see [9]). It is know that many problems of mathematical physics (for example, quantum theory) are reduced to the study of certain conditions of spectrum and essential spectrum.

The notion of Characteristic $[L]_A$ and Lower characteristic $[L]_a$ for bounded linear operator L was introduced into the functional analysis by A. Jürgen and all [15]. They investigate some basic properties of this concept and they find the connection with upper semi-Fredholm operators.

The aim of this work was to extend this result for a more general context, ie considering linear relations. We organize the paper in the following way. Section 2 contains preliminary and auxiliary properties that we will need to prove the main results of the other sections. In Section 3, we identify a characteristic and D-characteristic of linear relations and we provide certain outstanding results and certain prominent properties. In Section 4, we tackle the definiton of Lower characteristic of linear relations, we exhibit some pertinent results and eventually we enact a connection with upper semi-Fredholm relations.

2. Preliminary and auxiliary results

The notion of linear relations generalizes the concept of a linear operator to that of a multivalued linear operator. Linear relations emerged in functional analysis in J. von Neumann [14] triggered by the need to consider adjoints of non-densely defined operators invested in applications to the theory of generalized equations [11] as well as the need to consider the inverses of certain operators which are invested, for instance, in the investigation of certain Cauchy problems related to parabolic type equations in Banach spaces [1–8, 13]. Certain results that are confirmed in the case of linear operators need to be validated

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within the framework of linear relations, sometimes under supplementary conditions. Let X, Y, Z be vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We call a multivalued operator or a relation between X and Y, the mapping T defined on $\mathcal{D}(T) \subseteq X$ with a value in $2^Y \backslash \emptyset = \mathcal{P}(Y) \backslash \emptyset$. $\mathcal{D}(T) = \{x \in X : T(x) \neq \emptyset\}$ is called the domain of T. If T maps all the point of $\mathcal{D}(T)$ to singletons, then T is called a single-valued or simply an operator. A relation T is said to be a linear relation, if $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)\ \forall\, x, y \in \mathcal{D}(T)$ and $\alpha, \beta \neq 0$. We denote by $LR(X, Y)$, the class of all linear relation from X to Y. A linear relation $T \in LR(X, Y)$ is entirely defined by its graph, $G(T)$, which is expressed by

$$G(T) = \{(x, y) \in X \times Y : x \in \mathcal{D}(T), y \in Tx\}.$$

The linear relation T^{-1} is the inverse of T defined by

$$G(T^{-1}) = \{(y, x) \in Y \times X : (x, y) \in G(T)\}.$$

Let $M \subset X$, $T \in LR(X, Y)$. We call the range of M by T the set denoted $T(M)$ and defined by

$$T(M) = \bigcup_{m \in M \cap D(T)} T(m).$$

In particular, for $M = X$, $T(X) = R(T)$ is called the range of T. $T \in LR(X, Y)$ is said to be surjective if $R(T) = Y$. Let $T \in LR(X, Y)$, $\emptyset \neq H \subset Y$, we call a reciprocal range of H by T the set $T^{-1}(H)$ defined by

$$\begin{aligned} T^{-1}(H) &= \bigcup \{T^{-1}(y) : y \in \mathcal{D}(T^{-1}) \cap H\} \\ &= \{x \in \mathcal{D}(T) : T(x) \cap H \neq \emptyset\}. \end{aligned}$$

In particular, for $y \in R(T)$, we get $T^{-1}(y) = \{x \in \mathcal{D}(T), y \in Tx\}$

$$D(T^{-1}) = R(T), \ \ R(T^{-1}) = \mathcal{D}(T).$$

We call the kernel of T the subset of X indicated by

$$N(T) = \{x \in X : \ 0 \in Tx\} = T^{-1}(0).$$

If $N(T) = 0$, that is, T^{-1} is uni-value, we say that T is an injective relation. The identity relation defined on the subset E of X is denoted by I_E or simply I. It is represented in terms of

$$G(I_E) = \{(e, e) : e \in E\}.$$

Let $S, T \in LR(X, Y)$, $\lambda \in \mathbb{K}^*$. The relation $S + T$ is defined by

$$\forall \ x \in \mathcal{D}(S + T) \ \ (S + T)x = Sx + Tx.$$

$$\mathcal{D}(S + T) = \mathcal{D}(S) \cap \mathcal{D}(T).$$

$$G(S + T) = \{(x, y), x \in \mathcal{D}(S) \cap \mathcal{D}(T) : y = y_1 + y_2 : (x, y_1) \in G(S), (x, y_2) \in G(T)\}.$$

We define the relation λT by

$$\forall \ x \in \mathcal{D}(\lambda T) \ \ (\lambda T)x = \lambda Tx.$$

$$\mathcal{D}(\lambda T) = \mathcal{D}(T).$$

$$G(\lambda T) = \{(x, \lambda y) : (x, y) \in G(T)\}.$$

For $T \in LR(X, Y)$ and $S \in LR(Y, Z)$ where $R(T) \cap \mathcal{D}(S) \neq \emptyset$, the linear relation ST is the product of S and T defined by

$$ST(x) = S(Tx) \ (x \in X).$$

$$\begin{aligned} \mathcal{D}(ST) &= \{x \in X : S(Tx) \neq \emptyset\} \\ &= \{x \in X : Tx \cap \mathcal{D}(S) \neq \emptyset\} \\ &= T^{-1}(\mathcal{D}(S)). \end{aligned}$$

$$G(ST) = \{(x,z) \in X \times Z : \exists\, y \in Y : (x,y) \in G(T), (y,z) \in G(S)\}.$$

Let M be a subset of X such that $M \cap \mathcal{D}(T) \neq \emptyset$. The restriction of T to M denoted $T|_M$ is the relation in $LR(X,Y)$ defined by:

$$\begin{cases} T|_M x = Tx, \ \ x \in M \cap \mathcal{D}(T), \\ \mathcal{D}(T|_M) = \mathcal{D}(T) \cap M, \\ G(T|_M) = G(T) \cap (M \times Y) = \{(x,y) \in G(T) : \ x \in M\}. \end{cases}$$

We can easily infer that $T|_M = T|_{M \cap \mathcal{D}(T)}$.
For a given closed linear subspace E of X, let Q_E^X (or simply, Q_E) denote the natural quotient map with domain X and null space E. We shall denote $Q^Y_{\overline{T(0)}}$ by Q_T, or simply Q when T is understood. We define

$$\begin{cases} \|Tx\| := \|Q_T Tx\| \ \ (x \in \mathcal{D}(T)) \\ \|T\| := \|Q_T T\|. \end{cases}$$

Let A and B be nonempty subsets of a normed space. The distance between A and B is defined by the formula

$$d(A,B) = \inf\{\|y - z\| : y \in A, z \in B\}.$$

If $A = \{a\}$, then $d(a,B) = \inf\{\|a - z\| : z \in B\}$. We define the minimum modulus of T by

$$\gamma(T) = \inf\left\{\frac{\|Tx\|}{d(x, N(T))} : x \in \mathcal{D}(T), x \notin N(T)\right\}.$$

Conventionally, if $\mathcal{D}(T) \subset \overline{N(T)}$, then we get $\gamma(T) = +\infty$. If $\|T\| < \infty$, T is called continuous and if $\gamma(T) > 0$, T is said to be open. If $\mathcal{D}(T) = X$, $\|T\| < \infty$, then we said that T is bounded. We denote the class of bounded linear relations from X to Y by $BR(X,Y)$. The linear relation $\overline{T}$ is the closure of a linear relation T defined by

$$G(\overline{T}) = \overline{G(T)}.$$

We said that T is closed if its graph $G(T)$ is closed in $X \times Y$, or equivalently, if $T = \overline{T}$. We denote by $CR(X,Y)$ the class of closed linear relations from X to Y. $T \in LR(X,Y)$ is said to be compact if $\overline{Q_T B_X}$ is compact, where $B_X := \{x \in X : \|x\| < 1\}$. We denote by X', the norm dual of a normed linear space X, i.e., the space of all continuous functional x' expressed on X, with norm

$$\|x'\| = \inf\{\lambda : |x'x| \leq \lambda \|x\| \ \forall\, x \in X\}.$$

The linear relation T is invertible if T^{-1} is a bounded operator. We call the resolvent set of T the set $\rho(T)$ defined by:

$$\rho(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is invertible}\}.$$

The complement of $\rho(T)$ is called spectrum of T and is denoted
$\sigma(T) = \mathbb{C} \backslash \rho(T)$. A scalar λ such that $N(\lambda - T) \neq 0$ is called an eigenvalue of T.
Let λ be an eigenvalue of T. Then the non zero subspace $N(\lambda - T)$ is called the eigenspace of T corresponding to λ. The non zero vectors in $N(\lambda - T)$ are called eigenvectors. Clearly, if λ is an eigenvalue of T, the $\lambda \in \sigma(T)$. The set $\sigma(T)$ is decomposed into following three adjoint sets:

$$P\sigma(T) := \{\lambda \in \mathbb{C} : \text{consisting of the eigenvalues of } T\}.$$

$$R\sigma(T) := \{\lambda \in \mathbb{C} : \lambda - T \text{ is injective and } \overline{R(\lambda - T)} \neq X\}.$$

$$C\sigma(T) := \{\lambda \in \mathbb{C} : \lambda - T \text{ is injective and } \overline{R(\lambda - T)} = X \text{ but is not open}\}.$$

The subsets $P\sigma(.)$ and $R\sigma(.)$ are the point and residual spectrum, and $C\sigma(.)$ denotes the continuous spectrum.

Let $T \in CR(X,Y)$. The graph norm $\|.\|_T$ of $x \in \mathcal{D}(T)$, is indicated by $\|x\|_T = \|x\| + \|Tx\|$. We have $X_T = (\mathcal{D}(T), \|.\|_T)$ is a Banach space.

Let $\widetilde{X}$ denote the completion of the normed space X and let $\widetilde{T}$ denote the linear relation in $LR(\widetilde{X},\widetilde{Y})$ whose graph is the completion of $G(T)$. Therefore we call $\widetilde{T}$ the completion (or complete closure) of T. Let $T \in LR(X,Y)$. We define the nullity of T by $\alpha(T) := \dim N(T)$, the deficiency of T by $\beta(T) := \dim Y/R(T)$, and the index of T by the quantity $i(T) := \alpha(T) - \beta(T)$ provided that $\alpha(T)$ and $\beta(T)$ are not both infinite.

$T \in LR(X,Y)$ is upper semi-Fredholm, if and only if there exists a closed, finite, codimensional subspace M of X, such that the restriction $T|M$ is injective and open. $T \in LR(X,Y)$ is said to be a lower semi-Fredholm linear relation if its conjugate T' is an upper semi-Fredholm linear relation. We denote the set of upper semi-Fredholm linear relations by $F_+(X,Y)$, which we abbreviate as F_+, and the set of lower semi-Fredholm linear relations by $F_-(X,Y)$ (or F_-). In the case when X and Y are Banach spaces, we extend the classes of closed single-valued Fredholm type operators given earlier to include closed multivalued operators. Note that the definitions of the classes $F_+(X,Y)$ and $F_-(X,Y)$ are consistent, respectively with

$$\phi_+(X,Y) := \{T \in CR(X,Y) : \alpha(T) < \infty \text{ and } R(T) \text{ is closed in } Y\}, \quad \textit{and}$$

$$\phi_-(X,Y) := \{T \in CR(X,Y) : \beta(T) < \infty \text{ and } R(T) \text{ is closed in } Y\},$$

$$\phi(X,Y) = \phi_+(X,Y) \cap \phi_-(X,Y).$$

Remark 2.1. *[12] For $T \in LR(X,Y)$,*
(i)$T \in F_+ \Leftrightarrow Q_T \in F_+$.
(ii)$T \in F_- \Leftrightarrow Q_T \in F_-$.

Lemma 2.2. *[12] Let X, Y be two linear spaces and $T \in LR(X,Y)$. Therefore,*
(i) for $x \in \mathcal{D}(T)$, we get $y \in Tx \Longleftrightarrow Tx = y + T(0)$.
In particular, $0 \in Tx \Longleftrightarrow Tx = T(0)$.
(ii) For $x_1, x_2 \in \mathcal{D}(T)$, we have the following equivalence:

$$Tx_1 \cap Tx_2 \neq \emptyset \Longleftrightarrow Tx_1 = Tx_2.$$

Lemma 2.3. *[12] Let $T \in LR(X,Y)$ and $S \in LR(Y,Z)$ where X, Y and Z are linear spaces. Thus,*

$$(ST)^{-1} = T^{-1}S^{-1}.$$

Lemma 2.4. *[12]Let X, Y be two linear spaces and $T \in LR(X,Y)$. Therefore,*
(i) $T(0) = TT^{-1}(0)$ and $T^{-1}(0) = T^{-1}T(0)$.
(ii) $T^{-1}Tx = x + T^{-1}(0)\ \ \forall\, x \in \mathcal{D}(T)$.
(iii) $TT^{-1}y = y + T(0)\ \ \forall\, y \in R(T)$.

Proposition 2.5. *[12] We have*
$N(T) \subset N(Q_T)$ with equality if $T(0)$ is relatively closed in $R(T)$.
$\gamma(T) \leq \gamma(Q_T)$ with equality if $T(0)$ is relatively closed in $R(T)$.
If T is open and $N(T)$ is closed, then $N(T) = N(Q_T)$ and $\gamma(T) = \gamma(Q_T)$.

Proposition 2.6. *[12] let $T \in LR(X,Y)$. Then,*
(i) $Q_T T$ is single-valued.
(ii) $\|Tx\| = d(y, T(0))$ for any $x \in \mathcal{D}(T), y \in Tx$.
(iii) $\|Tx\| = d(Tx, T(0)) = d(Tx, 0)\ \ (x \in \mathcal{D}(T))$.
(iv) $\|T\| = \sup\limits_{x \in B_X \cap \mathcal{D}(T)} \|Tx\|$.
(v) $\gamma(T) = \|T^{-1}\|^{-1}$.

Definition 2.7. *Let X be a Banach space and $M \subset X$ a bounded subset. The (Hausdorff) measure of noncompactness of M is defined by*

$$o(M) = inf\{\varepsilon : \varepsilon > 0, M has\ a\ finite\ \varepsilon - net\ in X\},$$

where by a finite ε-net for M we mean, as usual, a finite set $\{z_1, ..., z_m\} \subset X$ with the property that

$$M \subset \{z_1 + B_\varepsilon(X)\} \cup ... \cup \{z_m + B_\varepsilon(X)\}.$$

Here and throughout the following we use the notation

$$B_r(X) := \{x \in X : \|x\| \leq r\}$$

for the closed ball in X, and

$$S_r(X) := \{x \in X : \|x\| \leq r\}$$

for the corresponding sphere. In case $r = 1$ we simply write $B_1(X) =: B(X)$ and $S_1(X) =: S(X)$.

In the following Proposition we recall some properties of the (Hausdorff) measure of noncompactness.

Proposition 2.8. *[15] The measure of noncompactness has the following properties* $(M, N \subset X, \lambda \in \mathbb{K}, z \in X)$:

(a) $o(M) = 0$ *if and only if M is relatively compact, i.e. has compact closure.*

(b) $|o(M) - o(N)| \leq o(M + N) \leq o(M) + o(N)$.

(c) $o(\lambda M) = |\lambda| o(M)$.

(d) $o(M + \{z\}) = o(M)$.

(e) $o(\overline{co}(M)) = o(M)$, *where $\overline{co}(M)$ is the convex closure of M.*

(f) $o(M \cup N) = max\{o(M), o(N)\}$.

(g) $o(B_r(X)) = o(S_r(X)) = 0$ *if* $\dim X < \infty$ *and* $= r$ *if* $\dim X = \infty$.

(h) *If $M_1 \supseteq M_2 \supseteq ... \supseteq M_n \supseteq ...$ is a decreasing sequence of closed sets in X with $o(M_n) \to 0$ as $n \to \infty$, then $M_\infty := \bigcap_{n=1}^{\infty} M_n$ is nonempty and compact.*

3. Characteristic and D-Characteristic

3.1. Characteristic

Definition 3.1. *Let $T \in LR(X, Y)$ where X, Y are Banach space. Let M a bounded subset of $\mathcal{D}(T)$, $Q_T T(M)$ a bounded subset of $Y/\overline{T(0)}$. We define the characteristic by:*

$$[T]_A := \sup\{k : k > 0, o(Q_T T(M)) \leq k o(M)\}.$$

Remark 3.2. i) *In particular, in case $[T]_A < 1$ the relation T is called o-contractive. Intuitively speaking, the condition $[T]_A < 1$ means that the image $Q_T T(M)$ of any bounded set $M \subset X$ is more compact than M itself.*
ii) *We observe that this definition is equivalent to:*

$$[T]_A = \sup_{o(M)>0} \frac{o(Q_T T(M))}{o(M)}$$

if the space X is infinite dimensional. In finite dimensional spaces this does not make sense, since all bounded sets are precompact, and so there are no sets M satisfying $0 < o(M) < \infty$.

Proposition 3.3. *Let $T, L \in LR(X, Y)$ and $\lambda \in \mathbb{K}$. The characteristics have the following properties:*
(i) $[T]_A = 0$ *if and only if* T *is compact.*
(ii) $[T + L]_A \leq [T]_A + [L]_A$.
(iii) $[\lambda T]_A = |\lambda|[T]_A$.
(iv) $[T]_A \leq \|T\|$.

Proof. The properties (i), (ii) and (iii) are immediate consequences of the definitions 3.1 and the properties of the measure of noncompactness proved in Proposition 2.8.
(iv) From the definition 2.7 of the measure of noncompactness it follows then that $o(Q_T T(M)) \leq ko(M)$ for any bounded subset $M \subset X$, where at least $k = \|Q_T T\|$. In fact, if $\{z_1, ..., z_m\}$ is a finite ε-net for M, then obviously $\{Q_T T(z_1), ..., Q_T T(z_m)\}$ is a finite $\|Q_T T\|\varepsilon$-net for $Q_T T(M)$. We have $\|T\| = \|Q_T T\|$, then $o(Q_T T(M)) \leq \|T\| o(M)$. Thus $[T]_A \leq \|T\|$. □

Proposition 3.4. *Let $T \in LR(X)$ with $[T]_A < 1$, the following is true:*
(i) *For any $\varepsilon > 0$, the set $\{\lambda \in P\sigma(T) : |\lambda| \geq [T]_A + \varepsilon\}$ is finite.*
(ii) *If λ with $|\lambda| > [T]_A$ is not an eigenvalue of T, then $\lambda I - T : X \mapsto X$ is an isomorphism.*
(iii) *Every point $\lambda \in \sigma(T)$ with $|\lambda| > [T]_A$ is an eigenvalue of T.*

Proof. To prove (i), suppose that there exists a sequence $(\lambda_n)_n$ of distinct eigenvalues of T with $|\lambda_n| \geq [T]_A + \varepsilon$, and let $(x_n)_n$ be a corresponding sequence of eigenvectors. Since T is o-contractive, we may find $k \in \mathbb{N}$ such that ${[T]_A}^k < \frac{1}{2}$. Since all eigenvalues are distinct, the sequence of spaces X_n spanned by $\{x_1, ..., x_n\}$ is strictly increasing. By the well-known Riesz lemma we may find a sequence $(e_n)_n$ in $S(X_n)$ such that $\|x_n - e_n\| \leq \frac{1}{2}$ for all $x_n \in X_{n-1}$. From the fact that en lies in the linear hull of $\{x_1, ..., x_n\}$ it follows that

$$Q_T T^k(e_n) - {\lambda_n}^k e_n \in X_{n-1}$$

and

$$z_{n,m} := e_n - \frac{Q_T T^k(e_n)}{{\lambda_n}^k} + \frac{Q_T T^k(e_m)}{{\lambda_m}^k} \in X_{n-1}$$

for $n > m$. Consequently,

$$\|Q_T T^k(\frac{e_n}{{\lambda_n}^k}) - Q_T T^k(\frac{e_m}{{\lambda_m}^k})\| = \|e_n - z_{n,m}\| \geq \frac{1}{2} \ \ (n > m).$$

Now, since the set $M := \{{\lambda_1}^k e_1, {\lambda_2}^k e_2, {\lambda_3}^k e_3, ...\}$ is included in the closed ball $B_r(X)$ of radius $r = ([T]_A + \varepsilon)^{-1}$, we conclude that

$$o(Q_T T^k(M)) \leq [T]_A o(M) < \frac{1}{2}.$$

On the other hand, we shows that $o(Q_T T^k(M)) \geq \frac{1}{2}$, a contradiction. To prove (ii) we show first that the range $R(\lambda I - T)$ of $\lambda I - T$ is closed in X. Let $(y_n)_n$ be a sequence in $R(Q_T(\lambda I - T))$ which converges to some $y \in Y/T(0)$, and choose $x_n \in X$ with $\lambda x_n - Q_T T(x_n) = y_n$. We claim that the sequence $(x_n)_n$ is bounded. In fact, suppose that $\|x_n\| \to \infty$ and put $e_n := \frac{x_n}{\|x_n\|}$. Then $e_n \in S(X)$ and $\lambda e_n - Q_T T(e_n) = \frac{y_n}{\|x_n\|}$, hence

$$\|\lambda e_n - Q_T T(e_n)\| = \frac{\|y_n\|}{\|x_n\|} \to 0 (n \to \infty).$$

Moreover,

$$\begin{aligned} o(\{e_1, e_2, ...\}) &\leq \frac{[T]_A}{|\lambda|} o(\{e_1, e_2, ...\}) + \frac{1}{|\lambda|} o(\{y_1, y_2, ...\}) \\ &= \frac{[T]_A}{|\lambda|} o(\{e_1, e_2, ...\}), \end{aligned}$$

which implies that $o(\{e_1, e_2, e_3, ...\}) = 0$, by our assumption $|\lambda| > [T]_A$. So $(e_n)_n$ admits a convergent subsequence, say $e_{n_k} \to e \in S(X)$ as $k \to \infty$. The continuity of T implies that $\lambda e - Q_T T e = 0$, contradicting the fact that $\lambda \notin P\sigma(L)$. So we have proved that the sequence $(x_n)_n$ is bounded. Repeating the same reasoning as before for $(x_n)_n$ instead of $(e_n)_n$ we obtain a convergent subsequence $x_{n_k} \to x$ as $k \to \infty$, and again by continuity we get $y = \lambda x - Q_T T x \in R(Q_T(\lambda I - T))$. The boundedness of the resolvent operator $R(\lambda; Q_T T)$ on $R(Q_T(\lambda I - T))$ follows as usual from the closed graph theorem. So $R(Q_T(\lambda I - T))$ is closed. Thus $R(\lambda I - T)$ is closed. Now we show that $\lambda I - T$ is onto, so actually $R(\lambda I - T) = X$. Since $\lambda \notin \sigma(T)$, so $(\lambda I - T)^{-1}$ is a bounded operator. Then, $\lambda I - T$ is injective and surjective. The assertion (*iii*) is only a reformulation of (*ii*), and so the proof is complete.
□

3.2. D-Characteristic

Definition 3.5. *Let $T \in LR(X, Y)$ where X, Y are Banach space. Let M a bounded subset of $\mathcal{D}(T)$, $Q_T T(M)$ a bounded subset of $Y/\overline{T(0)}$ and D is a closed subspace of $N(T)$. We define the D-characteristic by:*

$$[T]_{A_D} := \sup\{k : k > 0, o(Q_T T(M)) \leq k o(Q_D(M))\}.$$

Remark 3.6. *i) In particular, in case $[T]_{A_D} < 1$ the relation T is called o-D-contractive (D-condensing see [10]).*
ii) We observe that this definition is equivalent to:

$$[T]_A = \sup_{o(Q_D(M))>0} \frac{o(Q_T T(M))}{o(Q_D(M))}$$

if the space X is infinite dimensional. In finite dimensional spaces this does not make sense, since all bounded sets are precompact, and so there are no sets $Q_D(M)$ satisfying $0 < o(Q_D(M)) < \infty$.
iii) The D-characteristic $[T]_{A_D}$ satisfy the properties (i), (ii) and (iii) in Proposition 3.3.

4. Lower characteristic

Definition 4.1. *Let $T \in LR(X, Y)$ where X, Y are Banach space. Let M a bounded subset of $\mathcal{D}(T)$, $Q_T T(M)$ a bounded subset of $Y/\overline{T(0)}$. We define the lower characteristic by:*

$$[T]_a := \sup\{k : k > 0, o(Q_T T(M)) \geq k o(M)\}.$$

Remark 4.2. *We observe that this definition is equivalent to:*

$$[T]_a = \inf_{o(M)>0} \frac{o(Q_T T(M))}{o(M)}$$

if the space X is infinite dimensional. In finite dimensional spaces this does not make sense, since all bounded sets are precompact, and so there are no sets M satisfying $0 < o(M) < \infty$.

Proposition 4.3. *Let $T \in LR(X, Y)$. $[T]_a > 0$ if and only if $T \in \phi_+(X, Y)$.*

Proof. Suppose that $[T]_a > 0$, and fix $k \in]0, [T]_a[$. Since the set
$M := N(Q_T T) \cap B(X)$ is mapped into $Q_T T(M) = \{0\}$, we get

$$o(M) \leq \frac{1}{k} o(Q_T T(M)) = 0,$$

which shows that M is precompact, and hence $N(Q_T T)$ is finite dimensional. Thus, $N(T)$ is finite dimensional. We prove now that the range $R(T)$ of T is closed. Since $dim N(Q_T T) < \infty$, there exists a closed subspace $X_0 \subset X$ such that $X = X_0 \oplus N(Q_T T)$. Let $(y_n)_n$ be a sequence in $R(Q_T T)$ converging to some $y_* \in Y$, and choose

$(x_n)_n$ in X with $Q_T T(x_n) = y_n$. Now we distinguish two cases. First, suppose that $(x_n)_n$ is bounded. With $k > 0$ as before we get then

$$o(x_1, x_2, x_3, ...) \leq \frac{1}{k}\, o(y_1, y_2, y_3, ...) = 0,$$

and hence $x_{n_k} \to x_*$ for some subsequence $(x_{n_k})_k$ of $(x_n)_n$ and suitable $x_* \in X$. By continuity we see that $Q_T T(x_*) = y_*$, and so $y_* \in R(Q_T T)$. On the other hand, suppose that $\|x_n\| \to \infty$. Set $e_n := \frac{x_n}{\|x_n\|}$ and $E := \{e_1, e_2, e_3, ...\}$. Then clearly $E \subset S(X)$ and

$$Q_T T(e_n) = \frac{Q_T T(x_n)}{\|x_n\|} = \frac{y_n}{\|x_n\|} \to 0 \ \ (n \to \infty),$$

hence $o(Q_T T(E)) = 0$. On the other hand, $o(Q_T T(E)) \geq k o(E)$, by definition, and thus $o(E) = 0$. Without loss of generality we may assume that the sequence $(e_n)_n$ converges to some element $e \in S(X_0)$. So $Q_T T(e) = 0$, contradicting the fact that $X_0 \cap N(Q_T T) = \{0\}$. So $R(Q_T T)$ is closed and hence $R(T)$ is closed. Now we prove that the closedness of $R(T)$ and the fact that $N(T)$ is finite dimensional imply that $[T]_a > 0$. Since $dim N(T) < \infty$ we may find a closed subspace X_0 of X with $X = X_0 \oplus N(T)$. The projection $P : X \longrightarrow X_0$ satisfies $[P]_a = 1$, since $I - P$ is compact. Consider the canonical isomorphism $L : X_0 \longrightarrow R(T)$. Since $T = LP$ and $[L]_a > 0$, we conclude that also $[T]_a \geq [L]_a [P]_a > 0$. □

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